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On tails of stationary measures on a class of solvable groups

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Abstract

Let G be a subgroup of $\mathrm{GL}(\mathbb{R}, d)$ and let (Q_n, M_n) be a sequence of i.i.d. random variables with values in $\mathbb{R}^d \rtimes G$ and law μ. Under some natural conditions there exists a unique stationary measure ν on $\mathbb{R}^d$ of the process $X_n = M_n X_{n-1} + Q_n$. Its tail properties, i.e. behavior of $\nu\{x\colon |x| > t\}$ as t tends to infinity, were described some over thirty years ago by H. Kesten, whose results were recently improved by B. de Saporta, Y. Guivarc'h and E. Le Page. In the present paper we study the tail of ν in the situation when the group G_0 is Abelian and $\mathbb{R}^d$ is replaced by a more general nilpotent Lie group N. Thus the tail behavior of ν is described for a class of solvable groups of type NA, i.e. being semi-direct extension of a simply connected nilpotent Lie group N by an Abelian group isomorphic to $\mathbb{R}^d$. Then, due to A. Raugi, (N, ν) can be interpreted as the Poisson boundary of (NA, μ).

Résumé

Soit G un sous groupe de $\mathrm{GL}(\mathbb{R}, d)$ et soit $(Q_n, M_n) \in \mathbb{R}^d \rtimes G$ une suite de variables aléatoires indépendantes de loi μ. Sous des hypothèses convenables il y a une unique mesure stationnaire ν sur $\mathbb{R}^d$ pour le processus auto-régressif linéaire $X_n = M_n X_{n-1} + Q_n$. Les propriétés asymptotiques de la queue $\nu\{x\colon |x| > t\}$, $t \to \infty$, ont été étudiées par H. Kesten il y a 30 ans et plus récemment de nouveaux résultats ont été obtenus par B. de Saporta, Y. Guivarc'h and E. Le Page. Dans cet article on étudie le cas où G est abélien et $\mathbb{R}^d$ est remplacé par un groupe de Lie nilpotent N. On obtient alors le comportement à l'infini de la queue de ν pour une classe particulière de groupes de tyle NA produits semi-direct d'un groupe N simplement connexe N avec $G = \mathbb{R}^d$. Dans ce cas pariculier (N, ν) est un bord de Poisson au sens de A. Raugi.

Keywords: Solvable Lie groups; Stationary measure; Poisson kernel

1. Introduction

We study random recursions on solvable Lie groups S, which satisfy the following assumptions

- S is the semi-direct product of an Abelian group A, isomorphic to $\mathbb{R}^d$, acting on a simply connected nilpotent Lie group N,

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- there exists a contracting element $a_0 \in A$, i.e. for every $x \in N$: $\lim_{k\to\infty} \delta_{a_0}^k(x) = 0$, where δ_{a_0} stands for the action of a_0 on N, and 0 is the unit element in N.

Various classical objects like symmetric spaces, bounded homogeneous domains in $\mathbb{C}^n$ and manifolds of negative curvature admit simply transitive actions of such groups and therefore they are of considerable interest from many points of view [1,16,20].

Given a probability measure μ on S we define a random walk

$$S_n = X_n \cdots X_1,$$

where $\{X_i\}_{i=1}^{\infty}$ is a sequence of independent identically distributed (i.i.d.) random variables with law μ.

We write

$$X_1 = QM,$$

with $Q = \pi_N(X_1) \in N$, $M = \pi_A(X_1) \in A$, where π_N and π_A denote canonical projections of S onto N and A, respectively. We shall assume that

- μ is mean-contracting, that is the element of the group corresponding to the vector $\int_S \log M \, \mathrm{d}\mu(Q,M)$ is contracting;
- $\int_S (|\log \|M\|| + \log^+ |Q|) \, \mathrm{d}\mu(Q,M) < \infty$, for convenient norms on A and N that will be defined in Section 2.

Under these hypotheses the limit R of $\pi_N(S_n)$ exists in law (A. Raugi [21]) and gives rise to the measure ν that is the only stationary measure for the Markov chain $\pi_N(S_n)$ i.e.

$$\mu * \nu = \nu. \tag{1.1}$$

This means that for every positive, Borel measurable function f on N, we have

$$\mu * \nu(f) = \int f\big(\pi_N(g\cdot x)\big)\mu(\mathrm{d}g)\nu(\mathrm{d}x) = \nu(f).$$

Moreover, if μ is spread out (i.e. some power of μ is nonsingular with respect to the Haar measure on S) and its support generates the group S, A. Raugi [21] proved that (N, ν) is the Poisson boundary of this process, i.e. using the stationary measure ν one can reconstruct bounded μ-harmonic functions on S, knowing their boundary value on N.

Our aim is to study behavior of

$$\nu\{x\colon |x| > t\} = \mathbb{P}\big[|R| > t\big]$$

as t tends to infinity, provided some further hypothesis on μ.

When the Abelian group is one dimensional, i.e. $A = \mathbb{R}^+$, the tail behavior is well understood. If $N = \mathbb{R}$, it was observed by H. Kesten [17] that the tail behavior of ν is strictly related to properties of the Laplace transform of $\pi_A(\mu)$ and that under natural conditions there exists $\alpha > 0$ such that

$$\lim_{t\to\infty} t^\alpha \mathbb{P}\big[|R| > t\big] = C,$$

for some positive constant C. His proof was later essentially simplified by A.K. Grincevičius [12] and Ch. Goldie [11]. The general situation of solvable groups being extensions of nilpotent groups by one-dimensional Abelian group of automorphisms was studied in [2], where similar results were obtained. Much more can be said about ν when the measure μ comes from a second-order, subelliptic, left-invariant differential operator $\mathcal{L}$ on S, i.e. when instead of μ we consider a semigroup of measures μ_t, whose infinitesimal generator is $\mathcal{L}$, and the measure ν satisfies

$$\check{\mu}_t * \nu = \nu \quad \text{for every } t.$$

Then, the measure ν has a density and its behavior along some rays tending to infinity has been described in [6] and [3].

The situation when the group A acting on N is multidimensional is much more complicated. In the context of general solvable groups, the only results we know, concerning behavior at infinity of the stationary measure, were obtained in some particular cases when the measure μ is connected with an subelliptic operator on S (compare above).

If $X = G/K$ is a noncompact symmetric space, S is the solvable part of the Iwasawa decomposition of $G = SK$ and $\mathcal{L}$ is the Laplace–Beltrami operator, ν has a smooth density m, called Poisson kernel, which can be explicitly computed (see e.g. [8]). The formulas however are not very transparent as far as the pointwise decay at ∞ is concerned.

More general situation was studied by E. Damek and A. Hulanicki [4,5]. They considered on solvable groups $S = NA$, with diagonal action of A on N, a large class of left-invariant second order, degenerate elliptic operators $\mathcal{L}$ and identified the Poisson boundary of $(S, \mathcal{L})$ with (N_1, ν), where N_1 is some normal subgroup of N. Then the stationary measure ν on N_1 has again smooth density m and they proved, without knowing an explicit formula for m that

$$\int_{N_1} \tau_{N_1}(x)^{\varepsilon} m(x)\,\mathrm{d}x < \infty,$$

for some positive ε, where τ_{N_1} is the Riemannian distance of x from the identity, and dx is the Haar measure on N.

The case when N is an Euclidean space, but the measure is general (not coming from a differential operator) was studied by many authors. Assume $N = \mathbb{R}^m$ and there exists a group of matrices G (not necessarily Abelian) acting on $\mathbb{R}^m$. Consider the stochastic recursion

$$R_{n+1} = M_{n+1} R_n + Q_{n+1},$$

where (Q_n, M_n) is a sequence of i.i.d., $\mathbb{R}^m \times G$ valued random variables distributed according to the given probability measure μ. Then under suitable assumptions R_n converges to a random variable R, whose distribution ν is μ-invariant. Asymptotic properties of R were studied by several authors [17–19,7,14]. Their main assumptions (except mean-contractivity and finiteness of some exponential moments) were proximality and (or) irreducibility. Let $\bar{\mu}$ be the canonical projection of μ onto G. Then proximality means that the semigroup generated by the support of $\bar{\mu}$ contains a proximal element, i.e. a matrix having a unique real dominant eigenvalue (i.e. the corresponding eigenspace is one-dimensional). The action is called irreducible if there does not exist a finite union of proper subspaces of $\mathbb{R}^m$, which is invariant under the action of the support of $\bar{\mu}$.

In this paper we study the reducible situation on general solvable groups. Our assumptions are natural generalization of one-dimensional situation, i.e. first of all we require finiteness of some exponential moments of $\pi_A(\mu)$. The main results of the paper are presented in Section 3.4 as Main Theorem A and Main Theorem B. In full generality we prove that there exists a constant χ_0 such that for any $\varepsilon > 0$

$$C_1 t^{-\chi_0} \leqslant \mathbb{P}\big[|R| > t\big] \leqslant C_\varepsilon t^{-(\chi_0 - \varepsilon)},$$

where C_1 and C_ε are positive constants, and C_ε depends on ε. Notice that the result is new even in the case when an Abelian group of matrices $A = G$ acts on $N = \mathbb{R}^m$ and the measure μ does not satisfies to the assumptions of proximality and irreducibility required by the papers mentioned above.

We obtain more detailed description of the tail of the measure ν, when the action of A is fully reducible, i.e. A acts diagonally on N. This corresponds to the classical situations of symmetric spaces and bounded homogeneous domains. Then we prove, without assuming proximality of μ, the existence of positive constants χ_0 and C_2 such that

$$C_1 t^{-\chi_0} \leqslant \mathbb{P}\big[|R| > t\big] \leqslant C_2 t^{-\chi_0}.$$

If we assume existence of a dominant root (see Section 3 for precise definitions), that in some sense substitutes the notion of proximality, we show

$$\lim_{t\to\infty} t^{\chi_0} \mathbb{P}\big[|R| > t\big] = C_3,$$

for some $C_3 > 0$.

The outline of the paper is as follows. In Section 2 we introduce a class of solvable Lie groups for which our results holds and describe precisely their structure. In Section 3 we include a brief account of random walks on solvable groups: existence of an invariant measure and its properties in the case when the group A is one-dimensional. Then we describe our assumptions and state the main results of the article. Their proofs are contained in Sections 4 and 5, respectively.

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2. A class of solvable Lie groups

Let A be an Abelian group isomorphic to $\mathbb{R}^d$, acting on a nilpotent, connected and simply connected Lie group N, i.e.

$$\delta_a(xy) = \delta_a(x)\delta_a(y), \quad a \in A,\ x, y \in N, \tag{2.1}$$

where δ_a denotes the action of $a \in A$ on N.

The semi-direct product $N \rtimes A$ is a solvable Lie group denoted by S. We shall denote by $\circ$ the action of the group S on N, i.e.

$$(x, a) \circ y = x \cdot \delta_a(y), \quad \text{for } (x, a) \in S \text{ and } y \in N.$$

Then the group multiplication in S is given by

$$(x, a) \cdot (y, b) = \big((x, a) \circ y, ab\big).$$

Let e (0, I respectively) be the neutral element of S (N, A respectively).

