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Ann. I. H. Poincaré – PR 41 (2005) 703–724

ANNALES DE L'INSTITUT HENRI POINCARÉ
PROBABILITÉS ET STATISTIQUES

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Approximations of the Brownian rough path with applications to stochastic analysis

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Received 25 August 2003; received in revised form 15 March 2004; accepted 11 May 2004

Available online 11 September 2004

Abstract

A geometric p-rough path can be seen to be a genuine path of finite p-variation with values in a Lie group equipped with a natural distance. The group and its distance lift $(\mathbb{R}^d, +, 0)$ and its Euclidean distance.

This approach allows us to easily get a precise modulus of continuity for the Enhanced Brownian Motion (the Brownian Motion and its Lévy Area).

As a first application, extending an idea due to Millet and Sanz-Solé, we characterize the support of the Enhanced Brownian Motion (without relying on correlation inequalities). Secondly, we prove Schilder's theorem for this Enhanced Brownian Motion. As all results apply in Hölder (and stronger) topologies, this extends recent work by Ledoux, Qian, Zhang [Stochastic Process. Appl. 102 (2) (2002) 265–283]. Lyons' fine estimates in terms of control functions [Rev. Mat. Iberoamericana 14 (2) (1998) 215–310] allow us to show that the Itô map is still continuous in the topologies we introduced. This provides new and simplified proofs of the Stroock–Varadhan support theorem and the Freidlin–Wentzell theory. It also provides a short proof of modulus of continuity for diffusion processes along old results by Baldi.

Résumé

Un p-rough path est un chemin de p-variation finie à valeurs dans un groupe de Lie muni d'une distance sous-riemannienne. Le groupe et sa distance géneralisent $(\mathbb{R}^d, +, 0)$ et la distance euclidienne.

Cette approche nous permet d'obtenir un module de continuité tres précis pour le rough path brownien (le mouvement brownien et son aire de Lévy). Pour ce dernier, nous prouvons un théorème du support (adaptant une idée de Millet et Sans-Solé) et un théorème de Schilder. Comme tous les résultats sont prouvés en utilisant des topologies de type Hölder ou plus fines, cela géneralise le papier de Ledoux, Qian, Zhang [Stochastic Process. Appl. 102 (2) (2002) 265–283]. Les résultats de T. Lyons [Rev. Mat. Iberoamericana 14 (2) (1998) 215–310] permettent de prouver rapidement que l'application d'Itô est continue pour les topologies que nous avons introduites. Cela nous donne de nouvelles preuves du théorème du support de Stroock–Varadhan

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0246-0203/$ – see front matter
doi:10.1016/j.anihpb.2004.05.003

et de la théorie de Freidlin–Wentzell. Nous obtenons au passage une preuve simple du module de continuité pour les processus de diffusions, obtenu précédemment par Baldi.

MSC: 60F10; 60H10

Keywords: Rough paths; Support theorem; Large deviation; Modulus of continuity of diffusions

1. Introduction

Starting with [20], Terry Lyons developed a general theory of integration and differential equations of the form

$$dy_t = f(y_t)\,dx_t. \tag{1}$$

To include the important example of stochastic differential equations, x is allowed to be "rough" in some sense. Standard Hölder regularity of Brownian motion, for instance, implies finite p-variation only for $p > 2$. Another issue was to explain (deterministically) the difference between stochastic differential equations based on Stratonovich versus Itô integration. Last but not least, motivated from examples like Fractional Brownian motion, driving signals much rougher than Brownian motion should be included.

All this has been accomplished in a beautiful way and the reader can nowadays find the general theory exposed in [21,23,24].

Loosely speaking, for general $p \geqslant 1$, one needs to "enhance" the driving signal x, with values in some Banach space V, to $X \in V \oplus V^{\otimes 2} \cdots \oplus V^{\otimes [p]}$ such that the resulting object X satisfies certain algebraic[1] and analytic conditions. For x of finite variation, this enhancement will simply consist of all the iterated integrals of x,

$$X^k_{s,t} := \int\limits_{s<u_1<\cdots<u_k<t} dx_{u_1} \otimes \cdots \otimes dx_{u_k}, \quad k = 1, \ldots, [p].$$

These are the *Smooth Rough Paths*. Consider a time horizon of $[0, 1]$ (valid for the rest of the paper) and introduce the p-variation metric, defined as

$$d(X, Y) = \max_{k=1,\ldots,[p]} \left(\sup_D \sum_l |X^k_{t_{l-1},t_l} - Y^k_{t_{l-1},t_l}|^{p/k} \right)^{k/p},$$

where $\sup_D$ runs over all finite divisions of $[0, 1]$. Here $|\cdot|$ denotes (compatible) tensor norms in $V^{\otimes k}$. Closure of Smooth Rough Paths with respect to this metric yields the class of *Geometric Rough Paths*, denoted by $G\Omega_p(V)$. The solution map, also called *Itô map*, to (1) is then a continuous map from $G\Omega_p(V) \to G\Omega_p(W)$, provided $f : W \to L(V, W)$ satisfies mild regularity conditions. This is Lyons' celebrated *Universal Limit Theorem*. In particular, smooth approximations $X(n)$ which converge in p-variation to $X \in G\Omega_p(V)$ will cause the corresponding solutions $Y(n)$ to converge to Y in p-variation. Hence, one deals with some kind of generalized Stratonovich theory.

However, the so important case of $p \in (2, 3)$, on which this paper will focus, allows for more. For the sake of concreteness, we will set $V = \mathbb{R}^d$ from here on. Following [24, p. 149] and also [23] the driving signal only needs to be a *Multiplicative functional of finite p-variation*. By definition, this is a continuous map

$$(s, t) \to (X^1_{s,t}, X^2_{s,t}) \in \mathbb{R}^d \oplus (\mathbb{R}^d)^{\otimes 2} =: T^2,$$

where $0 \leqslant s \leqslant t \leqslant 1$, satisfying the algebraic *Chen condition*

$$X_{s,u} = X_{s,t} \otimes X_{t,u} \quad \Leftrightarrow \quad X^1_{s,u} = X^1_{s,t} + X^1_{t,u}, \qquad X^2_{s,u} = X^2_{s,t} + X^2_{t,u} + X^1_{s,t} \otimes X^1_{t,u}, \tag{2}$$

[1] For *algebraic* convenience X is often enhanced to $\mathbb{R} \oplus V \oplus V^{\otimes 2} \cdots \oplus V^{\otimes [p]}$ with scalar component constant 1.

whenever $s \leqslant t \leqslant u$, and the analytic condition $d_{p\text{-var}}(X,0) < \infty$ i.e.

$$\sup_{(0 \leqslant t_0 < \ldots < t_n \leqslant 1)} \sum_l |X^k_{t_{l-1},t_l}|^{p/k} < \infty, \quad k = 1, 2. \tag{3}$$

(Often $k = 1, 2$ are referred to as *first* resp. *second level*.) The class of such rough paths is denoted $\Omega_p(\mathbb{R}^d)$. Condition (2) is known as *Chen relation* and expresses simple additive properties whenever X^2 is obtained as *some* iterated integral including the cases of Stratonovich resp. Itô Enhanced Brownian Motion. Whenever a first order calculus underlies this integration (which is the case for Stratonovich integration), one has

$$Symm(X^2) = \frac{1}{2} X^1 \otimes X^1. \tag{4}$$

Geometric p-Rough Paths satisfy this condition, but paths satisfying this condition and which satisfy the condition $d_{p\text{-var}}(X,0)$ forms a set slightly bigger than $G\Omega_p(V)$ [13]. Clearly,

$$\{\text{Smooth Rough Paths}\} \subset G\Omega_p(\mathbb{R}^d) \subset \Omega_p(\mathbb{R}^d).$$

One can indeed choose in which space to work with and the Lyons theory will provide meaning, existence and uniqueness to the purely deterministic *rough differential equation*

$$dY = V(y_0 + Y^1_{0t})\, dX,$$

where $V = (V_1, \ldots, V_d)$ are, in general non-commuting, vector fields with mild regularity conditions. As before, the Itô map $X \mapsto Y$ is, continuous under p-variation topology. This rough differential equation indeed generalizes ordinary and stochastic (Stratonovich and Itô) differential equations. For instance, it is known (and also follows from the results in this paper) that a.s. the Stratonovich Enhanced Brownian Motion (EBM) $\mathbf{B} \in G\Omega_p(\mathbb{R}^d)$ for all $p \in (2,3)$. Choosing $X = \mathbf{B}$ the projection of the rough path Y to its first level will solve the associated Stratonovich stochastic differential equation. That is

$$y_0 + Y^1_{0t} \quad \text{solves} \quad dy = \sum_i V_i(y) \circ d\beta^i.$$

For the rest of the paper, p denotes a fixed real in $(2,3)$. The contributions of this paper may be summarized as follows:

(a) We look at geometric p-rough paths from a new angle. Observe that $G := \{X \in T^2\colon (4)\text{ holds}\}$ is the free nilpotent Lie group of step 2 [23,31], a simply connected Lie group which lifts $(\mathbb{R}^d, +, 0)$. Chen's condition is equivalent to the fact that $\mathbf{x}_t = X_{0,t}$ is a G-valued path such that $X_{s,t} = \mathbf{x}_s^{-1} \otimes \mathbf{x}_t = \mathbf{x}_{s,t}$. We put a homogenous, sub-additive norm on $(G, \otimes)$. Geometric p-rough paths are then easily seen to be the closure of "smooth" G-valued paths under the p-variation metric. Standard proofs for Kolmogorov's criterion or the Garsia, Rumsey, Rodemich inequality adapt with no changes from $(\mathbb{R}^d, +)$-valued to $(G, \otimes)$-valued processes. With this observation, regularity results for the EBM $\mathbf{B}$, as Hölder continuity and Lévy modulus of continuity, follow after simple moment estimates. Sometimes, it will be convenient to work in the associated Lie algebra of the group G. For instance, the EBM viewed through this chart is nothing else than the well studied Gaveau diffusion [14]. Its importance in the context of limit theorems was already highlighted in Malliavin's book, [26].

(b) We introduce a number of different topologies on $G\Omega_p(\mathbb{R}^d)$, effectively reducing this space to geometric rough paths for which the associated norms are finite. For instance, we are able to deal with Hölder and "modulus type" norms. Exploiting fine estimates in Lyons' Limit Theorem we have continuity of the Itô map in all these topologies. In former applications of rough path theory to stochastic analysis result were always obtained in p-variation topology, leaving open a gap between (usually well known) results in Hölder and stronger topologies.

(c) Lyons' Universal Limit Theorem implies a Lévy modulus of continuity for diffusions, along the results by Baldi [2].

(d) We establish convergence of several different approximations to the EBM. The EBM is usually defined as the limit of a sequence of smooth rough paths, which is shown to be Cauchy [24]. Here, we define directly the

EBM, and its regularity allows us to prove convergence of some sequences of smooth rough paths to the EBM. The first idea, common to works in [22,26,17,12], is that approximations are obtained by conditioning with respect to dyadic filtrations. After establishing uniform regularity of approximations, easily obtained by Doob's inequality, we can use an interpolation argument to show convergence in interesting topologies of some sequences of smooth rough paths to the EBM.

(e) If $\mathbb{W}$ is the Wiener measure on the Wiener space equipped with the, say, α-Hölder topology, $\alpha < 1/2$, the support $\mathbb{W}$ is the closure of smooth path under the α-Hölder metric. One proof, as noticed in [28], reduces to some convergence results of adapted linear approximation ω^n, which we take to be the dyadic approximation of ω time shifted of -2^{-n} (to make it adapted). Then ω^n converges to ω in the α-Hölder topology, which shows that the support of $\mathbb{W}$ is contained in the closure of smooth path under the α-Hölder metric. Reciprocally, given a smooth path g, $g+\omega-\omega^n$ converges to g, which shows the inverse inclusion, using the Cameron–Martin theorem. We extend this proof to the "Wiener measure" on the free nilpotent group of step 2, i.e. the law of the Enhanced Brownian motion. The main difficulty is to introduce a translation operator T, to make sense of $g+\omega-\omega^n$ for group valued paths. By means of continuity of the Itô map, from (b), this immediately implies the support theorem for diffusions. In the context of rough paths this improves work by Ledoux, Qian, Zhang (support theorem in p-variation topology, [25]) and by the first cited author (Hölder topology, [12]).

As for the history of the support theorem, it was originally obtained by Stroock, Varadhan [37] in sup-topology, then by Ben Arous, Gruadinaru, Ledoux [7] and Millet, Sanz-Solé [28] in Hölder norm of exponent less than 1/2. Probably the more abstract proof of this result can be found in [1]. Extension to modulus space has been obtain in [16] and to Orlicz–Besov space in [27]. We limit ourselves to "modulus norm", in the spirit of [16], and we will not recover fully the results in [16,27]. On the other hand, we have a description of the support of the Stratonovich enhanced diffusion the goes beyond the last quoted results.

(f) Schilder's theorem for EBM is obtained. As before, continuity of the Itô map will give the Freidlin–Wentzell large deviation result in the topologies mentioned in (b). Strassen's law is obtained as corollary. Again, we improve [25] and recover well known large deviations results in Hölder and modulus norm [5]. Once again though, we do not deal with Orlicz–Besov metrics (see [10]).

Constants in this paper may vary from line to line.

2. Rough paths

2.1. Free nilpotent Lie group of step 2

We fix the dimension d ($d \geqslant 2$ to avoid trivialities) and we denote by $\mathcal{L}(\mathbb{R}^d) = \mathbb{R}^d \oplus so(d)$, where $so(d) \simeq \mathbb{R}^{d(d-1)/2}$ is the space of real antisymmetric $d \times d$ matrices. With the bracket

$$
\begin{aligned}
&[,]: \mathcal{L}(\mathbb{R}^d) \times \mathcal{L}(\mathbb{R}^d) \to \mathcal{L}(\mathbb{R}^d),\\
&\big((a^1, a^2), (b^1, b^2)\big) \to [a, b] = (0, a^1 \otimes b^1 - b^1 \otimes a^1),
\end{aligned}
$$

$\mathcal{L}(\mathbb{R}^d)$ becomes a (step 2 nilpotent) Lie algebra. The group multiplication in the associated simply connected Lie Group $G(\mathbb{R}^d) = \exp(\mathcal{L}(\mathbb{R}^d))$,[31,39,41], is given by the Baker–Campbell–Hausdorff formula.[2]

$$
\begin{aligned}
&\otimes: G(\mathbb{R}^d) \times G(\mathbb{R}^d) \to G(\mathbb{R}^d),\\
&\quad \exp(a) \otimes \exp(b) \to \exp\left(a + b + \frac{1}{2}[a, b]\right).
\end{aligned}
$$

[2] Due to step 2 nilpotency, only the first bracket appears.

Its neutral element is $\exp(0)$, and the inverse of $\exp(a)$ is $\exp(-a)$. We can identify $G(\mathbb{R}^d)$ with the nonlinear submanifold of $\mathbb{R}^d \oplus \mathbb{R}^{d\times d}$ given by

$$\left\{g = (g^1, g^2) \in \mathbb{R}^d \oplus (\mathbb{R}^d)^{\otimes 2}\text{: symmetric part of } g^2 \text{ equals } \frac{1}{2} g^1 \otimes g^1\right\}$$

with usual (truncated) tensor multiplication, that is,

$$g \otimes h = (g^1 + h^1, g^2 + g^1 \otimes h^1 + h^2).$$

$(G(\mathbb{R}^d), \otimes, \exp(0))$ is the free nilpotent group of step 2 over $\mathbb{R}^d$, [23,31]. Note that $G(\mathbb{R}^d)$ is invariant under the dilation operator δ_t, for $t \in \mathbb{R}$, δ_t being defined by

$$\begin{aligned} G(\mathbb{R}^d) &\to G(\mathbb{R}^d), \\ \exp(a^1, a^2) &\to \exp(ta^1, t^2 a^2). \end{aligned}$$

We define on the group

$$\|g\| = \inf_{\substack{x^1,\dots,x^n \in \mathbb{R}^d \\ \bigotimes_{i=1}^n \exp(x^i) = g}} \sum_{i=1}^n |x_i|_{\mathbb{R}^d}, \tag{5}$$

where $|.|_{\mathbb{R}^d}$ is the Euclidean norm on $\mathbb{R}^d$. For all g, $\|g\|$ is finite by Chow theorem [15,29], although it can be seen quite directly here with the use of the Baker–Campbell–Hausdorff formula. $\|.\|$ is a sub-additive, symmetric homogeneous norm [11] on $G(\mathbb{R}^d)$, that is

(i) $\|g\|$ if and only if $g = \exp(0)$,
(ii) for all $g \in G(\mathbb{R}^d)$ and $t \in \mathbb{R}$, $\|\delta_t g\| = |t|\,\|g\|$,
(iii) for all $g, h \in G(\mathbb{R}^d)$, $\|g \otimes h\| \leqslant \|g\| + \|h\|$,
(iv) for all g, $\|g\| = \|g^{-1}\|$.

