Available online at www.sciencedirect.com

SCIENCE DIRECT®

Ann. I. H. Poincaré – PR 41 (2005) 953–970

ANNALES DE L'INSTITUT HENRI POINCARÉ
PROBABILITÉS ET STATISTIQUES

www.elsevier.com/locate/anihpb

Thick points for the Cauchy process

Olivier Daviaud [1]

Department of Mathematics, Stanford University, Stanford, CA 94305, USA

Received 13 June 2003; received in revised form 11 August 2004; accepted 15 October 2004

Available online 23 May 2005

Abstract

Let $\mu(x,\epsilon)$ denote the occupation measure of an interval of length 2ϵ centered at x by the Cauchy process run until it hits $(-\infty,-1]\cup[1,\infty)$. We prove that $\sup_{|x|\leqslant 1}\mu(x,\epsilon)/(\epsilon(\log\epsilon)^2)\to 2/\pi$ a.s. as $\epsilon\to 0$. We also obtain the multifractal spectrum for thick points, i.e. the Hausdorff dimension of the set of α-thick points x for which $\lim_{\epsilon\to 0}\mu(x,\epsilon)/(\epsilon(\log\epsilon)^2)=\alpha>0$.

Résumé

Soit $\mu(x,\epsilon)$ la mesure d'occupation de l'intervalle $[x-\epsilon,x+\epsilon]$ par le processus de Cauchy arrêté à sa sortie de $(-1,1)$. Nous prouvons que $\sup_{|x|\leqslant 1}\mu(x,\epsilon)/(\epsilon(\log\epsilon)^2)\to 2/\pi$ p.s. lorsque $\epsilon\to 0$. Nous obtenons également un spectre multifractal de points épais en montrant que la dimension de Hausdorff des points x pour lesquels $\lim_{\epsilon\to 0}\mu(x,\epsilon)/(\epsilon(\log\epsilon)^2)=\alpha>0$ est égale à $1-\alpha\pi/2$.

MSC: 60J55

Keywords: Thick points; Multi-fractal analysis; Cauchy process

1. Introduction

Let $X=(X_t,\ t\geqslant 0)$ be a Cauchy process on the real line $\mathbb{R}$, that is a process starting at 0, with stationary independent increments with the Cauchy distribution:

$$\mathbf{P}\big(X_{t+s}-X_t\in(x-\mathrm{d}x,x+\mathrm{d}x)\big)=\frac{s\,\mathrm{d}x}{\pi(s^2+x^2)},\quad s,t>0,\ x\in\mathbb{R}.$$

E-mail address: olivier.daviaud@gmail.com (O. Daviaud).

[1] Research partially supported by NSF grant #DMS-0072331.

0246-0203/$ – see front matter
doi:10.1016/j.anihpb.2004.10.001

Next, let

$$\mu_{\bar\theta}^X(A) := \int_0^{\bar\theta} \mathbf{1}_A(X_s)\,\mathrm{d}s$$

be the occupation measure of a measurable subset A of $\mathbb{R}$ by the Cauchy process run until $\bar\theta := \inf\{s\colon |X_s| \geqslant 1\}$. Let $I(x,\epsilon)$ denote the interval of radius ϵ centered at x. Our first theorem follows:

Theorem 1.1.

$$\lim_{\epsilon\to 0+} \sup_{x\in\mathbb{R}} \frac{\mu_{\bar\theta}^X(I(x,\epsilon))}{\epsilon(\log\epsilon)^2} = 2/\pi \quad \textit{a.s.} \tag{1.1}$$

This should be compared to the following analog of Ray's result [12]: for some constant $0 < c < \infty$

$$\limsup_{\epsilon\to 0+} \frac{\mu_{\bar\theta}^X(I(0,\epsilon))}{\epsilon \log(1/\epsilon) \log\log\log(1/\epsilon)} = c \quad \text{a.s.}$$

The above is a spin-off result of this paper (in combination with [12]) whose proof we shall not include.

Next, it follows from the previous theorem (or more simply from [11, Lemma 2.3]) that for almost all paths,

$$\liminf_{\epsilon\to 0+} \frac{\log \mu_{\bar\theta}^X(I(x,\epsilon))}{\log\epsilon} \geqslant 1$$

for *all* points x in the range $\{X_t \mid 0 \leqslant t \leqslant \bar\theta\}$. On the other hand, this fact together with [2, Chapter VIII, Theorem 5] and Fubini's theorem imply that for $\mathbf{P} \times \mathcal{L}eb$-almost every (ω, t) in $\Omega \times \{0 \leqslant t \leqslant \bar\theta\}$

$$\lim_{\epsilon\to 0+} \frac{\log \mu_{\bar\theta}^X(I(X_t,\epsilon))}{\log\epsilon} = 1.$$

Hence, standard multifractal analysis must be refined in order to obtain a non-degenerate dimension spectrum for thick points. This leads us to

Theorem 1.2. *For any $a \leqslant 2/\pi$,*

$$\dim\left\{x \in \mathbb{R}\colon \lim_{\epsilon\to 0+} \frac{\mu_{\bar\theta}^X(I(x,\epsilon))}{\epsilon(\log\epsilon)^2} = a\right\} = 1 - a\pi/2 \quad \textit{a.s.} \tag{1.2}$$

The results obtained here are the analogues of those in [6], when replacing the planar Brownian motion (which is a stable process of index 2 in dimension 2) by the Cauchy process (which is a stable process of index 1 in dimension 1). Theorems 1.1 and 1.2 answer the first part of open problem (6) of that paper (also implicitly present in [11]), the second part being solved in [7]. Our work relies heavily on the techniques developed in [6] and [7], and therefore owes a substantial debt to these papers.

The Cauchy process is a symmetric stable process of index $\alpha = 1$. Thick points for one-dimensional stable processes have been studied in several papers. In [14], a multifractal spectrum of thick points is obtained for stable subordinators of index $\alpha < 1$. A one-sided version of this result also follows from [10, Theorem A], which deals with fast points of local times (recall that by [15] all stable subordinators of index $\alpha < 1$ appear as inverses of appropriate local times; in this analogy fast points of local times correspond to thick points of subordinators). The case of transient symmetric stable processes (i.e. symmetric stable processes with index $\alpha < 1$) is treated in [4] and can be seen as a generalization of the results on thick points for spatial Brownian motion obtained in [5]. Note that all these processes are transient (unlike the Cauchy process), and therefore the techniques used in these papers differ significantly from the ones we shall use to study the case of the Cauchy process. When $\alpha > 1$ the process is recurrent, and thick points are easier to understand, since for such processes there exists a bi-continuous local time (e.g. see [3]). Thus in that case Theorem 1.1 would hold, but with a different scaling (simply ϵ) and $2/\pi$ would be replaced by a random variable (more precisely: the supremum of the local time). In conclusion, our findings apply only at the border of transience and recurrence of stable processes.

The main difficulty in obtaining results similar to those in [6] is that the Cauchy process is not continuous. Indeed, the proof of the lower bounds in [6] relies on the idea that unusually high occupation measures in the neighborhood of a point x are the result of an unusually high number of excursions of all scales around this point. But defining the notion of excursion is not clear when it comes to a non-continuous process. Our proof avoids this problem essentially by working with the Brownian representation of the Cauchy process: up to a time-change, the Cauchy process can be seen as the intersection of a two-dimensional Brownian motion and, say, the x-axis. Using this framework we obtain lower bounds by adapting the strategy in [7]. The same strategy could be used to derive upper bound results. However, because of its independent interest we use the following proposition as the key to our proof of the upper bounds:

Proposition 1.3. *Fix $r > 0$, and let $\bar\theta = \inf\{t\colon |X_t| \geqslant r\}$. For any $x_0 \in (-r, r)$ and any bounded Borel measurable function $f : [-r, r] \to \mathbb{R}$,*

$$\mathbb{E}^{x_0} \int_0^{\bar\theta} f(X_s)\,\mathrm{d}s = \int_{-r}^{r} f(x) G(x_0, x)\,\mathrm{d}x \tag{1.3}$$

where G is given by

$$G(x_0, x) = -\frac{1}{2\pi} \log\left|\frac{h(x/r) - h(x_0/r)}{1 - h(x/r)h(x_0/r)}\right|, \tag{1.4}$$

and

$$h(x) = \frac{\sqrt{(1+x)/(1-x)} - 1}{\sqrt{(1+x)/(1-x)} + 1}. \tag{1.5}$$

Remarks.

