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Stochastic flows associated to coalescent processes II: Stochastic differential equations

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Dedicated to the memory of Paul-André Meyer

Abstract

We obtain precise information about the stochastic flows of bridges that are associated with the so-called Λ-coalescents. When the measure Λ gives no mass to 0, we prove that the flow of bridges is generated by a stochastic differential equation driven by a Poisson point process. On the other hand, the case $\Lambda = \delta_0$ of the Kingman coalescent gives rise to a flow of coalescing diffusions on the interval $[0, 1]$. We also discuss a remarkable Brownian flow on the circle which has close connections with the Kingman coalescent.

Résumé

Nous étudions les flots de ponts associés aux processus de coagulation appelés Λ-coalescents. Quand la mesure Λ ne charge pas 0, nous montrons que le flot de ponts est engendré par une équation différentielle stochastique conduite par un processus de Poisson ponctuel. Au contraire, le cas $\Lambda = \delta_0$ du coalescent de Kingman fait apparaître un flot de diffusions coalescentes sur l'intervalle $[0, 1]$. Nous étudions aussi un flot brownien remarquable sur le cercle, qui est étroitement lié au coalescent de Kingman.

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1. Introduction

In a previous work [1], we obtained a surprising connection between the class of exchangeable coalescents and certain remarkable stochastic flows on the interval $[0,1]$. The main purpose of the present paper is to derive more explicit information about these flows, and in particular to represent them as solutions of stochastic differential equations.

Exchangeable coalescents, also called coalescents with simultaneous multiple collisions by Schweinsberg [14], are processes taking values in the set $\mathcal{P}$ of all partitions of $\mathbb{N}$, which appear as asymptotic models for phenomena of coagulation that occur when studying the genealogy of large populations. They have been studied recently by Möhle, Sagitov, Pitman and Schweinsberg [11–14]. Roughly speaking, an exchangeable coalescent is a Markov process $\Pi = (\Pi_t,\ t \geqslant 0)$ in $\mathcal{P}$, which satisfies the following two conditions. Firstly, for every $s \leqslant t$, the partition Π_s is finer than Π_t (blocks coagulate as time increases). Secondly, the semigroup of Π satisfies a natural exchangeability property saying that in the coagulation phenomenon all blocks play the same role. See [1] for a more precise definition.

The main result of [1] gives a one-to-one correspondence between exchangeable coalescents and flows of bridges on $[0,1]$. By definition, a *bridge* is a real-valued random process $(B(r),\ r \in [0,1])$ with $B(0) = 0$ and $B(1) = 1$, which has right-continuous nondecreasing sample paths and exchangeable increments. A *flow of bridges* is then a collection $(B_{s,t},\ -\infty < s \leqslant t < \infty)$ of bridges, satisfying the flow property $B_{s,u} = B_{s,t} \circ B_{t,u}$ for every $s \leqslant t \leqslant u$, and the usual stationarity and independence of "increments" property (see Section 2.1 below for the precise definition). These flows, or more precisely the dual flows $\widehat{B}_{s,t} = B_{-t,-s}$, fit in the general framework of Le Jan and Raimond [9].

Let us briefly describe the basic connection between exchangeable coalescents and flows of bridges [1], which may be viewed as an infinite-dimensional version of Kingman's famous theorem on the structure of exchangeable partitions of $\mathbb{N}$. Start with a flow of bridges $(B_{s,t},\ -\infty < s \leqslant t < \infty)$ and consider an independent sequence $(V_j)_{j\in\mathbb{N}}$ of i.i.d. uniform $[0,1]$ variables. Write $\mathcal{R}(B_{s,t})$ for the closed range of $B_{s,t}$. For every $t \geqslant 0$ define a random partition Π_t of $\mathbb{N}$ by declaring that two distinct integers i and j belong to the same block of Π_t if and only if V_i and V_j belong to the same connected component of $[0,1] \setminus \mathcal{R}(B_{0,t})$. Then, $(\Pi_t,\ t \geqslant 0)$ is an exchangeable coalescent and conversely any exchangeable coalescent can be obtained in this way from a (unique in law) flow of bridges.

In the present paper, we focus on the flows associated with an important subclass of exchangeable coalescents, namely the Λ-coalescents. Roughly speaking, Λ-coalescents are those exchangeable coalescents where only one subcollection of blocks can coagulate at a time. The law of such a process is characterized by a finite measure Λ on $[0,1]$ (see Section 2.2 for more details). Important special cases are the Kingman coalescent ($\Lambda = \delta_0$) and the Bolthausen–Sznitman coalescent (Λ is Lebesgue measure on $[0,1]$). The class of Λ-coalescents was introduced and studied by Pitman [12], under the name of coalescents with multiple collisions.

Let us now outline the main contributions of the present work. We let $B = (B_{s,t})_{-\infty<s\leqslant t<\infty}$ be the flow of bridges associated with a Λ-coalescent in the sense of [1]. Sections 3 and 4 below are devoted to the study of the Markov process

$$F_t = \bigl(B_{-t,0}(x),\ x \in [0,1]\bigr),$$

and particularly of the p-point motion $(F_t(r_1), \ldots, F_t(r_p))$, where $r_1 < \cdots < r_p$ are p fixed points in $[0,1]$. Assuming that $\Lambda(\{0\}) = 0$ we prove in Section 3 that

$$\bigl(F_t(r_1), \ldots, F_t(r_p)\bigr)_{t\geqslant 0} \overset{(\mathrm{d})}{=} (X_t^1, \ldots, X_t^p)_{t\geqslant 0},$$

where $(X^1, \ldots, X^p)$ is the (unique in law) solution of the stochastic differential equation

$$X_t^i = r_i + \int\limits_{[0,t]\times]0,1[\times]0,1]} M(\mathrm{d}s, \mathrm{d}u, \mathrm{d}x)\, x\bigl(1_{\{u\leqslant X_{s-}^i\}} - X_{s-}^i\bigr), \quad i = 1, \ldots, p,$$

which is driven by a Poisson point measure M on $\mathbb{R}_+\times]0,1[\times]0,1]$ with intensity $\mathrm{d}s\,\mathrm{d}u\,x^{-2}\Lambda(\mathrm{d}x)$. The integral with respect to M should be understood as a stochastic integral with respect to a compensated Poisson measure. A key intermediate step towards this representation is to obtain a martingale problem characterizing the law of the p-point motion $(F_t(r_1),\dots,F_t(r_p))$.

In Section 4 we consider the case of the celebrated Kingman coalescent [8] (i.e. when Λ is the Dirac point mass at 0). Then the p-point motion $(F_t(r_1),\dots,F_t(r_p))$ is a diffusion process in

$$\mathcal{D}_p := \{x=(x_1,\dots,x_p):\ 0\leqslant x_1\leqslant x_2\leqslant\cdots\leqslant x_p\leqslant 1\}$$

with generator

$$\mathcal{A}g(x)=\frac{1}{2}\sum_{i,j=1}^{p} x_{i\wedge j}(1-x_{i\vee j})\frac{\partial^2 g}{\partial x_i\partial x_j}(x),$$

for $g\in\mathcal{C}^2(\mathcal{D}_p)$. Note that the components of this diffusion process coalesce when they meet, and are also absorbed at 0 and 1.

The results of Sections 3 and 4 give insight in the behavior of the bridges $B_{s,t}$ when s decreases (recall that $F_t=B_{-t,0}$). What can be said about $B_{s,t}$ when t increases? To answer this question it is convenient to introduce the flow of inverses

$$\Gamma_{s,t}(r)=\inf\{u\in[0,1]:\ B_{s,t}(u)>r\},\quad r\in[0,1[,$$

and $\Gamma_{s,t}(1)=\Gamma_{s,t}(1-)$. Section 5 studies the corresponding (Markovian) p-point motions $(\Gamma_t(r_1),\dots,\Gamma_t(r_p))$, where $\Gamma_t=\Gamma_{0,t}$. For a general measure Λ such that $\Lambda(\{0\})=0$, we show that the law of the p-point motion satisfies a martingale problem analogous to the one obtained in Section 3 for F_t. In the Kingman case, we prove that $(\Gamma_t(r_1),\dots,\Gamma_t(r_p))$ is a diffusion process in $\mathcal{D}_p$ with generator

$$\tilde{\mathcal{A}}g(x)=\frac{1}{2}\sum_{i,j=1}^{p} x_{i\wedge j}(1-x_{i\vee j})\frac{\partial^2 g}{\partial x_i\partial x_j}(x)+\sum_{i=1}^{p}\left(\frac{1}{2}-x_i\right)\frac{\partial g}{\partial x_i}.$$

Again components of this diffusion process coalesce when they meet, but in contrast to the diffusion with generator $\mathcal{A}$ they never reach 0 or 1.

Together with Section 4, this gives a fairly complete picture of the flow associated with the Kingman coalescent. For every $s<t$, $B_{s,t}$ is a step function, that is a nondecreasing function taking only finitely many values. When t increases, the vector of jump times evolves like a diffusion process with generator $\tilde{\mathcal{A}}$, but the sizes of the jumps remain constant until the first moment when two jump times coalesce (yielding a "coagulation" of the corresponding jumps). Conversely, when s decreases, the vector of values taken by $B_{s,t}$ evolves like a diffusion process with generator $\mathcal{A}$, but the vector of jump times remains constant, until the moment when two among the values taken by $B_{s,t}$ coalesce (or one of them hits 0 or 1) thus provoking the disappearance of one jump.

Finally, Section 6 discusses closely related flows on the circle $\mathbb{T}=\mathbb{R}/\mathbb{Z}$ rather than on $[0,1]$. In the easy case where $\int x^{-2}\Lambda(\mathrm{d}x)<\infty$, corresponding to the simple flows in [1], we briefly explain how the Poissonian construction of [1] can be adapted to give flows on $\mathbb{T}$ which are associated with Λ-coalescents. A suitable limiting procedure then leads to a flow $\Theta=(\Theta_t,\ t\geqslant 0)$ which is associated with the Kingman coalescent. Precisely, Θ is a Brownian flow (in the sense of Harris [4]) on $\mathbb{T}$, with covariance function

$$b(y,y')=\frac{1}{12}-\frac{1}{2}d(y,y')\big(1-d(y,y')\big),$$

where d is the distance on $\mathbb{T}$. The connection with the Kingman coalescent can then be stated as follows. For every $t>0$, the range $\mathcal{S}_t$ of Θ_t is finite. For every $y\in\mathcal{S}_t$ we can define the mass of y at time t as the Lebesgue measure of $\{x\in\mathbb{T}:\ \Theta_t(x)=y\}$. Then, as a process in the variable t, the vector of masses of elements of $\mathcal{S}_t$ is distributed as the frequencies of blocks in the Kingman coalescent. Alternative formulations and more precise results about the flow Θ can be found in Section 6.

2. Preliminaries

2.1. Flows of bridges and exchangeable coalescents

To start with, we recall the basic correspondence between bridges on $[0,1]$ and exchangeable random partitions of $\mathbb{N} := \{1,2,\ldots\}$, which is a slight variation of a fundamental theorem of Kingman.

A mass-partition is a sequence $\beta = (\beta^i,\ i \in \mathbb{N})$ with

$$\beta^1 \geqslant \beta^2 \geqslant \cdots \geqslant 0 \quad \text{and} \quad \sum_{i=1}^{\infty} \beta^i \leqslant 1.$$

Following Kallenberg [6], given a random mass partition β and an independent sequence $(U^i,\ i \in \mathbb{N})$ of i.i.d. variables with uniform distribution over $[0,1]$, we may define a stochastic process $B = (B(r),\ r \in [0,1])$ with exchangeable increments by

$$B(r) = \left(1 - \sum_{i=1}^{\infty} \beta^i\right) r + \sum_{i=1}^{\infty} \beta^i 1_{\{U^i \leqslant r\}}, \quad r \in [0,1]. \tag{1}$$

Observe that B has right-continuous increasing paths with $B(0) = 0$ and $B(1-) = 1$, and that the ranked sequence of the jump sizes of B is given by the mass partition β.

In the sequel, we shall call *bridge* any process which can be expressed in the form (1). This is equivalent to the definition given in [1] or in the introduction above. It is easy to check that the composition of two independent bridges is again a bridge (this is essentially Bochner's subordination), which motivates the following definition. A *flow of bridges* is a collection $(B_{s,t},\ -\infty < s \leqslant t < \infty)$ of bridges such that:

(i) For every $s < t < u$, $B_{s,u} = B_{s,t} \circ B_{t,u}$ a.s.
(ii) The law of $B_{s,t}$ only depends on $t-s$. Furthermore, if $s_1 < s_2 < \cdots < s_n$, the bridges $B_{s_1,s_2}, B_{s_2,s_3}, \ldots, B_{s_{n-1},s_n}$ are independent.
(iii) $B_{0,0} = \mathrm{Id}$ and $B_{0,t} \to \mathrm{Id}$ in probability as $t \downarrow 0$, in the sense of Skorokhod's topology.

Recall that $\mathcal{P}$ denotes the set of all partitions of $\mathbb{N}$. We also denote by $\mathcal{P}_n$ the (finite) set of all partitions of $\{1,\ldots,n\}$. The set $\mathcal{P}$ is equipped with the smallest topology for which the restriction maps from $\mathcal{P}$ onto $\mathcal{P}_n$ are continuous, when $\mathcal{P}_n$ is equipped with the discrete topology. A random partition (of $\mathbb{N}$) is a random variable with values in $\mathcal{P}$. It is said *exchangeable* if its distribution is invariant under the natural action of the permutations of $\mathbb{N}$ on $\mathcal{P}$.

