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Ann. I. H. Poincaré – AN 21 (2004) 795–805

ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE

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Symmetry properties for the extremals of the Sobolev trace embedding ☆

Propriétés de symétrie des extrémales de l'immersion de traces de Sobolev

Julian Fernández Bonder [a], Enrique Lami Dozo [b,*], Julio D. Rossi [a]

[a] *Departamento de Matemática, FCEyN UBA (1428) Buenos Aires, Argentina*
[b] *CONICET-Univ. de Buenos Aires and Univ. Libre de Bruxelles, Argentina*

Received 14 August 2002; received in revised form 7 April 2003; accepted 22 September 2003

Available online 8 June 2004

Abstract

In this article we study symmetry properties of the extremals for the Sobolev trace embedding $H^1(B(0,\mu)) \hookrightarrow L^q(\partial B(0,\mu))$ with $1 \leqslant q \leqslant 2(N-1)/(N-2)$ for different values of μ. These extremals u are solutions of the problem

$$\begin{cases} \Delta u = u & \text{in } B(0,\mu), \\ \frac{\partial u}{\partial \eta} = \lambda |u|^{q-2} u & \text{on } \partial B(0,\mu). \end{cases}$$

We find that, for $1 \leqslant q < 2(N-1)/(N-2)$, there exists a unique normalized extremal u, which is positive and has to be radial, for μ small enough. For the critical case, $q = 2(N-1)/(N-2)$, as a consequence of the symmetry properties for small balls, we conclude the existence of radial extremals. Finally, for $1 < q \leqslant 2$, we show that a radial extremal exists for every ball.

Résumé

Dans cet article nous étudions des propriétés de symétrie des extrémales de l'immersion de Sobolev $H^1(B(0,\mu)) \hookrightarrow L^q(\partial B(0,\mu))$, où $1 \leqslant q \leqslant 2(N-1)/(N-2)$ en fonction du rayon μ. Ces extrémales sont solutions du problème

$$\begin{cases} \Delta u = u & \text{dans } B(0,\mu), \\ \frac{\partial u}{\partial \eta} = \lambda |u|^{q-2} u & \text{sur } \partial B(0,\mu). \end{cases}$$

Nous trouvons que, pour $1 \leqslant q < 2(N-1)/(N-2)$, il existe une extrémale normalisée unique u, qui est positive et radiale, pour μ suffisamment petite. Dans le cas critique $q = 2(N-1)/(N-2)$, comme conséquence des propriétés de symétrie pour des petits rayons, nous déduisons l'existence d'extrémales. Finalement, pour $1 < q \leqslant 2$, nous montrons qu'une extrémale radiale existe pour toute boule.

☆ Supported by CONICET and by ANPCyT PICT No. 05009.
* Corresponding author.
E-mail addresses: jfbonder@dm.uba.ar (J. Fernández Bonder), elami@dm.uba.ar (E. Lami Dozo), jrossi@dm.uba.ar (J.D. Rossi).

0294-1449/$ – see front matter
doi:10.1016/j.anihpc.2003.09.005

MSC: 35J65; 35B33

Keywords: Nonlinear boundary conditions; Sobolev trace embedding

1. Introduction

The aim of this article is to study of the following problem: Given a ball of radius μ, $B(0,\mu)$, in $\mathbb{R}^N$, $N \geqslant 3$, decide whether or not there exists a radial extremal for the embedding

$$H^1\big(B(0,\mu)\big) \hookrightarrow L^q\big(\partial B(0,\mu)\big).$$

First, let us introduce our motivation. Let $\Omega \subset \mathbb{R}^N$ be a bounded smooth domain. Relevant for the study of boundary value problems for differential operators are the two following Sobolev inequalities. For each $1 \leqslant q \leqslant 2(N-1)/(N-2) \equiv 2_*$, we have a continuous inclusion $H^1(\Omega) \hookrightarrow L^q(\partial\Omega)$, and for each $1 \leqslant p \leqslant 2N/(N-2) \equiv 2^*$, $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$, hence the following inequalities hold:

$$S\|u\|^2_{L^q(\partial\Omega)} \leqslant \|u\|^2_{H^1(\Omega)},$$
$$\bar{S}\|u\|^2_{L^p(\Omega)} \leqslant \|u\|^2_{H_0^1(\Omega)}.$$