- By the scaling property of the Cauchy process, for any deterministic $0 < r < \infty$, Theorems 1.1 and 1.2 still hold if we replace $\bar\theta$ by $\bar\theta_r = \inf\{t\colon |X_t| \geqslant r\}$. As a consequence, these results also hold if one replaces $\bar\theta$ by any deterministic $T < \infty$, or any almost surely finite stopping time.
- In the course of our study, we will prove (see Eq. (3.5)) that almost surely

  $$\dim\left\{x \in I(0,1)\colon \limsup_{\epsilon\to 0} \frac{\mu^X_{\bar\theta}(I(x,\epsilon))}{\epsilon(\log\epsilon)^2} \geqslant a\right\} \leqslant 1 - \frac{a\pi}{2}.$$

  Using this fact, Theorem 1.2 still holds if in Eq. (1.2) one replaces lim by lim sup or lim inf, and/or '$= a$' by '$\geqslant a$'. Note that this was also the case in [6]. By contrast [4,5] and [14] only deal with limit superior.
- Exactly as in [6], one can obtain the following result for the coarse multi-fractal spectrum: for every $a < 2/\pi$,

  $$\lim_{\epsilon\to 0} \frac{\log \mathcal{L}eb(x\colon \mu^X_{\bar\theta}(I(x,\epsilon)) \geqslant a\epsilon(\log\epsilon)^2)}{\log\epsilon} = a\frac{\pi}{2}, \quad \text{a.s.}$$

It is quite natural to consider also the discrete analogues of the results presented here. For example, let (X_i) be a sequence of i.i.d. variables with distribution:

$$\mathbf{P}(X_i = n) = \frac{C}{1+n^2}, \quad n \in \mathbb{Z}$$

where C is a normalizing constant. Let $S_n = \sum_{i=1}^n X_i$,

$$L_n^S(x) := \#\{i:\ S_i = x,\ 0 \leqslant i \leqslant n\},$$

be the number of visits to $x \in \mathbb{Z}$ during the first n steps of the walk and

$$T_n^X := \max_{x \in \mathbb{Z}} L_n^X(x),$$

its maximal value. Then we conjecture that there exists a constant α such that

$$\lim_{n\to\infty} \frac{T_n^X}{(\log n)^2} = \alpha, \quad \text{a.s.}$$

The source of the difficulty here is the absence of strong approximation theorems (such results were used to prove the Erdős–Taylor conjecture in [6]). Another integer-valued random variable for which we expect similar asymptotic results is the following one: if $X_n = (X_n^1, X_n^2)$ is a simple random walk in $\mathbb{Z}^2$, then we define $Y_n := X_{t_n}^1$ where t_n is the time at which 0 is visited for the n-th time by X^2. This is the discrete time analog of the Brownian representation of the Cauchy process, so our techniques should apply here. More generally, we suspect the existence of similar results for random variables in the domain of attraction of the Cauchy distribution.

In the next section, we prove Proposition 1.3 using the Brownian representation of the Cauchy process and the solution to some Dirichlet problem. In Section 3, we use this result to prove upper bounds for both theorems. In Section 4 we prove the lower bounds, using a well defined system of excursions analogous to the one which appears in [7]. Finally Section 5 establishes the connection between occupation measure and excursions.

2. Green function for the Cauchy process

This section will be devoted to proving Proposition 1.3. The proof is based on the Brownian representation of the Cauchy process: if (B^1, B^2) is a planar Brownian motion, and τ_t is the (right-continuous) inverse of the local time of B^2 at 0, then $B_{\tau_t}^1$ is a Cauchy process. In what follows whenever $x_0 \in \mathbb{R}$ we let $x_0^c := (x_0, 0)$. Let $\bar\sigma := \inf\{t:\ B_t^2 = 0,\ |B_t^1| \geqslant r\}$ and L be the local time of B^2 at 0 (more generally we will use the notation L^a for the local time of B^2 at $a \in \mathbb{R}$, and will typically omit the superscript when denoting the local time at 0). We start with the following lemma:

Lemma 2.1. *Let $\bar\theta^B := \inf\{s:\ |B_{\tau_s}^1| \geqslant r\}$. A.s. for every bounded measurable function $f : [-r, r] \to \mathbb{R}$,*

$$\int_0^{\bar\theta^B} f(B_{\tau_s}^1)\,\mathrm{d}s = \int_0^{\bar\sigma} f(B_u^1)\,\mathrm{d}L_u. \tag{2.1}$$

Proof. We first show that $\bar\theta^B = L_{\bar\sigma}$ almost surely. Since neither the Cauchy process nor the planar Brownian motion hit points, we have

$$\bar\theta^B := \inf\{s:\ |B_{\tau_s}^1| > r\} \quad \text{and} \quad \bar\sigma := \inf\{t:\ B_t^2 = 0,\ |B_t^1| > r\} \quad \text{a.s.}$$

We need the following result, taken from [13, p. 242]:

Lemma 2.2 [13]. *Let Z be the random set $\{t \geqslant 0:\ B_t^2 = 0\}$. Then*

$$\mathbf{P}(\forall s \geqslant 0,\ B_{\tau_s}^2 = B_{\tau_{s-}}^2 = 0) = 1.$$

Conversely, for any $u \in Z$, either $u = \tau_s$ or $u = \tau_{s-}$.

Therefore with probability one

$$\bar{\sigma} = \inf\{t\colon\ B_t^2 = 0,\ |B_t^1| > r\} = \inf\{\tau_u\colon\ |B_{\tau_u}^1| > r\} \wedge \inf\{\tau_{u^-}\colon\ |B_{\tau_{u^-}}^1| > r\}. \tag{2.2}$$

Now (by definition) $\tau_{u^-} = \lim_{s\to u^-} \tau_s$. So by continuity of the Brownian motion, for all u such that $|B_{\tau_{u^-}}^1| > r$ there exists $s < u$ such that $|B_{\tau_s}^1| > r$. Thus, the first infimum in (2.2) is less than or equal to the second one, and therefore $\bar{\sigma} = \inf\{\tau_u\colon\ |B_{\tau_u}^1| > r\}$. Since τ is non-decreasing and right-continuous, this implies that $\bar{\sigma} = \tau_{\bar{\theta}^B}$, which in turn gives $L_{\bar{\sigma}} = \bar{\theta}^B$ (since it follows from the definition of τ and the continuity of L that for any $x \geqslant 0$, $L_{\tau_x} = x$). In particular this yields

$$\int_0^{\bar{\theta}^B} f(B_{\tau_s}^1)\,\mathrm{d}s = \int_0^{L_{\bar{\sigma}}} f(B_{\tau_s}^1)\,\mathrm{d}s \quad \text{a.s.} \tag{2.3}$$

Next, note that since τ has at most a countable number of discontinuities (τ is monotonous and finite),

$$\int_0^{L_{\bar{\sigma}}} f(B_{\tau_s}^1)\,\mathrm{d}s = \int_0^{L_{\bar{\sigma}}} f(B_{\tau_s}^1)\mathbf{1}_{\{\tau_{s^-}=\tau_s\}}\,\mathrm{d}s \quad \text{a.s.} \tag{2.4}$$

Now recall that by Lemma 2.2, any $u \in Z$ is of the form τ_s or τ_{s^-} for some s. Since $\tau_{s^-} = \inf\{t\colon\ L_t \geqslant s\}$, we have that in both cases $L_u = s$. Hence $\tau_{L_u} = u$ provided that $\tau_s = \tau_{s^-}$. Therefore, the change of variable $s = L_u$ gives

$$\int_0^{L_{\bar{\sigma}}} f(B_{\tau_s}^1)\mathbf{1}_{\{\tau_{s^-}=\tau_s\}}\,\mathrm{d}s = \int_0^{\bar{\sigma}} f(B_u^1)\mathbf{1}_{\{\tau_{L_u^-}=\tau_{L_u}\}}\,\mathrm{d}L_u \quad \text{a.s.} \tag{2.5}$$

The same change of variable and countability argument gives that

$$0 = \int_0^{L_{\bar{\sigma}}} \mathbf{1}_{\{\tau_{s^-}\neq\tau_s\}}\,\mathrm{d}s = \int_0^{\bar{\sigma}} \mathbf{1}_{\{\tau_{L_u^-}\neq\tau_{L_u}\}}\,\mathrm{d}L_u \quad \text{a.s.}$$

and therefore (2.5) becomes

$$\int_0^{L_{\bar{\sigma}}} f(B_{\tau_s}^1)\mathbf{1}_{\{\tau_{s^-}=\tau_s\}}\,\mathrm{d}s = \int_0^{\bar{\sigma}} f(B_u^1)\,\mathrm{d}L_u \quad \text{a.s.} \tag{2.6}$$

Combining (2.3), (2.4) and (2.6) finishes the proof of Lemma 2.1. □

In particular, since B_{τ_t} is a Cauchy process, the above result implies that

$$\mathbb{E}^{x_0} \int_0^{\bar{\theta}} f(X_s)\,\mathrm{d}s = \mathbb{E}^{x_0^c} \int_0^{\bar{\sigma}} f(B_u^1)\,\mathrm{d}L_u. \tag{2.7}$$

From here on, without loss of generality we assume f to be continuous with compact support in $(-r, r)$ (by the monotone class theorem, Proposition 1.3 will then follow for any bounded measurable function). We now need the following

Lemma 2.3. *Let $g_\delta : \mathbb{R} \to \mathbb{R}^+$ be a family of continuous functions with support in $(-\delta, \delta)$ and such that $\int g_\delta = 1$. Then*

$$\mathbb{E}^{x_0^c} \int_0^{\bar{\sigma}} f(B_u^1)\,\mathrm{d}L_u^0 = \lim_{\delta \to 0} \mathbb{E}^{x_0^c} \int_0^{\bar{\sigma}} f(B_u^1) g_\delta(B_u^2)\,\mathrm{d}u. \tag{2.8}$$

Let us postpone the proof of this lemma, and continue with the proof of Proposition 1.3. We first rewrite the quantity in the right-hand side of (2.8). Using for example [9, Exercise 2.25, p. 253], we know it is equal to $u_\delta(x_0, 0)$ where u_δ is defined as follows:

Lemma 2.4. *The unique solution to the partial differential equation*:

$$\begin{cases} -(1/2)\Delta u_\delta(x_1, x_2) = g_\delta(x_2) f(x_1) & \text{on } D_r, \\ u_\delta(x_1, x_2) = 0 & \text{on } \partial D_r \end{cases}$$

where $D_r := \mathbb{R}^2 \setminus ((-\infty, r] \cup [r, \infty))$, is given by

$$u_\delta(x) = \int_{D_r} 2 g_\delta(z_2) f(z_1) G(x, z)\,\mathrm{d}z \tag{2.9}$$

where G is the Green function of D_r. G is given by a complex analog of (1.4), *i.e.*

$$G(z_0, z) = -\frac{1}{2\pi} \log \left| \frac{h(z/r) - h(z_0/r)}{1 - h(z/r)\overline{h(z_0/r)}} \right|, \tag{2.10}$$

where, as in (1.5),

$$h(z) = \frac{\sqrt{(1+z)/(1-z)} - 1}{\sqrt{(1+z)/(1-z)} + 1}. \tag{2.11}$$

Proof of Lemma 2.4. $h(z)$ can be written $u(v(z))$ where

$$v(z) := \sqrt{\frac{1+z}{1-z}}, \quad \text{and} \quad u(z) := \frac{z-1}{z+1}.$$

v is a conformal mapping of D_1 to the upper half-plane, and u is a conformal mapping of the upper-half plane to the unit disk. Thus h maps D_1 conformally to the unit disk, and sends 0 to 0. Hence for any $z_0 \in D_r$,

$$\frac{h(z/r) - h(z_0/r)}{1 - h(z/r)\overline{h(z_0/r)}}$$

is a conformal mapping of D_r to the unit disk, which sends z_0 to 0. Now the Green function of the unit disk with pole at 0 is $-(1/2\pi)\log z$. Since Green functions are conformally invariant (e.g. see [1, p. 257]), $G(z_0, z)$ as defined in (2.10) is indeed the Green function of D_r with pole at z_0. Then (2.9) is simply the Green's representation formula (e.g. see [8, Chapter 2]). □

So the expectation in the right-hand side of (2.8) becomes

$$u_\delta(x_0^c) = 2 \int_{-\infty}^{\infty} g_\delta(z_2) \left(\int_{-r}^{r} f(z_1) G\big(x_0^c, (z_1, z_2)\big)\,\mathrm{d}z_1 \right) \mathrm{d}z_2.$$

Now by dominated convergence, the second integral is a continuous function of z_2. Indeed, since h is one-to-one and analytic, for z in a compact subset K of D_r there exists M such that

$$\left|\frac{h(z/r)-h(x_0^c/r)}{1-h(z/r)\overline{h(x_0^c/r)}}\right| \geqslant M|z-x_0^c|.$$

Thus,

$$G(x_0^c, z) \leqslant -\frac{1}{2\pi}\log(M) - \frac{1}{2\pi}\log|x_0 - z_1| \tag{2.12}$$

which does not depend on z_2 and is integrable as a function of z_1. We have proved that

$$\lim_{\delta\to 0}\mathbb{E}^{x_0^c}\int_0^{\bar\sigma} f(B_u^1)g_\delta(B_u^2)\,\mathrm{d}u = 2\int_{-r}^{r} f(z_1)G\big(x_0^c,(z_1,0)\big)\,\mathrm{d}z_1.$$

This, together with (2.7) and Lemma 2.3 yields

$$\mathbb{E}^{x_0}\int_0^{\bar\theta} f(X_s)\,\mathrm{d}s = 2\int_{-r}^{r} f(z_1)G\big((x_0,0),(z_1,0)\big)\,\mathrm{d}z_1$$

which completes the proof of Proposition 1.3. $\square$

Proof of Lemma 2.3. We will prove the following stronger version of (2.8):

$$\int_0^{\bar\sigma} f(B_u^1)\,\mathrm{d}L_u^0 = \lim_{\delta\to 0}\int_0^{\bar\sigma} f(B_u^1)g_\delta(B_u^2)\,\mathrm{d}u \tag{2.13}$$

where the limit is taken with respect to the norm $\|\cdot\|_2$. To this aim, we first notice that

$$\begin{aligned}\int_0^{\bar\sigma} f(B_u^1)g_\delta(B_u^2)\,\mathrm{d}u &= \int_0^{\bar\sigma} f(B_u^1)\,\mathrm{d}\left(\int_0^{u} g_\delta(B_t^2)\,\mathrm{d}t\right) = \int_0^{\bar\sigma} f(B_u^1)\,\mathrm{d}\left(\int_{-\infty}^{\infty} g_\delta(a)L_u^a\,\mathrm{d}a\right)\\ &= \int_{-\infty}^{\infty} g_\delta(a)\left(\int_0^{\bar\sigma} f(B_u^1)\,\mathrm{d}L_u^a\right)\mathrm{d}a\end{aligned}$$

where the second step follows from the occupation time formula [13, Corollary (1.6) p. 224], and the last one by Fubini's theorem. Therefore (2.13) will follow once we prove that

$$\int_0^{\bar\sigma} f(B_u^1)\,\mathrm{d}L_u^0 = \lim_{a\to 0}\int_0^{\bar\sigma} f(B_u^1)\,\mathrm{d}L_u^a \tag{2.14}$$

in $\|\cdot\|_2$. Now let $a>0$. By Tanaka's formula (see e.g. [13], p. 222), for any $t \geqslant 0$

$$\frac{1}{2}L_t^a = (B_t^2-a)^+ - (B_0^2-a)^+ - \int_0^t \mathbf{1}_{(B_s^2>a)}\,\mathrm{d}B_s^2 = -\int_0^t \mathbf{1}_{(B_s^2>a)}\,\mathrm{d}B_s^2.$$

Therefore for any $t \geqslant 0$

$$L_t^a - L_t^0 = 2\int_0^t \mathbf{1}_{(0<B_s^2\leqslant a)}\,\mathrm{d}B_s^2,$$

hence

$$\int_0^{\bar\sigma} f(B_s^1)\,\mathrm{d}L_s^a - \int_0^{\bar\sigma} f(B_s^1)\,\mathrm{d}L_s^0 = 2\int_0^{\bar\sigma} f(B_s^1)\mathbf{1}_{(0<B_s^2\leqslant a)}\,\mathrm{d}B_s^2.$$

By L^2-isometry we obtain

$$\begin{aligned}\mathbb{E}^{x_0^c}\left[\left(\int_0^{\bar\sigma} f(B_s^1)\,\mathrm{d}(L_s^a - L_s^0)\right)^2\right] &= \mathbb{E}^{x_0^c}\left[4\int_0^{\bar\sigma} f^2(B_s^1)\mathbf{1}_{(0<B_s^2\leqslant a)}\,\mathrm{d}s\right] \leqslant 4\|f^2\|_\infty \mathbb{E}^{x_0^c}\left[\int_0^{\bar\sigma} \mathbf{1}_{(0<B_s^2\leqslant a)}\,\mathrm{d}s\right] \\ &= 4\|f^2\|_\infty \int_0^a \int_{-\infty}^{\infty} G\big(x_0^c,(x,y)\big)\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

A careful study of the inner integral on the last line reveals that it is finite, and continuous as a function of y. Indeed, let us fix $\epsilon > 0$ and define

$$\begin{aligned}\int_{-\infty}^{\infty} G\big(z,(x,y)\big)\,\mathrm{d}x = &\int_{-\infty}^{-r-\epsilon} G\big(z,(x,y)\big)\,\mathrm{d}x + \int_{-r-\epsilon}^{-r+\epsilon} G\big(z,(x,y)\big)\,\mathrm{d}x + \int_{-r+\epsilon}^{r-\epsilon} G\big(z,(x,y)\big)\,\mathrm{d}x \\ &+ \int_{r-\epsilon}^{r+\epsilon} G\big(z,(x,y)\big)\,\mathrm{d}x + \int_{r+\epsilon}^{\infty} G\big(z,(x,y)\big)\,\mathrm{d}x \\ = &\, I_1 + I_2 + I_3 + I_4 + I_5.\end{aligned}$$

By choosing ϵ appropriately, I_2 and I_4 can be made arbitrarily small. And by (2.12), I_3 is a continuous function of y. Finally, on $|x| \geqslant r + \epsilon$, $G(z,(x,y))$ can be seen to be dominated by C/x^2 for some constant $C > 0$ uniformly on y (y small enough). Thus by dominated convergence, $I_1 + I_5$ tends to 0 as $y \to 0$. These facts put together prove the continuity and the finiteness of the inner integral in (2.15). Thus the right-hand side of (2.15) tends to 0 when $a \to 0^+$. The case $a \to 0^-$ being similar, we have proved (2.14), and hence (2.13), which completes the proof of Lemma 2.3. □

3. Upper bounds

Throughout this section, fix $0 < r_1 \leqslant r_3/2$, let X be a Cauchy process, $I(x,\epsilon)$ the open interval of radius ϵ centered at x, $\bar\theta := \inf\{t > 0\colon |X_t| \geqslant r_3\}$ and define

$$\mu_{\bar\theta}^X(r_1) := \int_0^{\bar\theta} \mathbf{1}_{I(0,r_1)}(X_s)\,\mathrm{d}s.$$

Lemma 3.1. *There exists $c > 0$ such that for all $r_1 \leqslant r_3/2$ and $|x_0| \leqslant r_3$ we have*

$$\mathbb{E}^{x_0}\big(\mu_{\bar\theta}^X(r_1)\big) \leqslant r_1\left[c + \frac{2}{\pi}\log\left(\frac{r_3}{r_1}\right)\right] \tag{3.1}$$

and for all $k \geqslant 0$

$$\mathbb{E}^{x_0}\big(\mu_{\bar\theta}^X(r_1)\big)^k \leqslant k! r_1^k\left[c + \frac{2}{\pi}\log\left(\frac{r_3}{r_1}\right)\right]^k. \tag{3.2}$$