From this sub-additive, symmetric homogeneous norm, we construct a left invariant distance on $G(\mathbb{R}^d)$ (which is a Carnot–Caratheodory distance [15,29]) by defining

$$d(g, h) = \|h^{-1} \otimes g\|.$$

If $|\cdot|_{\mathbb{R}^d \otimes \mathbb{R}^d}$ denotes a norm on $\mathbb{R}^d \otimes \mathbb{R}^d$, then

$$\big|\!\big|\!\big| \exp(a^1, a^2) \big|\!\big|\!\big| = |a^1|_{\mathbb{R}^d} + \sqrt{|a^2|_{\mathbb{R}^d \otimes \mathbb{R}^d}}$$

defines another homogeneous norm on $G(\mathbb{R}^d)$ (that is a norm satisfying (i) and (ii)), and as all homogeneous norms are equivalent [18], one can find some positive constants c_1, c_2 such that for all $g \in G(\mathbb{R}^d)$

$$c_1 |\!|\!| g |\!|\!| \leqslant \|g\| \leqslant c_2 |\!|\!| g |\!|\!|. \tag{6}$$

This implies the following:

Corollary 2. *For some constant C,*

$$\|h^{-1} \otimes g \otimes h\| \leqslant C\big(\|g\| + \sqrt{\|h\|\|g\|}\big), \tag{7}$$

and for any $k \geqslant 2$,

$$d\left(\bigotimes_{i=1}^k g_i, \bigotimes_{i=1}^k h_i\right) \leqslant C \sum_{i=1}^k \left(d(g_i, h_i) + \sqrt{d(g_i, h_i) \left\| \bigotimes_{j=i+1}^k h_j \right\|}\right). \tag{8}$$

Proof. If $g = e^b, h = e^a, h^{-1} \otimes g \otimes h = e^b \otimes e^{-[a,b]}$, hence

$$|||h^{-1} \otimes g \otimes h||| \leqslant |||g||| + |||e^{-[a,b]}||| \leqslant |||g||| + \sqrt{|||g||| \cdot |||h|||}.$$

The inequality is then proved using inequality (6). The second inequality is a consequence of the first one. We show it for $k = 2$, the general case follows in exactly the same way, by induction.

$$\begin{aligned} d(g_1 \otimes g_2, h_1 \otimes h_2) &= \|h_2^{-1} \otimes h_1^{-1} \otimes g_1 \otimes g_2\| \\ &= \|h_2^{-1} \otimes h_1^{-1} \otimes g_1 \otimes h_2 \otimes h_2^{-1} \otimes g_2\| \\ &\leqslant \|h_2^{-1} \otimes h_1^{-1} \otimes g_1 \otimes h_2\| + \|h_2^{-1} \otimes g_2\| \\ &\leqslant C\bigl(d(g_1, h_1) + \sqrt{d(g_1, h_1)\|h_2\|}\bigr) + d(g_2, h_2). \qquad \square \end{aligned}$$

$(G(\mathbb{R}^d), \otimes, \exp(0))$ equipped with a homogeneous norm is a simple generalization of $(\mathbb{R}^d, +, 0)$ equipped with a norm.

We let $C_0([0,1], G(\mathbb{R}^d))$ to be the space of continuous function from $[0,1]$ to $G(\mathbb{R}^d)$ such that their value at time 0 is $\exp(0)$. With a slight abuse, we will call such elements $G(\mathbb{R}^d)$-valued paths. If $\mathbf{x} \in C_0([0,1], G(\mathbb{R}^d))$ and $s < t$, we will denote by $\mathbf{x}_{s,t}$ the element $\mathbf{x}_s^{-1} \otimes \mathbf{x}_t$.

Remark 3. Let $|.|_{\mathbb{R}^d \otimes \mathbb{R}^d}$ be a compatible tensor norm,[3] invariant under matrix-transposition. Then an explicit norm satisfying (i)–(iv) is given by

$$\max\left\{|a^1|_{\mathbb{R}^d}, \sqrt{\left|a^2 + \frac{1}{2} a^1 \otimes a^1\right|_{\mathbb{R}^d \otimes \mathbb{R}^d}}\right\},$$

where $(a^1, a^2) \in \mathcal{L}(\mathbb{R}^d)$.

2.2. p-variation

Let $(G, \otimes, e)$ be a group equipped with a homogeneous norm $\|.\|$. Here, we think of G being either $(\mathbb{R}^d, +, 0)$ or $(G(\mathbb{R}^d), \otimes, \exp(0))$. A path $x : [0,1] \to G$ is said to have finite p-variation if

$$\sup_{(0 \leqslant t_0 < \cdots < t_n \leqslant 1)} \sum_i \|x_{t_i, t_{i+1}}\|^p < \infty.$$

Note that a path x is continuous and of finite p-variation if and only if (see [24])

$$\|x_{s,t}\|^p \leqslant \omega(s,t) \quad \text{for all } s \leqslant t$$

for some control function ω. By definition, this means

(i) $\omega : \{(s,t), 0 \leqslant s \leqslant t \leqslant 1\} \to \mathbb{R}^+$ is continuous near the diagonal.
(ii) ω is super-additive, i.e.

$$\forall s < t < u, \quad \omega(s,t) + \omega(t,u) \leqslant \omega(t,u). \tag{9}$$

(iii) $\omega(t,t) = 0$ for all $t \in [0,1]$.

We will say in such case that x has finite p-variation controlled by ω. We will construct control functions in the following way,

[3] That is $|a \otimes b|_{\mathbb{R}^d \otimes \mathbb{R}^d} \leqslant |a|_{\mathbb{R}^d} |b|_{\mathbb{R}^d}$.

Proposition 4. *Consider a continuous map $f:\mathbb{R}^+\to\mathbb{R}^+$, increasing, convex, $f(0)=0$. Then $(s,t)\mapsto f(t-s)$ is a control function.*

Example 5. $f(t-s)=c(t-s)$ for some constant $c>0$. This is equivalent to $1/p$-Hölder continuity for the controlled path.

We define a p-variation distance between two G-valued paths x and y:

$$d_{p\text{-var}}(x,y)=\sup_{(0\leqslant t_0<\cdots<t_n\leqslant 1)}\sum_i d(x_{t_i,t_{i+1}},y_{t_i,t_{i+1}})^p.$$

2.3. Definition of a geometric p-rough path

We will denote by π_1 resp. π_2 the natural injection from $G(\mathbb{R}^d)$ onto $\mathbb{R}^d$ resp. onto $so(d)$. If $\mathbf{x}$ is a $G(\mathbb{R}^d)$-valued path of finite p-variation, then clearly, $\pi_1(\mathbf{x}):t\to\pi_1(\mathbf{x}_t)$ is a $\mathbb{R}^d$-valued path of finite p-variation. We will say that $\mathbf{x}$ lies above $\pi_1(\mathbf{x})$. Conversely, assume x is a $\mathbb{R}^d$-valued path of finite p-variation. If $p<2$, then

$$S(x):t\to\exp\left(x_t+\frac{1}{2}\int_0^t(x_u\otimes dx_u-dx_u\otimes x_u)\right)\tag{10}$$

is the unique $G(\mathbb{R}^d)$-valued path of finite p-variation lying above x ($\pi_2(S(x))_t$ is the Lévy area of x between time 0 and t; such integrals are well defined Young integrals [42]). The path $t\to S(x)_t$ is called the signature of the path x.

If $p\in(2,3)$, then there exists a $G(\mathbb{R}^d)$-valued path $\mathbf{x}$ lying above x [40], but uniqueness is not true anymore [23,40].

Definition 6. For $p\in[2,3)$, the set of geometric p-rough path is the closure of $\{S(x),\ x$ of finite 1-variation$\}$ under the p-variation metric $d_{p\text{-var}}$. Such class is denoted by $G\Omega_p(\mathbb{R}^d)$.

We see in particular that any $G(\mathbb{R}^d)$-valued path of finite q-variation, for $q>p$, is a p-rough path [13]. Once again, the set of $G(\mathbb{R}^d)$-valued path of finite p-variation, that we denote $G\Omega_{p^+}(\mathbb{R}^d)$ is strictly bigger than $G\Omega_p(\mathbb{R}^d)$.

Note that if $\mathbf{x}$ is a $G(\mathbb{R}^d)$-valued path, then $(s,t)\to\mathbf{x}_s^{-1}\otimes\mathbf{x}_t$ is a geometric multiplicative functional, in the sense of [23]. Reciprocally, if $\mathbf{y}_{s,t}$ is a multiplicative functional, then $\mathbf{x}_t=\mathbf{y}_{0,t}$ is a $G(\mathbb{R}^d)$ path starting at $\exp(0)$ and $\mathbf{y}_{s,t}=\mathbf{x}_s^{-1}\otimes\mathbf{x}_t$. Geometric multiplicative functionals and $G(\mathbb{R}^d)$-valued path are the same objects. The main theorem in rough path theory is the continuity of the Itô map.