These inequalities are known as the Sobolev trace theorem and the Sobolev embedding theorem respectively. The best constants for these embeddings are the largest S and $\bar{S}$ such that the above inequalities hold, that is,

$$S = \inf_{v \in H^1(\Omega)\setminus H_0^1(\Omega)} \frac{\int_\Omega |\nabla v|^2 + |v|^2\,dx}{(\int_{\partial\Omega} |v|^q\,d\sigma)^{2/q}} \tag{1.1}$$

and

$$\bar{S} = \inf_{v \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla v|^2\,dx}{(\int_\Omega |v|^p\,dx)^{2/p}}. \tag{1.2}$$

Along this paper, we denote by dx $(d\sigma)$ the N dimensional ($N-1$ dimensional) Hausdorff measure.

The main difference between these two quantities, is the fact that $\bar{S}$ is homogeneous under dilatations of the domain, that is, if we define $\mu\Omega = \{\mu x \mid x \in \Omega\}$, taking $v(x) = u(\mu x)$ in 1.2 and changing variables we get

$$\bar{S}(\mu\Omega) = \mu^{(pN-2p-2N)/p}\bar{S}(\Omega).$$

On the other hand, S is not homogeneous under dilatations. In fact we have

$$S_\mu \equiv S(\mu\Omega) = \mu^\beta \inf_{v \in H^1(\Omega)\setminus H_0^1(\Omega)} \frac{\int_\Omega \mu^{-2}|\nabla v|^2 + |v|^2\,dx}{(\int_{\partial\Omega} |v|^q\,d\sigma)^{2/q}}, \tag{1.3}$$

where $\beta = (Nq - 2N + 2)/q$.

For $1 \leqslant q < 2_*$ and $1 \leqslant p < 2^*$ the embeddings are compact, so we have existence of extremals, i.e. functions where the infimum is attained. These extremals are weak solutions of the following problems

$$\begin{cases} \Delta u = u & \text{in } \Omega, \\ \frac{\partial u}{\partial \eta} = \lambda |u|^{q-2}u & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where $\frac{\partial}{\partial \eta}$ is the outer unit normal derivative, and

$$\begin{cases} -\Delta u = \lambda |u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.5}$$

The asymptotic behavior of $S(\mu\Omega)$ in expanding ($\mu \to \infty$) and contracting domains ($\mu \to 0$), was studied in [4] and [6]. In [4] it is proved that for expanding domains and $q > 2$, $S(\mu\Omega) \to S(\mathbb{R}^N_+)$. In [6] it is shown that

$$\lim_{\mu\to 0+} \frac{S_\mu}{\mu^\beta} = \frac{|\Omega|}{|\partial\Omega|^{2/q}}. \tag{1.6}$$

The behavior of the extremals for (1.1) in expanding and contracting domains is also studied in [4] and [6]. For expanding domains, it is proved in [4] that the extremals develop a peak near a point where the mean curvature of the boundary is a maximum. For contracting domains, we have that the extremals, when rescaled to the original domain as $v(x) = u(\mu x)$, $x \in \Omega$, and normalized with $\|v\|_{L^q(\partial\Omega))} = 1$, are nearly constant in the sense that

$$\lim_{\mu\to 0} v = \frac{1}{|\partial\Omega|^{1/q}} \quad \text{in } H^1(\Omega).$$

Another big difference between the Sobolev trace theorem and the Sobolev embedding theorem arises in the behavior of extremals. Namely, if Ω is a ball, $\Omega = B(0,\mu)$, as the extremals do not change sign, from results of [7] the extremals for (1.2) are radial while, if q exceeds 2 and μ is large, extremals for (1.1) are not, since they develop a peaking concentration phenomena as is described in [4].

The above discussion leads naturally to the purpose of this article: the study of the symmetry properties for the extremals of the Sobolev trace embedding in small balls. We find that the symmetry properties of the extremals depend on the size of the ball. Our main result describes when there exists a radial extremal.