Proof of Lemma 3.1. Recall that by Proposition 1.3,

$$\mathbb{E}^{x_0}\int_0^{\bar\theta} \mathbf{1}_{|X_s|\leqslant r_1}\,\mathrm{d}s = 2\int_{-r_1}^{r_1} G\big(x_0^c, (z_1, 0)\big)\,\mathrm{d}z_1.$$

To prove Lemma 3.1 we therefore need to find an upper bound for the right-hand side of the above equation. We have

$$\begin{aligned} 2\int_{-r_1}^{r_1} G\big(x_0^c, (z_1, 0)\big)\,\mathrm{d}z_1 &= -\int_{-r_1}^{r_1} \frac{1}{\pi}\log\left|\frac{h(z_1/r_3) - h(x_0/r_3)}{1 - h(z_1/r_3)h(x_0/r_3)}\right|\mathrm{d}z_1 \\ &\leqslant -\int_{-r_1}^{r_1} \frac{1}{\pi}\log\left|h\left(\frac{z_1}{r_3}\right) - h\left(\frac{x_0}{r_3}\right)\right|\mathrm{d}z_1 + \frac{2}{\pi} r_1 \log 2. \end{aligned}$$

because h takes values in the unit disk. We will treat the cases $|x_0| \leqslant (3/4)r_3$ and $|x_0| \geqslant (3/4)r_3$ independently, and start with the former. The function h, when restricted to the compact set $[-3/4, 3/4]$ is smooth and its derivative does not vanish. Since $|z_1|/r_3 \leqslant 1/2$ when $z_1 \in (-r_1, r_1)$, this yields

$$\left|h\left(\frac{z_1}{r_3}\right) - h\left(\frac{x_0}{r_3}\right)\right| \geqslant \left|\frac{z_1 - x_0}{r_3}\right| M$$

for some constant $M > 0$. Hence

$$-\int_{-r_1}^{r_1} \frac{1}{\pi}\log\left|h\left(\frac{z_1}{r_3}\right) - h\left(\frac{x_0}{r_3}\right)\right|\mathrm{d}z_1 \leqslant -\int_{-r_1}^{r_1} \frac{1}{\pi}\log\left|\frac{z_1 - x_0}{r_3}\right|\mathrm{d}z_1 - \int_{-r_1}^{r_1} \frac{1}{\pi}\log M\,\mathrm{d}z_1 \leqslant r_1\left(\frac{2}{\pi}\log r_3 + A\right)$$

for some A uniformly in $|x_0| \leqslant (3/4)r_3$. It remains to treat the case $|x_0| \geqslant (3/4)r_3$. But then $|x_0/r_3| \geqslant 3/4$ while again $|z_1/r_3| \leqslant 1/2$. Therefore, h being continuous and one-to-one, there exists a constant $B > 0$ uniform in r_3 such that when $x_0 \geqslant (3/4)r_3$, $|h(z_1/r_3) - h(x_0/r_3)| \geqslant B$. So that in that case

$$-\int_{-r_1}^{r_1} \frac{1}{\pi}\log\left|h\left(\frac{z_1}{r_3}\right) - h\left(\frac{x_0}{r_3}\right)\right|\mathrm{d}z_1 \leqslant C r_1$$

for some constant C uniformly in r_3. This finishes the proof of equation (3.1). (3.2) will then follow from the strong Markov property for the Cauchy process. Indeed,

$$\mathbb{E}^{x_0}\big(\mu_{\bar\theta}^X(r_1)\big)^k = k!\mathbb{E}^{x_0}\left(\int_{0\leqslant s_1\cdots\leqslant s_k\leqslant\bar\theta} \prod_{i=1}^k \mathbf{1}_{I(0,r_1)}(X_{s_i})\,\mathrm{d}s_1\cdots\mathrm{d}s_k\right)$$

$$\leqslant k!\mathbb{E}^{x_0}\biggl(\int\limits_{0\leqslant s_1\leqslant\cdots\leqslant s_{k-1}\leqslant\bar\theta}\prod_{i=1}^{k-1}\mathbf{1}_{I(0,r_1)}(X_{s_i})r_1\biggl(c+\frac{2}{\pi}\log\biggl(\frac{r_3}{r_1}\biggr)\biggr)\,\mathrm{d}s_1\cdots\mathrm{d}s_{k-1}\biggr)$$
$$= r_1\biggl(c+\log\biggl(\frac{r_3}{r_1}\biggr)\biggr)k\mathbb{E}^{x_0}\bigl(\mu^X_{\bar\theta}(r_1)\bigr)^{k-1},$$

proving (3.2) by induction on k. □

This leads us to

Lemma 3.2. *We use the same notation as in Lemma* 3.1. *For* $0\leqslant\lambda<[(2/\pi)r_1\log(r_3/r_1)+cr_1]^{-1}$,

$$\mathbb{E}^{x_0}\bigl(\mathrm{e}^{\lambda\mu^X_{\bar\theta}(r_1)}\bigr)\leqslant\biggl(1-\lambda r_1\biggl[\frac{2}{\pi}\log\biggl(\frac{r_3}{r_1}\biggr)+c\biggr]\biggr)^{-1},\tag{3.3}$$

which implies that for $t>0$

$$\mathbf{P}\bigl(\mu^X_{\bar\theta}(r_1)\geqslant t\bigr)\leqslant t\biggl(r_1\biggl[\frac{2}{\pi}\log\biggl(\frac{r_3}{r_1}\biggr)+c\biggr]\biggr)^{-1}\exp\biggl(1-t\biggl(r_1\biggl[\frac{2}{\pi}\log\biggl(\frac{r_3}{r_1}\biggr)+c\biggr]\biggr)^{-1}\biggr).\tag{3.4}$$

Proof of Lemma 3.2. (3.4) follows from (3.3) by Chebyshev's inequality. (3.3) is a straightforward consequence of (3.2). □

In the remainder of this section, we use Lemma 3.1 to prove the upper bounds in Theorems 1.1 and 1.2. Namely if we define

$$\mathsf{Thick}_{\geqslant a}:=\biggl\{x\in I(0,1)\colon\ \limsup_{\epsilon\to0}\frac{\mu^X_{\bar\theta}(I(x,\epsilon))}{\epsilon(\log\epsilon)^2}\geqslant a\biggr\},$$

(where $\bar\theta=\inf\{t\colon\ |X_t|\geqslant1\}$), then we will show that for any $a\in(0,2/\pi]$,

$$\dim(\mathsf{Thick}_{\geqslant a})\leqslant 1-a\pi/2,\quad \text{a.s.},\tag{3.5}$$

and

$$\limsup_{\epsilon\to0}\sup_{|x|<1}\frac{\mu^X_{\bar\theta}(I(x,\epsilon))}{\epsilon(\log\epsilon)^2}\leqslant\frac{2}{\pi},\quad\text{a.s.}\tag{3.6}$$

Note that (3.5) will give the correct upper-bound for Theorem 1.2, since

$$\mathsf{Thick}_{\geqslant a}\supset\mathsf{Thick}_a:=\biggl\{x\in I(0,1)\colon\ \lim_{\epsilon\to0}\frac{\mu^X_{\bar\theta}(I(x,\epsilon))}{\epsilon(\log\epsilon)^2}=a\biggr\}.$$

Our proof follows [6]. Set $h(\epsilon)=\epsilon|\log\epsilon|^2$ and

$$z(x,\epsilon):=\frac{\mu_{\bar\theta}(I(x,\epsilon))}{h(\epsilon)}.$$

Fix $0<\delta<1$ and choose a sequence $\tilde\epsilon_n\downarrow0$ as $n\to\infty$ in such a way that $\tilde\epsilon_n<e^{-2}$ and

$$h(\tilde\epsilon_{n+1})=(1-\delta)h(\tilde\epsilon_n),\tag{3.7}$$

implying that ϵ_n is monotone decreasing in n. Since, for $\tilde\epsilon_{n+1}\leqslant\epsilon\leqslant\tilde\epsilon_n$ we have

$$z(x,\tilde\epsilon_n)=\frac{h(\tilde\epsilon_{n+1})}{h(\tilde\epsilon_n)}\frac{\mu_{\bar\theta}(I(x,\tilde\epsilon_n))}{h(\tilde\epsilon_{n+1})}\geqslant(1-\delta)z(x,\epsilon),\tag{3.8}$$

it is easy to see that for any $a > 0$,

$$\mathsf{Thick}_{\geqslant a} \subseteq D_a := \Big\{ x \in I(0,1) \ \Big| \ \limsup_{n\to\infty} z(x, \tilde{\epsilon}_n) \geqslant (1-\delta)a \Big\}.$$

Let $\{x_j\colon\ j = 1, \dots, \overline{K}_n\}$ denote a maximal collection of points in $I(0,1)$ such that $\inf_{\ell \neq j} |x_\ell - x_j| \geqslant \delta\tilde{\epsilon}_n$. Let $\bar{\theta}_2 := \inf\{t\colon\ |X_t| \geqslant 2\}$ and $\mathcal{A}_n$ be the set of $1 \leqslant j \leqslant \overline{K}_n$, such that

$$\mu^X_{\bar{\theta}}\big(I\big(x_j, (1+\delta)\tilde{\epsilon}_n\big)\big) \geqslant (1-2\delta)ah(\tilde{\epsilon}_n). \tag{3.9}$$