2.4. The Itô map

Theorem 7. *Let $\mathbf{x}\in G\Omega_{p^+}(\mathbb{R}^d)$, $\varepsilon>0$ and $\mathcal{V}$ be a linear map from $\mathbb{R}^d$ into the Lip$[p+\varepsilon,\mathbb{R}^n]$ vector fields.*[4] *There exists a solution $\mathbf{y}\in G\Omega_{p^+}(\mathbb{R}^n)$ to the rough differential equation*

$$d\mathbf{y}_t=\mathcal{V}\big(\pi_1(\mathbf{y}_t)\big)\,d\mathbf{x}_t,\quad y_0^1=a,\tag{11}$$

i.e. there exists an extension of $\mathbf{x}$ to $\mathbf{z}\in G\Omega_{p^+}(\mathbb{R}^d\oplus\mathbb{R}^n)$ such that $\mathbf{z}$ projects onto

$$\mathbf{z}_t=(\mathbf{x}_t,\mathbf{y}_t),$$

[4] A function f which is $(k+\varepsilon)$-Lipschitz on $\mathbb{R}^d$, for $k\in\mathbb{N}$ is a k times differentiable function whose kth derivative is ε-Hölder, using the classical definition of Hölder functions with parameter in $[0,1)$. See [23,34].

and $\mathbf{z}$ *satisfies* $\mathbf{z} = \int h(\pi_1(\mathbf{z}))\,d\mathbf{z}$*, with*

$$h:\mathbb{R}^d \oplus \mathbb{R}^n \to Hom(\mathbb{R}^d \oplus \mathbb{R}^n, \mathbb{R}^d \oplus \mathbb{R}^n),$$
$$(x,y) \to \big((dX,dY) \to (dX, f(y)dY)\big).$$

If the p*-variation of* $\mathbf{x}$ *is controlled by* ω*, then the* p*-variation of* $\mathbf{z}$ (*and hence* $\mathbf{y}$) *is controlled by* $C\omega$*, where* C *depends on* p, ε*, the* $(p+\varepsilon)$*-Lipschitz norm of* $\mathcal{V}$ *and the supremum of* ω *on the consider interval. Moreover, for all* $s<t$ *such that* $\omega(s,t)<1$,

$$\|\mathbf{z}_{s,t}\|^p \leqslant K_{p,\varepsilon,f}\,\omega(s,t),$$

where $K_{p,\varepsilon,\mathcal{V}}$ *is a constant which only depends on* p*,*ε*, and the* $(p+\varepsilon)$*-Lipschitz norm of* $\mathcal{V}$.

If $\mathbf{x}$ *and* $\tilde{\mathbf{x}}$ *are two elements of* $G\Omega_{p^+}(\mathbb{R}^d)$ *such that* $\|\mathbf{x}_{s,t}^{-1} \otimes \tilde{\mathbf{x}}_{s,t}\|^p \leqslant \varepsilon\omega(s,t)$*, then the corresponding solution of Eq.* (11) $\mathbf{z}$ *and* $\tilde{\mathbf{z}}$ *satisfy* $\|\mathbf{z}_{s,t}^{-1} \otimes \tilde{\mathbf{z}}_{s,t}\|^p \leqslant \eta(\varepsilon)\omega(s,t)$ *where* η *is a continuous function such that* $\eta(0)=0$ (*i.e. the map* $\mathbf{x} \to \mathbf{z}$ *is continuous, and hence the Itô map* $\mathbf{x} \to \mathbf{y}$ *is continuous*).

Proof. A simple translation of the first and second level estimates in [23] or [24] to our norm $|||\cdot|||$ or $\|\cdot\|$. □

A simple corollary of it, observed in [12], is the continuity of the Itô map in "Hölder type" norm. The same simple argument gives the continuity of the Itô map in "modulus topologies". First, we let

$$\Xi_p = \big\{\varphi:[0,1] \to \mathbb{R}^+, \text{ with } \varphi(0)=0 \text{ and } \varphi^p \text{ is strictly increasing and convex}\big\}.$$

Such set is obviously not empty, $t \to t^{1/p}$ being one example of an element of Ξ_p. Let us look at some more complicated one.

Example 8. Let $\alpha>0$. Then, the function $x \to (x(-\ln x)^a)^{p/2}$ is strictly increasing and convex in a neighborhood of 0 (it can be checked by differentiating it twice). Let $\chi_{a,p} = \inf_{x>0} \frac{d^2(x(-\ln x)^a)^{p/2}}{dx^2} < 0$. Then define

$$\phi_{a,p}(x) = \begin{cases} \sqrt{x(-\ln x)^a} & \text{if } x \in [0, \chi_{a,p}], \\ \big(\phi_{a,p}^p(\chi_{a,p}) + (\phi_{a,p}^p)'(\chi_{a,p})(x-\chi_{a,p})\big)^{1/p} & \text{if } x \in [\chi_{a,p}, 1]. \end{cases}$$

In other words, $\phi_{a,p}^p(x)$ is the smallest convex function dominating $(x(-\ln x)^a)^{p/2}$. Remark that for $a>1$,

$$\lim_{x\to 0} \frac{\phi_{1,p}(x)}{\phi_{a,p}(x)} = 0.$$

For a function $\varphi \in \Xi_p$, we define a distance between two $G(\mathbb{R}^d)$-valued paths $\mathbf{x}$ and $\tilde{\mathbf{x}}$

$$d_\varphi(\mathbf{x},\tilde{\mathbf{x}}) = \sup_{0\leqslant s<t\leqslant 1} \frac{\|\mathbf{x}_{s,t}^{-1} \otimes \tilde{\mathbf{x}}_{s,t}\|}{\varphi(t-s)}.$$

For a single $G(\mathbb{R}^d)$-valued path $\mathbf{x}$, we let

$$\|\mathbf{x}\|_\varphi = \sup_{0\leqslant s<t\leqslant 1} \frac{\|\mathbf{x}_{s,t}\|}{\varphi(t-s)}.$$

We also let $d_\infty(\mathbf{x},\tilde{\mathbf{x}}) = \sup_{0\leqslant s<t\leqslant 1} \|\mathbf{x}_{s,t}^{-1} \otimes \tilde{\mathbf{x}}_{s,t}\|$ and $\|\mathbf{x}\|_\infty = \sup_{0\leqslant s<t\leqslant 1} \|\mathbf{x}_{s,t}\|$.

It is straightforward to check that d_φ and d_∞ are distances on the space $C_0(G(\mathbb{R}^d))$.

Corollary 9. *Let $\mathbf{x},\tilde{\mathbf{x}} \in G\Omega_{p^+}(\mathbb{R}^d)$, $\varepsilon > 0$ and $\mathcal{V}$ be a linear map from $\mathbb{R}^d$ into the Lip$[p+\varepsilon,\mathbb{R}^n]$ vector fields, and $\mathbf{z},\tilde{\mathbf{z}} \in G\Omega_{p^+}(\mathbb{R}^d \oplus \mathbb{R}^n)$ the corresponding solution of Eq.* (11)*. There exists a continuous function δ satisfying $\delta(0)=0$, such that*

$$d_\varphi(\mathbf{x},\tilde{\mathbf{x}}) \leqslant \varepsilon \Rightarrow d_\varphi(\mathbf{z},\tilde{\mathbf{z}}) \leqslant \delta(\varepsilon).$$

Remark 10. The requirement that $\varphi \in \Xi_p$ to define d_φ is only for convenience (so that $(s,t) \to \varphi^p(t-s)$ is a control). Indeed, if $\tilde{\varphi}$ is another increasing function such that $\tilde{\varphi}$ is equivalent to φ at 0, then the topologies on $G(\mathbb{R}^d)$-valued paths space induced by d_φ and $d_{\tilde{\varphi}}$ are identical.

2.5. The translation operator on rough path space

We define the translation operator, first introduced in a more general situation in [23]. Let q be real such that $1/q+1/p>1$.

The following definition is motivated by replacing x by $x+f$ in (10).

Definition 11. Let $\mathbf{x} \in G\Omega_{p^+}(\mathbb{R}^d)$ and f be a $\mathbb{R}^d$-valued path of finite q-variation. We let $x_t = \pi_1(\mathbf{x}_t)$. Then we define $T_f(\mathbf{x})$ by

$$\pi_1\big(T_f(\mathbf{x})\big)_{s,t} = f_{s,t} + x_{s,t}$$

and

$$\pi_2\big(T_f(\mathbf{x})_{s,t}\big) = \pi_2(\mathbf{x}_{s,t}) + \frac{1}{2}\pi_2\left(\int_s^t f_{s,u}\otimes df_u\right) + \frac{1}{2}\int_s^t f_{s,u}\otimes dx_u + \frac{1}{2}\left(\int_s^t x_{s,u}\otimes df_u - \int_s^t df_u\otimes x_{s,u} - \int_s^t dx_u \otimes f_{s,u}\right),$$

where the integrals are well defined Young integrals.