Theorem 1.1. *Let $2_* = 2(N-1)/(N-2)$ be the critical exponent for the Sobolev trace immersion. Concerning symmetry properties of the extremals for the embedding*

$$H^1\big(B(0,\mu)\big) \hookrightarrow L^q\big(\partial B(0,\mu)\big)$$

there holds,

(1) *Let $1 < q \leqslant 2$. For every $\mu > 0$ there exists a radial extremal.*
(2) *Let $2 < q < 2_*$. There exists $\mu_0 > 0$ such that for every $\mu < \mu_0$ there is a unique positive extremal, u, normalized such that $\|u(\mu x)\|_{L^q(\partial B(0,1))} = 1$, moreover this extremal is a radial function. However, for large values of μ there is no radial extremal.*
(3) *Let $q = 2_*$. There exists $\mu_0 > 0$ such that for every $\mu < \mu_0$ there is a positive radial extremal.*

The main ingredient of the proof of the symmetry result for small balls is the implicit function theorem. We remark that the moving planes technique cannot be applied to obtain symmetry results in this case, as the extremals for large μ are not radial.

For the critical exponent $2_* = 2(N-1)/(N-2)$, we prove existence of extremals, which turns out to be radial functions, for small balls. We remark that the existence of extremals for the critical exponent is not trivial, this is due to the lack of compactness. This result has to be compared with the case of the immersion $H^1_0(B(0,\mu)) \to L^{2^*}(B(0,\mu))$ where it is well known that, by Pohozaev identity, there is no positive solution of (1.5) regardless the size of the ball for the critical exponent $2^* = 2N/(N-2)$. However, there exist solutions for topologically nontrivial domains.

For the existence of extremals in this critical case for general domains Ω, see [3,5].

The rest of the paper is organized as follows: in Section 2 we prove our symmetry result for small balls and subcritical q. In Section 3 we find a bound on the critical radius below which there exists radial extremals. In Section 4 we deal with the existence of radial extremals with critical exponent and finally in Section 5 we prove that there exists a radial extremal for every ball if $1 < q \leqslant 2$.

2. Symmetry for small balls and subcritical q

In this section we use the implicit function theorem to show that there exists a unique minimizer for μ small.

As observed in the introduction, making the change of variables $v(x) = u(\mu x)$, we get

$$S_\mu = \mu^\beta \inf_{v\in H^1(B(0,1))\setminus H_0^1(B(0,1))} \frac{\int_{B(0,1)} \mu^{-2}|\nabla v|^2 + |v|^2\,dx}{(\int_{\partial B(0,1)} |v|^q\,d\sigma)^{2/q}},$$

where $\beta = (qN - 2N + 2)/q$. As $q < 2_*$ the extremals exist and are solutions of

$$\begin{cases} \Delta v = \mu^2 v & \text{in B(0, 1)},\\ \frac{\partial v}{\partial \eta} = \mu^2 \frac{S_\mu}{\mu^\beta}|v|^{q-2}v & \text{on } \partial B(0,1). \end{cases} \tag{2.1}$$

We denote

$$\mathcal{S} = \left\{ v \in H^1\big(B(0,1)\big);\ \int_{\partial B(0,1)} |v|^q\,d\sigma = 1\right\}.$$

Let us consider the functional

$$F : \mathcal{S}\times[0,1] \to \big(H^1\big(B(0,1)\big)\big)^*$$

given by

$$F(v,\mu)(\phi) = \int_{B(0,1)} \nabla v\nabla\phi\,dx + \mu^2\int_{B(0,1)} v\phi\,dx - \mu^2\frac{S_\mu}{\mu^\beta}\int_{\partial B(0,1)} v^{q-1}\phi\,d\sigma.$$

This functional F is continuous and C^1 with respect to v.

We observe that

$$v_0 \equiv \frac{1}{|\partial B(0,1)|^{1/q}} \in \mathcal{S}$$

and satisfies

$$F\left(\frac{1}{|\partial B(0,1)|^{1/q}}, 0\right) = 0.$$

We want to use the implicit function theorem to show that there exists a unique solution, $v = v(\mu)$ to the equation $F(v,\mu) = 0$, defined for small values of μ near $v_0 = 1/|\partial B(0,1)|^{1/q}$.