There is a simple procedure to construct a random exchangeable partition from a bridge B, which is a variant of Kingman's paintbox process. Let $\mathcal{R} = \{B(r),\ r \in [0,1]\}^{\mathrm{cl}}$ be the closed range of B, so $\mathcal{R}^{\mathrm{c}} = [0,1] \setminus \mathcal{R}$ is a random open set which has a canonical decomposition into disjoint open intervals, called the interval components of $\mathcal{R}^{\mathrm{c}}$. Introduce a sequence of i.i.d. uniform variables on $[0,1]$, $(V_i,\ i \in \mathbb{N})$, which is independent of the bridge B. We define a random partition $\pi(B)$ of $\mathbb{N}$ by declaring that the indices $i \in \mathbb{N}$ such that $V_i \in \mathcal{R}$ are the singletons of $\pi(B)$, and two indices $i \neq j$ belong to the same block of $\pi(B)$ if and only if V_i and V_j belong to the same interval component of $\mathcal{R}^{\mathrm{c}}$. By the strong law of large numbers, the sizes β^k of the jumps of B correspond to the asymptotic frequencies of the blocks of $\pi(B)$. Obviously $\pi(B)$ is exchangeable, and conversely, any exchangeable random partition π is distributed as $\pi(B)$ for a certain bridge B.

The basic result in [1] stems from the observation that, informally, the sequence of jump sizes of a compound bridge $B = B_1 \circ B_2$ can be expressed as a certain coagulation of the jump sizes of B_1, where the coagulation mechanism is encoded by B_2. This entails that when one applies the above paintbox construction to a flow of bridges, one obtains a Markov process with values in $\mathcal{P}$, which starts from the partition of $\mathbb{N}$ into singletons, and is such that blocks of partitions coagulate as time passes. To be specific, let $(B_{s,t})_{-\infty<s\leqslant t<\infty}$ be a flow of bridges, and suppose that the sequence $(V_i,\ i \in \mathbb{N})$ introduced above is independent of the flow. Then, the process $(\pi(B_{0,t}),\ t \geqslant 0)$ is a $\mathcal{P}$-valued Markov process belonging to the class of *exchangeable coalescents* (see Definition 1 in [1] for a precise definition). Conversely, any exchangeable coalescent can be obtained by this procedure (see Theorem 1 in [1]).

2.2. *Λ-coalescents and generalized Fleming–Viot processes*

Pitman [12] and Sagitov [13] have pointed at an important class of exchangeable coalescents whose laws can be characterized by an arbitrary finite measure Λ on $[0,1]$. Specifically, a Λ-coalescent is a Markov process $\Pi = (\Pi_t,\ t \geqslant 0)$ on $\mathcal{P}$ started from the partition into singletons, whose evolution can be described as follows (see Theorem 1 in [12]).

First, one introduces the rates

$$\beta_{p,k} = \int \Lambda(\mathrm{d}x) x^{k-2}(1-x)^{p-k}, \tag{2}$$

for every integers $2 \leqslant k \leqslant p$. Next, for every integer n and every time $t \geqslant 0$, denote by Π_t^n the restriction of the partition Π_t to $\{1, \dots, n\}$. Then each process $(\Pi_t^n,\ t \geqslant 0)$ is a continuous time Markov chain with values in the (finite) set $\mathcal{P}_n$. The law of this Markov chain is characterized by its transition rates: Starting from a partition in $\mathcal{P}_n$ with p nonempty blocks, for each $k = 2, \dots, p$, every possible merging of k blocks (the other $p-k$ blocks remaining unchanged) occurs at rate $\beta_{p,k}$, and no other transition is possible. This description of the restricted processes Π^n determines the law of the Λ-coalescent Π.

In this work, we shall be interested in the flow of bridges $(B_{s,t},\ -\infty < s \leqslant t < \infty)$ corresponding to a Λ-coalescent in the sense explained above. In Sections 3 and 4 below, we will study the process

$$F_t := B_{-t,0}, \quad t \geqslant 0, \tag{3}$$

which takes values in the set of all right-continuous nondecreasing functions from $[0,1]$ into $[0,1]$. This process will be called the *Λ-process*. From properties (i) and (ii) of a flow, it is immediate to see that for every integer $p \geqslant 1$ and every $(x_1, \dots, x_p) \in [0,1]^p$, the p-point motion $(F_t(x_1), \dots, F_t(x_p))$ is Markovian with a Feller semigroup (see also the discussion in Section 5.1 of [1]).

For each $t \geqslant 0$, the function $F_t : [0,1] \to [0,1]$ can be viewed as the distribution function of a random probability measure ρ_t on $[0,1]$:

$$F_t(x) = \rho_t\big([0,x]\big), \quad x \in [0,1].$$

Note that $\rho_0 = \lambda$ is Lebesgue measure on $[0,1]$. The measure-valued process $(\rho_t,\ t \geqslant 0)$, which can be interpreted as a generalized Fleming–Viot process (see e.g. Chapter 1 of Etheridge [2] for an introduction to Fleming–Viot measure-valued processes), is studied in Section 5 of [1]. In the next subsection, we recall some basic properties of this process that play a crucial role in the present work.

2.3. *Martingales for the generalized Fleming–Viot process*

We first present a characterization of the law of the measure-valued process $(\rho_t,\ t \geqslant 0)$ as the solution to a martingale problem which is expressed in terms of the rates (2). In this direction, we first need to introduce some notation.

For every probability measure μ on $[0,1]$ and every bounded measurable function $g : [0,1] \to \mathbb{R}$, we write

$$\mu(g) := \int_{[0,1]} \mu(\mathrm{d}x) g(x).$$

Let $p \geqslant 1$ be an integer. For every $i = 1, \ldots, p$, let $h_i : [0,1] \to \mathbb{R}$ be a bounded measurable function. We consider the function $h : [0,1]^p \to \mathbb{R}$ defined by

$$h(x) := \prod_{i=1}^{p} h_i(x_i), \quad x = (x_1, \ldots, x_p). \tag{4}$$

Next, for every subset of indices $I \subseteq \{1, \ldots, p\}$ with $|I| \geqslant 2$, we write $h_I : [0,1]^p \to \mathbb{R}$ for the function defined by

$$h_I(x) := \prod_{i \in I} h_i(x_\ell) \times \prod_{j \notin I} h_j(x_j), \quad x = (x_1, \ldots, x_p),$$

where $\ell = \min I$. Finally we set

$$G_h(\mu) := \int h \, \mathrm{d}\mu^{\otimes p} = \prod_{i=1}^{p} \mu(h_i); \tag{5}$$

observe that

$$G_{h_I}(\mu) = \mu\Big(\prod_{i \in I} h_i\Big) \prod_{j \notin I} \mu(h_j).$$

Recall that Λ is a finite measure on $[0,1]$ and that the numbers $\beta_{p,k}$ defined in (2) are the transition rates of the Λ-coalescent. We introduce an operator L acting on functions of the type G_h:

$$LG_h(\mu) := \sum_{I \subseteq \{1,\ldots,p\},\ |I| \geqslant 2} \beta_{p,|I|}\big(G_{h_I}(\mu) - G_h(\mu)\big). \tag{6}$$

The following statement essentially rephrases Theorem 3(i) in [1]. The functions considered in [1] are supposed to be continuous rather than bounded and measurable. However the general case follows from a standard argument (see e.g. Proposition 4.2, page 111 of [3]).

Theorem 1. *The law of the process $(\rho_t,\ t \geqslant 0)$ is characterized by the following martingale problem. We have $\rho_0 = \lambda$ and, for every integer $p \geqslant 1$ and every bounded measurable functions $h_i : [0,1] \to \mathbb{R}$, $i = 1, \ldots, p$, the process*

$$G_h(\rho_t) - \int_0^t \mathrm{d}s \, LG_h(\rho_s)$$

is a martingale, where h is defined by (4), *G_h by* (5), *and LG_h by* (6).

Uniqueness for the martingale problem of Theorem 1 follows from a duality argument. To be specific, the process $(\rho_t,\ t \geqslant 0)$ can be interpreted as a measure-valued dual to the Λ-coalescent $(\Pi_t^p,\ t \geqslant 0)$ in $\mathcal{P}_p$, and we have the explicit formula

$$\mathbb{E}\big[G_h(\rho_t)\big] = \mathbb{E}\Big[\prod_{A \text{ block of } \Pi_t^p} \lambda\Big(\prod_{i \in A} h_i\Big)\Big] \tag{7}$$

(see formula (18) in [1]). Specializing to the case $h_i = 1_{[0,x]}$, we see that

$$\mathbb{E}\big[F_t(x)^p\big] = \mathbb{E}\big[x^{\#\Pi_t^p}\big], \tag{8}$$

where $\#\Pi_t^p$ denotes the number of blocks in Π_t^p.

3. A Poissonian SDE for Λ-processes

In this section, we assume that Λ is a finite measure on $[0,1]$ which has no atom at 0, i.e. $\Lambda(\{0\})=0$. Our goal is to get a representation of the Λ-process F as the solution to a stochastic differential equation driven by a Poisson point process.

As a first step, we shall see that in the easy case when the measure Λ fulfils the condition

$$\int_{[0,1]} x^{-2}\Lambda(\mathrm{d}x)<\infty, \tag{9}$$

the Λ-process solves a simple Poissonian SDE which derives directly from an explicit construction of F given in [1]. In the general case, this Poissonian SDE still makes sense thanks to the notion of stochastic integral with respect to a compensated point measure (see e.g. Jacod [5]). We prove that the Λ-process is a weak solution of the Poissonian SDE, and that weak uniqueness holds for this SDE. As a key tool, we establish that the law of the p-point motion is characterized by a martingale problem.

3.1. The simple case

We start by recalling the Poissonian construction of the Λ-process in the special case when (9) holds (see [1], Section 4). We denote by $m(\mathrm{d}u,\mathrm{d}x)$ the measure on $]0,1[\times]0,1]$ defined by $m(\mathrm{d}u,\mathrm{d}x)=\mathrm{d}u\otimes x^{-2}\Lambda(\mathrm{d}x)$. Consider a Poisson random measure on $\mathbb{R}_+\times]0,1[\times]0,1]$,

$$M=\sum_{i=1}^{\infty}\delta_{(t_i,u_i,x_i)},$$

with intensity $\mathrm{d}t\otimes m(\mathrm{d}u,\mathrm{d}x)$. Here the atoms $(t_1,u_1,x_1),(t_2,u_2,x_2),\dots$ of M are listed in the increasing order of their first coordinate, which is possible since the measure m is finite by our assumption (9). Next, for every $u\in]0,1[$ and $x\in]0,1]$, we introduce the elementary function

$$b_{u,x}(r)=(1-x)r+x\mathbf{1}_{\{u\leqslant r\}},\quad r\in[0,1].$$

The Λ-process $(F_t,\ t\geqslant 0)$ can then be obtained by composing to the left the elementary functions b_{u_i,x_i} as atoms (t_i,u_i,x_i) are found in the Poisson measure M. Specifically, we set $F_t=\mathrm{Id}_{[0,1]}$ when $t\in[0,t_1[$, and then for every integer $k\geqslant 1$ and $t\in[t_k,t_{k+1}[$

$$F_t=b_{u_k,x_k}\circ\cdots\circ b_{u_1,x_1}. \tag{10}$$

It is straightforward to check from (10) that for every $y\in[0,1]$, the process $(F_t(y),\ t\geqslant 0)$ can also be described as the unique solution to the following Poissonian stochastic differential equation

$$F_t(y)=y+\int_{[0,t]\times]0,1[\times]0,1]} M(\mathrm{d}s,\mathrm{d}u,\mathrm{d}x)x\Psi\big(u,F_{s-}(y)\big), \tag{11}$$

where for every $u\in]0,1[$ and $r\in[0,1]$,

$$\Psi(u,r)=\mathbf{1}_{\{u\leqslant r\}}-r. \tag{12}$$

3.2. A martingale problem for the p-point motion

From now on, we come back to the general case where Λ is a finite measure on $[0,1]$ which does not charge 0. Our purpose here is to characterize the law of the p-point motion of the Λ-process as the unique solution to a martingale problem. In this direction, we first introduce some notation.

Fix an integer $p \geqslant 1$. For every $y = (y_1, \dots, y_p) \in [0,1]^p$ and every function $g : [0,1]^p \to \mathbb{R}$ of class $\mathcal{C}^2$, we write, for $u \in]0,1[$ and $x \in]0,1]$,

$$y + x\Psi(u, y) := \big(y_1 + x\Psi(u, y_1), \dots, y_p + x\Psi(u, y_p)\big),$$

and then

$$\Delta_{u,x} g(y) := g\big(y + x\Psi(u, y)\big) - g(y) - x\Psi(u, y) \cdot \nabla g(y),$$

where

$$\Psi(u, y) \cdot \nabla g(y) := \sum_{i=1}^{p} \Psi(u, y_i)\partial_i g(y_1, \dots, y_p).$$

Next, observing that $|\Delta_{u,x} g(y)| \leqslant Cx^2$ for some constant $C > 0$ depending only on g, we set

$$\mathcal{L}g(y) := \int_{]0,1]} \Lambda(\mathrm{d}x)x^{-2} \int_0^1 \mathrm{d}u\, \Delta_{u,x} g(y).$$

Recall that

$$\mathcal{D}_p := \big\{x = (x_1, \dots, x_p):\ 0 \leqslant x_1 \leqslant x_2 \leqslant \dots \leqslant x_p \leqslant 1\big\}. \tag{13}$$

By construction, if $y = (y_1, \dots, y_p) \in \mathcal{D}_p$, the p-point motion $(F_t(y_1), \dots, F_t(y_p))$ lives in $\mathcal{D}_p$. We already noticed that it has a Feller semigroup, so that we can assume that it has càdlàg sample paths.

We will now characterize the distribution of the p-point motion by a martingale problem, which is clearly related to Theorem 1 above.