Remark that it is easily checked that $T_f(\mathbf{x})_{s,t} = T_f(\mathbf{x})_{0,s}^{-1}\otimes T_f(\mathbf{x})_{0,t}$.

Theorem 12. *Let $\mathbf{x}$ be a $G(\mathbb{R}^d)$-valued path of finite p-variation controlled by $\varepsilon^p\omega(s,t)$, and f be a $\mathbb{R}^d$-valued path of finite q-variation controlled by $\varkappa(s,t)$. Assume moreover that $\varkappa(s,t)^{1/q} \leqslant C\omega(s,t)^{1/p}$. Then for all $s<t$ and $\varepsilon<1$,*

$$d\big(T_f(\mathbf{x})_{s,t}, S(f)_{s,t}\big) \leqslant C\sqrt{\varepsilon}\,\omega(s,t)^{1/p}.$$

Proof. For all $s<t$, we have

$$S(f)_{s,t}^{-1}\otimes T_f(\mathbf{x})_{s,t} = \exp\left(x_{s,t} - \frac{1}{2}[f_{s,t},x_{s,t}] + \pi_2(\mathbf{x}_{s,t})\right)\otimes\exp\left(\frac{1}{2}\int_s^t f_{s,u}\otimes dx_u + \frac{1}{2}\int_s^t x_{s,u}\otimes df_u\right) \otimes \exp\left(-\frac{1}{2}\int_s^t df_{s,u}\otimes x_u - \frac{1}{2}\int_s^t dx_{s,u}\otimes f_u\right).$$

Hence, by inequality (6) and Young inequality [42] (which says that for all $s<t$, $|\int_s^t f_{s,u}\otimes dx_u| \leqslant C\varepsilon\varkappa(s,t)^{1/q}\times\omega(s,t)^{1/p}$ and similar inequalities for the other Young integrals), we get that

$$\big\|S(f)_{s,t}^{-1}\otimes T_f(\mathbf{x})_{s,t}\big\| \leqslant \varepsilon\omega^{1/p}(s,t) + \sqrt{\varepsilon^2\omega^{2/p}(s,t) + C\varepsilon\varkappa(s,t)^{1/q}\omega(s,t)^{1/p}} \leqslant C\sqrt{\varepsilon}\,\omega(s,t)^{1/p}. \qquad \square$$

3. The enhanced Brownian motion

In what follows we will lift Brownian motion as $(\mathbb{R}^d,+)$-valued to a $(G(\mathbb{R}^d),\otimes)$-valued process. Early work by Gaveau [14] and in particular the presentation in [26] use related algebraic ideas. See also [30].

3.1. Two classical properties of the Brownian motion

Let $(C_0(\mathbb{R}^d),\mathcal{F},(\mathcal{F}_t)_t,\mathbb{P})$ be the Wiener space. The evaluation operator B is then under $\mathbb{P}$ a Brownian motion starting at 0.

Applying Garsia, Rodemich and Rumsey inequality, it is not too difficult to see [36,38] that for all $s<t$,

$$\|B_{s,t}\| \leqslant \frac{4}{\sqrt{\alpha}} \int_0^{t-s} \sqrt{\frac{\log(1+\frac{4F}{u^2})}{u}}\,du,$$

where F is a L^1-random variable and a constant $\alpha>0$, sufficiently small.

We denote by $W^{1,2}$ the Cameron–Martin space

$$\left\{h:[0,1]\to\mathbb{R}^d,\ h(t)=\int_0^t h'(t)\,dt \text{ with } h'\in L^2([0,1])\right\}.$$

Cameron–Martin theorem, e.g. [19,32], states that if $f\in W^{1,2}$ is $(\mathcal{F}_t)_t$-adapted, then the law of $(B_t)_{0\leqslant t\leqslant 1}$ (i.e. the Wiener probability $\mathbb{P}$) and the law of $(B_t+f(t))_{0\leqslant t\leqslant 1}$ (that we will denote $\mathbb{P}^f$) are equivalent.

We will now extend B to a $G(\mathbb{R}^d)$-valued path of finite p-variation, for all $p>2$ (and hence a geometric p-rough path), and show that our "enhanced" Brownian motion has a similar modulus of continuity and extend Cameron–Martin theorem to our enhanced Brownian motion.

3.2. Their extensions to the enhanced Brownian motion

3.2.1. Definition of the enhanced Brownian motion

The enhanced Brownian motion was first defined in [33]. See also [24].

For $n\in\mathbb{N}$ and x a $\mathbb{R}^d$-valued path, we denote by x^n the path which agrees with x at the points $\frac{k}{2^n}$, $k=0,\dots,2^n$, and which is linear in the intervals $[\frac{k}{2^n},\frac{k+1}{2^n}]$, $k=0,\dots,2^n-1$. As x^n has finite 1-variation, we can define $\mathbf{x}^n=S(x^n)$, that is the natural $G(\mathbb{R}^d)$-valued path lying above x^n.

We denote by Γ the almost surely defined map

$$B\to\big(t\to\lim_{n\to\infty}S(B^n)_t\big),$$

and we let $(\mathbf{B}_t)_{0\leqslant t\leqslant 1}$ be the $G(\mathbb{R}^d)$-valued path $\Gamma(B)$. Note that [24]

$$\mathbf{B}_t =_{\text{a.s.}} \exp\left(B_t+\frac{1}{2}\int_s^t (B_u\otimes\circ dB_u-\circ dB_u\otimes B_u)\right),$$

where we have used Stratonovich integration. We will call $(\mathbf{B}_t)_{t\geqslant 0}$ the enhanced Brownian motion.

Remark 13. Almost surely, $\Gamma\circ\pi_1(\mathbf{B})=\mathbf{B}$ and $\pi_1\circ\Gamma(B)=B$.

Remark 14. The inverse of exp, denoted by log, provides a global chart for $G(\mathbb{R}^d)$. Gaveau's diffusion, [26], is exactly our EBM seen through this chart.

3.2.2. Modulus of continuity for the enhanced Brownian motion

We now cite Garsia, Rodemich and Rumsey inequality, but with the function f below being $G(\mathbb{R}^d)$-valued (while usually, f takes values in a normed vector space). Using the properties $\|\cdot\|$ on $G(\mathbb{R}^d)$, the proofs of the following theorem is the same proof than the classical Garsia, Rodemich and Rumsey theorem.

Theorem 15. *Let Ψ and p be continuous strictly increasing functions on $[0,\infty)$ with $p(0)=\Psi(0)=0$ and $\Psi(x)\to\infty$ as $x\to\infty$. Given $f\in C_0([0,1],G(\mathbb{R}^d))$, if*

$$\int_0^1\int_0^1 \Psi\left(\frac{f(s)^{-1}\otimes f(t)}{p(|t-s|)}\right)ds\,dt \leqslant F, \tag{12}$$

then for $0\leqslant s<t\leqslant 1$,

$$\left\|f(s)^{-1}\otimes f(t)\right\| \leqslant 8\int_0^{t-s}\Psi^{-1}\left(\frac{4F}{u^2}\right)dp(u).$$

Applying the Garsia, Rodemich and Rumsey inequality, we obtain a modulus of continuity for the enhanced Brownian motion

Theorem 16. *Define*

$$\zeta(x)=\frac{1}{2\sqrt{2}}\int_0^x\sqrt{\frac{\log(1+1/u^2)}{u}}\,du.$$

Then there exists a random variable $Z\geqslant 1$ $a.s.$ and in L^1, and a constant $C>0$ such that for all $s<t$,

$$\|\mathbf{B}_{s,t}\|\leqslant CZ^{1/4}\zeta\left(\frac{t-s}{\sqrt{Z}}\right). \tag{13}$$

Proof. As in the proof of Lévy's modulus of continuity in [36,38] we use the Garsia, Rodemich and Rumsey inequality with $f(t)=\mathbf{B}_t$, $p(x)=\sqrt{x}$ and $\Psi(x)=\exp(\alpha x^2)-1$. We obtain

$$\|\mathbf{B}_{s,t}\|\leqslant C\int_0^{t-s}\sqrt{\frac{\log(1+\frac{4F}{u^2})}{u}}\,du,$$

where F is the (now random) left-hand side of (12). Since $\|\mathbf{B}_{s,t}\| \overset{\text{law}}{=} \sqrt{t-s}\|\mathbf{B}_{0,1}\|$ the expectation of F is estimated by

$$\mathbb{E}\big(\exp\big(\alpha\|\mathbf{B}_{0,1}\|^2\big)\big). \tag{14}$$

We claim that this last expression is finite for a small enough $\alpha>0$. Remark that $\pi_1(\mathbf{B})$, resp. $\pi_2(\mathbf{B})$ are some elements of the first (resp. second) Wiener–Itô Chaos. By general integrability properties of the Wiener–Itô chaos [32, p. 207], there exists $\tilde\alpha>0$

$$\begin{aligned}\mathbb{E}\big(\exp\big(\tilde\alpha\big|\pi_1(\mathbf{B}_{0,1})\big|^2_{\mathbb{R}^d}\big)\big)&<\infty,\\ \mathbb{E}\big(\exp\big(\tilde\alpha\big|\pi_2(\mathbf{B}_{0,1})\big|_{\mathbb{R}^d\otimes\mathbb{R}^d}\big)\big)&<\infty.\end{aligned}$$