To this end we state the following lemmas.

Lemma 2.1. *The tangent space to $\mathcal{S}$ at v_0 is given by*

$$T_{v_0}(\mathcal{S}) = \left\{ z \in H^1\big(B(0,1)\big);\ \int_{\partial B(0,1)} z\,d\sigma = 0\right\}.$$

Proof. First let us prove that

$$T_{v_0}(\mathcal{S}) \subset \left\{ z \in H^1\big(B(0,1)\big);\ \int_{\partial B(0,1)} z\,d\sigma = 0\right\}.$$

Each curve $\gamma : (-1,1) \to \mathcal{S}$ with $\gamma(0) = v_0$ satisfies

$$\int_{\partial B(0,1)} \big|\gamma(t)\big|^q\,d\sigma = 1.$$

Differentiating at $t = 0$ we get

$$q \int_{\partial B(0,1)} \gamma(0)^{q-1} \gamma'(0)\, d\sigma = q v_0^{q-1} \int_{\partial B(0,1)} \gamma'(0)\, d\sigma = 0.$$

Now let us prove the reverse inclusion

$$T_{v_0}(\mathcal{S}) \supset \left\{ z \in H^1\bigl(B(0,1)\bigr);\ \int_{\partial B(0,1)} z\, d\sigma = 0 \right\}.$$

Let z be such that

$$\int_{\partial B(0,1)} z\, d\sigma = 0,$$

and consider the following curve

$$\gamma(t) = \frac{v_0 + tz}{(\int_{\partial B(0,1)} |v_0 + tz|^q\, d\sigma)^{1/q}}.$$

This curve verifies $\gamma(t) \in \mathcal{S}$, $\gamma(0) = v_0$, $\gamma'(0) = z$. This ends the proof. $\square$

Lemma 2.2. *Let*

$$A = \bigl\{ \varphi \in \bigl(H^1\bigl(B(0,1)\bigr)\bigr)^* : \langle \varphi, 1 \rangle = 0 \bigr\}.$$

The derivative of F with respect to v at the point $(v_0, 0)$ is given by

$$\frac{\partial F}{\partial v}(v_0, 0) : T_{v_0}(\mathcal{S}) \to A,$$
$$\frac{\partial F}{\partial v}(v_0, 0)(w)(\phi) = \int_{B(0,1)} \nabla w \nabla \phi\, dx.$$

Proof. The result follows directly from the fact that

$$F(v, 0)(\phi) = \int_{B(0,1)} \nabla v \nabla \phi\, dx. \qquad \square$$

Next we prove that $\frac{\partial F}{\partial v}(v_0, 0)$ has a continuous inverse.

Lemma 2.3. *Given $\varphi \in A$ there exists a unique $w \in T_{v_0}(\mathcal{S})$ such that*

$$\int_{B(0,1)} \nabla w \nabla \phi\, dx = \langle \varphi, \phi \rangle, \quad \forall \phi \in H^1\bigl(B(0,1)\bigr).$$

Moreover the map $\varphi \mapsto w$ is continuous.

Proof. Observe that $T_{v_0}(\mathcal{S})$ is a Hilbert space with the inner product given by

$$(u, v) = \int_{B(0,1)} \nabla u \nabla v\, dx.$$

Then the lemma follows from the Riesz representation theorem. □

Theorem 2.1. *Suppose that* $1 \leqslant q < 2_*$. *There exists* $\mu_0 > 0$ *such that for every* $\mu < \mu_0$ *there exists a unique positive extremal,* u, *for the embedding* $H^1(B(0,\mu)) \hookrightarrow L^q(\partial B(0,\mu))$ *normalized such that* $\|u(\mu x)\|_{L^q(\partial B(0,1))} = 1$. *Moreover this extremal is a radial function.*

Proof. From the previous lemmas we get that F verifies the hypothesis of the implicit function theorem, see for example [1]. Hence there exists μ_0 such that for every $\mu \in [0,\mu_0]$ there exists a unique solution $v = v(\mu)$ of

$$F(v,\mu) = 0$$

with v near v_0. Therefore there exists a unique weak solution of (2.1) near v_0 for small values of μ. By the results of [6] the extremals are weak solutions of (2.1) that converges, as μ goes to zero, to v_0 in $H^1(B(0,1))$, the uniqueness of the extremal follows.