Our main assumption on S is that the action of A on N is contractive i.e. that there exists an element $a \in A$ such that

$$\lim_{k\to\infty} \delta_a^k(x) = 0, \quad \text{for every } x \in N. \tag{2.2}$$

The Lie algebras of A, N, S are denoted by $\mathcal{A}, \mathcal{N}$ and $\mathcal{S}$ respectively. Then $\mathcal{S} = \mathcal{N} \oplus \mathcal{A}$ and of course for every $H \in \mathcal{A}$, $\operatorname{ad} H$ preserves $\mathcal{N}$. The exponential maps are global diffeomorphisms both between $\mathcal{N}$ and N, and between $\mathcal{A}$ and A. Their inverse will be denoted by log. Then for any $X \in \mathcal{N}$

$$\delta_a\big(\exp(X)\big) = \exp\big(\mathrm{e}^{\operatorname{ad}(\log a)} X\big). \tag{2.3}$$

We shall denote the foregoing action of the group A on the Lie algebra $\mathcal{N}$, using the same symbol $\delta_a(X)$. Let $\mathcal{N}^{\mathbb{C}}$ ($N^{\mathbb{C}}$) be the complexification of $\mathcal{N}$ (N respectively). For any λ in the set $(\mathcal{A}^*)^{\mathbb{C}}$ of continuous homomorphisms from $\mathcal{A}$ to $(\mathbb{C}, +)$ define

$$\mathcal{N}_\lambda^{\mathbb{C}} = \big\{Z \in \mathcal{N}^{\mathbb{C}}\colon \text{there exists } k \text{ such that } (\operatorname{ad} H - \lambda I)^k Z = 0, \text{ for any } H \in \mathcal{A}\big\}. \tag{2.4}$$

Then, it is known that for $\lambda_1, \lambda_2 \in (\mathcal{A}^*)^{\mathbb{C}}$

$$\big[\mathcal{N}_{\lambda_1}^{\mathbb{C}}, \mathcal{N}_{\lambda_2}^{\mathbb{C}}\big] \subset \mathcal{N}_{\lambda_1+\lambda_2}^{\mathbb{C}}. \tag{2.5}$$

Moreover any space $\mathcal{N}_\lambda^{\mathbb{C}}$ is preserved by the action of the group A, i.e.

$$\delta_a(Z) \in \mathcal{N}_\lambda^{\mathbb{C}}, \quad \text{for } Z \in \mathcal{N}_\lambda^{\mathbb{C}}. \tag{2.6}$$

We shall say that λ is a root if the appropriate space $\mathcal{N}_\lambda^{\mathbb{C}}$ is nonempty. The set of all roots will be denoted by Δ. Then, of course, if $\lambda \in \Delta$ then also $\bar{\lambda} \in \Delta$ and

$$\mathcal{N}^{\mathbb{C}} = \bigoplus_{\lambda\in\Delta} \mathcal{N}_\lambda^{\mathbb{C}}.$$

Let $i_\lambda = \dim_{\mathbb{C}} \mathcal{N}_\lambda^{\mathbb{C}}$. For any λ choose a basis $\{Z_{\lambda,1}, \ldots, Z_{\lambda,i_\lambda}\}$ of $\mathcal{N}_\lambda^{\mathbb{C}}$, such that with respect to this basis $\mathcal{A}$ acts triangularly, i.e. for any $H \in \mathcal{A}$

$$\operatorname{ad} H(Z_{\lambda,j}) = \lambda(H) Z_{\lambda,j} + W_{\lambda,j-1}, \tag{2.7}$$

for some $W_{\lambda,j-1} \in \operatorname{span}\{Z_{\lambda,1}, \ldots, Z_{\lambda,j-1}\}$. Then $i_\lambda = i_{\bar{\lambda}}$, moreover may assume that $\overline{Z_{\lambda,j}} = Z_{\bar{\lambda},j}$ and if λ is real then all the vectors $Z_{\lambda,j}$ are real.

For a chosen basis $\{H_1, \ldots, H_d\}$ of $\mathcal{A}$ introduce coordinates in A: any element H of $\mathcal{A}$ can be uniquely written as $H = \sum t_i(H) H_i$. Notice that one can compute the action of A on $\mathcal{N}^{\mathbb{C}}$, taking (2.3) and (2.7) into account, we obtain

$$\delta_{\exp H}(Z_{\lambda,k}) = \mathrm{e}^{\lambda(H)} \cdot \sum_{j\leqslant k} P_{\lambda,k,j}(H) Z_{\lambda,j}, \tag{2.8}$$

where $P_{\lambda,k,k} = 1$, and $P_{\lambda,k,j}$ for j smaller that k are some polynomials of $t_i(H)$. One can easily see that the polynomials depend on $t_i(H)$ only if $\lambda(H_i) \neq 0$.

Thus, the assumption that the action of A is contractive implies that the negative Weyl chamber

$$\mathcal{A}^{--} = \{H \in \mathcal{A}\colon \Re\lambda(H) < 0 \text{ for all } \lambda \in \Delta\} \tag{2.9}$$

is not empty. Let $\mathcal{A}^{++} = -\mathcal{A}^{--}$ be the positive Weyl chamber.

For any $z \in N^{\mathbb{C}}$ let $z_{\lambda,i}$ denotes its λ, i component, i.e.

$$z = \exp\Big(\sum z_{\lambda,i} Z_{\lambda,i}\Big).$$

A root λ_0 will be called simple if it cannot be written as a sum of other roots, i.e. for all possible choices of nonnegative integer numbers $\{c_\lambda\}_{\lambda\in\Delta}$, such that $\sum c_\lambda > 1$,

$$\lambda_0 \neq \sum_{\lambda\in\Delta} c_\lambda \lambda.$$

The set of all simple roots will be denoted by Δ_1.

For instance, let $A = \mathbb{R}^2$, choose two vector fields H_1, H_2 forming a basis of $\mathcal{A}$, and denote by λ_1, λ_2 two functionals on $\mathcal{A}$ such that $\lambda_i(H_j) = \delta_{ij}$. Then, if $\Delta = \{\lambda_1, \lambda_1/2, (\lambda_1+\lambda_2)/2, \lambda_1 + 2\lambda_2, \lambda_2\}$, the set of simple roots consists of three elements: $\Delta_1 = \{\lambda_1/2, (\lambda_1+\lambda_2)/2, \lambda_2\}$.

We have the following simple lemma

Lemma 2.10. *Any root λ_0 can be written in the form*

$$\lambda_0 = \sum_{\lambda\in\Delta_1} c_\lambda \lambda, \tag{2.11}$$

where c_λ are nonnegative integer numbers.

Proof. Suppose $H \in \mathcal{A}^{++}$ and let us number all the roots $\lambda_1, \lambda_2, \ldots, \lambda_k$ in the following way

$$\Re\lambda_1(H) \leqslant \Re\lambda_2(H) \leqslant \cdots \leqslant \Re\lambda_k(H).$$

We shall proceed by induction. Of course, λ_1 is a simple root and (2.11) holds with $c_{\lambda_1} = 1$. Assume the lemma holds for $\lambda_1, \ldots, \lambda_{i-1}$. If the root λ_i is simple then it satisfies (2.11). Otherwise, λ_i can be written as

$$\lambda_i = \sum_{\lambda\in\Delta} c_\lambda \lambda,$$

where c_λ are positive integer and $\sum c_\lambda > 1$. Therefore $\Re\lambda_i(H) > \Re\lambda(H)$ for any λ such that c_λ is nonzero. But this set contains either simple roots or other roots satisfying already (2.11). Therefore (2.11) also holds for λ_i. □

The group multiplication in N is given by the Campbell–Hausdorf formula:

$$\exp(X)\cdot\exp(Y) = \exp\big(X + Y + [X,Y]/2 + \cdots\big), \quad \text{for } X, Y \in \mathcal{N}. \tag{2.12}$$

Since the Lie algebra $\mathcal{N}$ is nilpotent, the sum above is finite. In particular if we fix a simple root λ_0, then in view of (2.5)

$$(x\cdot y)_{\lambda_0,i} = x_{\lambda_0,i} + y_{\lambda_0,i}, \tag{2.13}$$

for $x, y \in N$ and $i \leqslant i_{\lambda_0}$. We shall describe the Campbell–Hausdorf formula more precisely later in Section 5.

2.1. Norms on N and A

Now we are going to construct a norm on N adapted to the action of A. In the case A is one-dimensional and diagonalizable W. Hebisch and A. Sikora [15] have built on N a smooth outside zero norm, homogeneous on the action of one-dimensional group of dilations, i.e. satisfying $|\delta_a(x)| = a|x|$. Their ideas were used later in [2] to construct a homogeneous norm with respect to general one-dimensional group of dilations A. Here we shall adopt the construction for our purpose. Since we will need some further properties of the norm we give some details.

Fix $H_0 \in \mathcal{A}^{++}$ such that $\Re\lambda(H_0) > 1$ for all roots λ and let $A_0 = \{\exp tH_0,\ t \in \mathbb{R}\}$ be an one parameter subgroup of A. We change coordinates in A_0, identifying

$$A_0 \ni \exp tH_0 \sim \mathrm{e}^t \in \mathbb{R}^+.$$

For $b \in \mathbb{R}^+$ and $z \in N^{\mathbb{C}}$ define

$$\sigma_b(x) = \delta_{\exp(\log b)H_0}(z). \tag{2.14}$$

Then σ defines the action of $\mathbb{R}^+$ on $N^{\mathbb{C}}$, preserving N, and the semi-direct product $N \rtimes \mathbb{R}^+$ is a solvable group, belonging to the class of solvable Lie groups studied in [2]. A key step of the construction is the following lemma:

Lemma 2.15. ([15,2]) *There exists an open rectangle*

$$\Omega = \left\{ Z = \sum_{\lambda,i} z_{\lambda,i} Z_{\lambda,i} \in \mathcal{N}^{\mathbb{C}}\colon\ |z_{\lambda,i}| < c_{\lambda,i} \right\}, \tag{2.16}$$

where $c_{\lambda,i}$ are some positive constants, such that

$$\text{if } \log(z), \log(w) \in \Omega,\ \text{for } z, w \in N^{\mathbb{C}} \text{ and } 0 < b < 1 \text{ then } \log\bigl(\sigma_b(z)\sigma_{1-b}(w)\bigr) \in \Omega. \tag{2.17}$$

We define the norm on $N^{\mathbb{C}}$:

$$|z| = \inf\bigl\{b\colon\ \log\bigl(\sigma_{b^{-1}}(z)\bigr) \in \Omega\bigr\} = \inf\bigl\{\mathrm{e}^t\colon\ \log\bigl(\delta^{-1}_{\exp tH_0}(z)\bigr) \in \Omega\bigr\}.$$

One can easily check that this norm is continuous and satisfies to the following properties

- $|\cdot|$ is symmetric: $|z^{-1}| = |z|$;
- $|z| = 0$ if and only if $z = 0$;
- $|\cdot|$ is subadditive, i.e. $|z \cdot w| \leqslant |z| + |w|$;
- $|\sigma_b(z)| = b|z|$, for any $b \in \mathbb{R}^+$.

Finally, we define a norm on A:

$$\|a\| = \max_{|z|=1} \bigl|\delta_a(z)\bigr|.$$

Observe that

$$\bigl|\delta_a(z)\bigr| \leqslant \|a\||z| \quad \text{and} \quad \|a_1a_2\| \leqslant \|a_1\|\|a_2\|.$$

We shall often use the following constants being closely related to properties of the foregoing norms

$$d_\lambda = \Re\lambda(H_0), \quad \lambda \in \Delta,$$

and their simple property

$$\text{if } \lambda_0 = \sum c_\lambda \lambda,\ \text{then } d_{\lambda_0} = \sum c_\lambda d_\lambda. \tag{2.18}$$

A crucial step in the proof of our main results will be the following lemma:

Lemma 2.19. *There exist constants C and D such that*

$$\|a\| \leqslant C \max_{\lambda\in\Delta}\bigl\{\mathrm{e}^{\frac{\Re\lambda(H)}{d_\lambda}}\bigr\} \cdot \Bigl(1 \vee \max_i \bigl|t_i(H)\bigr|^D\Bigr)$$

for any $a = \exp H \in A$, where $t_i(H)$ denotes ith coordinate of H in the fixed basis of $\mathcal{A}$.