Lemma 1. *Let $p \geqslant 1$ and $(y_1, \dots, y_p) \in \mathcal{D}_p$. The law of the process $((F_t(y_1), \dots, F_t(y_p)),\ t \geqslant 0)$ is characterized by the following martingale problem. We have $(F_0(y_1), \dots, F_0(y_p)) = (y_1, \dots, y_p)$ and, for every function $g : \mathcal{D}_p \to \mathbb{R}$ of class $\mathcal{C}^2$, the process*

$$g\big(F_t(y_1), \dots, F_t(y_p)\big) - \int_0^t \mathrm{d}s\, \mathcal{L}g\big(F_s(y_1), \dots, F_s(y_p)\big), \quad t \geqslant 0,$$

is a martingale.

Proof. We start by proving that the p-point motion does solve the martingale problem of the lemma. Let $k_1, \dots, k_p$ be nonnegative integers and set $k = k_1 + \dots + k_p$. Set $j(i) = 1$ if and only if $1 \leqslant i \leqslant k_1$ and, for $j \in \{2, \dots, p\}$, set $j(i) = j$ if and only if $k_1 + \dots + k_{j-1} < i \leqslant k_1 + \dots + k_j$. If A is a nonempty subset of $\{1, \dots, k\}$, we also set

$$j(A) = \inf_{i \in A} j(i).$$

Define a function g on $\mathcal{D}_p$ by

$$g(z_1, \dots, z_p) = \prod_{j=1}^{p} (z_j)^{k_j}. \tag{14}$$

We start by calculating $\mathcal{L}g$. Noting that $\int_0^1 \mathrm{d}u\Psi(u, y) = 0$ for every $y \in [0,1]$, we have

$$\mathcal{L}g(z_1,\ldots,z_p)=\int\limits_{]0,1]}\Lambda(\mathrm{d}x)x^{-2}\left(\int\limits_0^1\mathrm{d}u\left(\prod_{j=1}^p\bigl((1-x)z_j+x\mathbf{1}_{\{u\leqslant z_j\}}\bigr)^{k_j}-\prod_{j=1}^p(z_j)^{k_j}\right)\right)$$
$$=\sum_{I\subset\{1,\ldots,k\},\ |I|\geqslant 2}\beta_{k,|I|}\left(\left(\prod_{j=1}^p(z_j)^{k_j-k_j^I}\right)z_{j(I)}-\prod_{j=1}^p(z_j)^{k_j}\right),\tag{15}$$

where $k_j^I=|\{i\in I:\ j(i)=j\}|$ for every nonempty subset I of $\{1,\ldots,k\}$ and every $j\in\{1,\ldots,p\}$. The last equality is obtained by expanding the first product in the preceding line, in a way very similar to [1], p. 281.

Now define a function h on $[0,1]^k$ by

$$h(x_1,\ldots,x_k)=\prod_{i=1}^k 1_{[0,y_{j(i)}]}(x_i).$$

In the notation of Section 2.3 we have, for every $s\geqslant 0$,

$$G_h(\rho_s)=\prod_{j=1}^p\rho_s\bigl([0,y_j]\bigr)^{k_j}=g\bigl(F_s(y_1),\ldots,F_s(y_p)\bigr).\tag{16}$$

We can also compute $LG_h(\mu)$ from formula (6):

$$LG_h(\mu)=\sum_{I\subset\{1,\ldots,k\},\ |I|\geqslant 2}\beta_{k,|I|}\bigl(G_{h_I}(\mu)-G_h(\mu)\bigr)\tag{17}$$

and

$$h_I(x_1,\ldots,x_k)=\left(\prod_{i\notin I}1_{[0,y_{j(i)}]}(x_i)\right)\times 1_{[0,y_{j(I)}]}(x_\ell)\tag{18}$$

with $\ell=\min I$.

By comparing (17) and (18) with (15), we get for every $s\geqslant 0$

$$LG_h(\rho_s)=\mathcal{L}g\bigl(F_s(y_1),\ldots,F_s(y_p)\bigr).\tag{19}$$

From (16), (19) and Theorem 1 we obtain the martingale problem of the lemma in the special case where g is of the type (14). The general case follows by a standard density argument.

It remains to prove uniqueness. To this end we use a duality argument analogous to the one presented in Section 5.2 of [1]. Recall that $\mathcal{P}_k$ denotes the space of all partitions of $\{1,\ldots,k\}$ and $(\Pi_t^k,\ t\geqslant 0)$ is the Λ-coalescent in $\mathcal{P}_k$. For every partition $\pi\in\mathcal{P}_k$, and every $(z_1,\ldots,z_p)\in\mathcal{D}_p$ we set

$$P\bigl((z_1,\ldots,z_p),\pi\bigr)=\prod_{A\text{ block of }\pi}z_{j(A)}.$$

If $\mathcal{L}^*$ denotes the generator of (Π_t^k), viewing $P((z_1,\ldots,z_p),\pi)$ as a function of π, we have

$$\mathcal{L}^*P\bigl((z_1,\ldots,z_p),\pi\bigr)=\sum_{I\subset\{1,\ldots,\#\pi\},\ |I|\geqslant 2}\beta_{k,|I|}\left(\prod_{A\text{ block of }c_I(\pi)}z_{j(A)}-\prod_{A\text{ block of }\pi}z_{j(A)}\right),$$

where if $A_1,A_2,\ldots$ are the blocks of π, $c_I(\pi)$ is the new partition obtained by coagulating the blocks A_i for $i\in I$. On the other hand, viewing $P((z_1,\ldots,z_p),\pi)$ as a function of $(z_1,\ldots,z_p)$ we can also evaluate $\mathcal{L}P((z_1,\ldots,z_p),\pi)$ from formula (15), and we easily obtain

$$\mathcal{L}^*P\bigl((z_1,\ldots,z_p),\pi\bigr)=\mathcal{L}P\bigl((z_1,\ldots,z_p),\pi\bigr).\tag{20}$$

Now suppose that $(Z_t^1, \dots, Z_t^p)$ is a $\mathcal{D}_p$-valued càdlàg process that solves the martingale problem of the lemma with initial value $(y_1, \dots, y_p)$, and let π_0 be the partition of $\{1, \dots, k\}$ in singletons. By standard arguments (see Section 4.4 in [3]) we deduce from (20) that

$$\mathbb{E}\left[\prod_{j=1}^{p} (Z_t^j)^{k_j}\right] = \mathbb{E}\big[P\big((Z_t^1, \dots, Z_t^p), \pi_0\big)\big] = \mathbb{E}\big[P\big((y_1, \dots, y_p), \Pi_t^k\big)\big]. \tag{21}$$

This is enough to show that the law of $(Z_t^1, \dots, Z_t^p)$ is uniquely determined. □

Remark. In the case where $(Z_t^1, \dots, Z_t^p) = (F_t(y_1), \dots, F_t(y_p))$, the identity (21) is of course a special case of (7).

3.3. Weak existence and uniqueness for a Poissonian SDE

The identity (11) in the simple case treated in Subsection 3.1 incites us to construct on a suitable filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$:

- an $(\mathcal{F}_t)$-Poisson point process M on $\mathbb{R}_+ \times]0,1[\times]0,1]$ with intensity $\mathrm{d}t \otimes m(\mathrm{d}u, \mathrm{d}x) := \mathrm{d}t \otimes \mathrm{d}u \otimes x^{-2} \Lambda(\mathrm{d}x)$,
- a collection $(X_t(r),\ t \geqslant 0)$, $r \in [0,1]$ of adapted càdlàg processes with values in $[0,1]$, in such a way that for every $r \in [0,1]$, a.s.

$$X_t(r) = r + \int_{[0,t] \times]0,1[\times]0,1]} M(\mathrm{d}s, \mathrm{d}u, \mathrm{d}x) x \Psi\big(u, X_{s-}(r)\big). \tag{22}$$

The Poissonian stochastic integral in the right-hand side should be understood with respect to the compensated Poisson measure M (see e.g. Chapter 3 of [5]). This makes sense as $|\Psi| \leqslant 1$ and $\int x^2 m(\mathrm{d}u, \mathrm{d}x) < \infty$. Recall also that $\int_0^1 \mathrm{d}u \Psi(u, r) = 0$ for all $r \in [0,1]$, so that roughly speaking, the compensation plays no role.

A pair $(M, (X_\cdot(r),\ r \in [0,1]))$ satisfying the above conditions will be called a *weak solution* of (22). The main result of this section is the following.

Theorem 2. *There exists a weak solution of* (22)*, which satisfies the additional property that* $X_t(r_1) \leqslant X_t(r_2)$ *for every* $t \geqslant 0$*, a.s. whenever* $0 \leqslant r_1 \leqslant r_2 \leqslant 1$*. Moreover, for every such solution* (M, X)*, every integer* $p \geqslant 1$ *and every* p*-tuple* $(r_1, \dots, r_p) \in \mathcal{D}_p$*, the process* $((X_t(r_1), \dots, X_t(r_p)),\ t \geqslant 0)$ *has the same distribution as the* p*-point motion of the* Λ*-process started at* $(r_1, \dots, r_p)$*.*

Proof. The second part of the theorem (weak uniqueness) is an easy consequence of Lemma 1. Recall the notation $m(\mathrm{d}u, \mathrm{d}x) = \mathrm{d}u \otimes x^{-2} \Lambda(\mathrm{d}x)$. Suppose that $((Z_t^1, \dots, Z_t^p),\ t \geqslant 0)$ is a $\mathcal{D}_p$-valued adapted process which satisfies the SDE

$$Z_t^i = r_i + \int_{[0,t] \times]0,1[\times]0,1]} M(\mathrm{d}s, \mathrm{d}u, \mathrm{d}x) x \Psi(u, Z_{s-}^i), \quad i = 1, \dots, p.$$

Recall the notation $\Delta Z_s^i = Z_s^i - Z_{s-}^i$ for the jumps of Z^i. From the very definition of the stochastic integral, Z^i is a purely discontinuous martingale and the compensator of its jump measure

$$\sum_{\Delta Z_s^i \neq 0} \delta_{(s, \Delta Z_s^i)}$$

is the image of $\mathrm{d}s \otimes m(\mathrm{d}u, \mathrm{d}x)$ under the mapping $(s, u, x) \to x \Psi(u, Z_{s-}^i)$. By applying Itô's formula in the discontinuous case (see e.g. Meyer [10]), we see that $(Z^1, \dots, Z^p)$ solves the martingale problem of Lemma 1, and hence is distributed as the p-point motion of the Λ-process.

It remains to establish the existence of a weak solution. We fix a sequence $(r_1, r_2, \ldots)$ of real numbers which is everywhere dense in $[0, 1]$. In the first part of the proof, we also fix an integer $p \geqslant 1$.

Set

$$Y_t = (Y_t^1, \ldots, Y_t^p) \quad \text{where } Y_t^i := F_t(r_i) \text{ for } i = 1, \ldots, p,$$

and recall Lemma 1. By comparison with Itô's formula, we see that for every function $g : [0,1]^p \to \mathbb{R}$ of class $\mathcal{C}^2$, the predictable projection (in the filtration generated by the Λ-process) of the finite variation process

$$\sum_{s \leqslant t,\ \Delta Y_s \neq 0} \bigl(g(Y_s) - g(Y_{s-}) - \Delta Y_s \cdot \nabla g(Y_{s-})\bigr)$$

is

$$\int_0^t \mathrm{d}s \int m(\mathrm{d}u, \mathrm{d}x)\bigl(g\bigl(Y_{s-} + x\Psi(u, Y_{s-})\bigr) - g(Y_{s-}) - x\Psi(u, Y_{s-}) \cdot \nabla g(Y_{s-})\bigr).$$

(In order to apply Lemma 1, we first need to reorder $r_1, \ldots, r_p$; still the preceding assertion holds without reordering.) By standard arguments, this entails that the dual predictable projection of the measure

$$\sum_{s \geqslant 0,\ \Delta Y_s \neq 0} \delta_{(s, \Delta Y_s)}$$

is $\nu(\omega, \mathrm{d}s, \mathrm{d}y)$ defined as the image of $\mathrm{d}s \otimes m(\mathrm{d}u, \mathrm{d}x)$ under the mapping

$$(s, u, x) \to \bigl(s, x\Psi(u, Y_{s-})\bigr).$$

Finally, we see that Y is a vector-valued semimartingale with characteristics $(0, 0, \nu)$.

We may now apply Theorem 14.80 of [5] (with $\check{w}(\omega, s, z) = x\Psi(u, \omega(s-))$ for $z = (u, x) \in D :=]0,1[\times]0,1]$ and $\omega \in \mathbb{D}([0, \infty[, \mathbb{R}^p))$ to see that we can define on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ an $(\mathcal{F}_t)$-Poisson point process $M(\mathrm{d}t, \mathrm{d}u, \mathrm{d}x)$ with intensity $\mathrm{d}t \otimes m(\mathrm{d}u, \mathrm{d}x)$ and a càdlàg adapted process $X_t = (X_t^1, \ldots, X_t^p)$ such that $(X^1, \ldots, X^p) \overset{\mathcal{L}}{=} (Y^1, \ldots, Y^p)$ and

$$X_t^i = r_i + \int_{[0,t] \times]0,1[\times]0,1]} M(\mathrm{d}s, \mathrm{d}u, \mathrm{d}x) x\Psi(u, X_{s-}^i) \tag{23}$$

for every $i \in \{1, \ldots, p\}$.