Now take an extremal u_1 in $B(0,\mu)$ and let R be any rotation, then $u_2(x) = u_1(Rx)$ is also an extremal. Since there is a unique extremal we must have $u_1 = u_2$ and we conclude that the unique extremal must be a radial function. □

Remark 2.1. This method also works for any domain Ω and hence there exists $\mu_0 > 0$ such that for every $\mu < \mu_0$ there exists a unique positive extremal, u, for the embedding $H^1(\mu\Omega) \hookrightarrow L^q(\partial\mu\Omega)$ normalized such that $\|u(\mu x)\|_{L^q(\partial\Omega)} = 1$.

Remark 2.2. Once we know that the extremals are radial we can obtain some monotonicity properties expanding them as power series. Indeed as radial extremals can be written in terms of Bessel functions, they have an expansion in powers of μ, $v(r) = \sum_{j=0}^{\infty} a_j(r)\mu^{2j}$. As an immediate consequence, we get some monotonicity properties for small values of μ, the normalized extremals are strictly decreasing as functions of μ, that is, if $\mu_1 < \mu_2$ then $v_{\mu_1} > v_{\mu_2}$ in $B(0,1)$. Moreover, S_μ/μ^β is decreasing as a function of μ, that is, if $\mu_1 < \mu_2$ then $S_{\mu_1}/\mu_1^\beta > S_{\mu_2}/\mu_2^\beta$.

Remark 2.3. In the case $N = 2$, Theorem 2.1 holds for $1 < q < \infty$.

3. Estimate for the critical radius

Theorem 2.1 says that for small balls (μ small) the extremals are radial, while the results of [4] say that this is not the case for large balls. Therefore we can define

$$\mu_0 = \sup\bigl\{\mu\colon \text{ there exists } u_\theta \text{ a radial extremal in } B(0,\theta), \forall\theta < \mu\bigr\}. \tag{3.1}$$

This value μ_0 is the critical size where we pass from radial extremals to nonradial ones. Our next result is an estimate on the value of μ_0.

Theorem 3.1. *The critical radius* μ_0 *defined by* (3.1) *verifies*

$$\mu_0 \geqslant \frac{|\partial B(0,1)|^{1/q}}{|B(0,1)|^{1/2}}\sqrt{\frac{\bar{c}_1}{q-1}}, \tag{3.2}$$

where

$$\bar{c}_1 = \inf\left\{\frac{\int_{B(0,1)}|\nabla v|^2\,dx}{\|v\|^2_{L^{2_*}(\partial B(0,1))}}\colon v \in H^1\bigl(B(0,1)\bigr), \int\limits_{\partial B(0,1)} v\,d\sigma = 0\right\} > 0. \tag{3.3}$$

Moreover, the set of parameters (μ, q) such that there is no radial extremal is open.

Proof. From now on we use the notation

$$Q(v,\mu,q)=\frac{\int_{B(0,1)}\mu^{-2}|\nabla v|^2+|v|^2\,dx}{(\int_{\partial B(0,1)}|v|^q\,d\sigma)^{2/q}},$$

for $v\in H^1(B(0,1))\setminus H_0^1(B(0,1))$, $\mu>0$ and $1<q\leqslant 2_*$.

Let us look at the linear part of the problem near any positive solution v of (1.4). The kernel of the linear part are the solutions of

$$\begin{cases}\Delta z=\mu^2 z & \text{in } B(0,1),\\ \frac{\partial z}{\partial\eta}=\mu^2\frac{S_\mu}{\mu^\beta}(q-1)v^{q-2}z & \text{on } \partial B(0,1),\end{cases}\tag{3.4}$$

with

$$\int_{\partial B(0,1)} z\,d\sigma=0.$$

Let us look for a bound on the value $\tilde\mu$ such that (3.4) has a nontrivial solution.