Proof. First, we shall prove that

$$\|a\| \leqslant \max_{\lambda\in\Delta} \sup_{\{z_\lambda\in N_\lambda^{\mathbb{C}}:\ |z_\lambda|=1\}} \left|\delta_a(z_\lambda)\right|. \tag{2.20}$$

In fact, every space $\mathcal{N}_\lambda^{\mathbb{C}}$ is invariant under the action of A (2.6), and writing any element of $N^{\mathbb{C}}$ as $z = \exp(\sum_{\lambda,i} z_{\lambda,i} Z_{\lambda,i})$ and using the fact that the action of A on $\mathcal{N}^{\mathbb{C}}$ is linear we have

$$\begin{aligned}\|a\| &= \sup_{|z|=1}\left|\delta_a(z)\right| = \sup_{|z|=1} \inf\left\{b:\ \sigma_{b^{-1}}\delta_a\left(\sum_{\lambda,i} z_{\lambda,i}Z_{\lambda,i}\right)\in\Omega\right\}\\ &= \sup_{|z|=1}\inf\left\{b:\ \sigma_{b^{-1}}\delta_a\left(\sum_i z_{\lambda,i}Z_{\lambda,i}\right)\in\Omega \text{ for all roots } \lambda\right\}\\ &\leqslant \max_{\lambda\in\Delta} \sup_{\{z_\lambda\in N_\lambda^{\mathbb{C}}:\ |z_\lambda|=1\}} \left|\delta_a(z_\lambda)\right|,\end{aligned}$$

which proves desired inequality (2.20).

Define the function $g(H) = \max_i |t_i(H)|$. In view of (2.20) it is enough to justify that for any root λ there exist constants C_λ, D_λ such that if $g(H) > C_\lambda$, then

$$\sigma_b^{-1}\delta_{\exp H}(Z_\lambda) \in \Omega$$

for $b = \exp\{\frac{\Re\lambda(H)}{d_\lambda}\}\cdot(1\vee g(H)^{D_\lambda})$ and any $Z_\lambda = \sum_i z_{\lambda,i}Z_{\lambda,i} \in \bar{\Omega}\cap\mathcal{N}_\lambda^{\mathbb{C}}$. In view of (2.8)

$$\delta_a(Z_\lambda) = \mathrm{e}^{\lambda(H)}\cdot\sum_k z_{\lambda,k}\left(\sum_{j\leqslant k} P_{\lambda,k,j}(H)Z_{\lambda,j}\right),$$

where $P_{\lambda,k,j}$ are some polynomials of $t_j(H)$ and $P_{\lambda,k,k} = 1$.

Next we have

$$\begin{aligned}\sigma_b^{-1}\delta_a(Z_\lambda) &= \delta_{\exp(-\log b)H_0}\delta_a(Z_\lambda)\\ &= \mathrm{e}^{-\log b\cdot\lambda(H_0)+\lambda(H)}\cdot\sum_k z_{\lambda,k}\left(\sum_j \bar{P}_{\lambda,k,j}(H,\log b)Z_{\lambda,j}\right),\end{aligned}$$

where $\bar{P}_{\lambda,k,j}$ are some polynomials of $t_j(H)$ and $\log b$. Substituting b in the formula above we obtain

$$\begin{aligned}\sigma_b^{-1}\delta_a(Z_\lambda) &= \left(1\vee g(H)\right)^{-D_\lambda\Re\lambda(H_0)}\cdot\mathrm{e}^{-\mathrm{i}\Im\lambda(H_0)(\frac{\Re\lambda(H)}{\Re\lambda(H_0)}+D_\lambda\log^+ g(H))}\cdot\mathrm{e}^{\mathrm{i}\Im\lambda(H)}\\ &\quad\times\sum_k\left(\sum_j \bar{\bar{P}}_{\lambda,k,j}\left(t_i(H), D_\lambda\log^+ g(H), \Re\lambda(H)\right)z_{\lambda,j}\right)Z_{\lambda,k},\end{aligned}$$

where $\bar{\bar{P}}_{\lambda,k,j}$ are polynomials coming from appropriately modified polynomials $\bar{P}_{\lambda,k,j}$ and degrees of these polynomials depends only on the structure of the solvable group S. Finally, choosing D_λ large enough, there exists C_λ such that if $g(H) > C_\lambda$ then for all k

$$\left(1\vee g(H)\right)^{-D_\lambda\Re\lambda(H_0)}\sum_j\left|\bar{\bar{P}}P_{\lambda,k,j}\left(t_i(H), D_\lambda\log^+ g(H), \Re\lambda(H)\right)\right||z_{\lambda,j}| \leqslant c_{\lambda,k},$$

which proves the lemma. □

3. Random walks on NA groups and main theorems

3.1. Random walks

Given a probability measure μ on S we define a random walk:

$$S_n = (Q_n, M_n)\cdots(Q_1, M_1),$$

where (Q_n, M_n) is a sequence of i.i.d. S-valued random variables with a distribution μ. The law of S_n is the nth-convolution μ^{*n} of μ.

Our aim is to study the N-component of S_n, i.e. the Markov chain on N generated by the random walk on S:

$$R_n = \pi_N(S_n) = (Q_n, M_n) \circ R_{n-1},$$
$$R_0 = \delta_0, \tag{3.1}$$

where π_N denotes the canonical projection $\pi_N : S \to S/A$. By π_A we shall denote the analogous projection of S onto A.

It was proved by A. Raugi [21] that when μ is mean-contracting, i.e.

$$\mathbb{E}\log M = \int \log M \mu_A(\mathrm{d}M) \in \mathcal{A}^{--}, \tag{3.2}$$

where $\mu_A = \pi_A(\mu)$, and under the following integrability condition

$$\mathbb{E}\bigl|\log\|M\| + \log^+|Q|\bigr| < \infty$$

(the norms used by A. Raugi were different, but his proof gives the result also in our case) R_n converges in law to a random variable R, whose distribution will be denoted by ν, and R does not depend on the choice of R_0. Moreover, ν is a unique stationary solution of the stochastic equation

$$\nu = \mu * \nu,$$

where

$$\mu * \nu(f) = \int f(g \circ x)\mu(\mathrm{d}g)\nu(\mathrm{d}x).$$

The above equation can be also written in the form

$$R =_d (Q, M) \circ R,$$

where R and (Q, M) are independent distributed according to ν and μ, respectively.

The random variable R is constructed as a pointwise limit of the "backward" process:

$$R_0^* = 0,$$
$$R_n^* = \pi_N\bigl((Q_1, M_1)\cdots(Q_n, M_n)\bigr) = Q_1 \cdot \delta_{\Pi_1}(Q_2)\cdots\delta_{\Pi_{n-1}}(Q_n), \tag{3.3}$$

where $\Pi_n = M_1 \cdots M_n$.

Our aim is to study, under some additional hypothesis, behavior of

$$\nu\bigl\{x\colon |x| > t\bigr\} = \mathbb{P}\bigl[|R| > t\bigr]$$

as t tends to infinity.

3.2. *Asymptotic behavior of R when* $\dim A = 1$

When the Abelian group A is one-dimensional, the behavior of the above sequence is well-known. The simplest example of a solvable group is the "ax + b" group, i.e. semi-direct product of $N = \mathbb{R}$ and $A = \mathbb{R}^+$, with the group action

$$(x, a)\cdot(y, b) = (x + ay, ab), \quad x, y \in \mathbb{R},\ a, b \in \mathbb{R}^+.$$

Then, Kesten [17] proved (under some further assumptions) that there exist positive constants α and C such that

$$\lim_{t\to\infty} t^\alpha \mathbb{P}\bigl[|R| > t\bigr] = C.$$

His proof was later essentially simplified by Grincevičius [12] and Goldie [11]. Their ideas were used in [2] to handle with general situation of homogeneous groups, when the group S is a semi-direct product of a nilpotent group N and of an one-dimensional group of dilations $A = \mathbb{R}^+$. In this case the norm $|\cdot|$ is homogeneous for the action of $\mathbb{R}^+$, i.e. $|\delta_a(x)| = a|x|$ for every $a \in \mathbb{R}^+$, $x \in N$, and we have the following theorem:

Theorem 3.4. ([2]) *Let* $S = N \rtimes \mathbb{R}^+$ *and assume that*

- $\mathbb{E}\log M < 0$,
- *there exists* $\alpha > 0$*, such that* $\mathbb{E}M^\alpha = 1$,
- *the law of* $\lambda \log M$ *is non-arithmetic, i.e. there does not exist* $a > 0$ *such that* $\lambda \log M \in a\mathbb{Z}$,
- $\mathbb{E}M^\alpha \log M < \infty$,
- $\mathbb{E}|Q|^\alpha < \infty$.

Then

$$\lim_{t\to\infty} t^\alpha \mathbb{P}\big[|R| > t\big] = C. \tag{3.5}$$

for some constant C*. Moreover, if the action of* $\mathbb{R}^+$ *on* N *is diagonalizable then the constant* C *is nonzero if and only if for every* $x \in N$,

$$\mathbb{P}\big[(Q, M) \circ x = x\big] < 1.$$

If the action is not diagonalizable, the constant C *is positive under the additional hypothesis that* $|Q|$ *is bounded almost surely.*

We shall often use description of asymptotic behavior of

$$\mathbb{P}\Big[\max_n\{M_1 \cdots M_n\} > t\Big],$$

where M_i are i.i.d. real valued random variables satisfying the assumptions of Theorem 3.4. It was observed by Kesten, that the sequence is strictly connected with asymptotic behavior of R. Then it is well known that there exists a positive constant C such that

$$\lim_{t\to\infty} t^\alpha \mathbb{P}\Big[\max_n\{M_1 \cdots M_n\} > t\Big] = C \tag{3.6}$$

(see [9] for more details).

3.3. Laplace transform

In order to describe the tail of R we shall need some further assumptions on μ. Consider the Laplace transform of the measure $\mu_A = \pi_A(\mu)$:

$$L(\alpha) = \int_A e^{\alpha(\log M)} \mu_A(dM) = \mathbb{E}\big[e^{\alpha(\log M)}\big]$$

where $\alpha \in \mathcal{A}^*$. We assume that

$$\text{for any } \lambda \in \Delta \text{ there exists } \chi_\lambda > 0 \text{ such that } L\left(\frac{\chi_\lambda \Re\lambda}{d_\lambda}\right) = \mathbb{E}\big[e^{\frac{\chi_\lambda \Re\lambda(\log M)}{d_\lambda}}\big] = 1. \tag{3.7}$$

Then it is known that the Laplace transform is well defined for all functionals on A belonging to the convex hull V of 0 and $\chi_\lambda \Re\lambda/d_\lambda$ for all roots $\lambda \in \Delta$. Furthermore L is convex on V and because of (3.2) and (3.7) it is strictly smaller than 1 on the set

$$V_0 = \left\{\alpha \in \mathcal{A}^*\colon \alpha = \sum_{\lambda\in\Delta} c_\lambda \cdot \frac{\chi_\lambda \Re\lambda}{d_\lambda}, \text{ for nonnegative numbers } c_\lambda \text{ satisfying } 0 < \sum c_\lambda < 1\right\},$$

i.e.

$$\text{if } \alpha \in V_0 \text{ then } L(\alpha) < 1. \tag{3.8}$$

Define

$$\chi_0 = \min_{\lambda\in\Delta}\{\chi_\lambda\},$$

then the following holds

Lemma 3.9. *Let $\lambda_0 = \sum c_\lambda \lambda$ for some nonnegative numbers c_λ. Assume that for some root λ_1 the constant c_{λ_1} is nonzero and $\chi_{\lambda_1} > \chi_0$. Then $\chi_{\lambda_0} > \chi_0$.*

Proof. Let us write

$$\frac{\chi_0 \Re\lambda_0}{d_{\lambda_0}} = \sum_\lambda \frac{\chi_0 d_\lambda c_\lambda}{d_{\lambda_0}\chi_\lambda} \cdot \frac{\chi_\lambda \Re\lambda}{d_\lambda},$$

and notice that because of our assumptions and (2.18) we have

$$\sum_\lambda \frac{\chi_0 d_\lambda c_\lambda}{d_{\lambda_0}\chi_\lambda} < \frac{1}{d_{\lambda_0}} \cdot \sum d_\lambda c_\lambda = 1.$$

Therefore by (3.8)

$$L\left(\frac{\chi_0 \Re\lambda_0}{d_{\lambda_0}}\right) < 1,$$

which implies $\chi_0 < \chi_{\lambda_0}$. □

Corollary 3.10. *There exists a simple root λ_0 such that $\chi_{\lambda_0} = \chi_0$.*

We conclude that to compute χ_0 it suffices to consider only simple roots:

$$\chi_0 = \min_{\lambda \in \Delta_1} \{\chi_\lambda\}. \tag{3.11}$$

3.4. Main theorems

For any root λ and $j \leqslant i_\lambda$, let $V_{\lambda,j}$ be the real subspace on $\mathcal{N}$ spanned by $Z_{\lambda,j}$ if λ is real and by $\Re Z_{\lambda,j}$ and $\Im Z_{\lambda,j}$, otherwise. Then for $X \in \mathcal{N}$, by $X|_{V_{\lambda,j}}$ we shall denote the projection of X on $V_{\lambda,j}$.