Now write $\mathbb{Q}_p$ for the distribution of $(M, X^1, \ldots, X^p, 0, 0, \ldots)$ on the product space $\mathcal{M}_r(\mathbb{R}_+ \times D) \times \mathbb{D}(\mathbb{R}_+, [0,1])^{\mathbb{N}}$ (here $\mathcal{M}_r(\mathbb{R}_+ \times D)$ is the space of Radon measures on $\mathbb{R}_+ \times D$ equipped with the usual weak topology). Notice that this product space is Polish, and that the sequence $(\mathbb{Q}_p)$ is tight (the one-dimensional marginals of $\mathbb{Q}_p$ do not depend on p whenever p is large enough). Hence we can find a subsequence $(\mathbb{Q}_{p_n})$ that converges weakly to $\mathbb{Q}_\infty$.

We abuse the notation by writing $M, X^1, X^2, \ldots$ for the coordinate process on $\mathcal{M}_r(\mathbb{R}_+ \times D) \times \mathbb{D}(\mathbb{R}_+, [0,1])^{\mathbb{N}}$, and let $(\mathcal{G}_t)_{t \geqslant 0}$ be the canonical filtration on this space. Plainly, under $\mathbb{Q}_\infty$, M is a $(\mathcal{G}_t)$-Poisson random measure with intensity $\mathrm{d}t\, m(\mathrm{d}u, \mathrm{d}x)$. Moreover a careful passage to the limit shows that Eq. (23) still holds $\mathbb{Q}_\infty$-a.s. for every $i = 1, 2, \ldots$.

Recall that for every $p \geqslant 1$, $(X_t^1, \ldots, X_t^p)_{t \geqslant 0}$ has the same distribution under $\mathbb{Q}_\infty$ as the p-point motion $(F_t(r_1), \ldots, F_t(r_p))_{t \geqslant 0}$. If $r \in [0, 1]$ is fixed, we can therefore set

$$X_t(r) := \lim_{r_i \downarrow r} \downarrow X_t^i,$$

and the process $(X_t(r),\ t \geqslant 0)$ has the same distribution as $(F_t(r),\ t \geqslant 0)$ so that in particular it has a càdlàg modification. A second moment calculation shows that

$$\lim_{r_i \downarrow r} \int_0^t \int_D M(\mathrm{d}s, \mathrm{d}u, \mathrm{d}x) x \Psi(u, X^i_{s-}) = \int_0^t \int_D M(\mathrm{d}s, \mathrm{d}u, \mathrm{d}x) x \Psi\big(u, X_{s-}(r)\big)$$

in $L^2(\mathbb{Q}_\infty)$. From (23) we now infer that (22) holds for every $r \in [0, 1]$. This completes the proof. □

4. The Kingman flow

Throughout this section we suppose $\Lambda = \delta_0$. Then the Λ-coalescent is Kingman's coalescent [8]. Indeed, the rates (2) are simply

$$\beta_{p,k} = \begin{cases} 1 & \text{if } k = 2, \\ 0 & \text{if } k > 2. \end{cases}$$

Proposition 1. *For every $x \in [0, 1]$, the process $(F_t(x),\ t \geqslant 0)$ has a continuous version which is distributed as the unique strong solution to the SDE*

$$X_t = x + \int_0^t \sqrt{X_s(1 - X_s)}\, \mathrm{d}W_s, \tag{24}$$

where $(W_s,\ s \geqslant 0)$ is a standard one-dimensional Brownian motion.

Proof. By applying Theorem 1 with $h_i = 1_{[0,y]}$ for every i, we obtain that

$$F_t(y)^p - \frac{p(p-1)}{2} \int_0^t \mathrm{d}s \big(F_s(y)^{p-1} - F_s(y)^p\big) \quad \text{is a martingale} \tag{25}$$

for every integer $p \geqslant 1$. Hence (or as a consequence of (8)), we have

$$\mathbb{E}\big[F_t(y)^p\big] = y^p + \frac{p(p-1)}{2}(y^{p-1} - y^p)t + \mathrm{o}(t), \tag{26}$$

where the remainder $\mathrm{o}(t)$ is uniform in y as $t \to 0$. Next, writing

$$\big(F_t(y) - y\big)^4 = F_t(y)^4 - 4yF_t(y)^3 + 6y^2F_t(y)^2 - 4y^3F_t(y) + y^4,$$

and applying again (25), we get

$$\mathbb{E}\big[\big(F_t(y) - y\big)^4\big] = \int_0^t \mathrm{d}s\, \mathbb{E}\big[6\big(F_s(y)^3 - F_s(y)^4\big) - 12y\big(F_s(y)^2 - F_s(y)^3\big) + 6y^2\big(F_s(y) - F_s(y)^2\big)\big].$$

Invoking (26), we deduce that there is some finite constant c (which does not depend of y) such that

$$\mathbb{E}\big[\big|F_t(y) - y\big|^4\big] \leqslant ct^2.$$

By the Markov property of the one-point motion $F_t(x)$, we see that Kolmogorov's criterion is fulfilled, which ensures the existence of a continuous version. That the latter can be expressed as a solution to (24) is now a standard consequence of (25) for $p = 1, 2$, see for instance Proposition 4.6 in Chapter 5 of [7].

The dispersion coefficient $x \to \sqrt{x - x^2}$ is Hölder continuous with exponent $1/2$ on the interval $[0, 1]$, so that we can apply the well-known Yamada–Watanabe criterion which gives pathwise uniqueness for (24). We note that 0 and 1 are absorbing points for X. $\square$

We now turn our attention to the p-point motion of the flow. Recall the notation (13) and for $x = (x_1, \ldots, x_p) \in \mathcal{D}_p$ introduce the dispersion matrix $\sigma(x) = (\sigma_{i,j}(x)\colon\ 1 \leqslant i \leqslant p,\ 1 \leqslant j \leqslant p+1)$ defined by

$$\sigma_{i,j}(x) = \begin{cases} (1 - x_i)\sqrt{x_j - x_{j-1}} & \text{if } i \geqslant j, \\ -x_i\sqrt{x_j - x_{j-1}} & \text{if } i < j, \end{cases} \tag{27}$$

where $x_0 = 0$, $x_{p+1} = 1$ by convention. It is easily checked that for every $x = (x_1, \ldots, x_p) \in \mathcal{D}_p$, the coefficients $(a_{i,j}(x))_{1 \leqslant i,j \leqslant p}$ of the matrix $\sigma(x)\sigma^*(x)$ are given for $x \in \mathcal{D}_p$ by

$$a_{i,j}(x) = x_{i \wedge j}(1 - x_{i \vee j}). \tag{28}$$

We also introduce the operator

$$\mathcal{A}g(x) = \frac{1}{2}\sum_{i,j=1}^{p} x_{i \wedge j}(1 - x_{i \vee j})\frac{\partial^2 g}{\partial x_i \partial x_j}(x), \tag{29}$$

for $g \in \mathcal{C}^2(\mathcal{D}_p)$.

Theorem 3. *For every integer $p \geqslant 1$ and $x = (x_1, \ldots, x_p) \in \mathcal{D}_p$, the p-point motion*

$$\big(F_t(x_1), \ldots, F_t(x_p)\big), \quad t \geqslant 0,$$

has a continuous version which solves the martingale problem: For every $g \in \mathcal{C}^2(\mathcal{D}_p)$,

$$g\big(F_t(x_1), \ldots, F_t(x_p)\big) - \int_0^t \mathcal{A}g\big(F_s(x_1), \ldots, F_s(x_p)\big)\,\mathrm{d}s$$

is a martingale. Furthermore the process $(F_t(x_1), \ldots, F_t(x_p))$ is distributed as the unique strong solution to the SDE

$$X_t = x + \int_0^t \sigma(X_s)\,\mathrm{d}W_s, \tag{30}$$

where $(W_s,\ s \geqslant 0)$ is a standard $(p+1)$-dimensional Brownian motion and σ is defined by (27).

Proof. The existence of a continuous version of the p-point motion follows from Proposition 1. Next, fix two integers $1 \leqslant k \leqslant \ell \leqslant p$ and set $h_1 = \mathbf{1}_{[0,x_k]}$, $h_2 = \mathbf{1}_{[0,x_\ell]}$, so that $\rho_t(h_1) = F_t(x_k)$ and $\rho_t(h_2) = F_t(x_\ell)$. Note also that $h_1 h_2 = h_1$. Just as in the proof of Proposition 1, we deduce from Theorem 1 that

$$F_t(x_k)F_t(x_\ell) - \int_0^t F_s(x_k)\big(1 - F_s(x_\ell)\big)\,\mathrm{d}s, \quad t \geqslant 0,$$

is a martingale. We conclude using Proposition 4.6 in Chapter 5 of [7] that (the continuous version of) the process $(F_t(x_1), \ldots, F_t(x_p))$ can be expressed as a solution to (30), and the martingale problem of the theorem follows readily.

It remains to prove pathwise uniqueness for (30). Let $X = (X_t,\ t \geqslant 0)$ be a solution to (30). It is convenient to introduce the p-dimensional simplex

$$\Delta_p = \left\{ y = (y_1, \ldots, y_{p+1}):\ 0 \leqslant y_i \leqslant 1 \text{ for } i = 1, \ldots, p+1 \text{ and } \sum_{i=1}^{p+1} y_i = 1 \right\}$$

and the increments

$$Y_t = (Y_t^1, \ldots, Y_t^{p+1}) \quad \text{where } Y_t^i = X_t^i - X_t^{i-1},\ i = 1, \ldots, p+1,$$

with the convention $X_t^0 \equiv 0$ and $X_t^{p+1} \equiv 1$. Then Y is a continuous process which lives in Δ_p and solves the SDE

$$Y_t = y + \int_0^t \tau(Y_s)\,\mathrm{d}W_s, \tag{31}$$

where $y = (x_1, x_2 - x_1, \ldots, x_p - x_{p-1}, 1 - x_p)$ and the dispersion matrix $\tau(y) = (\tau_{i,j}(y):\ 1 \leqslant i, j \leqslant p+1)$ is defined for $y \in \Delta_p$ by

$$\tau_{i,j}(y) = \begin{cases} -y_i\sqrt{y_j} & \text{if } i \neq j, \\ (1 - y_i)\sqrt{y_i} & \text{if } i = j. \end{cases}$$

We shall establish by induction on p that (31) has a unique solution, where by a solution we mean a Δ_p-valued continuous adapted process such that (31) holds. For $p = 1$, this is easy, so we assume from now on that $p \geqslant 2$ and that uniqueness of the solution of (31) has been established at order $p - 1$. The following argument is related to the proof of Lemma 3.2 in [4].

Suppose first that the starting point $y = (y_1, \ldots, y_{p+1})$ lies on the boundary of the simplex

$$\partial\Delta_p = \left\{ y \in \Delta_p:\ y_i = 0 \text{ for some } i \in \{1, \ldots, p+1\} \right\}.$$

So there is some index i such that the martingale Y_t^i starts from 0, and since it takes values in [0, 1], we have $Y_t^i = 0$ for all $t \geqslant 0$. Consider the process $\widetilde{Y}$ (respectively, $\widetilde{W}$) obtained from Y (respectively, W) by suppressing the i-th coordinate, viz.

$$\widetilde{Y}_t = (Y_t^1, \ldots, Y_t^{i-1}, Y_y^{i+1}, \ldots, Y_t^{p+1}), \qquad \widetilde{W}_t = (W_t^1, \ldots, W_t^{i-1}, W_y^{i+1}, \ldots, W_t^{p+1}).$$

It is immediate that

$$\widetilde{Y}_t = \tilde{y} + \int_0^t \tilde{\tau}(\widetilde{Y}_s)\,\mathrm{d}\widetilde{W}_s,$$

where the dispersion matrix $\tilde{\tau}$ is obtained from τ by removing the i-th column and i-th row. Since $\widetilde{W}$ is a standard p-dimensional Brownian motion, this SDE is that corresponding to (31) for the $(p-1)$-point motion and we conclude that uniqueness holds in that case.

We now suppose that the starting point y belongs to the interior $\Delta_p \setminus \partial\Delta_p$ of the simplex. Since the dispersion matrix τ is smooth in $\Delta_p \setminus \partial\Delta_p$, the solution exists and is clearly unique up to the first hitting time of $\partial\Delta_p$ by Y. By the strong Markov property of W at this first hitting time, we are reduced to the case when the starting point lies on the boundary $\partial\Delta_p$, for which we already know that uniqueness holds.

We have thus shown the existence of a unique solution for (31), and pathwise uniqueness for (30) readily follows. This completes the proof. □

Corollary 1. *The family of rescaled processes*

$$t^{-1/2}\big(F_t(x) - x\big), \quad x \in [0, 1],$$

converges in the sense of finite-dimensional distributions to a Brownian bridge when $t \to 0+$.

Proof. One easily deduces from Theorem 3 that for every integer $p \geqslant 1$ and $x = (x_1, \dots, x_p) \in \mathcal{D}_p$, the p-tuple

$$\frac{1}{\sqrt{t}}\big(F_t(x_1) - x_1, \dots, F_t(x_p) - x_p\big)$$

converges in distribution to a centered Gaussian variable $(G(x_1), \dots, G(x_p))$ with covariance matrix $\sigma(x)\sigma^*(x)$. From (28), we recognize the p-marginal of a standard Brownian bridge. $\square$

Remark. In the terminology of Harris [4], Section 11, we may say that the Brownian bridge is the generating field of the flow (F_t).

5. The flow of inverses

In this section, we consider a finite measure Λ on $[0, 1]$ and the flow of bridges $(B_{s,t})_{-\infty<s\leqslant t<\infty}$ associated with the Λ-coalescent. The dual flow is $\widehat{B}_{s,t} = B_{-t,-s}$. Recall that the Λ-coalescent $(\Pi_t,\ t \geqslant 0)$ in $\mathcal{P}$ may be constructed by the formula $\Pi_t = \pi(B_{0,t})$ (cf. Section 2).