This value $\tilde\mu$ can be estimated as follows, multiply by z (3.4) and integrate by parts to get

$$\int_{B(0,1)}|\nabla z|^2\,dx+\mu^2\int_{B(0,1)}z^2\,dx=\mu^2\frac{S_\mu}{\mu^\beta}(q-1)\int_{\partial B(0,1)}v^{q-2}z^2\,d\sigma.$$

Now, by Hölder's inequality,

$$\int_{B(0,1)}|\nabla z|^2\,dx+\mu^2\int_{B(0,1)}z^2\,dx\leqslant\mu^2\frac{S_\mu}{\mu^\beta}(q-1)\|v\|^{q-2}_{L^q(\partial B(0,1))}\|z\|^2_{L^q(\partial B(0,1))}.$$

We define $c_1=c_1(q)$ as

$$c_1(q)=\inf\left\{\frac{\int_{B(0,1)}|\nabla v|^2\,dx}{\|v\|^2_{L^q(\partial B(0,1))}}:\ v\in H^1\big(B(0,1)\big),\ \int_{\partial B(0,1)}v\,d\sigma=0\right\},$$

and we get

$$c_1(q)\leqslant\mu^2\frac{S_\mu}{\mu^\beta}(q-1).$$

Also we have that, by [6] and Remark 2.2,

$$\frac{S_\mu}{\mu^\beta}\leqslant\frac{|B(0,1)|}{|\partial B(0,1)|^{2/q}}.$$

Then, if

$$c_1(q)>\mu^2\frac{|B(0,1)|}{|\partial B(0,1)|^{2/q}}(q-1),$$

we can not have a nontrivial solution v. Therefore we get that

$$\tilde\mu\geqslant\frac{|\partial B(0,1)|^{1/q}}{|B(0,1)|^{1/2}}\sqrt{\frac{c_1(q)}{q-1}}.$$

We observe that $c_1(q) \geqslant c_1(2_*) = \bar{c}_1$.

To finish the proof of the estimate it remains to show that $\tilde{\mu} \leqslant \mu_0$.

Assume that $\mu_0 < \tilde{\mu}$. Then, there exists a sequence $\mu_n \in \mathbb{R}$ and $v_n \in H^1(B(0,1))$ such that $\mu_n \to \mu_0 < \tilde{\mu}$, v_n is an extremal for S_{μ_n} with $\|v_n\|_{L^q(B(0,1))} = 1$. Then, it follows that $\|v_n\|_{H^1(B(0,1))} \leqslant C$ and so, there exists a subsequence (that we still call v_n) and a function $v \in H^1(B(0,1))$ such that

$$v_n \rightharpoonup v \quad \text{weakly in } H^1\big(B(0,1)\big), \tag{3.5}$$

$$v_n \to v \quad \text{in } L^2\big(B(0,1)\big), \tag{3.6}$$

$$v_n \to v \quad \text{in } L^q\big(\partial B(0,1)\big). \tag{3.7}$$

Let us see that v is an non-radial extremal for S_{μ_0}. By (3.7), $\|v\|_{L^q(\partial B(0,1))} = 1$ so $v \neq 0$.

Now, if v is a radial function, as there exists a unique radial solution of (1.4), (v, μ_0) must be a bifurcation point from the branch that starts with $(v_0, 0)$ which contradicts the definition of $\tilde{\mu}$, then v is not a radial function.

By (3.5)–(3.6), we have

$$Q(v, \mu_0, q) = \int\limits_{B(0,1)} \mu_0^{-2} |\nabla v|^2 + |v|^2 \, dx \leqslant \liminf Q(v_n, \mu_n, q).$$

Also, if there exists a function $\tilde{v} \in H^1(B(0,1))$ with $Q(\tilde{v}, \mu_0, q) < Q(v, \mu_0, q)$, we get a contradiction from the fact that $Q(\tilde{v}, \mu_n, q) < Q(v_n, \mu_n, q)$. Therefore, the limit v must be a non-radial extremal.