Now we can state the main results of the paper

Main Theorem A. *Assume*

(A1) $\mathbb{E} \log M \in \mathcal{A}^{--}$;
(A2) *for any root λ there exists a positive number χ_λ such that* $\mathbb{E}[\mathrm{e}^{\frac{\chi_\lambda \Re\lambda(\log M)}{d_\lambda}}] = 1$;
(A3) *the Laplace transform of the measure μ_A is finite in some neighborhood U of 0 in $\mathcal{A}^*$ i.e. if $\alpha \in U$, then $L(\alpha) < \infty$*;
(A4) $\mathbb{E}|Q|^{\chi_0} < \infty$, *for χ_0 defined in* (3.11).

Assume moreover that there exists a simple root λ_0 such that $\chi_{\lambda_0} = \chi_0$ satisfying

(A5) *the law of* $\Re\lambda_0(\log M)$ *is non-arithmetic*;
(A6) $\mathbb{E}[\mathrm{e}^{\frac{\chi_{\lambda_0} \Re\lambda_0(\log M)}{d_{\lambda_0}}} |\Re\lambda_0(\log M)|] < \infty$;
(A7) *for any* $X \in V_{\lambda_0, i_{\lambda_0}}$

$$\mathbb{P}\big[\log\big((Q, M) \circ \exp X\big)\big|_{V_{\lambda_0, i_{\lambda_0}}} = X\big] < 1.$$

Then there exists a positive constant C_1 and for any $\varepsilon > 0$ there exists C_ε such that

$$C_1 t^{-\chi_0} \leqslant \mathbb{P}\big[|R| > t\big] \leqslant C_\varepsilon t^{-(\chi_0 - \varepsilon)}.$$

A simple root λ_0 is called dominant if

$$\chi_{\lambda_0} = \chi_0$$

and if $\chi_\lambda = \chi_0$ for some other root λ, then there exists a constant c_λ larger than 1 such that $\lambda = c_\lambda \lambda_0$.

Of course it may happen that dominant root does not exists, i.e. for two different simple roots λ_1, λ_2, such that $\lambda_1 \neq c\lambda_2$ for any constant c, we have $\chi_0 = \chi_{\lambda_1} = \chi_{\lambda_2}$.

Main Theorem B. *Assume that the action of A on N is diagonalizable and*

(B1) $\mathbb{E}\log M \in \mathcal{A}^{--}$;
(B2) *for any root λ there exists a positive number χ_λ such that* $\mathbb{E}[e^{\frac{\chi_\lambda \lambda(\log M)}{d_\lambda}}] = 1$;
(B3) *for any root* λ, $\mathbb{E}[e^{\frac{\chi_\lambda \lambda(\log M)}{d_\lambda}}|\lambda(\log M)|] < \infty$;
(B4) $\mathbb{E}|Q|^{\chi_0} < \infty$;

where χ_0 was defined in (3.11). *Assume moreover that there exists a simple root λ_0 such that $\chi_{\lambda_0} = \chi_0$ satisfying*

(B5) *the law of $\lambda_0(\log M)$ is non-arithmetic*;
(B6) *there exists $i \leqslant i_{\lambda_0}$ such that for every $X \in V_{\lambda_0,i}$,* $\mathbb{P}[\log((Q,M)\circ \exp X)|_{V_{\lambda_0,i}} = X] < 1$.

Then there exists a positive number C_1 such that

$$\frac{1}{C_1} t^{-\chi_0} \leqslant \mathbb{P}\big[|R| > t\big] \leqslant C_1 t^{-\chi_0}.$$

Moreover if there exists in Δ_1 a dominant root λ_0 satisfying both (B5) *and* (B6)*, then*

$$\lim_{t\to\infty} t^{\chi_0}\mathbb{P}\big[|R| > t\big] = C_2,$$

for some positive number C_2.

4. Proof of Main Theorem A

4.1. Upper estimates

In order to prove the upper bound of the tail of R, we shall use Lemma 2.19 and prove existence of χth moment of R for any χ satisfying $0 < \chi < \chi_0$ and then the estimates follows immediately (Corollary 4.9).

Lemma 4.1. *Under the hypothesis* (A1)–(A4) *the stationary measure of R has all moments smaller than χ_0, i.e.*

$$\mathbb{E}|R|^\chi < \infty$$

for all χ satisfying $0 < \chi < \chi_0$.

Proof. Fix χ' such that $\chi < \chi' < \chi_0$, then by definition of χ_0

$$L\left(\frac{\chi' \Re\lambda}{d_\lambda}\right) < 1, \quad \lambda \in \Delta.$$

For any root λ let us define a positive number

$$a_\lambda = \begin{cases} \chi, & \text{if } L\left(\frac{\chi\Re\lambda}{d_\lambda}\right) > L\left(\frac{\chi'\Re\lambda}{d_\lambda}\right), \\ \chi', & \text{otherwise.} \end{cases}$$

Then, since the Laplace transform is convex

$$L\left(\frac{\beta\Re\lambda}{d_\lambda}\right) < L\left(\frac{a_\lambda\Re\lambda}{d_\lambda}\right) \tag{4.2}$$

for any root λ and $\beta \in (\chi, \chi')$. We may choose positive δ satisfying

$$0 < \delta < \frac{1}{L(a_\lambda \Re\lambda / d_\lambda)} - 1, \quad \text{for any } \lambda \in \Delta. \tag{4.3}$$

Consider the function

$$f(s)=\mathbb{E}\big[\mathrm{e}^{s\sum_i |t_i(\log M)|}\big].$$

For any sequence σ of 0 and 1's having the length d define the element of $\mathcal{A}^*$ by the formula

$$\alpha_\sigma(H)=\sum_{i=1}^{d}(-1)^{\sigma(i)}t_i(H),\qquad H\in\mathcal{A},$$

and notice that f can be dominated by the sum

$$f(s)\leqslant\sum_{\sigma\in\{0,1\}^d}L(s\alpha_\sigma).$$

By (A3) for small values of s the Laplace transform $L(st_i)$ is well-defined, moreover it is continuous as a function of s and tends to 1 as t goes to 0. Therefore also f is continuous and tends to 1. So, there exists θ, such that

$$f(s)<1+\delta,\quad \text{for } s\leqslant\theta. \tag{4.4}$$

Next, choose a positive number ε satisfying

$$\varepsilon<\min\left\{\frac{\theta(\chi'-\chi)}{\chi\chi'},\frac{\theta}{2\chi}\right\}. \tag{4.5}$$

Finally, define

$$q=\frac{\theta}{\varepsilon\chi},\qquad p=\frac{\theta}{\theta-\varepsilon\chi}. \tag{4.6}$$

Then notice that $\frac{1}{p}+\frac{1}{q}=1$, by (4.5)

$$q>2\quad\text{and}\quad p<2 \tag{4.7}$$

and moreover

$$\chi<p\chi<\chi'. \tag{4.8}$$

Recall that R was constructed as the limit in distribution of R_n. Therefore it is enough to estimate χth moment of R_n independently on n. We have

$$\begin{aligned}
\big(\mathbb{E}|R_n|^\chi\big)^{\frac{1}{\chi}}&=\big[\mathbb{E}\big|Q_n\cdot\delta_{M_n}(Q_{n-1})\cdots\delta_{M_1\ldots M_n}(Q_0)\big|^\chi\big]^{\frac{1}{\chi}}\\
&\leqslant\left(\mathbb{E}\left[\sum_{k=0}^{n-1}\|M_{k+1}\cdots M_n\||Q_k|\right]^\chi\right)^{\frac{1}{\chi}}+\big(\mathbb{E}|Q_n|^\chi\big)^{\frac{1}{\chi}}\\
&\leqslant\sum_{k=0}^{n-1}\big(\mathbb{E}\big[\|M_{k+1}\cdots M_n\||Q_k|\big]^\chi\big)^{\frac{1}{\chi}}+\big(\mathbb{E}|Q_n|^\chi\big)^{\frac{1}{\chi}}\\
&\leqslant\big(\mathbb{E}|Q|^\chi\big)^{\frac{1}{\chi}}\left(1+\sum_{k=1}^{\infty}\big(\mathbb{E}\|M_1\cdots M_k\|^\chi\big)^{\frac{1}{\chi}}\right).
\end{aligned}$$

Thus, we have to prove that the series

$$\sum_{k=1}^{\infty}\big(\mathbb{E}\|M_1\cdots M_k\|^\chi\big)^{\frac{1}{\chi}}$$

is convergent.

For this purpose, observe that by Lemma 2.19, the Hölder inequality and (4.6)

$$\begin{aligned}
\mathbb{E}\|M_1\cdots M_k\|^\chi &\leqslant C\mathbb{E}\Big[\max_\lambda\big\{e^{\frac{\chi\Re\lambda(\log\Pi_k)}{d_\lambda}}\big\}\cdot\Big(1\vee\max_i\big|t_i(\log\Pi_k)\big|^{\chi D}\Big)\Big]\\
&\leqslant C\Big(\mathbb{E}\Big[\max_\lambda\big\{e^{\frac{p\chi\Re\lambda(\log\Pi_k)}{d_\lambda}}\big\}\Big]\Big)^{\frac{1}{p}}\cdot\Big(\mathbb{E}\Big[1\vee\max_i\big|t_i(\log\Pi_k)\big|^{q\chi D}\Big]\Big)^{\frac{1}{q}}\\
&\leqslant C'\Big(\mathbb{E}\Big[\sum_\lambda e^{\frac{p\chi\Re\lambda(\log\Pi_k)}{d_\lambda}}\Big]\Big)^{\frac{1}{p}}\cdot\Big(\mathbb{E}\Big[\prod_i e^{\varepsilon q\chi\sum_{j\leqslant k}|t_i(\log M_j)|}\Big]\Big)^{\frac{1}{q}}\\
&\leqslant C''\sum_\lambda\big(\mathbb{E}\big[e^{\frac{p\chi\Re\lambda(\log M)}{d_\lambda}}\big]\big)^{\frac{k}{p}}\cdot\big(\mathbb{E}\big[e^{\varepsilon q\chi\sum_i|t_i(\log M)|}\big]\big)^{\frac{k}{q}}.
\end{aligned}$$

Therefore, applying (4.6), (4.8) and (4.2) we obtain

$$\begin{aligned}
\sum_{k=1}^\infty\big(\mathbb{E}\|M_1\cdots M_k\|^\chi\big)^{\frac{1}{\chi}} &\leqslant C\sum_{k=1}^\infty\bigg(\sum_\lambda L\bigg(\frac{p\chi\Re\lambda}{d_\lambda}\bigg)^{\frac{k}{p}}\cdot f(\varepsilon q\chi)^{\frac{k}{q}}\bigg)^{\frac{1}{\chi}}\\
&\leqslant C'\sum_\lambda\sum_{k=1}^\infty\bigg[L\bigg(\frac{a_\lambda\Re\lambda}{d_\lambda}\bigg)^{\frac{1}{p}}\cdot(1+\delta)^{\frac{1}{q}}\bigg]^{\frac{k}{\chi}}\\
&\leqslant C'\sum_\lambda\sum_{k=1}^\infty\bigg[\bigg(L\bigg(\frac{a_\lambda\Re\lambda}{d_\lambda}\bigg)\cdot(1+\delta)\bigg)^{\frac{1}{2\chi}}\bigg]^k,
\end{aligned}$$

where for the last inequality we used (4.7).