For every $s \leqslant t$, we set

$$\Gamma_{s,t}(u) = \inf\big\{r \geqslant 0:\ B_{s,t}(r) > u\big\}, \quad \text{if } u \in [0, 1[,$$

and $\Gamma_{s,t}(1) = \Gamma_{s,t}(1-)$. The function $u \to \Gamma_{s,t}(u)$ is then nondecreasing and right-continuous from $[0, 1]$ into $[0, 1]$. Note that in contrast to bridges we may have $\Gamma_{s,t}(0) > 0$ or $\Gamma_{s,t}(1) < 1$. If $r \leqslant s \leqslant t$, the identity $B_{r,t} = B_{r,s} \circ B_{s,t}$ implies

$$\Gamma_{r,t} = \Gamma_{s,t} \circ \Gamma_{r,s}, \quad \text{a.s.} \tag{32}$$

To simplify notation, we set $\Gamma_t = \Gamma_{0,t}$.

Theorem 4. *Let $p \geqslant 1$. For every $(x_1, \dots, x_p) \in \mathcal{D}_p$, the process $(\Gamma_t(x_1), \dots, \Gamma_t(x_p))$ is a Markov process taking values in $\mathcal{D}^p$ with a Feller semigroup.*

Proof. If follows from (32) that for every $0 \leqslant s \leqslant t$ we have $\Gamma_t = \widetilde{\Gamma}_{t-s} \circ \Gamma_s$, where $\widetilde{\Gamma}_{t-s}$ is independent of Γ_s and distributed as Γ_{t-s}. This entails that the process $(\Gamma_t(x_1), \dots, \Gamma_t(x_p))$ is Markov with semigroup Q_t characterized as follows: For $(y_1, \dots, y_p) \in \mathcal{D}_p$, $Q_t((y_1, \dots, y_p), \cdot)$ is the distribution of $(\Gamma_t(y_1), \dots, \Gamma_t(y_p))$. We know that $B_{0,s}$ converges in probability to the identity mapping Id as $s \to 0$, in the sense of the Skorokhod topology. It follows that the same property holds for Γ_s as $s \to 0$. Therefore $Q_t((y_1, \dots, y_p), \cdot)$ converges weakly to the Dirac measure $\delta_{(y_1,\dots,y_p)}$ as $t \to 0$. To complete the proof of the Feller property, we need to verify that the mapping $(y_1, \dots, y_p) \to Q_t((y_1, \dots, y_p), \cdot)$ is continuous for the weak topology. To this end, it is enough to prove that $\Gamma_t(y)$ tends to $\Gamma_t(x)$ a.s. as $y \to x$, or equivalently that $\Gamma_t(x-) = \Gamma_t(x)$ a.s., for every fixed $x \in]0, 1[$ (when $x = 1$ we just use the definition of $\Gamma_t(1)$).

We argue by contradiction, supposing that there exists $t > 0$ and $x \in]0, 1[$ such that $\mathbb{P}[\Gamma_t(x-) < \Gamma_t(x)] > 0$. Equivalently, with positive probability there is a nonempty open interval $]a, b[\subset]0, 1[$ such that $B_{0,t}(r) = x$ for every $x \in]a, b[$. Obviously this is possible only if the bridge $B_{0,t}$ has zero drift (equivalently the partition Π_t has no singletons) and finitely many jumps (equivalently Π_t has finitely many blocks). By known facts about the Λ-coalescent (see Sections 3.6 and 3.7 of [12]), the previous two properties then hold a.s. for $B_{0,r}$ and Π_r, for every $r > 0$.

From the connection between bridges and coalescents, we see that on the event $\{\Gamma_t(x-) < \Gamma_t(x)\}$, there is a subcollection of blocks of Π_t whose union has asymptotic frequency x. Using the Markov property at time t, we get that with positive probability the partition Π_{t+1} consists of two blocks with respective frequencies x and $1-x$. Replacing t by $t+1$ and x by $1-x$ (if necessary) we obtain that

$$\mathbb{P}\big[|\Pi_t| = (x, 1-x, 0, 0, \ldots)\big] > 0, \tag{33}$$

where $|\pi|$ denotes the ranked sequence of frequencies of the partition π.

To get a contradiction, let $\varepsilon > 0$ and recall that

$$\Pi_{t+\varepsilon} \stackrel{\text{(d)}}{=} c_{\widetilde{\Pi}_t}(\Pi_\varepsilon), \tag{34}$$

where $\widetilde{\Pi}_t$ is a copy of Π_t which is independent of $(\Pi_r, r \geqslant 0)$ and $c_{\widetilde{\Pi}_t}(\Pi_\varepsilon)$ denotes the coagulation of Π_ε by $\widetilde{\Pi}_t$ (see [1], Section 2.2). We will verify that

$$\mathbb{P}\big[\big|c_{\widetilde{\Pi}_t}(\Pi_\varepsilon)\big| = (x, 1-x, 0, 0, \ldots)\big] \underset{\varepsilon\to 0}{\longrightarrow} 0. \tag{35}$$

Together with (34) this clearly gives a contradiction with (33). Write $\#\pi$ for the number of blocks of the partition π. Since $\#\Pi_\varepsilon$ converges to ∞ in probability as $\varepsilon \to 0$, it is immediate to see that

$$\mathbb{P}\big[\#\big(c_{\widetilde{\Pi}_t}(\Pi_\varepsilon)\big) = 2 \text{ and } \#\widetilde{\Pi}_t \neq 2\big] \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

Therefore we can concentrate on the case $\#\widetilde{\Pi}_t = 2$ and we denote by $\mathbb{P}^*$ the conditional probability $\mathbb{P}[\cdot \mid \#\widetilde{\Pi}_t = 2]$. Since $\widetilde{\Pi}_t$ is an exchangeable partition, the distribution of $\widetilde{\Pi}_t$ under $\mathbb{P}^*$ must be of the following type: There is a random variable q with values in $]0,1[$ such that, under the probability measure $\mathbb{P}^*$,

$$\widetilde{\Pi}_t \stackrel{\text{(d)}}{=} \big(\{i \geqslant 1:\ X_i = 1\}, \{i \geqslant 1:\ X_i = 0\}\big),$$

where conditionally given q the variables X_i are independent Bernoulli variables with parameter q (and we may also assume that the X_i's are independent of $(\Pi_r,\ r \geqslant 0)$). Write $|\Pi_\varepsilon| = (a_1^\varepsilon, a_2^\varepsilon, \ldots, a_{n_\varepsilon}^\varepsilon, 0, 0, \ldots)$ for the ranked sequence of frequencies of Π_ε. Then the ranked frequencies of $c_{\widetilde{\Pi}_t}(\Pi_\varepsilon)$ are distributed under $\mathbb{P}^*$ as the decreasing rearrangement of $(Y_\varepsilon, 1-Y_\varepsilon, 0, 0, \ldots)$, where

$$Y_\varepsilon = \sum_{i=1}^{n_\varepsilon} a_i^\varepsilon X_i.$$

Note that $\sum_{i=1}^{n_\varepsilon} a_{n_\varepsilon}^\varepsilon = 1$ and that $\sup_{i\geqslant 1} a_i^\varepsilon$ converges a.s. to 0 as $\varepsilon \to 0$. Also denote by

$$V_\varepsilon = \left(q(1-q)\sum_{i=1}^{n_\varepsilon}(a_i^\varepsilon)^2\right)^{1/2}$$

the square root of the conditional variance of Y_ε knowing q and $(\Pi_r,\ r \geqslant 0)$. By well-known limit theorems for triangular arrays, the conditional distribution given q and $(\Pi_r,\ r \geqslant 0)$ of

$$Z_\varepsilon := \frac{Y_\varepsilon - q}{V_\varepsilon}$$

converges as $\varepsilon \to 0$ to the standard normal distribution on the line. It follows that

$$\begin{aligned}
\mathbb{P}^*\big[\big|c_{\widetilde{\Pi}_t}(\Pi_\varepsilon)\big| = (x, 1-x, 0, 0, \ldots)\big] &= \mathbb{P}^*[Y_\varepsilon = x \text{ or } Y_\varepsilon = 1-x] \\
&= \mathbb{P}^*\left[Z_\varepsilon = \frac{x-q}{V_\varepsilon} \text{ or } Z_\varepsilon = \frac{1-x-q}{V_\varepsilon}\right] \\
&= \mathbb{E}^*\left[\mathbb{P}^*\left[Z_\varepsilon = \frac{x-q}{V_\varepsilon} \text{ or } Z_\varepsilon = \frac{1-x-q}{V_\varepsilon}\,\middle|\, q, (\Pi_r)_{r\geqslant 0}\right]\right]
\end{aligned}$$

which tends to 0 as $\varepsilon \to 0$. This completes the proof of (35) and gives the desired contradiction. □

In a way analogous to Lemma 1, we will now discuss the martingale problem satisfied by the process $(\Gamma_t(x_1), \ldots, \Gamma_t(x_p))$.

Theorem 5. *Suppose that $\Lambda(0)=0$. For every function $F\in\mathcal{C}^2(\mathcal{D}_p)$ and every $(y_1,\dots,y_p)\in\mathcal{D}_p$, set*

$$\widetilde{\mathcal{L}}F(y_1,\dots,y_p)=\int\Lambda(\mathrm{d}z)z^{-2}\left(\int_0^1\mathrm{d}v\big(F\big(\psi_{z,v}(y_1),\dots,\psi_{z,v}(y_p)\big)-F(y_1,\dots,y_p)\big)\right),$$

where

$$\psi_{z,v}(y)=1_{\{v>y\}}\left(\left(\frac{y}{1-z}\right)\wedge v\right)+1_{\{v\leqslant y\}}\left(\left(\frac{y-z}{1-z}\right)\vee v\right)$$

if $0<z<1$, and $\psi_{1,v}(y)=v$. Then, for every $(u_1,\dots,u_p)\in\mathcal{D}_p$,

$$F\big(\Gamma_t(u_1),\dots,\Gamma_t(u_p)\big)-\int_0^t\widetilde{\mathcal{L}}F\big(\Gamma_s(u_1),\dots,\Gamma_s(u_p)\big)\,\mathrm{d}s$$

is a martingale.

Remark. By using the Taylor expansion for F in the neighborhood of $(y_1,\dots,y_p)$, it is not hard to verify that the integral with respect to $\Lambda(\mathrm{d}z)$ in the definition of $\widetilde{\mathcal{L}}F$ is absolutely convergent, and moreover the function $\widetilde{\mathcal{L}}F$ is bounded over $[0,1]^p$.

Proof. First observe that for every $s\geqslant 0$ and $u,x\in[0,1[$,

$$\big\{\Gamma_s(u)<x\big\}=\big\{B_{0,s}(x)>u\big\},\qquad\text{a.s.}\tag{36}$$

The inclusion $\{\Gamma_s(u)<x\}\subset\{B_{0,s}(x)>u\}$ is obvious by definition. Conversely, since $B_{0,s}$ is continuous at x, a.s., the condition $B_{0,s}(x)>u$ also implies that $\Gamma_s(u)<x$ a.s.

Let g be a polynomial function on $[0,1]^p$ and let $f\in\mathcal{C}^\infty([0,1]^p)$. Also set

$$G(t_1,\dots,t_p)=\int_0^{t_1}\mathrm{d}u_1\int_0^{t_2}\mathrm{d}u_2\cdots\int_0^{t_p}\mathrm{d}u_p\,g(u_1,\dots,u_p),$$
$$F(t_1,\dots,t_p)=\int_{t_1}^1\mathrm{d}x_1\int_{t_2}^1\mathrm{d}x_2\cdots\int_{t_p}^1\mathrm{d}x_p\,f(x_1,\dots,x_p).$$

From (36), we get that for every $u_1,\dots,u_p,x_1,\dots,x_p\in[0,1[$,

$$\begin{aligned}\mathbb{P}\big[\Gamma_s(u_1)<x_1,\dots,\Gamma_s(u_p)<x_p\big]&=\mathbb{P}\big[B_{0,s}(x_1)>u_1,\dots,B_{0,s}(x_p)>u_p\big]\\&=\mathbb{P}\big[\widehat{B}_{0,s}(x_1)>u_1,\dots,\widehat{B}_{0,s}(x_p)>u_p\big].\end{aligned}$$

Integrating with respect to the measure $g(u_1,\dots,u_p)f(x_1,\dots,x_p)\,\mathrm{d}u_1\cdots\mathrm{d}u_p\mathrm{d}x_1\cdots\mathrm{d}x_p$, we arrive at

$$\begin{aligned}&\int_{[0,1[^p}\mathrm{d}u_1\cdots\mathrm{d}u_p\,g(u_1,\dots,u_p)\mathbb{E}\big[F\big(\Gamma_s(u_1),\dots,\Gamma_s(u_p)\big)-F(u_1,\dots,u_p)\big]\\&\quad=\int_{[0,1[^p}\mathrm{d}x_1\cdots\mathrm{d}x_p\,f(x_1,\dots,x_p)\mathbb{E}\big[G\big(\widehat{B}_{0,s}(x_1),\dots,\widehat{B}_{0,s}(x_p)\big)-G(x_1,\dots,x_p)\big].\end{aligned}\tag{37}$$

Denote by $A_{F,G}$ the right-hand side of (37). We can evaluate $A_{F,G}$ from the knowledge of the generator $\mathcal{L}$ for the process $(F_s(x_1),\dots,F_s(x_p))=(\widehat{B}_{0,s}(x_1),\dots,\widehat{B}_{0,s}(x_p))$ (strictly speaking we should reorder $x_1,\dots,x_p$ since the process $(F_s(x_1),\dots,F_s(x_p))$ and its generator were discussed above in the case when $(x_1,\dots,x_p)\in\mathcal{D}_p$; we will leave this trivial reduction to the reader). Denoting by P_t the semigroup of this process, we have

$$\mathbb{E}\big[G\big(F_s(x_1),\dots,F_s(x_p)\big)-G(x_1,\dots,x_p)\big]=\int_0^s \mathrm{d}t\,\mathcal{L}P_tG(x_1,\dots,x_p).$$