By inequality (6), the finiteness of the expectation of $\exp(\alpha\|\mathbf{B}_{0,1}\|^2)$ is easily obtain.[5] It remains just to define $Z=\max\{4F,1\}$ (we want $Z\geqslant 1$ for technical convenience later on), and do a change of variable to obtain inequality (13). □

Remark 17. As $\zeta(x)\sim_{x\to 0} C\sqrt{-x\ln x}$,

$$\varlimsup_{\delta\to 0}\sup_{\substack{0\leqslant s<t\leqslant 1\\ |t-s|\leqslant\delta}}\frac{\|\mathbf{B}_{s,t}\|}{\sqrt{-\delta\ln\delta}}\leqslant C. \tag{15}$$

On the other hand we can trivially get a deterministic lower bound by noting $|B_{s,t}|_{\mathbb{R}^d}\leqslant\|\mathbf{B}_{s,t}\|$ and using Lévy's result. All this is known (with an equality) for hypoelliptic diffusions on Nilpotent group in [2] and elliptic diffusions in [3].

Lemma 18. *There exists a constant C such that for all $x, y\in[0,1]$,*

$$\zeta(xy)\leqslant C\zeta(x)\zeta(y).$$

Proof. For a small enough, there exists constants K_1 and K_2 such that if $x\in(0,a]$

$$K_1\sqrt{-x\ln x}\leqslant\zeta(x)\leqslant K_2\sqrt{-x\ln x}.$$

Hence, from the inequality $\forall x,y\in(0,a]$, $-\ln(xy)\leqslant\frac{-2\ln a}{\ln^2 a}\ln(x)\ln(y)$, we obtain that for all $x,y\in[0,a]$, $\zeta(xy)\leqslant C\zeta(x)\zeta(y)$ for a constant C.

For a fixed b, it is easily seen that $0<\inf_{x\in(0,1/b)}\frac{\zeta(xb)}{\zeta(x)}<\sup_{x\in(0,1/b)}\frac{\zeta(xb)}{\zeta(x)}<\infty$. Hence, if $x,y\in[0,1]$,

$$\begin{aligned}\zeta(xy)&=\zeta\left(\frac{axay}{a^2}\right)\leqslant\sup_{z\in(0,a^2)}\frac{\zeta(z/a^2)}{\zeta(z)}\zeta(axay)\\&\leqslant C\sup_{z\in(0,a^2)}\frac{\zeta(z/a^2)}{\zeta(z)}\zeta(ay)\zeta(ay)\\&\leqslant C\frac{\sup_{z\in(0,a^2)}\frac{\zeta(z/a^2)}{\zeta(z)}}{(\inf_{z\in(0,1/a)}\frac{\zeta(za)}{\zeta(z)})^2}\zeta(x)\zeta(y).\end{aligned}$$ □

Proposition 19.

$$\|\mathbf{B}_{s,t}\|\leqslant M\zeta(t-s), \tag{16}$$

where M is a random variable for which there exists a constant $\lambda>0$ such that $E(\exp(\lambda M^2))<\infty$.

Proof. From the previous lemma, we can set

$$M=CZ^{1/4}\zeta\left(\frac{1}{\sqrt{Z}}\right)=2C\int_0^1\sqrt{\ln\left(1+\frac{Z}{v^4}\right)}\,dv.$$

Hence, by Jensen inequality,

[5] One could prove this directly as we know the density of $\mathbf{B}_{0,1}$ [22].

$$E\left(\exp(\lambda M^2)\right) \leqslant E\left(\int_0^1 \exp\left(4\lambda C^2 \ln\left(1+\frac{Z}{v^4}\right)\right) dv\right) \leqslant E\left(\int_0^1 \left(1+\frac{Z}{v^4}\right)^{4\lambda C^2} dv\right)$$
$$\leqslant E\left(\int_0^1 \left(\frac{2Z}{v^4}\right)^{4\lambda C^2} dv\right) < \infty$$

for λ small enough. □

The last estimate looks like a control but ζ^p is not convex on the entire interval $[0,1]$. We define ϕ_p to be pth-root of the smallest convex function dominating $x \to \zeta(x)^p$ (remark that ϕ_p is very similar to $\phi_{1,p}$ of Example 8, as $\frac{\zeta(x)}{\sqrt{-x \ln x}} \to_{x\to 0} 1$).

Corollary 20. *The p-variation of* $\mathbf{B}$ *is controlled by* $(s,t) \to C^p Z^{p/4} \phi_p^p(\frac{t-s}{\sqrt{Z}})$ *and also by* $(s,t) \to M^p \phi_p^p(t-s)$.

3.2.3. A Cameron–Martin theorem on the group

This section, despite being short and quite trivial, will be crucial in the proof of the support theorem. For $f \in W^{1,2}$, the translation of $\mathbf{B}$ by f is $T_f(\mathbf{B})$, and it is well defined as f has finite 1-variation and $\mathbf{B}$ has almost surely finite p-variation, $2 < p < 3$.

Theorem 21. *Let $f \in W^{1,2}$ be an $(\mathcal{F}_t)$-adapted path. Then the law of* $\mathbf{B}$ *is equivalent to the law of* $T_f(\mathbf{B})$.

Proof. By the Definition 11 and properties of Young and Stratonovich integral, $T_f(\mathbf{B}) = \Gamma(f + B)$. Hence, the law of $\mathbf{B}$ is $\mathbb{P} \circ \Gamma^{-1}$ while the law of $T_f(\mathbf{B})$ is $\mathbb{P}^f \circ \Gamma^{-1}$. Hence, by the Cameron–Martin theorem, these two laws are equivalent. □

4. Modulus of continuity for solution of SDEs

Theorem 22. *Let y_t be the solution of the Stratonovich differential equation*

$$dy_t = f_0(y_t)\,dt + f(y_t) \circ dB_t,$$

where f_0 is $(1+\varepsilon)$-Lipschitz and f is $(2+\varepsilon)$-Lipschitz. For $h \in W^{1,2}$, we denote by $F(h)$ the solution of the ordinary differential equation[6]

$$dF(h)_t = f_0\left(F(h)_t\right) dt + f\left(F(h)_t\right) dh_t.$$

We also denote by $\mathbf{F}$ *the extension of the Itô map F to the set of $G(\mathbb{R}^d)$-valued path of finite p-variation. There exists a constant C such that*

$$\overline{\lim_{h\to 0}} \sup_{\substack{0\leqslant s<t\leqslant 1\\ |t-s|\leqslant \delta}} \frac{\|\mathbf{F}(\mathbf{B})_{s,t}\|}{\sqrt{-\delta \ln \delta}} \leqslant C \quad a.s.$$

In particular,

$$\overline{\lim_{h\to 0}} \sup_{\substack{0\leqslant s<t\leqslant 1\\ |t-s|\leqslant \delta}} \frac{|y_t - y_s|_{\mathbb{R}^d}}{\sqrt{-\delta \ln \delta}} \leqslant C \quad a.s.$$

[6] For $h \in W^{1,2}$ but not piecewise C^1 this ODE still makes sense as rough differential equation with driving signal of finite $p = 1$ variation, see [21].

Proof. Assume first that $f_0 = 0$. By Theorem 7 and Corollary 20, for $s < t$ such that $C^p Z^{p/4} \phi_p^p (\frac{t-s}{\sqrt{Z}}) < 1$,

$$\left\|\mathbf{F}(\mathbf{B})_{s,t}\right\| \leqslant K_{\varepsilon,V} C Z^{1/4} \phi_p\left(\frac{t-s}{\sqrt{Z}}\right),$$

and we obtain our theorem by remarking once again that $\phi_p(x) \sim_{x\to 0} \sqrt{-x\ln(x)}$. The second inequality is obvious from the fact that $\pi_1(\mathbf{F}(\mathbf{B})_{s,t}) = y_t - y_s$.