Finally by (4.3)

$$\bigg(L\bigg(\frac{a_\lambda\Re\lambda}{d_\lambda}\bigg)\cdot(1+\delta)\bigg)^{\frac{1}{2\chi}}<1,$$

therefore the series above converges. □

Corollary 4.9. *For any ε there exists C_ε such that*

$$\mathbb{P}\big[|R|>t\big]\leqslant C_\varepsilon t^{-(\chi_0-\varepsilon)}$$

Proof. We have

$$t^{\chi_0-\varepsilon}\mathbb{P}\big[|R|>t\big]\leqslant\int_{\{x:\,|x|>t\}}|x|^{\chi_0-\varepsilon}\nu(\mathrm{d}x)\leqslant\mathbb{E}|R|^{\chi_0-\varepsilon}$$

and by the lemma above the value is finite. □

4.2. Lower estimates

To prove the lower estimate we choose a simple root λ_0 such that $\chi_{\lambda_0}=\chi_0$, satisfying (A5)–(A7), and then study projection of the random walk R_n on a suitable one or two dimensional linear subspace of $\mathcal{N}_{\lambda_0}$ or $\mathcal{N}^{\mathbb{C}}_{\lambda_0}\oplus\mathcal{N}^{\mathbb{C}}_{\bar\lambda_0}$, respectively, depending whether λ_0 is real or complex. In both cases the projected random walk can be explicitly computed. If λ_0 is real we obtain just a random walk on $\mathbb{R}$ generated by the action of "ax + b" group on $\mathbb{R}$, described in Section 3.2, and we conclude the result from Theorem 3.4. The case when λ_0 is complex is more complicated. Then we obtain a random walk on $\mathbb{R}^2$ generated by the action of $\mathbb{R}^+\times O(2)$ on $\mathbb{R}^2$, as studied in [2]. But our assumptions are different and we cannot apply the results proved there, so we shall give here a complete proof based on some ideas of A.K. Grincevičius [13] and Ch. Goldie [11].

Fix a simple root λ_0 satisfying all the assumptions (A1)–(A7). We shall consider two cases.

Case I. λ_0 is real.

Then we have the following lemma.

Lemma 4.10. *If λ_0 is real then there exists a positive constant C such that*

$$\mathbb{P}\big[|R|>t\big]\geqslant Ct^{-\chi_0}.$$

Proof. Notice that by (2.8)

$$\big(\delta_a(x)\big)_{\lambda_0,i_{\lambda_0}}=\mathrm{e}^{\lambda_0(\log a)}x_{\lambda_0,i_{\lambda_0}},$$

for $x=\exp(\sum x_{\lambda,i}Z_{\lambda,i})$. We shall prove that

$$|x|\geqslant\left(\frac{|x_{\lambda_0,i_{\lambda_0}}|}{c_{\lambda_0,i_{\lambda_0}}}\right)^{\frac{1}{d_{\lambda_0}}}\quad\text{for } x\in N. \tag{4.11}$$

Note that if for some $x\in N$

$$\big|x_{\lambda_0,i_{\lambda_0}}\big|\geqslant c_{\lambda_0,1}t^{d_{\lambda_0}}$$

then

$$\big|\big(\sigma_{t^{-1}}(x)\big)_{\lambda_0,i_{\lambda_0}}\big|=\big|\big(\delta_{(\exp(-\log t)H_0)}(x)\big)_{\lambda_0,i_{\lambda_0}}\big|=t^{-d_{\lambda_0}}|x_{\lambda_0,i_{\lambda_0}}|\geqslant c_{\lambda_0,i_{\lambda_0}}$$

hence $|x|\geqslant t$, which gives (4.11).

Thus, we have

$$\mathbb{P}\big[|R|>t\big]\geqslant\mathbb{P}\big[|R_{\lambda_0,i_{\lambda_0}}|>c_{\lambda_0,i_{\lambda_0}}t^{d_{\lambda_0}}\big]. \tag{4.12}$$

But notice that because λ_0 is simple, in view of (2.13), we have

$$(R_n)_{\lambda_0,i_{\lambda_0}}=\bar{M}_n(R_{n-1})_{\lambda_0,i_{\lambda_0}}+\bar{Q}_n,$$

where

$$\bar{M}_n=\mathrm{e}^{\lambda_0(\log M_n)},$$
$$\bar{Q}_n=(Q_n)_{\lambda_0,i_{\lambda_0}}.$$

The foregoing formula defines a random walk on $\mathbb{R}$, generated by the action of the "ax + b" group on $\mathbb{R}$, which is a special case of the situation described in Section 3.2. Moreover all the assumptions of Theorem (3.4) are satisfied for $\bar{\chi}=\chi_0/d_{\lambda_0}$:

$$\mathbb{E}\bar{M}^{\bar{\chi}}=\mathbb{E}\big[\mathrm{e}^{\frac{\chi_0\lambda_0(\log M)}{d_{\lambda_0}}}\big]=1$$

and by (4.11)

$$\mathbb{E}\big|\bar{Q}\big|^{\bar{\chi}}\leqslant C\mathbb{E}|Q|^{\chi_0}<\infty.$$

$R_{\lambda_0,i_{\lambda_0}}$ is the limit in law of $(R_n)_{\lambda_0,i_{\lambda_0}}$, therefore there exists a positive constant C such that

$$\mathbb{P}\big[|R_{\lambda_0,i_{\lambda_0}}|>c_{\lambda_0,i_{\lambda_0}}t^{d_{\lambda_0}}\big]\geqslant Ct^{-\bar{\chi}\cdot d_{\lambda_0}}=Ct^{-\chi_0}.$$

Combining the inequality above with (4.12) we obtain the lemma. □

Case II. λ_0 *is complex.*

To simplify our notation we shall write Z instead of $Z_{\lambda_0,i_{\lambda_0}}$, then $\bar{Z}=Z_{\bar{\lambda}_0,i_{\lambda_0}}$. Define

$$X=\frac{1}{2}\big(Z+\bar{Z}\big),$$
$$Y=-\frac{\mathrm{i}}{2}\big(Z-\bar{Z}\big).$$

For any $x\in N$ let $x|_V$ denotes the projection of $\log x$ onto the real space V, spanned by X and Y. Let $|\cdot|_0$ be the usual Euclidean norm on V, i.e. $|v|_0=\sqrt{\alpha^2+\beta^2}$ for $v=\alpha X+\beta Y\in V$.

Lemma 4.13. *We have*

$$\mathbb{P}\big[|R| > t\big] \geqslant \mathbb{P}\big[|R|_V|_0 \geqslant 2c_{\lambda_0,i_{\lambda_0}} t^{d_{\lambda_0}}\big].$$

Proof. It is enough to prove that for any $x \in N$

$$|x| \geqslant \left(\frac{|x|_V|_0}{2c_{\lambda_0,i_{\lambda_0}}}\right)^{\frac{1}{d_{\lambda_0}}}. \tag{4.14}$$

Assume

$$|x|_V|_0 \geqslant 2c_{\lambda_0,i_{\lambda_0}} \cdot t^{d_{\lambda_0}} \quad \text{and} \quad x|_V = \alpha X + \beta Y.$$

By (2.8) we have

$$\begin{aligned} \delta_a(\exp Z) &= \exp\big(e^{\lambda_0(\log a)} Z + W\big),\\ \delta_a\big(\exp \bar Z\big) &= \exp\big(e^{\bar\lambda_0(\log a)} \bar Z + \bar W\big), \end{aligned} \tag{4.15}$$

where $W \in \operatorname{span}\{Z_{\lambda_0,1}, \dots, Z_{\lambda_0,i_{\lambda_0}-1}\}$.

Then

$$\begin{aligned} \sigma_{t^{-1}}(x) &= \sigma_{t^{-1}}\left(\exp\left(\frac{\alpha - i\beta}{2}\cdot Z + \frac{\alpha + i\beta}{2}\cdot \bar Z + W'\right)\right)\\ &= \exp\left(\frac{\alpha - i\beta}{2}\cdot t^{-\lambda_0(H_0)} Z + \frac{\alpha + i\beta}{2}\cdot t^{-\lambda_0(H_0)}\bar Z + W''\right). \end{aligned}$$

Notice that

$$\left|\frac{\alpha - i\beta}{2}\cdot t^{-\lambda_0(H_0)}\right| = \frac{|x|_V|_0}{2}\cdot t^{-d_{\lambda_0}} \geqslant c_{\lambda_0,i_{\lambda_0}},$$

therefore $\log \sigma_{t^{-1}}(x) \notin \Omega$, which implies $|x| \geqslant t$ and proves (4.14). □

The lemma reduces the problem to prove existence of a positive constant C such that

$$\mathbb{P}\big[|R|_V|_0 \geqslant t\big] \geqslant C t^{-\frac{\chi_0}{d_{\lambda_0}}}. \tag{4.16}$$

Let us restrict the random walk R_n to the linear space V, defining $\bar R_n = R_n|_V$. Then the following holds

Lemma 4.17. *The random variables $\bar R_n$ satisfy the following stochastic recursion*

$$\bar R_n = \bar M_n \bar R_{n-1} + \bar Q_n,$$

where $\bar Q_n = Q_n|_V$ and $\bar M_n = r(\bar M_n) O(\bar M_n)$, where

$$r\big(\bar M_n\big) = e^{\Re\lambda_0(\log M_n)}$$

is an element of one parameter group of dilations of $\mathbb{R}^2$, and

$$O\big(\bar M_n\big) = \begin{pmatrix} \cos(\Im\lambda_0(\log M_n)) & \sin(\Im\lambda_0(\log M_n))\\ -\sin(\Im\lambda_0(\log M_n)) & \cos(\Im\lambda_0(\log M_n)) \end{pmatrix}$$

belongs to the orthogonal group $O(2)$.

Proof. By (4.15)

$$\begin{aligned} \delta_a(\exp X) + i\delta_a(\exp Y) = \delta_a(\exp Z) &= \exp\big(e^{\lambda_0(\log a)} Z + W\big)\\ &= \exp\big(e^{\Re\lambda_0(\log a)}\big(\cos(\Im\lambda_0(\log a))X - \sin\big(\Im\lambda_0(\log a)\big)Y\big)\\ &\quad + i e^{\Re\lambda_0(\log a)}\big(\cos\big(\Im\lambda_0(\log a)\big)Y + \sin\big(\Im\lambda_0(\log a)\big)X\big) + W\big). \end{aligned}$$

Since the action of A is real we have

$$\delta_a(\exp X)|_V = \mathrm{e}^{\Re\lambda_0(\log a)}\big(\cos\big(\Im\lambda_0(\log a)\big)X - \sin\big(\Im\lambda_0(\log a)\big)Y\big),$$
$$\delta_a(\exp Y)|_V = \mathrm{e}^{\Re\lambda_0(\log a)}(\cos\big(\Im\lambda_0(\log a)\big)Y + \sin\big(\Im\lambda_0(\log a)\big)X).$$

which, in view of (2.13), implies the lemma. □

Denote $\bar{S} = V \rtimes (\mathbb{R}^+ \times O(2))$ to be the semi-direct product of V and $\mathbb{R}^+ \times O(2)$, then $\bar{R}_n$ define a random walk on V analogous to (3.1) i.e. $\bar{R}_n$ is a projection onto V of $(\bar{Q}_n, \bar{M}_n)\cdots(\bar{Q}_1, \bar{M}_1)$ where $(\bar{Q}_n, \bar{M}_n)$ are i.i.d. $\bar{S}$-valued random variables.

Projecting our assumptions (A1)–(A7) onto V we obtain

- $\mathbb{E}\log r(\bar{M}) < 0$,
- $\mathbb{E}r(\bar{M})^{\bar{\chi}} = 1$, for $\bar{\chi} = \frac{\chi_0}{d_{\lambda_0}}$,
- $\mathbb{E}[r(\bar{M})^{\bar{\chi}}|\log r(\bar{M})|] < \infty$,
- $\mathbb{E}|\bar{Q}|_0^{\bar{\chi}} < \infty$.