To simplify notation, set $x=(x_1,\dots,x_p)$ and $\Psi(v,x)=(\Psi(v,x_1),\dots,\Psi(v,x_p))$ as in Section 3.2. From the formula for $\mathcal{L}$ (Section 3.2), the last displayed quantity is equal to

$$\int_0^s \mathrm{d}t\int \Lambda(\mathrm{d}z)z^{-2}\int_0^1 \mathrm{d}v\big(P_tG\big(x+z\Psi(v,x)\big)-P_tG(x)\big).$$

From the explicit formula for P_tG when G is a polynomial function (see (21)), we see that P_tG is again a polynomial function and moreover we can get a uniform bound on the second derivatives of P_tG. Using Taylor's formula, and the fact that $\int_0^1 \mathrm{d}v\,\Psi(v,x)=0$, we get

$$\left|\int_0^1 \mathrm{d}v\big(P_tG\big(x+z\Psi(v,x)\big)-P_tG(x)\big)\right|\leqslant Cz^2$$

with a constant C independent of t,x,z. This allows us to apply Fubini's theorem in order to get

$$\begin{aligned}A_{F,G}&=\int_{[0,1[^p}\mathrm{d}x\,f(x)\int_0^s\mathrm{d}t\int\Lambda(\mathrm{d}z)z^{-2}\int_0^1\mathrm{d}v\big(P_tG\big(x+z\Psi(v,x)\big)-P_tG(x)\big)\\&=\int_0^s\mathrm{d}t\int\Lambda(\mathrm{d}z)z^{-2}\int_{[0,1[^p}\mathrm{d}x\,f(x)\int_0^1\mathrm{d}v\big(P_tG\big(x+z\Psi(v,x)\big)-P_tG(x)\big).\end{aligned}$$

Then, from the definition of G and the fact that $\widehat{B}_{0,t}\overset{(\mathrm{d})}{=}B_{0,t}$,

$$\begin{aligned}&P_tG\big(x+z\Psi(v,x)\big)-P_tG(x)\\&\quad=\mathbb{E}\big[G\big(\widehat{B}_{0,t}\big(x_1+z\Psi(v,x_1)\big),\dots,\widehat{B}_{0,t}\big(x_p+z\Psi(v,x_p)\big)\big)-G\big(\widehat{B}_{0,t}(x_1),\dots,\widehat{B}_{0,t}(x_p)\big)\big]\\&\quad=\mathbb{E}\left[\int_{[0,1[^p}\mathrm{d}u_1\cdots\mathrm{d}u_p\,g(u_1,\dots,u_p)\left(\prod_{i=1}^p 1_{\{u_i<B_{0,t}(x_i+z\Psi(v,x_i))\}}-\prod_{i=1}^p 1_{\{u_i<B_{0,t}(x_i)\}}\right)\right].\end{aligned}$$

At this point we use (36) with u replaced by u_i and x replaced by x_i, or by $x_i+z\Psi(v,x_i)$. We also observe that the condition $x_i+z\Psi(v,x_i)>\Gamma_t(u_i)$ holds if and only if $x_i>\psi_{z,v}(\Gamma_t(u_i))$, or possibly $x_i=v$ in the case when $\psi_{z,v}(\Gamma_t(u_i))=v$. Since the case $x_i=v$ obviously gives no contribution when we integrate with respect to $\mathrm{d}v$, we get

$$\int_0^1\mathrm{d}v\big(P_tG\big(x+z\Psi(v,x)\big)-P_tG(x)\big)=\int_0^1\mathrm{d}v\int_{[0,1[^p}\mathrm{d}u\,g(u)\mathbb{E}\left[\prod_{i=1}^p 1_{\{x_i>\psi_{z,v}(\Gamma_t(u_i))\}}-\prod_{i=1}^p 1_{\{x_i>\Gamma_t(u_i)\}}\right].$$

By substituting this in the preceding formula for $A_{F,G}$, we arrive at

$$A_{F,G}=\int_0^s \mathrm{d}t\int \Lambda(\mathrm{d}z)z^{-2}\int_0^1 \mathrm{d}v\int_{[0,1[^p} \mathrm{d}u\, g(u)\mathbb{E}\bigl[F\bigl(\psi_{z,v}\bigl(\Gamma_t(u_1)\bigr),\ldots\bigr)-F\bigl(\Gamma_t(u_1),\ldots\bigr)\bigr]$$

$$=\int_{[0,1[^p} \mathrm{d}u\, g(u)\int_0^s \mathrm{d}t\int \Lambda(\mathrm{d}z)z^{-2}\int_0^1 \mathrm{d}v\,\mathbb{E}\bigl[F\bigl(\psi_{z,v}\bigl(\Gamma_t(u_1)\bigr),\ldots\bigr)-F\bigl(\Gamma_t(u_1),\ldots\bigr)\bigr],$$

where the last application of Fubini's theorem is easily justified by observing that there exists a constant C such that for every $z\in\,]0,1]$ and $y_1,\ldots,y_p\in[0,1]$,

$$\left|\int_0^1 \mathrm{d}v\bigl(F\bigl(\psi_{z,v}(y_1),\ldots,\psi_{z,v}(y_p)\bigr)-F(y_1,\ldots,y_p)\bigr)\right|\leqslant Cz^2.$$

From the Feller property of the process $(\Gamma_t(u_1),\ldots,\Gamma_t(u_p))$ and the previous bound, we get that the mapping

$$(u_1,\ldots,u_p)\longrightarrow\int_0^s \mathrm{d}t\int \Lambda(\mathrm{d}z)z^{-2}\int_0^1 \mathrm{d}v\,\mathbb{E}\bigl[F\bigl(\psi_{z,v}\bigl(\Gamma_t(u_1)\bigr),\ldots\bigr)-F\bigl(\Gamma_t(u_1),\ldots\bigr)\bigr]$$

is continuous. By comparing with (37), we conclude that

$$\mathbb{E}\bigl[F\bigl(\Gamma_s(u_1),\ldots,\Gamma_s(u_p)\bigr)-F(u_1,\ldots,u_p)\bigr]=\int_0^s \mathrm{d}t\,\mathbb{E}\bigl[\widetilde{\mathcal{L}}F\bigl(\Gamma_t(u_1),\ldots,\Gamma_t(u_p)\bigr)\bigr].$$

This gives the martingale problem stated in the theorem, at least for functions F of the type considered above. The general case follows from an easy induction on p together with a density argument to go from $\mathcal{C}^\infty$ functions to $\mathcal{C}^2$ functions. □

A natural question is uniqueness for the martingale problem stated in Theorem 5 (compare with Lemma 1). This does not seem to follow directly from our approach. Instead we will turn to the case of the Kingman coalescent, where the law of the flow of inverses can be made more explicit. Recall that the domain $\mathcal{D}_p$ has been defined in (13).

Theorem 6. *Suppose that $\Lambda=\delta_0$. Let $(u_1,\ldots,u_p)\in\mathcal{D}_p$. Then the process $(\Gamma_t(u_1),\ldots,\Gamma_t(u_p))$ is a diffusion process in $\mathcal{D}_p$ with generator*

$$\tilde{\mathcal{A}}g(x)=\frac{1}{2}\sum_{i,j=1}^p x_{i\wedge j}(1-x_{i\vee j})\frac{\partial^2 g}{\partial x_i\partial x_j}(x)+\sum_{i=1}^p\left(\frac{1}{2}-x_i\right)\frac{\partial g}{\partial x_i}(x),$$

for $g\in\mathcal{C}^2(\mathcal{D}_p)$.

Proof. This can be deduced from the martingale problem for the process $(F_t(x_1),\ldots,F_t(x_p))$ in a way similar to the proof of Theorem 5. We will treat the case $p=1$ and leave details of the general case to the reader. Let $f\in\mathcal{C}^2([0,1])$ and let g be a polynomial function on $[0,1]$. As in the proof of Theorem 5, we set

$$F(x)=\int_x^1 f(y)\,\mathrm{d}y,\qquad G(x)=\int_0^x g(u)\,\mathrm{d}u.$$

As in (37), we have

$$\int_0^1 \mathrm{d}u\, g(u)\mathbb{E}\bigl[F\bigl(\Gamma_s(u)\bigr) - F(u)\bigr] = \int_0^1 \mathrm{d}x\, f(x)\mathbb{E}\bigl[G\bigl(\widehat{B}_{0,s}(x)\bigr) - G(x)\bigr], \tag{38}$$

and

$$\mathbb{E}\bigl[G\bigl(\widehat{B}_{0,s}(x)\bigr) - G(x)\bigr] = \int_0^s \mathrm{d}t\, \mathcal{A}P_t G(x). \tag{39}$$

Fix $t > 0$ and set $h = P_t G$. Recall from (29) that $\mathcal{A}h(x) = \frac{1}{2}x(1-x)h''(x)$. Note that $h(0) = 0$ and $h(1) = G(1) = \int_0^1 g(u)\,\mathrm{d}u$. Also set $\phi(x) = \frac{1}{2}x(1-x)f'(x) + (\frac{1}{2} - x)f(x)$. Using two integrations by parts, we get

$$\begin{aligned}
\int_0^1 \mathrm{d}x\, f(x)\mathcal{A}h(x) &= -\int_0^1 \mathrm{d}x\, \phi(x)h'(x) = -\phi(1)h(1) + \int_0^1 \mathrm{d}x\, \phi'(x)h(x) \\
&= -\phi(1)h(1) + \int_0^1 \mathrm{d}x\, \phi'(x)\mathbb{E}\Biggl[\int_0^{\widehat{B}_{0,t}(x)} \mathrm{d}u\, g(u)\Biggr] \\
&= -\phi(1)h(1) + \int_0^1 \mathrm{d}u\, g(u)\int_0^1 \mathrm{d}x\, \phi'(x)\mathbb{P}\bigl[\Gamma_t(u) < x\bigr] = -\int_0^1 \mathrm{d}u\, g(u)\mathbb{E}\bigl[\phi\bigl(\Gamma_t(u)\bigr)\bigr].
\end{aligned}$$

By combining this with (38) and (39) we arrive at

$$\mathbb{E}\bigl[F\bigl(\Gamma_s(u)\bigr) - F(u)\bigr] = -\int_0^s \mathrm{d}t\, \mathbb{E}\bigl[\phi\bigl(\Gamma_s(u)\bigr)\bigr].$$

The case $p = 1$ of the theorem easily follows. □

Remark. By arguments similar to the proof of Theorem 3, it is easy to verify that uniqueness holds for the martingale problem associated with the generator $\tilde{\mathcal{A}}$. Moreover the process $(\Gamma_s(x_1), \ldots, \Gamma_s(x_p))$ can be obtained as the unique strong solution of a stochastic differential equation analogous to (30). In the case $p = 1$ in particular, $(\Gamma_t(x),\ t \geqslant 0)$ has the same law as the process $(X_t,\ t \geqslant 0)$ solving the equation

$$X_t = x + \int_0^t \sqrt{X_s(1 - X_s)}\,\mathrm{d}W_s + \int_0^t \left(\frac{1}{2} - X_s\right)\mathrm{d}s,$$

where W is a standard linear Brownian motion. If $x \notin \{0, 1\}$, then X_t never hits 0 or 1. This property, which is in contrast with the diffusion process of Theorem 3, can be seen as follows. If $T_0 := \inf\{t \geqslant 0\colon\ X_t = 0\}$, an application of Itô's formula shows that, for $t \in [0, T_0[$,

$$\log X_t = \log x + \int_0^t \sqrt{\frac{1 - X_s}{X_s}}\,\mathrm{d}W_s - \frac{t}{2}.$$

Hence $\frac{t}{2} + \log X_t$ is a local martingale on the stochastic interval $[0, T_0[$, and cannot converge to $-\infty$ as $t \to T_0$. This proves that $T_0 = \infty$ a.s., and a similar argument applies to the hitting time of 1.

5.1. More about the Kingman flow

Let us summarize the various results we have obtained for the flow associated with the Kingman coalescent. Fix $s, t \in \mathbb{R}$ with $s < t$. Then, we know that the number $N_{s,t}$ of jumps of the bridge $B_{s,t}$ is distributed as the number of blocks in the Kingman coalescent at time $t - s$. Furthermore, conditionally on $\{N_{s,t} = p\}$, we may write

$$B_{s,t}(r) = \sum_{i=1}^{p-1} Y_{s,t}^i 1_{[Z_{s,t}^i, Z_{s,t}^{i+1}[}(r) + 1_{[Z_{s,t}^p, 1]}(r), \tag{40}$$

where the random vectors $(Z_{s,t}^1, \ldots, Z_{s,t}^p)$ and $(Y_{s,t}^1, \ldots, Y_{s,t}^{p-1})$ are independent, $(Z_{s,t}^1, \ldots, Z_{s,t}^p)$ is distributed as the ordered statistics of p independent uniform variables on $[0, 1]$ and $(Y_{s,t}^1, \ldots, Y_{s,t}^{p-1})$ is distributed as the ordered statistics of $p - 1$ independent uniform variables on $[0, 1]$ (this last property is needed only if $p > 1$). The first two properties follow from general facts about bridges. The last one follows from the known distribution of block frequencies in the Kingman coalescent (see [8]).