Applying (3.4) with $r_1 = (1+\delta)\tilde{\epsilon}_n$ and $r_3 = 2$ gives

$$\mathbf{P}^x\big(\mu_{\bar{\theta}_2}\big(I\big(0, (1+\delta)\tilde{\epsilon}_n\big)\big) \geqslant (1-2\delta)ah(\tilde{\epsilon}_n)\big) \leqslant c\tilde{\epsilon}_n^{a(1-5\delta)\pi/2},$$

for some $c = c(\delta) < \infty$, and any $x \in I(0,1)$. Note that for all $x \in I(0,1)$ and $\epsilon, b \geqslant 0$

$$\mathbf{P}\big(\mu_{\bar{\theta}}\big(I(x,\epsilon)\big) \geqslant b\big) \leqslant \mathbf{P}^{-x}\big(\mu_{\bar{\theta}_2}\big(I(0,\epsilon)\big) \geqslant b\big).$$

Thus for any j and $a > 0$,

$$\mathbf{P}(j \in \mathcal{A}_n) \leqslant c\tilde{\epsilon}_n^{a(1-5\delta)\pi/2},$$

implying that

$$\mathbb{E}|\mathcal{A}_n| \leqslant c'\tilde{\epsilon}_n^{a(1-5\delta)\pi/2-1} \tag{3.10}$$

(by definition of $\overline{K}_n$). Let $\mathcal{V}_{n,j} = I(x_j, \delta\tilde{\epsilon}_n)$. For any $x \in I(0,1)$ there exists $j \in \{1, \dots, \overline{K}_n\}$ such that $x \in \mathcal{V}_{n,j}$, hence $I(x, \tilde{\epsilon}_n) \subseteq I(x_j, (1+\delta)\tilde{\epsilon}_n)$. Consequently, $\bigcup_{n \geqslant m} \bigcup_{j \in \mathcal{A}_n} \mathcal{V}_{n,j}$ forms a cover of D_a by sets of maximal diameter $2\delta\tilde{\epsilon}_m$. Fix $a \in (0, 2/\pi]$. Since $\mathcal{V}_{n,j}$ have diameter $2\delta\tilde{\epsilon}_n$, it follows from (3.10) that for $\gamma = 1 - \pi a(1-6\delta)/2 > 0$,

$$\mathbb{E} \sum_{n=m}^{\infty} \sum_{j \in \mathcal{A}_n} |\mathcal{V}_{n,j}|^\gamma \leqslant c'(2\delta)^\gamma \sum_{n=m}^{\infty} \tilde{\epsilon}_n^{\delta a\pi/2} < \infty.$$

Thus, $\sum_{n=m}^{\infty} \sum_{j \in \mathcal{A}_n} |\mathcal{V}_{n,j}|^\gamma$ is finite a.s. implying that $\dim(D_a) \leqslant \gamma$ a.s. Taking $\delta \downarrow 0$ completes the proof of the upper bound (3.5).

Turning to prove (3.6), set $a = 2(1+\delta)/(\pi(1-5\delta))$ noting that by (3.10)

$$\sum_{n=1}^{\infty} \mathbf{P}\big(|\mathcal{A}_n| \geqslant 1\big) \leqslant \sum_{n=1}^{\infty} \mathbb{E}|\mathcal{A}_n| \leqslant c' \sum_{n=1}^{\infty} \tilde{\epsilon}_n^\delta < \infty.$$

By Borel–Cantelli, it follows that a.s. $\mathcal{A}_n$ is empty for all $n > n_0(\omega)$ and some $n_0(\omega) < \infty$. By (3.8) we then have

$$\sup_{\epsilon \leqslant \tilde{\epsilon}_{n_0(\omega)}} \sup_{|x|<1} \frac{\mu_{\bar{\theta}}(I(x,\epsilon))}{\epsilon(\log \epsilon)^2} \leqslant a\frac{1-2\delta}{1-\delta} \leqslant a,$$

and (3.6) follows by taking $\delta \downarrow 0$. $\square$

4. Lower bounds

In this section we adapt the proof of [7, Section 3]. While its authors studied the intersection local time for two independent Brownian motions, we are interested in the same quantity but for the intersection of a Brownian motion with a line. Throughout what follows we use notation similar to that in [7, Section 3].

Fixing $a < 2/\pi$, $c > 0$ and $\delta > 0$, let

$$\bar{\theta}_c := \inf\{t > 0\colon |X_t| \geqslant c\}, \qquad \Gamma_c = \Gamma_c(\omega) := \left\{x \in I(0,1)\colon \lim_{\epsilon \to 0} \frac{\mu_{\bar{\theta}_c}(I(x,\epsilon))}{\epsilon(\log \epsilon)^2} = a\right\},$$

and $\mathcal{E}_c := \{\omega\colon \dim(\Gamma_c(\omega)) \geqslant 1 - a\pi/2 - \delta\}$. In view of the results of Section 3, we will obtain Theorem 1.2 once we show that $\mathbf{P}(\mathcal{E}_1) = 1$ for any $a < 2/\pi$ and $\delta > 0$. Indeed, since $\mathsf{Thick}_a \subset \mathsf{Thick}_{\geqslant a}$ and since we have seen that

$$\dim(\mathsf{Thick}_{\geqslant a}) \leqslant 1 - \frac{a\pi}{2}, \quad \text{a.s.},$$

proving $\mathbf{P}(\mathcal{E}_1) = 1$ will imply

$$\dim \mathsf{Thick}_a = 1 - \frac{a\pi}{2}, \quad \text{a.s.}$$

Moreover, the inequality

$$\liminf_{\epsilon \to 0} \sup_{|x|<1} \frac{\mu_{\bar{\theta}}(I(x,\epsilon))}{\epsilon(\log \epsilon)^2} \geqslant \sup_{|x|<1} \liminf_{\epsilon \to 0} \frac{\mu_{\bar{\theta}}(I(x,\epsilon))}{\epsilon(\log \epsilon)^2}$$

then implies that for any $\eta > 0$,

$$\liminf_{\epsilon \to 0} \sup_{|x|<1} \frac{\mu_{\bar{\theta}}(I(x,\epsilon))}{\epsilon(\log \epsilon)^2} \geqslant 2/\pi - \eta, \quad \text{a.s.}$$

In view of (3.6), these lower bounds establish Theorem 1.1.

The bulk of this section and the next will be dedicated to showing that $\mathbf{P}(\mathcal{E}_1) > 0$. Assuming this for the moment, let us show that this implies $\mathbf{P}(\mathcal{E}_1) = 1$. Let us momentarily assume that X is the canonical version of the process. If $\omega := X$ denotes the sample path, and $X^c_t := c^{-1} X_{ct}$ we have that $c\bar{\theta}(\omega^c) = \inf\{ct\colon |c^{-1}X_{ct}| \geqslant 1\} = \bar{\theta}_c(\omega)$, and hence

$$\mu^{X^c}_{\bar{\theta}}\big(I(x,\epsilon)\big) = \int_0^{\bar{\theta}(\omega^c)} \mathbf{1}_{\{|X^c_s - x| \leqslant \epsilon\}}\,\mathrm{d}s = \int_0^{\bar{\theta}(\omega^c)} \mathbf{1}_{\{|X_{cs} - cx| \leqslant c\epsilon\}}\,\mathrm{d}s = \frac{1}{c}\int_0^{c\bar{\theta}(\omega^c)} \mathbf{1}_{\{|X_s - cx| \leqslant c\epsilon\}}\,\mathrm{d}s = \frac{1}{c}\mu^X_{\bar{\theta}_c}\big(I(cx, c\epsilon)\big).$$

Consequently, $\Gamma_c(\omega) = c\Gamma_1(\omega^c)$, so the Cauchy process' scaling property implies that $p = \mathbf{P}(\mathcal{E}_c)$ is independent of $c > 0$. Let

$$\mathcal{E} := \limsup_{n \to \infty} \mathcal{E}_{n^{-1}},$$

so that $\mathbf{P}(\mathcal{E}) \geqslant p$. The Cauchy process is a Feller process; hence if we let $(\mathcal{F}_t)$ be the usual augmentation of the natural filtration, it can be shown that $(\mathcal{F}_t)$ is right-continuous (e.g. see [13, III-2]). Therefore, since $\mathcal{E}_c \in \mathcal{F}_{\bar{\theta}_c}$, $\mathcal{E} \in \mathcal{F}_0$ which implies $\mathbf{P}(\mathcal{E}) \in \{0,1\}$. Thus, $p > 0$ yields $\mathbf{P}(\mathcal{E}) = 1$. We will see momentarily that the events $\mathcal{E}_c$ are essentially increasing in c, i.e.

$$\forall 0 < b < c \quad \mathbf{P}(\mathcal{E}_b \setminus \mathcal{E}_c) = 0. \tag{4.1}$$

Thus, $\mathbf{P}(\mathcal{E} \setminus \mathcal{E}_1) \leqslant \mathbf{P}(\bigcup_n \{\mathcal{E}_{n^{-1}} \setminus \mathcal{E}_1\}) = 0$, so that also $\mathbf{P}(\mathcal{E}_1) = 1$. To see (4.1), we proceed exactly as in [6, Section 3]. First notice that for $b < c$,

$$\Gamma_b(\omega) \setminus \{\omega_t\colon \bar{\theta}_b \leqslant t \leqslant \bar{\theta}_c\} \subset \Gamma_c(\omega).$$

Hence

$$\mathbf{P}(\mathcal{E}_b \setminus \mathcal{E}_c) \leqslant \mathbb{E}\mathbf{P}\big(\dim\big(\Gamma_b(\omega)\big) \neq \dim\big(\Gamma_b(\omega) \setminus \{\omega_t\colon \bar{\theta}_b \leqslant t \leqslant \bar{\theta}_c\}\big)\big|\mathcal{F}_{\bar{\theta}_b}\big).$$

Then applying the strong Markov property at time $\bar{\theta}_b$ and observing that the set $\Gamma_b(\omega)$ is Borel gives (4.1) exactly as in [6, Section 3, p. 247] (since the Cauchy process does not hit points).