For the general case, one just need to consider $(t, \mathbf{B}_t)$ as a path of mixed $(1, p)$-variation and use the continuity of $\mathbf{F}$, as proved in [22]. $\square$

There is a good hope that thanks to forthcoming paper of A.M. Davie, one will be able to consider the optimal case where f_0 is 1-Lipschitz and f is 2-Lipschitz. One should also be able to extend the rough path theory so that these functions could depend continuously on time. This is out of the scope of this paper.

5. On the support theorem

We are going to show a support theorem for the enhanced Brownian motion. Using the continuity of the Itô map, we will recover the classical support theorem (and even more). First, we need to look carefully at the convergence of piecewise linear approximation of our Brownian motion to the enhanced Brownian motion, in various topologies.

5.1. Convergence of some smooth paths to the enhanced Brownian motion

Proposition 23. *Let $\varphi, \phi_p \in \Xi_p$, such that $\lim_{x\to 0} \frac{\phi_p(x)}{\varphi(x)} = 0$. Let $\mathbf{x}, \mathbf{y}$ be two $G(\mathbb{R}^d)$-valued paths. Then, for all A big enough, with A in any case greater than $d_{\phi_p}(\mathbf{x}, \mathbf{y})$,*

$$d_\varphi(\mathbf{x}, \mathbf{y}) \leqslant A\left(\frac{\varphi^{-1}}{\phi_p^{-1}}\left(\frac{d_\infty(\mathbf{x}, \mathbf{y})}{A}\right)\right)^{1/p}.$$

Remark that the functions $\varphi = \phi_{a,p}$, $a > 1$ of example 8 satisfy the condition $\lim_{x\to 0} \frac{\phi_p(x)}{\varphi(x)} = 0$.

Proof. First note that as $\varphi \in \Xi_p$, $\lim_{x\to\infty} \varphi(x) = +\infty$, hence φ^{-1} is an increasing concave function from $\mathbb{R}^+$ onto $\mathbb{R}^+$. Idem for ϕ_p. With our assumption, one can see $\frac{\varphi^{-1}}{\phi_p^{-1}}$ is increasing and bounded by 1 on a neighborhood $(0, \tau)$ of 0. Let us choose a $A \geqslant C_{\varphi,\phi_p} d_{\phi_p}(\mathbf{x}, \mathbf{y})$, where C_{φ,ϕ_p} is a constant greater than 1 such that $\frac{d_\infty(\mathbf{x},\mathbf{y})}{A} \leqslant \tau$. Then, for all $s < t$,

$$\begin{aligned}
\varphi^{-1}\left(\frac{\|\mathbf{x}_{s,t}^{-1} \otimes \mathbf{y}_{s,t}\|}{A}\right) &= \frac{\varphi^{-1}}{\phi_p^{-1}}\left(\frac{\|\mathbf{x}_{s,t}^{-1} \otimes \mathbf{y}_{s,t}\|}{A}\right)\phi_p^{-1}\left(\frac{\|\mathbf{x}_{s,t}^{-1} \otimes \mathbf{y}_{s,t}\|}{A}\right)\\
&\leqslant \left(\frac{\varphi^{-1}}{\phi_p^{-1}}\left(\frac{d_\infty(\mathbf{x}, \mathbf{y})}{A}\right)\right)\phi_p^{-1}\left(\frac{d_{\phi_p}(\mathbf{x}, \mathbf{y})}{A}\phi_p(t-s)\right)\\
&\leqslant \left(\frac{\varphi^{-1}}{\phi_p^{-1}}\left(\frac{d_\infty(\mathbf{x}, \mathbf{y})}{A}\right)\right)(t-s).
\end{aligned}$$

So far, we have used the fact that $\frac{\varphi^{-1}}{\phi_p^{-1}}$ and ϕ_p^{-1} are increasing, and that $\frac{d_{\phi_p}(\mathbf{x},\mathbf{y})}{A} \leqslant 1$. For all $s < t$,

$$\frac{\|x_{s,t}^{-1} \otimes y_{s,t}\|}{A} \leqslant \varphi\left(\left(\frac{\varphi^{-1}}{\phi_p^{-1}}\left(\frac{d_\infty(\mathbf{x}, \mathbf{y})}{A}\right)\right)(t-s)\right) \leqslant \left(\frac{\varphi^{-1}}{\phi_p^{-1}}\left(\frac{d_\infty(\mathbf{x}, \mathbf{y})}{A}\right)\right)^{1/p}\varphi(t-s),$$

as φ^p is convex and $\frac{\varphi^{-1}}{\phi_p^{-1}}(\frac{d_\infty(\mathbf{x},\mathbf{y})}{A}) \leqslant 1$. Hence,

$$d_\varphi(\mathbf{x},\mathbf{y}) \leqslant A\left(\frac{\varphi^{-1}}{\phi_p^{-1}}\left(\frac{d_\infty(\mathbf{x},\mathbf{y})}{A}\right)\right)^{1/p}. \qquad \square$$

Corollary 24. *Let $\varphi \in \Xi_p$, such that $\lim_{x\to 0} \frac{\phi_p(x)}{\varphi(x)} = \lim_{x\to 0} \frac{\sqrt{-x\ln x}}{\varphi(x)} = 0$. Let $\mathbf{x}_n$ be a sequence of $G(\mathbb{R}^d)$-valued paths, which converges pointwise to another $G(\mathbb{R}^d)$-valued path $\mathbf{x}$. Assume that $\sup_n \|\mathbf{x}_n\|_{\phi_p} < \infty$. Then $\mathbf{x}_n$ converges to $\mathbf{x}$ in the topology induced by d_φ.*

Proof. First notice that by Arzela–Ascoli theorem,

$$\widetilde{d_\infty}(\mathbf{x}_n,\mathbf{x}) := \sup_{t\in[0,1]} \|\mathbf{x}_{n,t}^{-1}\otimes \mathbf{x}_t\| \to_{n\to\infty} 0.$$

But inequality (7) gives

$$\widetilde{d_\infty}(\mathbf{x}_n,\mathbf{x}) \leqslant d_\infty(\mathbf{x}_n,\mathbf{x}) \leqslant C\left(\widetilde{d_\infty}(\mathbf{x}_n,\mathbf{x}) + \sqrt{\widetilde{d_\infty}(\mathbf{x}_n,\mathbf{x})\|\mathbf{x}\|_\infty}\right), \tag{17}$$

hence $d_\infty(\mathbf{x}_n,\mathbf{x}) \to_{n\to\infty} 0$. Using Proposition 23 and its notations, we see that $d_\varphi(\mathbf{x}_n,\mathbf{x})$ is bounded by

$$2C_{\varphi,\phi_p}\sup_n \|\mathbf{x}_n\|_{\phi_p}\left(\frac{\varphi^{-1}}{\phi_p^{-1}}\left(\frac{d_\infty(\mathbf{x}_n,\mathbf{x})}{2C_{\varphi,\phi_p}\sup_n\|\mathbf{x}_n\|_{\phi_p}}\right)\right)^{1/p}$$

which goes to 0 when n tends to infinity. $\square$

This corollary is going to allow us to prove that various approximations of enhanced Brownian motion converge in the topology induced by d_φ, where $\varphi \in \Xi_p$ is such that $\phi_p(x) = \mathrm{o}(\varphi(x))$ as $x \to 0$. To obtain an accurate uniform control of $\|\mathbf{B}^n\|_{\phi_p}$, we first need the following result, in the spirit of Doob's martingale inequality.

Lemma 25. *Let X a random variable such that $E(\exp\lambda X^2) < \infty$, and $\mathcal{G}_n$ a sequence of σ-algebras. Define $X_n = E(X/\mathcal{G}_n)$. Then*

$$E\left(\sup_n \exp\lambda X_n^2\right) < \infty.$$

Proof. Using Fubini and Doob's L^p inequality, we obtain

$$\begin{aligned}
E\left(\sup_n \exp\lambda X_n^2\right) &= 1 + \sum_{k=1}^\infty \frac{\lambda^k}{k!} E\left(\sup_n X_n^{2k}\right)\\
&\leqslant 1 + \sum_{k=1}^\infty \frac{\lambda^k}{k!}\left(\frac{2k}{2k-1}\right)^{2k} E(X^{2k})\\
&\leqslant 2eE(\exp\lambda X^2). \qquad \square
\end{aligned}$$

Proposition 26. *$t \to \mathbf{B}_t$ and $t \to \mathbf{B}_t^n$ are almost surely of finite p-variation uniformly controlled by $(s,t) \to K^p\phi_p^p(t-s)$, where K is a random variable such that for λ small enough $E(\exp(\lambda K^2)) < \infty$.*