Moreover $\bar{R} = R|_V$ is the limit in distribution of $\bar{R}_n$. Random walks of this type were studied in [2], where asymptotic of their tails has been described:

$$\lim_{t\to\infty} t^{\bar{\chi}}\mathbb{P}\big[|\bar{R}|_0 > t\big] = C.$$

In order to prove positivity of the constant C the authors needed some additional hypothesis: boundedness of $|\bar{Q}|_0$ and larger moments of $r(\bar{M})$. The argument used there, based on a theorem of Landau, cannot be applied here. To prove positivity of C we shall apply to our settings an approach of Grincevičius [13] and Goldie [11], who considered the problem on the "ax + b" group.

Define the "backward" process $\bar{R}_n^*$:

$$\bar{R}_0^* = 0,$$
$$\bar{R}_n^* = \pi_V\big((\bar{Q}_1, \bar{M}_1)\cdots(\bar{Q}_n, \bar{M}_n)\big) = \bar{Q}_1 + \bar{\Pi}_1 Q_2 + \cdots + \bar{\Pi}_{n-1}\bar{Q}_n,$$

where

$$\bar{\Pi}_k = \bar{M}_1\cdots\bar{M}_k.$$

Recall that $\bar{R}_n^*$ converges pointwise to $\bar{R}$, and

$$\bar{R} = \bar{R}_n^* + \bar{\Pi}_n\bar{R}^{*,n}, \tag{4.18}$$

where

$$\bar{R}^{*,n} = \sum_{k=n+1}^{\infty}\big(\bar{M}_{n+1}\cdots\bar{M}_{k-1}\big)\bar{Q}_k,$$

hence for any n, $\bar{R}^{*,n}$ and $\bar{R}$ have the same distribution.

Lemma 4.19. *There exists a positive constant C such that*

$$\mathbb{P}\big[|\bar{R}|_0 > t\big] \geqslant Ct^{-\bar{\chi}}.$$

Proof. Fix two positive numbers η and δ. There exists a ball U in V centered at some point u of radius δ such that $\varepsilon = \mathbb{P}[\bar{R}\in U]$ is positive. Then by (4.18) we have

$$\mathbb{P}\Big[\inf_{x\in U}\big|\bar{R}_n^* + \bar{\Pi}_n x\big|_0 > t \text{ for some } n\Big] = \sum_n \mathbb{P}\Big[\max_{i<n}\inf_{x\in U}\big|\bar{R}_i^* + \bar{\Pi}_i x\big|_0 \leqslant t \text{ and } \inf_{x\in U}\big|\bar{R}_n^* + \bar{\Pi}_n x\big|_0 > t\Big]$$

$$= \frac{1}{\varepsilon} \sum_n \mathbb{P}\Big[\max_{i<n} \inf_{x\in U} \big|\bar{R}_i^* + \bar{\Pi}_i x\big|_0 \leqslant t \text{ and } \inf_{x\in U} \big|\bar{R}_n^* + \bar{\Pi}_n x\big|_0 > t\Big] \mathbb{P}\big[\bar{R}^{*,n} \in U\big]$$

$$= \frac{1}{\varepsilon} \sum_n \mathbb{P}\Big[\max_{i<n} \inf_{x\in U} \big|\bar{R}_i^* + \bar{\Pi}_i x\big|_0 \leqslant t \text{ and } \inf_{x\in U} \big|\bar{R}_n^* + \bar{\Pi}_n x\big|_0 > t \text{ and } |\bar{R}|_0 > t\Big]$$

$$\leqslant \frac{1}{\varepsilon} \mathbb{P}\big[\big|\bar{R}\big|_0 > t\big].$$

Define

$$U_n = \bar{R}_n^* + \bar{\Pi}_n u - \big(\bar{R}_{n-1}^* + \bar{\Pi}_{n-1} u\big) = \bar{\Pi}_{n-1}\big(\bar{Q}_n + \big(\bar{M}_n - I\big)u\big).$$

Then we have

$$\begin{aligned}
\mathbb{P}\big[|\bar{R}|_0 > t\big] &\geqslant \varepsilon \mathbb{P}\Big[\inf_{x\in U} \big|\bar{R}_n^* + \bar{\Pi}_n x\big|_0 > t \text{ for some } n\Big] \\
&\geqslant \varepsilon \mathbb{P}\big[\big|\bar{R}_n^* + \bar{\Pi}_n u\big|_0 - r\big(\bar{\Pi}_n\big)\delta > t \text{ for some } n\big] \\
&\geqslant \varepsilon \mathbb{P}\big[|U_n|_0 - \big(r\big(\bar{\Pi}_n\big) + r\big(\bar{\Pi}_{n-1}\big)\big)\delta > 2t \text{ for some } n\big] \\
&= \varepsilon \mathbb{P}\big[r\big(\bar{\Pi}_{n-1}\big)\big(\big|\bar{Q}_n + (\bar{M}_n - I)u\big|_0 - \big(r\big(\bar{M}_n\big) + 1\big)\delta\big) > 2t \text{ for some } n\big] \\
&\geqslant \varepsilon \mathbb{P}\big[\big|\bar{Q} + \big(\bar{M} - I\big)u\big|_0 - \big(r\big(\bar{M}\big) + 1\big)\delta > \eta\big] \mathbb{P}\Big[\max_n r\big(\bar{\Pi}_n\big) > 2t/\eta\Big] \\
&\geqslant C\mathbb{P}\big[\big|\bar{Q} + \big(\bar{M} - I\big)u\big|_0 - \big(r\big(\bar{M}\big) + 1\big)\delta > \eta\big] t^{-\bar{\chi}},
\end{aligned}$$

where the last inequality follows from (3.6). Finally we have to justify that for sufficiently small η and δ the constant above is positive.

By (A7) there exist positive numbers η, θ such that

$$\mathbb{P}\big[\big|\bar{Q} + \big(\bar{M} - I\big)u\big|_0 > 2\eta\big] = \theta.$$

Moreover by (A2) there is a large number N such that

$$\mathbb{P}\big[r\big(\bar{M}\big) \geqslant N\big] \leqslant \frac{\theta}{2},$$

hence taking $\delta = \frac{\eta}{N+1}$ we obtain

$$\begin{aligned}
\mathbb{P}\big[\big|\bar{Q} + \big(\bar{M} - I\big)u\big|_0 - \big(r\big(\bar{M}\big) + 1\big)\delta > \eta\big] &\geqslant \mathbb{P}\big[\big|\bar{Q} + \big(\bar{M} - I\big)u\big|_0 > 2\eta \text{ and } r\big(\bar{M}\big) < N\big] \\
&\geqslant \mathbb{P}\big[\big|\bar{Q} + \big(\bar{M} - I\big)u\big|_0 > 2\eta\big] - \mathbb{P}\big[r\big(\bar{M}\big) \geqslant N\big] \\
&\geqslant \frac{\theta}{2},
\end{aligned}$$

which finishes the proof. □

Finally, in view of Lemma 4.13, the foregoing result implies the lower estimate of the tail of R when λ_0 is complex.

5. Proof of Main Theorem B

5.1. Diagonal action of A on N

In this section we shall change slightly our notation. From now we shall assume that the action of A on N is diagonalizable. Then all the roots are real and the real vectors $\{Z_{\lambda,j}\}_{\substack{\lambda\in\Delta\\ j\leqslant i_\lambda}}$ form a basis of $\mathcal{N}$. Let us denote these vectors by $X_1, \ldots, X_{n_0}$ ($n_0 = \dim N$), then for any $H \in \mathcal{A}$

$$\operatorname{ad}(H)X_j = \lambda_j(H)X_j, \quad j = 1, \ldots, n_0,$$

for some root λ_j. In this notation it may of course happen that $\lambda_i = \lambda_j$ for $i \neq j$.

Then the action of A on N is given by

$$\delta_a(x)=\exp\left(\sum_j \mathrm{e}^{\lambda_j(H)}x_jX_j\right), \tag{5.1}$$

for $x=\exp(\sum x_jX_j)$.

We change also numeration of constants defined in previous chapters. If $X_j=X_{\lambda,i}$ then we define

$$\chi_j=\chi_\lambda,$$
$$c_j=c_{\lambda,i},$$
$$d_j=d_\lambda.$$

Notice that in this case both norms, on N and on A, can be explicitly computed:

$$|x|=\inf\{b\colon \log(\sigma_{b^{-1}}(x))\in\Omega\cap\mathcal{N}\}=\inf\left\{b\colon \sum_j b^{-\lambda_j(H_0)}x_jX_j\in\Omega\cap\mathcal{N}\right\}$$

$$=\inf\{b\colon |x_j|<c_jb^{d_j}\text{ for }j=1,\dots,n_0\}=\inf\left\{b\colon \frac{|x_j|^{\frac{1}{d_j}}}{c_j^{\frac{1}{d_j}}}<b\right\}=\max_j\{\bar c_j|x_j|^{\frac{1}{d_j}}\} \tag{5.2}$$

for $\bar c_j=c_j^{-\frac{1}{d_j}}$, and next

$$\|a\|=\max_{|x|=1}|\delta_a(x)|=\max_j\{\mathrm{e}^{\frac{\lambda_j(\log a)}{d_j}}\}. \tag{5.3}$$

Let us define the lower central sequence in $\mathcal{N}$

$$\mathcal{N}_0=\mathcal{N},$$
$$\mathcal{N}_{i+1}=[\mathcal{N}_i,\mathcal{N}],$$

for $i=1,\dots,m_0$ and $\mathcal{N}_{m_0+1}=\{0\}$. Then we may assume that there is a sequence $0=i_0\leqslant i_1\leqslant\cdots\leqslant i_{m_0}=n_0$ such that $X_{i_j+1},\dots,X_{i_{j+1}}$ are a basis of $\mathcal{N}_j/\mathcal{N}_{j+1}$.

Define

$$I_1=\{1,\dots,i_1\}$$

and notice the set of indices of simple roots is a subset of I_1.

We shall use the lower central sequence to obtain a better description of the Campbell–Hausdorf formula [10]. If $(x\cdot y)_i$ denotes ith coordinate of $x\cdot y$, for $x=\exp(\sum x_iX_i)$, $y=\exp(\sum y_iX_i)$ elements of N, then

$$(x\cdot y)_i=x_i+y_i\quad\text{for }i\in I_1,$$
$$(x\cdot y)_i=x_i+y_i+P_i(x,y)\quad\text{for }i\in I_p,\ \text{for }p>1, \tag{5.4}$$

where P_i are polynomials depending on $x_1,\dots x_{i_{p-1}},y_1,\dots y_{i_{p-1}}$ and they can be written as

$$P_i(x,y)=\sum_{\underline{\mathbf{a}},\underline{\mathbf{b}}}c_{\underline{\mathbf{a}},\underline{\mathbf{b}}}P_i^{\underline{\mathbf{a}},\underline{\mathbf{b}}}(x,y)=\sum_{\underline{\mathbf{a}},\underline{\mathbf{b}}}c_{\underline{\mathbf{a}},\underline{\mathbf{b}}}x^{\underline{\mathbf{a}}}y^{\underline{\mathbf{b}}}, \tag{5.5}$$

where $c_{\underline{\mathbf{a}},\underline{\mathbf{b}}}$ are some real constants (most of them are zero, then we assume $P_j^{\underline{\mathbf{a}},\underline{\mathbf{b}}}=0$, but for at least one pair $(\underline{\mathbf{a}},\underline{\mathbf{b}})$ the constant $c_{\underline{\mathbf{a}},\underline{\mathbf{b}}}$ is nonzero), $\underline{\mathbf{a}}$ and $\underline{\mathbf{b}}$ are multi-indexes of natural numbers of the length i_{p-1}, and we have used the notation (which will be used also in the rest of the paper without any saying):