Next what happens in the representation (40) if we vary s and t? First, if s is fixed, and t increases, the vector $(Y_{s,t}^1, \ldots, Y_{s,t}^{p-1})$ will remain constant as long as $N_{s,t} = p$. Meanwhile, Theorem 6 shows that the vector $(Z_{s,t}^1, \ldots, Z_{s,t}^p)$ evolves as a diffusion process with generator $\tilde{\mathcal{A}}$. Eventually, two successive coordinates of this process will meet and coalesce, thus corresponding to a coalescence in the Kingman coalescent. At the same time $N_{s,t}$ jumps from p to $p - 1$, and so on.

On the contrary, if we fix t and decrease s, the vector $(Z_{s,t}^1, \ldots, Z_{s,t}^p)$ will remain constant as long as $N_{s,t} = p$. Meanwhile, Theorem 3 shows that $(Y_{s,t}^1, \ldots, Y_{s,t}^{p-1})$ evolves as a diffusion process with generator $\mathcal{A}$. Eventually two successive coordinates of this process will coalesce, or the first one $Y_{s,t}^1$ will be absorbed at 0, or the last one $Y_{s,t}^{p-1}$ will be absorbed at 1 (in the genealogical interpretation of [1], each of these events corresponds to the extinction of a subpopulation consisting of descendants of one individual at the initial generation). At that moment, $N_{s,t}$ jumps from p to $p - 1$, and so on.

6. Flows on the circle

6.1. A Poissonian construction

Our goal in this section is to investigate certain flows on the circle which are associated with Λ-coalescents in a similar way to the flows on $[0, 1]$ considered in the previous sections. We will start with a Poissonian construction which is analogous to the one in Section 4 of [1]. For this reason we will skip some details of the proofs.

We consider the one-dimensional torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. We denote by $d(x, y)$ the distance on $\mathbb{T}$ and by σ Lebesgue measure on $\mathbb{T}$. If $x, y \in \mathbb{T}$, we will denote by $[\![x, y]\!]$ the counterclockwise arc going from x to y: If p is the canonical projection from $\mathbb{R}$ onto $\mathbb{T}$, and if x_1, resp. y_1, is the representative of x, resp. y, in $[0, 1[$, then $[\![x, y]\!] = p([x_1, y_1])$ if $x_1 \leqslant y_1$ and $[\![x, y]\!] = p([x_1, y_1 + 1])$ if $x_1 > y_1$. We also set $d^*(x, y) = \sigma([\![x, y]\!])$. Finally, for every $x \in \mathbb{T}$, we set $\bar{x} = x + \frac{1}{2}$ and if $y \in \mathbb{T}$ and $y \neq \bar{x}$, we denote by $[x, y]$ the shortest arc between x and y (that is the range of the unique geodesic from x to y).

Let $z \in \mathbb{T}$ and $a \in]0, 1]$. We denote by $f_{a,z}$ the unique continuous mapping from $\mathbb{T}$ into $\mathbb{T}$ such that

$$f_{a,z}(y) = z \quad \text{if } d(y, z) \leqslant \frac{a}{2}$$

and if $d(y, z) > \frac{a}{2}$, $f_{a,z}(y)$ is the unique element of $\mathbb{T}$ such that

$$d\big(\bar{z}, f_{a,z}(y)\big) = \frac{1}{1 - a} d(\bar{z}, y)$$

and

$$f_{a,z}(y) \in [y, z]$$

(the latter condition makes sense only if $y \neq \bar{z}$, which is the case where it is needed).

Note that the image of the restriction of σ to $\{y\colon d(y,z) > a/2\}$ under the mapping $f_{a,z}$ is $(1-a)\sigma$. This is the key property needed for the subsequent developments.

Let ν be a finite measure on $]0,1]$, and let $\mathcal{N}(\mathrm{d}t\,\mathrm{d}z\,\mathrm{d}a)$ be a Poisson point measure on $\mathbb{R} \times \mathbb{T} \times]0,1]$ with intensity $\mathrm{d}t\,\sigma(\mathrm{d}z)\nu(\mathrm{d}a)$. Then, for every $s, t \in \mathbb{R}$ with $s \leqslant t$, define

$$\Phi_{s,t} = f_{a_k,z_k} \circ f_{a_{k-1},z_{k-1}} \circ \cdots \circ f_{a_1,z_1}, \tag{41}$$

where $(t_1, z_1, a_1), \ldots, (t_k, z_k, a_k)$ are the atoms of $\mathcal{N}$ in $]s,t] \times \mathbb{T} \times]0,1]$, ordered in such a way that $t_1 < \cdots < t_k$. If $k = 0$, that is if there are no such atoms, we let $\Phi_{s,t}$ be the identity mapping of $\mathbb{T}$. By construction,

$$\Phi_{s,u} = \Phi_{t,u} \circ \Phi_{s,t}, \quad \text{if } s \leqslant t \leqslant u.$$

Finally, let $V_1, V_2, \ldots$ be a sequence of i.i.d. random variables which are uniformly distributed on $\mathbb{T}$. Also assume that this sequence is independent of the Poisson measure $\mathcal{N}$. For every $s \leqslant t$, define a random equivalence relation $\Pi_{s,t}$ on $\mathbb{N}$ by declaring that i and j are in the same block of $\Pi_{s,t}$ if and only if $\Phi_{s,t}(V_i) = \Phi_{s,t}(V_j)$.

Proposition 2. *The process $(\Pi_{0,t},\ t \geqslant 0)$ is a Λ-coalescent, with $\Lambda(\mathrm{d}x) = x^2\nu(\mathrm{d}x)$.*

This is very similar to Lemma 4 in [1], so that we will skip the proof. The crucial observation is the following. Let $a \in]0,1]$ and let Z be a random variable uniformly distributed over $\mathbb{T}$, independent of the sequence (V_j). For $n \geqslant 1$, set $K_n = |\{i \leqslant n\colon d(Z, V_i) \leqslant \frac{a}{2}\}|$. Then, conditionally on $K_n = k$, the distinct values taken by $f_{a,Z}(V_i)$, $i \leqslant n$, are distributed as $n - k + 1$ independent uniform variables on $\mathbb{T}$ (compare with Lemma 2 of [1]).

Note that our presentation is a bit different from the one in [1], because we consider the "flow of inverses" rather than the direct flow as in Section 4 of [1]. This explains the apparent difference between (41) and formula (13) of [1].

At this point it would be tempting to continue in the spirit of Theorem 2 of [1] and to consider a sequence (ν_n) such that the measures $x^2\nu_n(\mathrm{d}x)$ converge weakly to a given finite measure Λ on $[0,1]$. Denoting by Φ^n the flow associated with ν_n by the above construction, one expects that the sequence Φ^n converges in a suitable sense to a limiting flow associated with the Λ-coalescent. This convergence is indeed easy to obtain for the one-point motions, and because of rotational invariance of our construction, we see that the limiting one-point motions are Lévy processes on $\mathbb{T}$. However, proving the convergence of several points motions is harder because it does not seem easy to obtain a simple characterization of the limiting law. We will not address this general problem here, but in the next subsection we will concentrate on the case of the Kingman coalescent ($\Lambda = \delta_0$), which leads to a Brownian flow on $\mathbb{T}$.

6.2. A remarkable Brownian flow

For every $\varepsilon \in]0,1]$, let $\nu_\varepsilon = \varepsilon^{-2}\delta_\varepsilon$, and let $\Phi^\varepsilon = (\Phi^\varepsilon_{s,t})_{-\infty < s \leqslant t < \infty}$ be the Poissonian flow constructed in the preceding subsection with $\nu = \nu_\varepsilon$.

Proposition 3. *Let $z_1, \ldots, z_p \in \mathbb{T}$. Then the processes*

$$\big(\Phi^\varepsilon_{0,t}(z_1), \ldots, \Phi^\varepsilon_{0,t}(z_p)\big)_{t \geqslant 0}$$

converge in distribution as $\varepsilon \downarrow 0$, in the sense of weak convergence in the Skorokhod space $\mathbb{D}(\mathbb{R}_+, \mathbb{T}^p)$, towards a diffusion process with generator

$$\mathcal{B}g(y_1, \ldots, y_p) = \frac{1}{2}\sum_{i,j=1}^{p} b(y_i, y_j)\frac{\partial^2 g}{\partial y_i \partial y_j}(y_1, \ldots, y_p), \quad g \in C^2(\mathbb{T}^p),$$

where the function b is defined on $\mathbb{T}^2$ *by*

$$b(y,y') = \frac{1}{12} - \frac{1}{2} d(y,y')\big(1 - d(y,y')\big). \tag{42}$$

As the proof will show, uniqueness holds for the martingale problem associated with the generator $\mathcal{B}$, so that the limit in the proposition is well defined.

In the terminology of Harris [4], we can identify the limiting flow as the (coalescing) Brownian flow on $\mathbb{T}$ with covariance function b (note that b is translation invariant). In particular the one-point motions are (scaled) Brownian motions on $\mathbb{T}$.

Proof. First consider the case $p = 1$, $z_1 = z$. In that case, we observe that $\Phi^{\varepsilon}_{0,t}(z)$ is a continuous-time random walk on $\mathbb{T}$, with jump rate ε^2 and symmetric jump distribution π^{ε} given by

$$\int_{\mathbb{T}} \pi^{\varepsilon}(\mathrm{d}y)\varphi(y) = \int_{-\varepsilon/2}^{\varepsilon/2} \mathrm{d}a\, \varphi(a) + \int_{\varepsilon/2}^{1-\varepsilon/2} \mathrm{d}a\, \varphi\left(\frac{\varepsilon(a-1/2)}{1-\varepsilon}\right).$$

Notice that π^{ε} is supported on $[-\varepsilon/2, \varepsilon/2]$ and that we slightly abuse notation by identifying elements of $[-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]$ with their equivalent classes in $\mathbb{T}$. When $\varepsilon \to 0$, the second moment of π^{ε} behaves as

$$\varepsilon^2 \int_0^1 \left(a - \frac{1}{2}\right)^2 \mathrm{d}a = \frac{\varepsilon^2}{12}.$$

From well-known invariance principles, this is enough to conclude that the process $(\Phi^{\varepsilon}_{0,t}(z))_{t \geqslant 0}$ converges in distribution, in the sense of weak convergence in the Skorokhod space $\mathbb{D}(\mathbb{R}_+, \mathbb{T})$, towards a Brownian motion on $\mathbb{T}$ started at z (with generator $\frac{1}{24}\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ instead of the usual $\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}$).

Let us come back to the general case $p \geqslant 1$. From the case $p = 1$, we already know that the family of the distributions of the processes $(\Phi^{\varepsilon}_{0,t}(z_1), \ldots, \Phi^{\varepsilon}_{0,t}(z_p))_{t \geqslant 0}$ is tight as $\varepsilon \to 0$. To prove the desired convergence we need to characterize the sequential limits of this family. By construction, the process $(\Phi^{\varepsilon}_{0,t}(z_1), \ldots, \Phi^{\varepsilon}_{0,t}(z_p))$ is a continuous-time Markov chain with generator

$$\mathcal{B}^{\varepsilon} g(y_1, \ldots, y_p) = \varepsilon^{-2} \int \sigma(\mathrm{d}z)\big(g\big(f_{\varepsilon,z}(y_1), \ldots, f_{\varepsilon,z}(y_p)\big) - g(y_1, \ldots, y_p)\big).$$

Assume that $g \in C^2(\mathbb{T}^p)$. Then Taylor's expansion shows that as $\varepsilon \downarrow 0$,

$$\mathcal{B}^{\varepsilon} g(y_1, \ldots, y_p) = \frac{\varepsilon^{-2}}{2} \sum_{i,j=1}^{p} \frac{\partial^2 g}{\partial y_i \partial y_j}(y_1, \ldots, y_p) \int \big(f_{\varepsilon,z}(y_i) - y_i\big)\big(f_{\varepsilon,z}(y_j) - y_j\big)\sigma(\mathrm{d}z) + \mathrm{o}(1),$$

where we again abuse notation by writing $f_{\varepsilon,z}(y_i) - y_i$ for the representative of this element of $\mathbb{T}$ in the real interval $[-\varepsilon, \varepsilon]$. Elementary calculations show that for every $y, y' \in \mathbb{T}$,

$$\lim_{\varepsilon \to 0} \varepsilon^{-2} \int \big(f_{\varepsilon,z}(y) - y\big)\big(f_{\varepsilon,z}(y') - y'\big)\sigma(\mathrm{d}z) = b(y, y'),$$

where the function $b(y, y')$ is as in the statement of the theorem.

By a standard argument we obtain that any weak sequential limit $(\Gamma^1_t, \ldots, \Gamma^p_t)$ of the family $(\Phi^{\varepsilon}_{0,t}(z_1), \ldots, \Phi^{\varepsilon}_{0,t}(z_p))$ as $\varepsilon \downarrow 0$ solves the following martingale problem: For every $g \in C^2(\mathbb{T}^p)$,

$$g(\Gamma^1_t, \ldots, \Gamma^p_t) - \int_0^t \mathcal{B} g(\Gamma^1_s, \ldots, \Gamma^p_s)\, \mathrm{d}s$$

is a martingale. It remains to verify that this martingale problem is well-posed. To this end, let $\Gamma_t = (\Gamma_t^1, \ldots, \Gamma_t^p)$ be any continuous process that solves the preceding martingale problem with initial value $(z_1, \ldots, z_p)$. Fix $i, j \in \{1, \ldots, n\}$ and let

$$T_{i,j} = \inf\{t \geqslant 0:\ \Gamma_t^i = \Gamma_t^j\}.$$

We first prove that

$$\Gamma_t^i = \Gamma_t^j \quad \text{for every } t \geqslant T_{i,j}, \quad \text{a.s.} \tag{43}$$

Without loss of generality we may take $i = 1$ and $j = 2$. Let $z_0 \in \mathbb{T} \setminus \{z_1, z_2\}$ and

$$T_0 = \inf\{t \geqslant 0:\ \Gamma_t^1 = z_0 \text{ or } \Gamma_t^2 = z_0\}.$$

For every $t \geqslant 0$, set

$$X_t = d^*(z_0, \Gamma_t^1) - d^*(z_0, \Gamma_t^2)$$

(recall that $d^*(x, y)$ is the length of the counterclockwise arc from x to y). From the martingale problem for Γ, we easily deduce that for every $g \in C^2(\mathbb{R})$ the process

$$g(X_t) - \frac{1}{2}|X_t|\big(1 - |X_t|\big)g''(X_t)$$

is a local martingale on the stochastic interval $[0, T_0[$ (the restriction to $[0, T_0[$ is needed since the function $(x, y) \to d^*(z_0, x) - d^*(z_0, y)$ is C^2 only on $\mathbb{T} \setminus \{z_0\}$). Now notice that the diffusion process with generator $\frac{1}{2}|x|(1 - |x|)\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ (in the real interval $[-1, 1]$) is absorbed at the origin. We conclude that $\Gamma_t^1 = \Gamma_t^2$ for every $t \in [T_{1,2}, T_0[$, a.s. on $\{T_{1,2} < T_0\}$. Our claim (43) follows by applying a similar argument to the shifted process $(X_{s+t})_{t\geqslant 0}$ for any $s \geqslant 0$.