By the implicit function theorem, and by our assumption $\tilde{\mu} < \mu_0$ there exists a branch of non-radial solutions, v_μ, of (2.1) that passes through v and verify $\|v_\mu\|_{L^q(\partial B(0,1))} = 1$. Therefore, $\|v_\mu\|_{H^1(B(0,1))}$ is uniformly bounded, hence the branch cannot go to infinity for $\mu < \tilde{\mu}$ and cannot go to zero. The only possibility that remains is that for every $0 < \mu < \mu_0$ there exists a non-radial solution of (2.1), but this is a contradiction with the results of Section 2 that proves that the unique solution of (2.1) must be radial for μ small enough.

To finish the proof of the theorem, it remains to show that the set of parameters where the extremals are non-radial functions is open.

Let us define the set

$$A = \big\{(\mu, q): \text{ there is no radial extremal of (1.1)}\big\}.$$

We denote by $H^1_{rad}(B(0,1))$ the set of radial functions in $H^1(B(0,1))$. Let $(\mu_0, q_0) \in A$. We have that

$$\inf_{v \in H^1(B(0,1))} Q(v, \mu_0, q_0) < \inf_{v \in H^1_{rad}(B(0,1))} Q(v, \mu_0, q_0).$$

Now, if (μ, q) is close to (μ_0, q_0) by continuity of Q, we get that

$$\inf_{v \in H^1(B(0,1))} Q(v, \mu, q) < \inf_{v \in H^1_{rad}(B(0,1))} Q(v, \mu, q).$$

Hence $(\mu, q) \in A$ for every (μ, q) close to (μ_0, q_0) and the result follows. $\square$

4. Extremals for the critical exponent

In this section we focus on the existence of extremals for the critical exponent $2_* = 2(N-1)/(N-2)$.

Theorem 4.1. *For every μ small enough there exists a radial extremal for the immersion*

$$H^1\big(B(0,\mu)\big) \to L^{2_*}\big(\partial B(0,\mu)\big).$$

Proof. From (3.2) we get that there exists μ_1 such that for every $\mu < \mu_1$ the extremals are radial for every $2 < q < 2_*$. In order to prove existence of extremals in the critical case, the main idea is to take the limit as $q \nearrow 2_*$ of a sequence of radial extremals for $H^1(B(0,\mu)) \hookrightarrow L^q(\partial B(0,\mu))$.

Let $\mu < \mu_1$ be fixed, $q_j < 2_*$ be any sequence such that $q_j \nearrow 2_*$ and let v_j a radial extremal defined in $B(0,1)$. These v_j are solutions of

$$\begin{cases} \Delta v_j = v_j & \text{in } B(0,1), \\ v_j = |\partial B(0,1)|^{-1/q_j} & \text{on } \partial B(0,1). \end{cases} \tag{4.1}$$

As the boundary values converges uniformly

$$v_j|_{\partial B(0,1)} = \left|\partial B(0,1)\right|^{-1/q_j} \to \left|\partial B(0,1)\right|^{-1/2_*}$$

we get that the sequence v_j converges strongly in $H^1(B(0,1))$ and uniformly to some function v^*.

We claim that v^* is an extremal for 2_*. In fact, assume that there exists $w \in H^1(B(0,1))$ such that $Q(w,\mu,2_*) < Q(v^*,\mu,2_*)$. We arrive at a contradiction noticing that, $v_j \to v^*$ and $q_j \to 2_*$ imply, by the continuity of Q, $Q(w,\mu,q_j) < Q(v_j,\mu,q_j)$ for j large enough. □

5. Symmetry of extremals for $1 < q \leqslant 2$

Let us see that if there exists a radial extremal for some q_1 then there exists a radial extremal for every $1 < q < q_1$.