- $0^0=1$;
- if $\underline{\mathbf{c}}$ is a multi-index of the length i and z is a vector of length at least i (usually it will be longer than i) then

$$z^{\underline{\mathbf{c}}}=\prod_{j\leqslant i}z_j^{\underline{\mathbf{c}}_j}.$$

Moreover, we shall strongly rely on the following properties of the Campbell–Hausdorf formula: if $c_{\underline{\mathbf{a}},\underline{\mathbf{b}}}$ is nonzero then:

$$\text{both } \underline{\mathbf{a}} \text{ and } \underline{\mathbf{b}} \text{ are nonzero and } \sum_{j<i}(\underline{\mathbf{a}}_j+\underline{\mathbf{b}}_j)\lambda_j=\lambda_i. \tag{5.6}$$

In order to prove the last equation we shall use (5.1). Fix $H\in\mathcal{A}$, then for any $x, y\in N$ we have

$$\big(\delta_{\exp H}(xy)\big)_i=\mathrm{e}^{\lambda_i(H)}(x\cdot y)_i,$$

but on the other side, by (2.1) we write

$$\begin{aligned}\big(\delta_{\exp H}(xy)\big)_i&=\big(\delta_{\exp H}(x)\cdot\delta_{\exp H}(x)\big)_i\\&=\sum_{\underline{\mathbf{a}},\underline{\mathbf{b}}}c_{\underline{\mathbf{a}},\underline{\mathbf{b}}}\big(\delta_{\exp H}(x)\big)^{\underline{\mathbf{a}}}\big(\delta_{\exp H}(y)\big)^{\underline{\mathbf{b}}}=\sum_{\underline{\mathbf{a}},\underline{\mathbf{b}}}c_{\underline{\mathbf{a}},\underline{\mathbf{b}}}\mathrm{e}^{\sum_{j<i}(\underline{\mathbf{a}}_j+\underline{\mathbf{b}}_j)\lambda_j(H)}x^{\underline{\mathbf{a}}}y^{\underline{\mathbf{b}}}.\end{aligned}$$

Comparing last two equations we obtain (5.6).

5.2. *Proof of the first part of Main Theorem B*

To prove the theorem we shall compute explicitly R. Recall that R is the pointwise limit of the "backward" process R_n^*, by definition (3.3) and the Campbell–Hausdorf formula (5.4) we have

$$\begin{aligned}\big(R_{n+1}^*\big)_j&=\big(R_n^*\cdot\delta_{\Pi_n}(Q_{n+1})\big)_j\\&=\big(R_n^*\big)_j+\mathrm{e}^{\lambda_j(\log\Pi_n)}(Q_{n+1})_j+\sum_{\underline{\mathbf{a}},\underline{\mathbf{b}}}c_{\underline{\mathbf{a}},\underline{\mathbf{b}}}(R_n^*)^{\underline{\mathbf{a}}}\big(\delta_{\Pi_n}(Q_{n+1})\big)^{\underline{\mathbf{b}}}\\&=\sum_{k=0}^{n}\mathrm{e}^{\lambda_j(\log\Pi_k)}(Q_{k+1})_j+\sum_{\underline{\mathbf{a}},\underline{\mathbf{b}}}c_{\underline{\mathbf{a}},\underline{\mathbf{b}}}\sum_{k=0}^{n}\big(R_k^*\big)^{\underline{\mathbf{a}}}\big(\delta_{\Pi_k}(Q_{k+1})\big)^{\underline{\mathbf{b}}}.\end{aligned}$$

Hence

$$\big(R_{n+1}^*\big)_j=\begin{cases}T_n^j, & \text{for } j\in I_1,\\ T_n^j+\sum_{\underline{\mathbf{a}},\underline{\mathbf{b}}}c_{\underline{\mathbf{a}},\underline{\mathbf{b}}}\sum_{k=0}^{n}(R_k^*)^{\underline{\mathbf{a}}}(\delta_{\Pi_k}(Q_{k+1}))^{\underline{\mathbf{b}}}, & \text{for } j\notin I_1,\end{cases} \tag{5.7}$$

where

$$T_n^j=\sum_{k=0}^{n}\mathrm{e}^{\lambda_j(\log\Pi_k)}(Q_{k+1})_j.$$

Notice that T_n^j is the "backward" process for a random walk generated by i.i.d. random variables $(\mathrm{e}^{\lambda_j(\log M_k)},(Q_k)_j)$, which converges pointwise to some random variable T^j. We shall later estimate T_n^j by

$$\bar{T}_n^j=\sum_{k=0}^{n}\mathrm{e}^{\lambda_j(\log\Pi_k)}\big|(Q_{k+1})_j\big|,$$

then $\bar{T}_n^j$ is also the "backward process" generated by $(\mathrm{e}^{\lambda_j(\log M_k)},|(Q_k)_j|)$. $\bar{T}_n^j$ converges monotonously to a random variable $\bar{T}^j$, which by Theorem 3.4 satisfies

$$\mathbb{P}\big[\bar{T}^j>t^{d_j}\big]\leqslant Ct^{-\chi_j}. \tag{5.8}$$

Lemma 5.9. *For any i*

$$\mathbb{P}\big[\big|\big(R_k^*\big)_i\big|>t^{d_i}\ \text{for some } k\big]\leqslant Ct^{-\chi_0}.$$

Proof. For $j \in I_1$ we have

$$\mathbb{P}\big[\big|\big(R_n^*\big)_j\big| > t^{d_j} \text{ for some } n\big] = \mathbb{P}\big[\big|T_n^j\big| > t^{d_j} \text{ for some } n\big] \leqslant \mathbb{P}\big[\bar{T}_n^j > t^{d_j} \text{ for some } n\big]$$
$$= \mathbb{P}\big[\bar{T}^j > t^{d_j}\big] \leqslant Ct^{-\chi_j} \leqslant Ct^{-\chi_0}. \tag{5.10}$$

Fix $j \notin I_1$ and assume that the lemma holds for $i < j$. Then by (5.7), for $C = \sum |c_{\underline{\mathbf{a}},\underline{\mathbf{b}}}| + 1$, we have

$$\mathbb{P}\big[\big|\big(R_{n+1}^*\big)_j\big| > t^{d_j} \text{ for some } n\big]$$
$$\leqslant \mathbb{P}\left[\bar{T}^j > \frac{t^{d_j}}{C}\right] + \sum_{\{\underline{\mathbf{a}},\underline{\mathbf{b}}:\ c_{\underline{\mathbf{a}},\underline{\mathbf{b}}}\neq 0\}} \mathbb{P}\left[\sum_{k=0}^{n}\big|\big(R_k^*\big)^{\underline{\mathbf{a}}}\big(\delta_{\Pi_k}(Q_{k+1})\big)^{\underline{\mathbf{b}}}\big| > \frac{t^{d_j}}{C} \text{ for some } n\right]$$
$$\leqslant C't^{-\chi_j} + \sum_{\{\underline{\mathbf{a}},\underline{\mathbf{b}}:\ c_{\underline{\mathbf{a}},\underline{\mathbf{b}}}\neq 0\}} \mathbb{P}\left[\Big(\max_{k\leqslant n}\big|(R_k^*)^{\underline{\mathbf{a}}}\big|\Big)\cdot\left(\sum_{k=0}^{n}\big|\big(\delta_{\Pi_k}(Q_{k+1})\big)^{\underline{\mathbf{b}}}\big|\right) > \frac{t^{d_j}}{C} \text{ for some } n\right]$$
$$\leqslant C't^{-\chi_j} + \sum_{\{\underline{\mathbf{a}},\underline{\mathbf{b}}:\ c_{\underline{\mathbf{a}},\underline{\mathbf{b}}}\neq 0\}} \mathbb{P}\left[\left(\max_{1\leqslant k<\infty}\left|\prod_{i<j}(R_k^*)_i^{\underline{\mathbf{a}}_i}\right|\right)\cdot\left(\sum_{k=0}^{n}\prod_{i<j}\big(e^{\lambda_i(\log(\Pi_k))}\big|(Q_{k+1})_i\big|\big)^{\underline{\mathbf{b}}_i}\right) > \frac{t^{d_j}}{C} \text{ for some } n\right]$$
$$\leqslant C't^{-\chi_i} + \sum_{\{\underline{\mathbf{a}},\underline{\mathbf{b}}:\ c_{\underline{\mathbf{a}},\underline{\mathbf{b}}}\neq 0\}} \mathbb{P}\left[\prod_{i<j}\left(\Big(\max_{1\leqslant k\leqslant\infty}\big|\big(R_k^*\big)_i\big|\Big)^{\underline{\mathbf{a}}_i}\cdot\big(\bar{T}^i\big)^{\underline{\mathbf{b}}_i}\right) > \frac{t^{d_j}}{C}\right]. \tag{5.11}$$

If $c_{\underline{\mathbf{a}},\underline{\mathbf{b}}}$ is nonzero, then by (5.6) and (2.18)

$$d_j = \sum_{i<j}(\underline{\mathbf{a}}_i + \underline{\mathbf{b}}_i)d_i. \tag{5.12}$$

Therefore the above expression can be dominated by

$$C't^{-\chi_j} + \sum_{\{\underline{\mathbf{a}},\underline{\mathbf{b}}:\ c_{\underline{\mathbf{a}},\underline{\mathbf{b}}}\neq 0\}}\sum_{i<j}\left(\mathbb{P}\left[\max_k\big|\big(R_k^*\big)_i\big| > \frac{t^{d_i}}{C''}\right] + \mathbb{P}\left[\bar{T}^i > \frac{t^{d_i}}{C''}\right]\right) \leqslant Ct^{-\chi_0},$$

by the induction assumption and (5.8). □

Corollary 5.13. *There exists a positive constant C such that*

$$\frac{1}{C}t^{-\chi_0} \leqslant \mathbb{P}\big[|R| > t\big] \leqslant Ct^{-\chi_0}.$$

Proof. By the lemma above we have

$$\mathbb{P}\big[|R| > t\big] \leqslant \sum_j \mathbb{P}\big[|R_j| > c_j t^{d_j}\big] \leqslant \sum_j \mathbb{P}\big[\big|\big(R_n^*\big)_j\big| \geqslant c_j t^{d_j} \text{ for some } n\big] \leqslant Ct^{-\chi_0}.$$

On the other hand, choose j_0 such that λ_{j_0} is a simple root, $\chi_{j_0} = \chi_0$ and satisfies (A3) and (A4). Then by (5.2) and Theorem 3.4

$$\mathbb{P}\big[|R| > t\big] \geqslant \mathbb{P}\big[|R_{j_0}| \geqslant c_{j_0}t^{d_{j_0}}\big] \geqslant \mathbb{P}\big[\big|T^{j_0}\big| \geqslant c_{j_0}t^{d_{j_0}}\big] \geqslant Ct^{-\chi_0}. \qquad \square$$

5.3. Dominant root

Now, we shall assume that there exists in Δl_1 a dominant root. Let us denote it by λ_0 and let j_0 be an index such that $\lambda_{j_0} = \lambda_0$. Define

$$I_0 = \{j:\ \lambda_j \text{ is a multiple of } \lambda_0\}.$$

Then by Lemma 3.9

$$\mathbb{E}\big[\mathrm{e}^{\frac{\chi_0\lambda_j(\log M)}{d_j}}\big]=1,\quad \text{for } j\in I_0$$
$$\mathbb{E}\big[\mathrm{e}^{\frac{\chi_0\lambda_j(\log M)}{d_j}}\big]<1,\quad \text{for } j\notin I_0. \tag{5.14}$$

Let $\mathcal{N}_0$ be the Lie algebra defined by

$$\mathcal{N}_0=\text{Lie span}\{X_j\}_{j\in I_0}=\bigoplus_{\{\lambda:\ \lambda \text{ is a multiple of } \lambda_0\}}\mathcal{N}_\lambda.$$

For any $j\in I_0$ let s_j be the unique number such that $\lambda_j=s_j\lambda_0$. Put $N_0=\exp\mathcal{N}_0$. Notice that for $x\in N_0$ and any $a\in A$, by (5.2)

$$\delta_a(x)=\exp\bigg(\sum_{j\in I_0}\mathrm{e}^{\lambda_j(\log a)}x_jX_j\bigg)=\exp\bigg(\sum_{j\in I_0}b^{s_j\lambda_0(H_0)}x_jX_j\bigg)=\sigma_b(x), \tag{5.15}$$

where $b=\mathrm{e}^{\frac{\lambda_0(\log a)}{\lambda_0(H_0)}}$, hence the action of A on N_0 depends only on the value of $\lambda_0(\log a)$.