Since the covariance function b is smooth outside the diagonal, the desired uniqueness property easily follows from (43). See Lemma 3.2 in [4] for a similar argument. □

We now turn to a more detailed discussion of properties of the limiting flow. Note that the notion of a right-continuous function on $\mathbb{T}$ makes sense with an obvious meaning. A function $\varphi : \mathbb{T} \to \mathbb{T}$ is said to be monotone if the condition $y \in [\![x, z]\!]$ implies $\varphi(y) \in [\![\varphi(x), \varphi(z)]\!]$.

By adapting arguments of Harris [4] (Section 4), we may construct a collection $(\Theta_t(x))_{t\geqslant 0}$ indexed by $x \in \mathbb{T}$, of continuous processes with values in $\mathbb{T}$, in such a way that the following holds:

(i) For every $z_1, \ldots, z_p$, the process $(\Theta_t(z_1), \ldots, \Theta_t(z_p))$ is distributed as the solution of the martingale problem associated with $\mathcal{B}$ started at $(z_1, \ldots, z_p)$.
(ii) For every $t \geqslant 0$, the function $x \to \Theta_t(x)$ is right-continuous and monotone.
(iii) The mapping $t \to (\Theta_t(x),\ x \in \mathbb{T})$ is continuous with respect to the uniform norm on Borel functions from $\mathbb{T}$ into $\mathbb{T}$.
(iv) If $x, y \in \mathbb{T}$ and $S_{x,y} = \inf\{t \geqslant 0:\ \Theta_t(x) = \Theta_t(y)\}$ then $S_{x,y} < \infty$ and we have $\Theta_t(x) = \Theta_t(y)$ for every $t \geqslant S_{x,y}$.

From now on we deal with a collection $(\Theta_t(x))$ satisfying the above properties (i)–(iv).

Theorem 7. *Let $(V_1, V_2, \ldots)$ be a sequence of independent uniform variables on $\mathbb{T}$, which is also independent of the collection $(\Theta_t(x))$. For every $t \geqslant 0$, let Π_t be the random partition of $\mathbb{N}$ constructed by saying that i and j are in the same block of Π_t if and only if $\Theta_t(V_i) = \Theta_t(V_j)$. Then $(\Pi_t)_{t\geqslant 0}$ is a Kingman coalescent.*

Proof. Recall the Poissonian flow Φ^ε of the beginning of this subsection, and fix $p \geqslant 1$. As a consequence of Proposition 3, we know that

$$\big(\Phi_{0,t}^\varepsilon(V_1), \ldots, \Phi_{0,t}^\varepsilon(V_p)\big)_{t\geqslant 0} \longrightarrow \big(\Theta_t(V_1), \ldots, \Theta_t(V_p)\big)_{t\geqslant 0} \tag{44}$$

in the sense of weak convergence in the Skorokhod space. By using the Skorokhod representation theorem, we may and will assume that this convergence holds a.s. along a given subsequence $\varepsilon_k \to 0$. From now on we restrict our attention to values of ε belonging to this subsequence. For $i, j \in \{1, \ldots, p\}$ with $i \neq j$, set

$$T^{\varepsilon}_{i,j} = \inf\bigl\{t \geqslant 0\colon\ \Phi^{\varepsilon}_{0,t}(V_i) = \Phi^{\varepsilon}_{0,t}(V_j)\bigr\}.$$

Lemma 2. *We have*

$$\lim_{\varepsilon\to 0} T^{\varepsilon}_{i,j} = S_{V_i,V_j} \quad \text{in probability}$$

and the variable S_{V_i,V_j} is exponentially distributed with mean 1.

We postpone the proof of the lemma. For every $t \geqslant 0$, let $\Pi^{\varepsilon}_{0,t}$ be the random partition of $\mathbb{N}$ associated with Φ^{ε} as explained before Proposition 2. By Proposition 2, we know that the process $(\Pi^{\varepsilon}_{0,t})_{t\geqslant 0}$ is a Λ-coalescent with $\Lambda = \delta_{\varepsilon}$, and thus converges in distribution to the Kingman coalescent as $\varepsilon \to 0$ (see Theorem 1 in [12]).

On the other hand, it immediately follows from Lemma 2 and our definitions that the restriction of $\Pi^{\varepsilon}_{0,t}$ to $\{1, \ldots, p\}$ converges in probability to the restriction of Π_t. Hence we conclude that the restriction of $(\Pi_t)_{t\geqslant 0}$ to $\{1, \ldots, p\}$ is distributed as the Kingman coalescent. Since this holds for any p the proof is complete. □

Proof of Lemma 2. It is clear from the a.s. convergence (44) that we have

$$S_{V_i,V_j} \leqslant \liminf_{\varepsilon\to 0} T^{\varepsilon}_{i,j}, \quad \text{a.s.}$$

To get the first part of the lemma, it is then enough to prove that $\mathbb{E}[T^{\varepsilon}_{i,j}]$ converges to $\mathbb{E}[S_{V_i,V_j}]$ as $\varepsilon \to 0$. From Proposition 2 and the known properties of the Λ-coalescent (see e.g. [12], Example 19), or by a direct argument, it is easily checked that $T^{\varepsilon}_{i,j}$ has the same distribution as $U_1 + \cdots + U_{N_\varepsilon}$, where $U_1, \ldots$ are independent exponential variables with mean ε^2, and N_ε is independent of the sequence $U_1, \ldots$ and such that $\mathbb{P}[N_\varepsilon = k] = \varepsilon^2(1-\varepsilon^2)^{k-1}$ for every $k \in \mathbb{N}$. It immediately follows that $\mathbb{E}[T^{\varepsilon}_{i,j}] = 1$. Therefore the proof of the first assertion will be complete if we verify the second assertion, that is S_{V_i,V_j} is exponential with mean 1.

The following argument is related to Lemma 3.4 in Harris [4]. By using the martingale problem and arguments similar to the proof of Proposition 3, it is easy to check that the process $(d^*(\Theta_t(V_i), \Theta_t(V_j)),\ 0 \leqslant t < S_{V_i,V_j})$ is distributed as the diffusion with generator $\frac{1}{2}x(1-x)\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ with initial value uniform over $[0, 1]$, up to its first hitting time of $\{0, 1\}$ (notice that this is the same diffusion as in Corollary 1). Consequently, if U is a random variable with uniform distribution over $[0, 1]$, and W is a standard linear Brownian motion, then S_{V_i,V_j} has the same distribution as $T = \inf\{t \geqslant 0\colon\ Y_t = 0 \text{ or } 1\}$, where Y is the unique (strong) solution of the stochastic equation

$$Y_t = U + \int_0^t \sqrt{Y_s(1-Y_s)}\,\mathrm{d}W_s.$$

Note that Y is absorbed at 0 and 1 and that $\mathbb{E}[Y_t] = \frac{1}{2}$ for every $t \geqslant 0$. From Itô's formula, we get that for every integer $k \geqslant 2$,

$$\mathbb{E}[Y^k_t] = \frac{1}{k+1} + \frac{k(k-1)}{2}\int_0^t \bigl(\mathbb{E}[Y^{k-1}_s] - \mathbb{E}[Y^k_s]\bigr)\,\mathrm{d}s.$$

From this formula and an easy induction argument we get

$$\mathbb{E}[Y^k_t] = \frac{1}{2} - \frac{k-1}{2(k+1)}\,\mathrm{e}^{-t}.$$

The distribution of Y_t readily follows: By letting k go to ∞ and using symmetry, we have first $\mathbb{P}[Y_t = 0] = \mathbb{P}[Y_t = 1] = \frac{1}{2}(1 - \mathrm{e}^{-t})$ and we also see that, conditionally on $\{Y_t \notin \{0, 1\}\}$, Y_t is uniform on $]0, 1[$. This is more than enough for our needs. □

We observed in the preceding proof that the process $(d^*(\Theta_t(V_i), \Theta_t(V_j)),\ 0 \leqslant t < S_{V_i,V_j})$ is distributed as the diffusion process in Corollary 1. This is generalized in the following proposition, which provides a connection between the flow $(\Theta_t)_{t\geqslant 0}$ and the Kingman flow on the interval $[0, 1]$, thus shedding light on Theorem 7.

Recall our notation $(F_t)_{t\geqslant 0}$ for the Λ-process and take $\Lambda = \delta_0$. For every $x \in [0, 1]$, we can view $(F_t(x))_{t\geqslant 0}$ as a $\mathbb{T}$-valued process: This simply means that we identify the values 0 and 1.

Proposition 4. *Let $0 \leqslant x_1 < x_2 < \cdots < x_p < 1$. Then the $\mathbb{T}^p$-valued processes*

$$\big(d^*\big(\Theta_t(0), \Theta_t(x_1)\big), d^*\big(\Theta_t(0), \Theta_t(x_2)\big), \ldots, d^*\big(\Theta_t(0), \Theta_t(x_p)\big)\big)_{t\geqslant 0}$$

and

$$\big(F_t(x_1), F_t(x_2), \ldots, F_t(x_p)\big)_{t\geqslant 0}$$

have the same distribution.

Proof. The generator $\mathcal{A}$ of the Markov process $(F_t(x_1), F_t(x_2), \ldots, F_t(x_p))$ is known from Theorem 3. From the knowledge of the generator $\mathcal{B}$ for the process $(\Theta_t(0), \Theta_t(x_1), \ldots, \Theta_t(x_p))$ we can also identify the law of the process

$$\big(d^*\big(\Theta_t(0), \Theta_t(x_1)\big), d^*\big(\Theta_t(0), \Theta_t(x_2)\big), \ldots, d^*\big(\Theta_t(0), \Theta_t(x_p)\big)\big)_{t\geqslant 0}$$

(compare with Section 5 of Harris [4]). Precisely, we verify that the latter process solves the martingale problem associated with $\mathcal{A}$, at least up to the stopping time S_{0,x_p}, and we then use an induction argument. Details are left to the reader. □

As a consequence of Theorem 7 (or of the preceding proposition), we know that for every $t > 0$ the range $\mathcal{S}_t$ of Θ_t is finite, and more precisely $N_t = |\mathcal{S}_t|$ is distributed as the number of blocks in the Kingman coalescent at time t. Set

$$\mathcal{S}_t = \{U_1^t, \ldots, U_{N_t}^t\},$$

where U_t^1 is drawn uniformly at random from $\mathcal{S}_t$, and then the points $U_1^t, U_2^t, \ldots, U_{N_t}^t$ are listed in counterclockwise order. The next corollary is a simple consequence of Proposition 4 and the discussion at the end of Section 5.

Corollary 2. *Fix $t > 0$. Let*

$$M_t = \big(\sigma\big(\Theta_t^{-1}(U_1^t)\big), \sigma\big(\Theta_t^{-1}(U_2^t)\big), \ldots, \sigma\big(\Theta_t^{-1}(U_{N_t}^t)\big)\big)$$

be the vector of masses attached to the points in $\mathcal{S}_t$, and let

$$D_t = \big(d^*(U_1^t, U_2^t), d^*(U_2^t, U_3^t), \ldots, d^*(U_{N_t}^t, U_1^t)\big)$$

be the vector of lengths of the adjacent intervals to the points in $\mathcal{S}_t$. Then conditionally on $\{N_t = k\}$, the vectors M_t and D_t are independent and both uniformly distributed on the simplex $\{(x_1, \ldots, x_k) \in [0, 1]^k\colon\ x_1 + \cdots + x_k = 1\}$.

Remark. (i) There is in a sense more symmetry in the flow (Θ_t) than in the Kingman flow on the interval $[0, 1]$, for which the end points 0 and 1 play a special role. The fact that the random vectors M_t and D_t have the same distribution is clearly related to Theorem 10.5 and Corollary 10.6 in Harris [4], who deals with Brownian flows on the real line.

(ii) As a final observation, let us comment on the constant $\frac{1}{12}$ in formula (42) for the covariance function b. Let $a > 0$ and let β^a be a Brownian motion on $\mathbb{T}$ started at 0 with generator $\frac{a}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}$. Assume that β^a is independent of $(\Theta_t)_{t\geqslant 0}$ and for every $t \geqslant 0$ set

$$\Theta_t^a(y) = \Theta_t(y) + \beta_t^a, \quad y \in \mathbb{T}.$$

Then $(\Theta_t^a)_{t\geqslant 0}$ is a Brownian flow in $\mathbb{T}$ with covariance $b^a(y, y') = b(y, y') + a$. Obviously, Theorem 7, Proposition 4 and Corollary 2 remain valid if Θ is replaced by Θ^a.

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