Theorem 5.1. *If there exists a radial extremal of the embedding $H^1(B(0,\mu)) \hookrightarrow L^{q_1}(\partial B(0,\mu))$ and $1 < q \leqslant q_1$, then there exists a radial extremal for $H^1(B(0,\mu)) \hookrightarrow L^q(\partial B(0,\mu))$. Moreover, these extremals are multiples of each other.*

Proof. From Hölder's inequality

$$\left(\int_{\partial\Omega} |u|^{q_1}\, d\sigma\right)^{1/q_1} \leqslant |\partial\Omega|^{\frac{1}{q_1}-\frac{1}{q_2}} \left(\int_{\partial\Omega} |u|^{q_2}\, d\sigma\right)^{1/q_2}$$

for $1 < q_1 < q_2$, we get that

$$S_{q_2} \leqslant S_{q_1} |\partial\Omega|^{\frac{1}{q_1}-\frac{1}{q_2}}.$$

Now assume that there exists a radial extremal, u_r, for $q = q_2$. Using that u_r is constant on the boundary $\partial B(0,\mu)$

$$S_{q_2} = \frac{\int_{B(0,\mu)} |\nabla u_r|^2 + |u_r|^2\, dx}{(\int_{\partial B(0,\mu)} |u_r|^{q_2}\, d\sigma)^{2/q_2}} = \left|\partial B(0,\mu)\right|^{\frac{1}{q_1}-\frac{1}{q_2}} \frac{\int_{B(0,\mu)} |\nabla u_r|^2 + |u_r|^2\, dx}{(\int_{\partial B(0,\mu)} |u_r|^{q_1}\, d\sigma)^{2/q_1}} \leqslant \left|\partial B(0,\mu)\right|^{\frac{1}{q_1}-\frac{1}{q_2}} S_{q_1}.$$

Therefore

$$\frac{\int_{B(0,\mu)} |\nabla u_r|^2 + |u_r|^2\, dx}{(\int_{\partial B(0,\mu)} |u_r|^{q_1}\, d\sigma)^{2/q_1}} \leqslant S_{q_1}.$$

This finishes the proof. □

Now we prove that for every μ the extremal is radial for $q = 2$, see also [8] for a proof of this fact for the immersion $W^{1,p}(B(0,\mu)) \hookrightarrow L^p(\partial B(0,\mu))$.

Lemma 5.1. *Let $q = 2$, then for every $\mu > 0$ there exists a radial extremal for the immersion $H^1(B(0,\mu)) \hookrightarrow L^q(\partial B(0,\mu))$.*

Proof. It follows easily from the fact that the eigenfunctions of

$$\begin{cases} \Delta u = u & \text{in } B(0,\mu), \\ \frac{\partial u}{\partial \eta} = \lambda u & \text{on } \partial B(0,\mu), \end{cases} \tag{5.1}$$

can be expanded in terms of spherical harmonics. In fact, the eigenfunctions are given by

$$u_{kj}(x) = C_{kj}|x|^{1-\frac{N}{2}} I_{k+N/2-1}\big(|x|\big) Y_{kj}\left(\frac{x}{|x|}\right),$$

where C_{kj} is a constant; I_ν and Y_{kj} stand for the modified Bessel function of first kind and order ν and for any spherical harmonic of degree k. Here j labels the spherical harmonics of degree k. For a review on special functions, see [2,9].

The eigenvalues of (5.1) are given by

$$\lambda_k = \frac{1 - N/2}{\mu} + \frac{I'_{k+N/2-1}(\mu)}{I_{k+N/2-1}(\mu)}.$$

The eigenfunctions u_{kj} belongs to the eigenvalue λ_k.

As $I'_\nu(a)/I_\nu(a)$ increases when ν increases, the smallest eigenvalue is λ_0 that has associated a radial eigenfunction. □

Hence we get the following result.

Corollary 5.1. *For every $q \leqslant 2$ and every $\mu > 0$ there exists a radial extremal for the embedding $H^1(B(0,\mu)) \hookrightarrow L^q(\partial B(0,\mu))$.*

Acknowledgements

This research was started while the third author (JDR) was a visitor at Univ. Autónoma de Madrid and continued at Univ. Libre de Bruxelles. He is grateful to both institutions for their stimulating atmosphere.

The authors thank the referee for a remark on the presentation and for providing a simple proof of Lemma 5.1.

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