So we just have to show that $\mathbf{P}(\mathcal{E}_1) > 0$. To achieve this goal we will use the Brownian representation of the Cauchy process, and follow the strategy of [7]. More precisely, moving to a Brownian setting, we will now focus our attention on the “projected intersection local time measures”:

$$\mathcal{I}_t(A) := \int_0^t \mathbf{1}_{\{B_u \in A\}} \, \mathrm{d}L_u^0,$$

where $B = (B^1, B^2)$ is a planar Brownian motion, $L_\cdot^0$ is the local time of B^2 at 0 and A is any measurable subset of $\mathbb{R}^2$. $\mathcal{I}$ is simply the amount of local time spent in A before t. To see how this relates to the Cauchy process, note that, for example, for any set $A \subset \mathbb{R}^2$, $\mathcal{I}_{\bar\sigma}(A) = \mu_{\bar\theta}(A \cap x\text{−axis})$ where $\bar\sigma := \inf\{t\colon\ B_t^2 = 0 \text{ and } |B_t^1| \geqslant 1\}$, $\bar\theta := \inf\{t\colon\ |X_t| \geqslant 1\}$, X is the Cauchy process associated to the planar Brownian motion B and μ is the occupation measure for X. Indeed this follows from Lemma 2.1

We then reproduce the setting of [7, p. 248]: fix $a < 2$, $\epsilon_1 = 1/8$ and the square $S = S_1 = [\epsilon_1, 2\epsilon_1]^2 \subset D(0,1)$, where for $x \in \mathbb{R}^2$, $\rho > 0$, $D(x,\rho)$ denotes the closed disk of center x and radius ρ. Note that for all $x \in S$ and $y \in S \cup \{0\}$ both $0 \notin D(x,\epsilon_1)$ and $0 \in D(x,1/2) \subset D(y,1) \subset D(x,2)$. Let $\epsilon_k = \epsilon_1 (k!)^{-3} = \epsilon_1 \prod_{l=2}^k l^{-3}$. For $x \in S$, $k \geqslant 2$ and $\rho > \epsilon_1$, let $N_k^x(\rho)$ denote the number of excursions of $B_\cdot$ from $\partial D(x,\epsilon_k)$ to $\partial D(x,\epsilon_{k-1})$ prior to hitting $\partial D(x,\rho)$. Set $n_k = 3ak^2 \log k$. We will say that a point $x \in S$ is *n-perfect* if

$$n_k - k \leqslant N_k^x(1/2) \leqslant N_k^x(2) \leqslant n_k + k, \quad \forall k = 2,\dots,n.$$

For $n \geqslant 2$ we partition S into $M_n = \epsilon_1^2/(2\epsilon_n)^2 = (1/4)\prod_{l=1}^n l^6$ non-overlapping squares of edge length $2\epsilon_n = 2\epsilon_1/(n!)^3$, which we denote by $S(n,i)$; $i = 1,\dots,M_n$ with $x_{n,i}$ denoting the center of each $S(n,i)$. Let $Y(n,i)$; $i = 1,\dots,M_n$ be the sequence of random variables defined by

$$Y(n,i) = 1 \quad \text{if } x_{n,i} \text{ is } n\text{-perfect}$$

and $Y(n,i) = 0$ otherwise. Define

$$A_n = \bigcup_{i\colon Y(n,i)=1} S(n,i),$$

and

$$F = F(\omega) = \bigcap_m \overline{\bigcup_{n \geqslant m} A_n} := \bigcap_m F_m. \tag{4.2}$$

Note that each $x \in F$ is the limit of a sequence $\{x_n\}$ such that x_n is n-perfect. We finally rotate this picture by 45 degrees clockwise. S now intersects the x-axis; let D be this intersection. The next lemma will be proved in the next section.

Lemma 4.1. *Let* $\bar\theta := \inf\{t\colon\ |B_t| \geqslant 1\}$. *A.s. for all* $x \in F \cap D$

$$\lim_{\epsilon \to 0} \frac{\mathcal{I}_{\bar\theta}(D(x,\epsilon))}{\epsilon (\log \epsilon)^2} = \frac{2}{\pi} a.$$

Now Lemma 3.2 in [7] shows that for every $a < 1$, and every $\delta > 0$ such that $1 - a - \delta > 0$,

$$\mathbf{P}\big(\dim(F \cap D) \geqslant 1 - a - \delta\big) > 0. \tag{4.3}$$

This, together with Lemma 4.1 implies

$$\mathbf{P}\left(\dim\left(D \cap \left\{\lim_{\epsilon \to 0} \frac{\mathcal{I}_{\bar\theta}(D(x,\epsilon))}{\epsilon (\log \epsilon)^2} = \frac{2}{\pi} a\right\}\right) \geqslant 1 - a - \delta\right) > 0.$$

Now if $\bar\sigma$ denotes the first time that the planar Brownian motion hits the complement of $(-1,1)$ (on the real axis), we have by the strong Markov property

$$\begin{aligned}&\mathbf{P}\left(\dim\left(D\cap\left\{\lim_{\epsilon\to 0}\frac{\mathcal{I}_{\bar\sigma}(D(x,\epsilon))}{\epsilon(\log\epsilon)^2}=\frac{2}{\pi}a\right\}\right)\geqslant 1-a-\delta\right)\\&\quad\geqslant K\mathbf{P}\left(\dim\left(D\cap\left\{\lim_{\epsilon\to 0}\frac{\mathcal{I}_{\bar\theta}(D(x,\epsilon))}{\epsilon(\log\epsilon)^2}=\frac{2}{\pi}a\right\}\right)\geqslant 1-a-\delta\right)\end{aligned}$$

where

$$K=\inf_{|a|=1}\mathbf{P}^a\big(\text{Brownian motion hits } (-\infty,-1]\cup[1,\infty) \text{ before } D\big).$$

Since $K>0$, in view of Lemma 2.1 we have proved that $\mathbf{P}(\mathcal{E}_1)>0$, which concludes the section.

5. From excursions to intersection local time

This section follows (very) closely the argument developed in [7, Section 4]. The sets F and D are the same as in the previous section, and $h(\epsilon):=\epsilon(\log\epsilon)^2$. Lemma 4.1 will follow from the next two lemmas.

Lemma 5.1. *For every $\delta>0$, if $x\in F\cap D$ then*

$$\frac{2}{\pi}a(1-\delta)^6\leqslant\liminf_{\epsilon\to 0}\frac{\mathcal{I}_{\bar\theta}(D(x,\epsilon))}{h(\epsilon)}. \tag{5.1}$$

Lemma 5.2. *For every $\delta>0$, if $x\in F\cap D$ then*

$$\limsup_{\epsilon\to 0}\frac{\mathcal{I}_{\bar\theta}(D(x,\epsilon))}{h(\epsilon)}\leqslant\frac{2}{\pi}a(1+\delta)^5. \tag{5.2}$$

Proof of Lemma 5.1. We use the same notation as in [7]. Let ϵ_k be as in the previous section, $\delta_k:=\epsilon_k/k^6$ and let $\mathcal{D}_k$ be a δ_k-net of points in S. Let

$$\epsilon'_k=\epsilon_k e^{1/k^6},\qquad \epsilon''_{k-1}=\epsilon_{k-1}e^{-2/k^6},$$

so that for k large enough,

$$\epsilon'_k\geqslant\epsilon_k+\delta_k,\qquad \epsilon''_{k-1}\leqslant\epsilon_{k-1}-\delta_k. \tag{5.3}$$

We will say that a point $x'\in\mathcal{D}_k$ is *lower k-successful* if there are at least n_k-k excursions of B from $\partial D(x',\epsilon'_k)$ to $\partial D(x',\epsilon''_{k-1})$ prior to $\bar\theta$. Let

$$\epsilon_{k,j}=\epsilon_k\,\mathrm{e}^{-j/k},\quad j=0,1,\ldots,3k\log(k+1),$$

and let $\epsilon'_{k,j}=\epsilon_{k,j}\,\mathrm{e}^{-4/k^3}=\epsilon'_k\,\mathrm{e}^{-j/k}\,\mathrm{e}^{-4/k^3-1/k^6}$. By analogy with [7], we say that $x'\in\mathcal{D}_k$ is *lower k, δ-successful* if it is lower k-successful and in addition,

$$\frac{2}{\pi}(1-\delta)\epsilon'_{k,j}\leqslant\rho\big(D(x',\epsilon'_{k,j})\big),\quad\forall j=0,\ldots,3k\log(k+1) \tag{5.4}$$

where ρ denotes the measure supported on the real axis, and whose restriction to the real axis is $1/\pi$ times the Lebesgue measure. We recall Lemma 2.3 of [7], adapted to our situation. In what follows,

$$\bar\theta_{x,r}:=\inf\big\{t:\ |B_t-x|=r\big\},$$

and we write $a=b\pm c$ as a shorthand for $|a-b|\leqslant c$.