Let $S_0=N_0\rtimes\mathbb{R}^+$ be the semi-direct product of N_0 and $\mathbb{R}^+$ with the group multiplication

$$(x,b)\cdot(x',b')=\big(x\cdot\sigma_b(x'),bb'\big),\quad x,x'\in N_0,\ b,b'\in\mathbb{R}^+.$$

Denote by $|\cdot|_0$ the restriction of $|\cdot|$ to N_0, i.e. $|x|_0=|x|$ for $x\in N_0$, by (5.2)

$$|x|_0=\max_{j\in I_0}\big\{\bar{c}_j|x_j|^{\frac{1}{d_j}}\big\}.$$

For any $x=\exp(\sum x_{\lambda,j}X_{\lambda,j})\in N$ let $x|_{N_0}$ denotes its restriction to N_0, i.e.

$$x|_{N_0}=\exp\bigg(\sum_{j\in I_0}x_jX_j\bigg).$$

We shall prove two lemmas

Lemma 5.16. *We have*

$$\lim_{t\to\infty}t^{\chi_0}\mathbb{P}\big[\big|\bar{R}\big|_0>t^{d_{\lambda_0}}\big]=C_+,$$

for some positive constant C_+, where $\bar{R}=R|_{N_0}$.

Lemma 5.17. *For any j not being an element of I_0*

$$\lim_{t\to\infty}t^{\chi_0}\mathbb{P}\big[|R_j|^{\frac{1}{d_j}}>t\big]=0.$$

Notice, that the lemmas imply the second part of Main Theorem B.

Proof of Main Theorem B. We have

$$\begin{aligned}\mathbb{P}\big[|R|>t\big]&\leqslant\mathbb{P}\Big[\max_{j\in I_0}\big\{\bar{c}_j|R_j|^{\frac{1}{d_j}}\big\}>t\Big]+\mathbb{P}\Big[\max_{j\notin I_0}\big\{\bar{c}_j|R_j|^{\frac{1}{d_j}}\big\}>t\Big]\\&\leqslant\mathbb{P}\big[\big|\bar{R}\big|_0>t\big]+\sum_{j\notin I_0}\mathbb{P}\big[\bar{c}_j|R_j|^{\frac{1}{d_j}}>t\big]\end{aligned}$$

hence

$$\lim_{t\to\infty}t^{\chi_0}\mathbb{P}\big[|R|>t\big]\leqslant C_+.$$

On the other hand

$$\mathbb{P}\big[|R|>t\big]\geqslant\mathbb{P}\Big[\max_{j\in I_0}\big\{\bar{c}_j|R_j|^{\frac{1}{d_j}}\big\}>t\Big]=\mathbb{P}\big[|\bar{R}|_0>t\big],$$

which gives

$$\lim_{t\to\infty} t^{\chi_0}\mathbb{P}\big[|R|>t\big]\geqslant C_+. \qquad \square$$

5.4. Proofs of Lemmas 5.16 and 5.17

Lemma 5.18. *For any* $x, y \in N$

$$x|_{N_0}\cdot y|_{N_0} = (x\cdot y)|_{N_0}.$$

Proof. Suppose $j\in I_0$, then by the Campbell–Hausdorff formula (5.4)

$$(x\cdot y)_j = x_j + y_j + \sum c_{\underline{\mathbf{a}},\underline{\mathbf{b}}} x^{\underline{\mathbf{a}}} y^{\underline{\mathbf{b}}}.$$

To prove the Lemma we have to justify that in the formula above do not appear coordinates x_i and y_i for $i\notin I_0$. In other word we should infer that if $c_{\underline{\mathbf{a}},\underline{\mathbf{b}}}$ is not zero and $\underline{\mathbf{a}}_i+\underline{\mathbf{b}}_i\neq 0$ for some index i, then i belongs to the set I_0.

Assume $c_{\underline{\mathbf{a}},\underline{\mathbf{b}}}\neq 0$. Then by (5.6)

$$\lambda_j = \sum_{i<j}(\underline{\mathbf{a}}_i+\underline{\mathbf{b}}_i)\lambda_i.$$

λ_j is a multiple of λ_0, therefore $\chi_j=\chi_0$. By Lemma 3.9, this implies that if $\underline{\mathbf{a}}_i+\underline{\mathbf{b}}_i\neq 0$ then $\chi_i=\chi_0$, but (5.14) says that then χ_i is a multiple of λ_0 and we conclude the proof. $\square$

Proof of Lemma 5.16. For random variable (Q_j, M_j) as in (3.1) define

$$\bar{Q}_j = Q_j|_{N_0},$$
$$\bar{M}_j = \mathrm{e}^{\frac{\lambda_0(\log M_j)}{\lambda_0(H_0)}}.$$

Then $(\bar{Q}_j,\bar{M}_j)$ are i.i.d. S_0-valued random variables and satisfy assumptions of Theorem 3.4. Consider the random walk on S_0

$$\bar{S}_n = (\bar{Q}_n,\bar{M}_n)\cdots(\bar{Q}_1,\bar{M}_1),$$

by Lemma (5.18) and (5.15)

$$\bar{R}_j = \pi_{N_0}(\bar{S}_n) = R_j|_{N_0},$$

therefore $\bar{R}_j$ converges in law to $\bar{R}$, and we may apply Theorem 3.4, which finishes the proof. $\square$

Lemma 5.19. *If j does not belong to I_0, then there exists $\varepsilon>0$ such that*

$$\mathbb{P}\big[\big|(R_k^*)_j\big|>t^{d_j}\ \textit{for some}\ k\big]\leqslant Ct^{-(\chi_0+\varepsilon)}.$$

Proof. The idea of the proof is the same as of Lemma 5.9, but now we shall proceed more delicate.

If $j\in I_1$, then by (5.10)

$$\mathbb{P}\big[\big|(R_k^*)_j\big|>t^{d_j}\ \text{for some}\ k\big]\leqslant Ct^{-\chi_j}.$$

Next assume that $j\notin I_1\cup I_0$ and the Lemma holds for $i<j$. Then arguing as in (5.11) we obtain

$$\mathbb{P}\big[\big|(R_{n+1}^*)_j\big|>t^{d_j}\ \text{for some}\ n\big]\leqslant Ct^{-\chi_j}+\sum_{\{\underline{\mathbf{a}},\underline{\mathbf{b}}:\ c_{\underline{\mathbf{a}},\underline{\mathbf{b}}}\neq 0\}}\mathbb{P}\bigg[\prod_{i<j}\Big(\Big(\max_k\big|(R_k^*)_i\big|\Big)^{\underline{\mathbf{a}}_i}\cdot(\bar{T}^i)^{\underline{\mathbf{b}}_i}\Big)>\frac{t^{d_j}}{C}\bigg].$$

By (5.14) $\chi_j>\chi_0$, therefore it is enough to estimate, for any nonzero $c_{\underline{\mathbf{a}},\underline{\mathbf{b}}}$ the appropriate factor in the sum above.

The root λ_j is not a multiple of λ_0, therefore, if $c_{\underline{\mathbf{a}},\underline{\mathbf{b}}}\neq 0$, then by (5.6) there exists an index $i_0<j$ such that λ_{i_0} is not a multiple of λ_0 and $\underline{\mathbf{a}}_{i_0}+\underline{\mathbf{b}}_{i_0}>0$. We shall consider two cases

Case 1. $\underline{\mathbf{a}}_{i_0}>0$.

By the induction hypothesis there exists $\chi' > \chi_0$ such that

$$\mathbb{P}\big[\big|\big(R_k^*\big)_{i_0}\big| > t^{d_{i_0}} \text{ for some } k\big] \leqslant C t^{-\chi'}. \tag{5.20}$$

Then take any positive number δ satisfying

$$\delta < \frac{d_{i_0}(\chi' - \chi_0)}{\chi'} \tag{5.21}$$

and define

$$\delta' = \frac{\delta \underline{\mathbf{a}}_{i_0}}{\sum_{i \neq i_0, i<j} (\underline{\mathbf{a}}_i + \underline{\mathbf{b}}_i) + \underline{\mathbf{b}}_{i_0}}.$$

Then, in view of (5.12)

$$d_j = \underline{\mathbf{a}}_{i_0}(d_{i_0} - \delta) + \sum_{i \neq i_0, i<j} (\underline{\mathbf{a}}_i + \underline{\mathbf{b}}_i)(d_i + \delta') + \underline{\mathbf{b}}_{i_0}(d_{i_0} + \delta')$$

and by Lemma 5.9 and (5.20)

$$\begin{aligned}
&\mathbb{P}\bigg[\prod_{i<j}\Big(\Big(\max_k\big|\big(R_k^*\big)_i\big|\Big)^{\underline{\mathbf{a}}_i} \cdot \big(\bar{T}^i\big)^{\underline{\mathbf{b}}_i}\Big) > \frac{t^{d_j}}{C}\bigg] \\
&\quad \leqslant \mathbb{P}\bigg[\max_k\big|\big(R_k^*\big)_{i_0}\big| > \frac{t^{d_{i_0}-\delta}}{C'}\bigg] + \sum_{i \neq i_0, i<j} \mathbb{P}\bigg[\max_k\big|\big(R_k^*\big)_i\big| > \frac{t^{d_i+\delta'}}{C'}\bigg] + \sum_{i<j} \mathbb{P}\bigg[\bar{T}^i > \frac{t^{d_i+\delta'}}{C'}\bigg] \\
&\quad \leqslant C\bigg(t^{-\chi' \cdot \frac{d_{i_0}-\delta}{d_{i_0}}} + \sum_{i<j} t^{-\chi_0 \cdot \frac{d_i+\delta'}{d_i}}\bigg) \leqslant C t^{-\chi}
\end{aligned}$$

for some $\chi > \chi_0$.

Case 2. $\underline{\mathbf{b}}_{i_0} > 0$.

Then take δ as in (5.21) and define

$$\delta'' = \frac{\delta \underline{\mathbf{b}}_{i_0}}{\sum_{i \neq i_0, i<j} (\underline{\mathbf{a}}_i + \underline{\mathbf{b}}_i) + \underline{\mathbf{a}}_{i_0}}.$$

Arguing as above we obtain

$$\begin{aligned}
&\mathbb{P}\bigg[\prod_{i<j}\Big(\Big(\max_k\big|\big(R_k^*\big)_i\big|\Big)^{\underline{\mathbf{a}}_i} \cdot \big(\bar{T}^i\big)^{\underline{\mathbf{b}}_i}\Big) > \frac{t^{d_j}}{C}\bigg] \\
&\quad \leqslant \mathbb{P}\bigg[\bar{T}^{i_0} > \frac{t^{d_{i_0}-\delta}}{C'}\bigg] + \sum_{i<j} \mathbb{P}\bigg[\max_k\big|\big(R_k^*\big)_i\big| > \frac{t^{d_i+\delta''}}{C'}\bigg] + \sum_{i \neq i_0, i<j} \mathbb{P}\bigg[\bar{T}^i > \frac{t^{d_i+\delta''}}{C'}\bigg] \\
&\quad \leqslant C\bigg(t^{-\chi' \cdot \frac{d_{i_0}-\delta}{d_{i_0}}} + \sum_{i<j} t^{-\chi_0 \cdot \frac{d_i+\delta''}{d_i}}\bigg) \leqslant C t^{-\chi}
\end{aligned}$$

for some $\chi > \chi_0$. □

Proof of Lemma 5.17. The lemma is an immediate consequence of the above lemma. □

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