Lemma 5.3. *We can find $c<\infty$ such that for all $k\geqslant 1$, $r_1\leqslant r_2\leqslant r/2\leqslant 1/2$, x and x_0 with $|x_0-x|=r_2$,*

$$\mathbb{E}^{x_0}\big(\mathcal{I}_{\bar\theta_{x,r}}\big(D(x,r_1)\big)\big)^k\leqslant k!\bigg(\rho\big(D(x,r_1)\big)\log\bigg(\frac{r}{r_1}\bigg)+cr_1\bigg)^k, \tag{5.5}$$

and

$$\mathbb{E}^{x_0}\big(\mathcal{I}_{\bar\theta_{x,r}}\big(D(x,r_1)\big)\big)=\rho\big(D(x,r_1)\big)\log\bigg(\frac{r}{r_2}\bigg)\pm cr_1. \tag{5.6}$$

The above lemma can be seen as an analog of Lemma 3.1; the main difference lies in the fact that the Brownian motion is now stopped when it leaves a disk of radius r. We are now in a position to prove Lemmas 5.1 and 5.2. We will derive Lemma 5.1 from the following lemma, which is an analog of Lemma 4.3 in [7].

Lemma 5.4. *There exists a $k_0=k_0(\delta,\omega)$ such that for all $k\geqslant k_0$ and $x'\in\mathcal{D}_k$, if x' is lower k,δ-successful then*

$$\frac{2}{\pi}(1-\delta)^4h(\epsilon'_{k,j})\leqslant\mathcal{I}_{\bar\theta}\big(D(x',\epsilon'_{k,j})\big),\quad\forall j=0,1,\dots,3k\log(k+1). \tag{5.7}$$

The derivation of Lemma 5.1 from Lemma 5.4 is exactly the same as the derivation of Lemma 4.1 from Lemma 4.3 in [7]. For the reader's convenience, we repeat this argument here. First, for $x\in F$, there exists a sequence $\{y_n\}$ of n-perfect points which tends to x. Since n-perfect points are also k-perfect for $k\leqslant n$, we can find a sequence $\{\tilde x_k\}$ of k-perfect points such that $|x-\tilde x_k|\leqslant\delta_k$. Finally, by definition of $\mathcal{D}_k$ for any such point $\tilde x_k$ there exits $x_k\in\mathcal{D}_k$ such that $|\tilde x_k-x_k|\leqslant\delta_k$. Since $\tilde x_k$ is k-perfect, (5.3) and the fact that δ_k is decreasing guarantee that x_k is lower k-successful. Note that $|x-x_k|\leqslant 2\delta_k$. Thus, since $\delta_k/(\epsilon_{k+1}\,\mathrm{e}^{-4/k^3})\to 0$ as $k\to\infty$,

$$\frac{2}{\pi}(1-\delta)\epsilon'_{k,j}\leqslant\rho\big(D(x_k,\epsilon'_{k,j})\big),\quad\forall j=0,\dots,3k\log(k+1)$$

as soon as k is greater than some $k_1(\delta)$. Therefore, x_k is lower k,δ-successful as soon as $k\geqslant k_1$. Consequently, by Lemma 5.4,

$$\frac{2}{\pi}(1-\delta)^4h(\epsilon'_{k,j})\leqslant\mathcal{I}_{\bar\theta}\big(D(x_k,\epsilon'_{k,j})\big),\quad\forall j=0,1,\dots,3k\log(k+1) \tag{5.8}$$

as soon as k is greater than $k_2(\delta,\omega)=k_1\vee k_0$. Now notice that for k large enough (say greater than k_3), $\epsilon'_{k,j}+2\delta_k\leqslant\epsilon_{k,j}$ for all j in the range of interest. This implies

$$\mathcal{I}_{\bar\theta}\big(D(x,\epsilon_{k,j})\big)\geqslant\mathcal{I}_{\bar\theta}\bigg(D\bigg(x,\epsilon'_{k,j}+\frac{2\epsilon_k}{k^6}\bigg)\bigg)\geqslant\mathcal{I}_{\bar\theta}\big(D(x_k,\epsilon'_{k,j})\big).$$

Combined with (5.8) this gives

$$\mathcal{I}_{\bar\theta}\big(D(x,\epsilon_{k,j})\big)\geqslant\frac{2}{\pi}(1-\delta)^4h(\epsilon'_{k,j})\geqslant\frac{2}{\pi}(1-\delta)^5h(\epsilon_{k,j})$$

for all $j=0,1,\dots,3k\log(k+1)$ and k sufficiently large. Finally, for any $\epsilon_{k+1}\leqslant\epsilon\leqslant\epsilon_k$, let j be such that $\epsilon_{k,j+1}\leqslant\epsilon\leqslant\epsilon_{k,j}$. Then, using the monotonicity of h, we obtain that

$$\frac{\mathcal{I}_{\bar\theta}(D(x,\epsilon))}{h(\epsilon)}\geqslant\frac{\mathcal{I}_{\bar\theta}(D(x,\epsilon_{k,j+1}))}{h(\epsilon_{k,j})}\geqslant\frac{\mathcal{I}_{\bar\theta}(D(x,\epsilon_{k,j+1}))}{h(\epsilon_{k,j+1})}\bigg(1-\frac{2}{k}\bigg),$$

where the last step holds for k large enough. This completes the proof of Lemma 5.1. □

Proof of Lemma 5.4. Here again most of the proof is identical to the one of [7, Lemma 4.3]. Let $x'\in\mathcal{D}_k$ be lower k,δ-successful. Then B makes at least $n'_k=n_k-k$ excursions between $\partial D(x',\epsilon'_k)$ and $\partial D(x',\epsilon''_{k-1})$. For such a point we let $\tau_{l,k,j}$ denote the projected intersection local time measure of $D(x',\epsilon'_{k,j}) \subset D(x',\epsilon'_k)$ accumulated by B during its l-th excursion from $\partial D(x',\epsilon'_k)$ to $\partial D(x',\epsilon''_{k-1})$. Let

$$A(x',k,j) := \left\{\mathcal{I}_{\bar{\theta}}\big(D(x',\epsilon'_{k,j})\big) \leqslant \frac{2}{\pi}(1-\delta)^4 h(\epsilon'_{k,j})\right\}.$$

We define

$$P_{x',k,j} := \mathbf{P}\big(A(x',k,j), x' \text{ is lower } k,\delta\text{-successful}\big) \leqslant \mathbf{P}_{x',s}\left(\sum_{l=1}^{n'_k} \tau_{l,k,j} \leqslant \frac{2}{\pi} a(1-\delta)^4 h(\epsilon'_{k,j})\right)$$

where $\mathbf{P}_{x',s}(\cdot) := \mathbf{P}(\cdot | x' \text{ is lower } k,\delta\text{-successful})$. If we let $\mathbb{E}^l_{x',s}$ be the conditional expectation with respect to the measure $\mathbf{P}_{x',s}$ and given the trajectory of B up to the starting time of the l-th excursion, then

$$\mathbb{E}^l_{x',s}(\tau_{l,k,j}) \geqslant \frac{2}{\pi}(1-\delta)^2 \log\left(\frac{r}{r_2}\right) r_1 \quad \mathbf{P}_{x',s} \quad \text{a.s.} \tag{5.9}$$

where $r_1 := \epsilon'_{k,j}$, $r_2 := \epsilon'_k$, $r := \epsilon''_{k-1}$, and where we have used (5.6) in conjunction with (5.4). The extra $(1-\delta)$ factor comes from the fact that $\log(r/r_2)$ tends to infinity as k becomes large. Note that the left-hand side of (5.9) is random, while the right-hand side is not. Now

$$\big|\log(\epsilon'_{k,j})\big| \leqslant \left|\log(\epsilon_{k+1}) + \frac{4}{k^3}\right| = \left|\log\epsilon_1 - 3\log(k+1)! + \frac{4}{k^3}\right| = \big(1+\mathrm{o}(1)\big)3k\log k, \tag{5.10}$$

where the last step follows from Sterling's formula. On the other hand, since $n'_k = 3ak^2\log k - k$,

$$n'_k \log\left(\frac{r}{r_2}\right) = n'_k(3\log k - 3k^{-6}) = \big(1+\mathrm{o}(1)\big)a(3k\log k)^2. \tag{5.11}$$

Therefore, (5.9) (multiplied by n'_k) together with (5.10) and (5.11) gives

$$n'_k \mathbb{E}^l_{x',s}(\tau_{l,k,j}) \geqslant \frac{2}{\pi} a(1-\delta)^3 h(\epsilon'_{k,j}) \quad \mathbf{P}_{x',s} \quad \text{a.s.}$$

provided k is large enough. This results with

$$P_{x',k,j} \leqslant \mathbf{P}_{x',s}\left(\frac{1}{n'_k}\sum_{l=1}^{n'_k} \hat{\tau}_{l,k,j} \leqslant 1-\delta\right),$$

where $\hat{\tau}_{l,k,j} := \tau_{l,k,j}/\mathbb{E}^l_{x',s}(\tau_{l,k,j})$. By definition, $\mathbb{E}^l_{x',s}(\hat{\tau}_{l,k,j}) = 1$, so that, with $\tilde{\tau}_{l,k,j} := \hat{\tau}_{l,k,j} - \mathbb{E}^l_{x',s}(\hat{\tau}_{l,k,j})$ we have

$$P_{x',k,j} \leqslant \mathbf{P}_{x',s}\left(\frac{1}{n'_k}\sum_{l=1}^{n'_k} \tilde{\tau}_{l,k,j} \leqslant -\delta\right).$$

Since

$$\log\left(\frac{r}{r_2}\right) = 3\log k - 3k^{-6}$$

and

$$\log\left(\frac{r}{r_1}\right) \leqslant 3\log k(k+1) - 2k^{-6} + 4k^{-3},$$

Lemma 5.3 implies that $\mathbb{E}^l_{x',s}(\tilde{\tau}^2_{l,k,j}) \leqslant C$ for some $C > 0$. Now remark that there exists $D > 0$ such that for $x \leqslant 1$, $\mathrm{e}^x \leqslant 1 + x + Dx^2$. Since $\tilde{\tau}_{l,k,j} \geqslant -1$, it follows that for all $0 < \theta < 1$,

$$\mathbb{E}^l_{x',s}(\mathrm{e}^{-\theta\tilde{\tau}_{l,k,j}}) \leqslant 1 + D\theta^2\mathbb{E}^l_{x',s}(\tilde{\tau}^2_{l,k,j}) \leqslant 1 + CD\theta^2 \leqslant \mathrm{e}^{CD\theta^2}.$$

Taking $\theta = \delta/(2CD)$ (which is smaller than 1 provided that δ is small enough) and applying Chebyshev's inequality (together with the Markov property) shows that for some $\lambda = \lambda(a, \delta) > 0$, $C_1 < \infty$ and all $k, x' \in \mathcal{D}_k$, j,

$$P_{x',k,j} \leqslant C_1\,\mathrm{e}^{-\lambda k^2 \log k}.$$

Now since $|\mathcal{D}_k| \leqslant \mathrm{e}^{C_2 k \log k}$ for some $C_2 < \infty$ and all k, it follows that

$$\sum_{k=1}^{\infty} \sum_{j=0}^{3k\log(k+1)} \sum_{x'\in\mathcal{D}_k} P_{x',k,j} \leqslant \sum_{k=1}^{\infty} 3C_1 k \log(k+1)\,\mathrm{e}^{C_2 k \log k}\,\mathrm{e}^{-\lambda k^2 \log k} < \infty. \tag{5.12}$$

The Borel–Cantelli lemma completes the proof. □

Proof of Lemma 5.2. Again, we use the same notation as in [7]: we let

$$\bar{\epsilon}'_k = \epsilon_k\,\mathrm{e}^{-2/k^6}, \qquad \bar{\epsilon}''_{k-1} = \epsilon_{k-1}\,\mathrm{e}^{1/k^6},$$

so that for k large enough

$$\bar{\epsilon}'_k \leqslant \epsilon_k - \delta_k, \qquad \bar{\epsilon}''_{k-1} \geqslant \epsilon_{k-1} + \delta_k.$$

Let $\bar{\epsilon}'_{k,j} := \epsilon_{k,j}\,\mathrm{e}^{4/k^3}$. We now say that $x' \in \mathcal{D}_k$ is *upper k-successful* if there are at most $n''_k := n_k + k$ excursions of B from $\partial D(x', \bar{\epsilon}'_k)$ to $\partial D(x', \bar{\epsilon}''_{k-1})$ prior to $\bar{\theta}$, and if in addition

$$\frac{1}{\pi}\bar{\epsilon}'_{k,j} \leqslant \rho\bigl(D(x', \bar{\epsilon}'_{k,j})\bigr), \quad \forall j = 0, \ldots, 3k\log(k+1). \tag{5.13}$$

The constant $1/\pi$ on the left-hand side could be replaced by any positive constant smaller than $2/\pi$. Using the same argument as in the previous case, Lemma 5.2 can be derived from

Lemma 5.5. *There exists a $k_0 = k_0(\delta, \omega)$ such that for all $k \geqslant k_0$ and $x' \in \mathcal{D}_k$, if x' is upper k-successful then*

$$\frac{2}{\pi}a(1+\delta)^3 h(\bar{\epsilon}'_{k,j}) \geqslant \mathcal{I}_{\bar{\theta}}\bigl(D(x', \bar{\epsilon}'_{k,j})\bigr), \quad \forall j = 1, \ldots, 3k\log(k+1).$$

Proof of Lemma 5.5. Again, the proof is very similar to the one of [7, Lemma 4.4]. In a similar manner we let $\tau_{l,k,j}$ denote the projected intersection local time measure of $D(x', \bar{\epsilon}'_{k,j})$ accumulated by B during its l-th excursion from $\partial D(x', \bar{\epsilon}'_k)$ to $\partial D(x', \bar{\epsilon}''_{k-1})$ (note that $D(x', \bar{\epsilon}'_{k,j}) \subset D(x', \bar{\epsilon}'_k)$ for k large enough since we only consider $j \geqslant 1$). Let $\mathbb{E}^l$ denote the conditional expectation given the trajectory of B up to the starting time of the l-th excursion. Using the same line of arguments as in the proof of Lemma 5.4 (where r, r_1 and r_2 are set to $\bar{\epsilon}''_{k-1}$, $\bar{\epsilon}'_{k,j}$ and $\bar{\epsilon}'_k$ respectively, and where (5.9) is replaced by

$$\mathbb{E}^l_{x',s}(\tau_{l,k,j}) \leqslant \frac{2}{\pi}(1+\delta)\log\left(\frac{r}{r_2}\right) r_1 \quad \mathbf{P}_{x',s}\text{ a.s.} \tag{5.14}$$

since $\rho(D(x', \bar{\epsilon}'_{k,j}))$ is now simply bounded by $(2/\pi)\bar{\epsilon}'_{k,j}$) we obtain that if

$$B(x', k, j) := \left\{\mathcal{I}_{\bar{\theta}}\bigl(D(x', \bar{\epsilon}'_{k,j})\bigr) \geqslant \frac{2}{\pi}(1+\delta)^3 h(\bar{\epsilon}'_{k,j})\right\}$$

then

$$Q_{x',k,j} := \mathbf{P}\big(B(x',k,j), x' \text{ is upper } k\text{-successful}\big) = \mathbf{P}\left(\sum_{l=1}^{n_k''} \tau_{l,k,j} \geqslant \frac{2}{\pi} a(1+\delta)^3 h(\bar{\epsilon}_{k,j}')\right)$$
$$\leqslant \mathbf{P}\left(\frac{1}{n_k''}\sum_{l=1}^{n_k''} \tilde{\tau}_{l,k,j} \geqslant \delta\right).$$

Here $\tilde{\tau}_{l,k,j} := (\tau_{l,k,j}/\mathbb{E}^l \tau_{l,k,j}) - 1$ whenever there are at least l excursions and $\tilde{\tau}_{l,k,j} := 0$ otherwise. For any $a, b \geqslant 0$, $(a+b)^n \leqslant 2^n(a^n + b^n)$, and therefore by (5.5), (5.6) and (5.13)

$$\mathbb{E}^l(\tilde{\tau}_{l,k,j}^n) \leqslant n!AB^n \tag{5.15}$$

for some $A, B > 0$. Thus,

$$\mathbb{E}^l(\mathrm{e}^{\lambda\tilde{\tau}_{l,k,j}}) = 1 + \sum_{n=2}^{\infty} \frac{\lambda^n}{n!}\mathbb{E}^l(\tilde{\tau}_{l,k,j}^n) = 1 + \sum_{n=2}^{\infty} A(\lambda B)^n \leqslant 1 + C\lambda^2 \leqslant \mathrm{e}^{C\lambda^2}$$

for some $C > 0$, where the first inequality holds as soon as λ is small enough. As in the proof of Lemma 5.4, Chebyshev's inequality then implies that $Q_{x',k,j} \leqslant b\,\mathrm{e}^{-a\lambda k^2 \log k}$ for some constants a, b. A last argument of the form (5.12) finishes the proof. □

Acknowledgements

I am very grateful to Amir Dembo and Yuval Peres for suggesting this problem to me, and for many helpful discussions.

References

[1] L. Ahlfors, Complex Analysis, third ed., McGraw-Hill, 1979.
[2] J. Bertoin, Levy Processes, Cambridge University Press, New York, 1996.
[3] E.S. Boylan, Local time for a class of Markov processes, Illinois J. Math. 8 (1964) 19–39.
[4] A. Dembo, Y. Peres, J. Rosen, O. Zeitouni, Thick points for transient symmetric stable processes, Electronic J. Probab. 4 (10) (1999) 1–13.
[5] A. Dembo, Y. Peres, J. Rosen, O. Zeitouni, Thick points for spatial Brownian motion: Multifractal analysis of occupation measure, Ann. Probab. 28 (2000) 1–35.
[6] A. Dembo, Y. Peres, J. Rosen, O. Zeitouni, Thick points for planar Brownian motion and the Erdős–Taylor conjecture on random walk, Acta Math. 186 (2001) 239–270.
[7] A. Dembo, Y. Peres, J. Rosen, O. Zeitouni, Thick points for intersections of planar Brownian paths, Trans. Amer. Math. Soc. 354 (2002) 4969–5003.
[8] D. Gilbard, N. Trudinger, Elliptic Partial Differential Equations of Second Order , Springer-Verlag, Berlin, 1983.
[9] I. Karatzas, S.E. Shreve, Brownian Motion and Stochastic Calculus, second ed., Springer-Verlag, 1991.
[10] L. Marsalle, Slow points and fast points of local times, Ann. Probab. 27 (1999) 150–165.
[11] E.A. Perkins, S.J. Taylor, Uniform measure results for the image of subsets under Brownian motion, Probab. Theory Related Fields 76 (1987) 257–289.
[12] D. Ray, Sojourn times and the exact Hausdorff measure of the sample path for planar Brownian motion, Trans. Amer. Math. Soc. 106 (1963) 436–444.
[13] D. Revuz, M. Yor, Continuous Martingales and Brownian Motion, third ed. , Springer-Verlag, 1998.
[14] N.R. Shieh, S.J. Taylor, Logarithmic multifractal spectrum of stable occupation measure, Stochastic Process Appl. 79 (1998) 249–261.
[15] C.J. Stone, The set of zeros of a semi-stable process, Illinois J. Math. 7 (1963) 631–637.