Proof. $\mathbf{B}$ is controlled in p-variation by $C^pM^p\phi_p^p(t-s)$, where, for λ small enough $E(\exp(\lambda M^2)) < \infty$. By inequality (6), we see that for all $s < t$

$$|||\mathbf{B}_{s,t}||| \leqslant \frac{M}{c_1}\phi_p(t-s).$$

Now define $\mathcal{G}_n$ the σ-algebra generated by the random variables $B_{\frac{k}{2^n}}$. Then, $\log \mathbf{B}^n_{s,t} = \mathbb{E}(\log \mathbf{B}_{s,t}/\mathcal{G}_n)$ [26]. Hence,

$$\begin{aligned}
|||\mathbf{B}^n_{s,t}||| &= |||\exp \mathbb{E}(\log \mathbf{B}_{s,t}/\mathcal{G}_n)||| \\
&= \left|\mathbb{E}\big(\pi_1(\mathbf{B}_{s,t})/\mathcal{G}_n\big)\right|_{\mathbb{R}^d} + \sqrt{\left|\mathbb{E}\big(\pi_2(\mathbf{B}_{s,t})/\mathcal{G}_n\big)\right|_{\mathbb{R}^d\otimes\mathbb{R}^d}} \\
&\leqslant \sqrt{\mathbb{E}\big(\left|\pi_1(\mathbf{B}_{s,t})\right|^2_{\mathbb{R}^d}/\mathcal{G}_n\big)} + \sqrt{\mathbb{E}\big(\left|\pi_2(\mathbf{B}_{s,t})\right|_{\mathbb{R}^d\otimes\mathbb{R}^d}/\mathcal{G}_n\big)} \\
&\leqslant C\sqrt{\mathbb{E}\big(|||\mathbf{B}_{s,t}|||^2/\mathcal{G}_n\big)} \\
&\leqslant \frac{C}{c_1}\phi_p(t-s)\sqrt{\mathbb{E}(M^2/\mathcal{G}_n)}.
\end{aligned}$$

Define $\widetilde{K}^2 = \sup_n \mathbb{E}(M^2/\mathcal{G}_n)$. For all n and $s<t$, $|||\mathbf{B}^n_{s,t}||| \leqslant \frac{C\widetilde{K}}{c_1}\phi_p(t-s)$. By the previous lemma, for λ small enough $E(\exp(\lambda\widetilde{K}^2))<\infty$. A last use of (6) gives us the proposition. □

As a consequence of Proposition 26 and Corollary 24, we obtain the following corollaries.

Corollary 27. *Let $\varphi \in \Xi_p$, such that $\lim_{x\to 0}\frac{\sqrt{-x\ln(x)}}{\varphi(x)} = 0$. Then, $\mathbf{B}^n$ converges almost surely to $\mathbf{B}$ in the topology induced by d_φ.*

Corollary 28. *Let $\varphi \in \Xi_p$, such that $\lim_{x\to 0}\frac{\sqrt{-x\ln(x)}}{\varphi(x)} = 0$. We define $B^{(n)}$ to be the $(\mathcal{F}_t)$-adapted path such that*

$$B^{(n)}_t = B_{\frac{[2^n t]}{2^n}} + \left(t - \frac{[2^n t]}{2^n}\right)\left(B_{\frac{[2^n t]}{2^n}} - B_{\frac{[2^n t]-1}{2^n}\vee 0}\right).$$

Then, $\mathbf{B}^{(n)} = S(B^{(n)})$ converges almost surely to $\mathbf{B}$ in the topology induced by d_φ.

Proof. We have seen in Proposition 26 that, almost surely, $\sup_n \|\mathbf{B}^n\|_{\phi_p} < \infty$ a.s. Observe that $\mathbf{B}^{(n)}$ is essentially $\mathbf{B}^n$ shifted by $\varepsilon = 1/2^n$. More precisely,

$$\|\mathbf{B}^{(n)}_{s,t}\| = \|\mathbf{B}^n_{(s-\varepsilon)\wedge 0,(t-\varepsilon)\wedge 0}\| \leqslant \sup_n \|\mathbf{B}^n\|_{\phi_p}\phi_p(t-s).$$

By Corollary 24 it suffices to show that, almost surely, $\mathbf{B}^{(n)}_t \to \mathbf{B}_t$ for fixed t. But this simply follows from

$$\|\mathbf{B}^{-1}_t \otimes \mathbf{B}^{(n)}_t\| \leqslant \underbrace{\|\mathbf{B}^{-1}_t \otimes \mathbf{B}^n_t\|}_{\to 0} + \underbrace{\left\|(\mathbf{B}^n_t)^{-1}\otimes \mathbf{B}^n_{(t-\varepsilon)\wedge 0}\right\|}_{\leqslant C\phi_p(\varepsilon)\to 0 \text{ with } n\to\infty.}. \qquad \square$$

In particular, we recover the convergence in the $1/p$-Hölder distance (and hence in the p-variation topology) of $\mathbf{B}^n$ and $\mathbf{B}^{(n)}$ to $\mathbf{B}$.

5.2. Some more convergence results

Lemma 29. *Let $\varphi \in \Xi_p$, such that $\lim_{x\to 0}\frac{\sqrt{-x\ln(x)}}{\varphi(x)} = 0$ and $f \in W^{1,2}$ be an $(\mathcal{F}_t)$-adapted path. Then $\mathbb{P}$-almost surely, $T_{f-B^{(n)}}(\mathbf{B})$ converges in the topology induced by d_φ to $S(f)$.*

Proof. Note first that $f - B^{(n)} \in W^{1,2} \subset W^{1,1}$, which is precisely the set of path of bounded variation, hence $T_{f-B^{(n)}}(\mathbf{B})$ is well defined. Assume that we have shown that $T_{-B^{(n)}}(\mathbf{B})$ converges to $\exp(0)$ in the topology induced by d_φ, i.e. that there exists a sequence ε_n, which converges almost surely to $\exp(0)$, and such that $\varepsilon_n^p \varphi^p(t-s)$ controls the p-variation of $T_{-B^{(n)}}(\mathbf{B})$. Then, as

$$\int_s^t |f'_u|\,du \leqslant \sqrt{t-s}\sqrt{\int_0^1 |f'_u|^2\,du} \leqslant C\sqrt{\int_0^1 |f'_u|^2\,du\,\varphi(t-s)},$$

we obtain from theorem 12 and the equality $T_{f-B^{(n)}}(\mathbf{B}) = T_f[T_{-B^{(n)}}(\mathbf{B})]$ that $d_\varphi(T_{f-B^{(n)}}(\mathbf{B}), S(f)) \leqslant C\sqrt{\varepsilon_n}$. Hence, we can assume that $f=0$.

Note that

$$T_{-B^{(n)}}(\mathbf{B})_{s,t} = \exp\left(B_{s,u} - B^{(n)}_{s,t} + \int_s^t \left[B_{s,u} - B^{(n)}_{s,u}, \circ d\left(B_u - B^{(n)}_u\right)\right]\right).$$

Conditioning partially, i.e. only with respect to the values of the ith component of the Brownian motion at some fixed time, we obtain, similarly as in the previous section (and as in [12]), that, almost surely, $T_{-B^{(n)}}(\mathbf{B})$ converges pointwise to $\exp(0)$, and that for all n,

$$\left\|T_{-B^{(n)}}(\mathbf{B})_{s,t}\right\| \leqslant K'\phi_p(t-s),$$

where K' is a random variable such that for λ small enough, $E(\exp\lambda K'^2) < \infty$. The proof is then finished using Corollary 24. $\square$

5.3. Support theorem with refined norms

As observed in [12] we can combine the rough path approach of [25] with ideas from [28] and get an improved support theorem as a corollary.

Theorem 30. *The support of the law of* $\mathbf{B}$ *is the closure of* $S(W^{1,2})$ *in the topology induced by* d_φ*, where* $\varphi \in \Xi_p$ *is such that* $\lim_{x\to 0}\frac{\sqrt{-x\ln(x)}}{\varphi(x)} = 0$.

Proof. Corollary 28 implies classically that the support of the law of $\mathbf{B}$ is contained in the closure of $S(W^{1,2})$ in the d_φ topology. Reciprocally, our Cameron–Martin Theorem 21 implies it is enough to show that for a function $x \in W^{1,2}$, $T_{x-B^{(n)}}(\mathbf{B})$ converges in the d_φ topology to $S(x)$. But this was proven in Lemma 29. $\square$

Remark 31. As noted in [16], the previous theorem would not work in the topology induced by d_{ϕ_p}. Indeed, $\{\mathbf{x} \in C_0([0,1], G(\mathbb{R}^d)), \|\mathbf{x}\|_{\phi_p} < \infty\}$ is not separable (but the set of continuous $G(\mathbb{R}^d)$-valued path $\mathbf{x}$ such that $\|\mathbf{x}\|_{\phi_p} < \infty$ and such that $\lim_{\delta\to 0}\sup_{|t-s|\leqslant\delta}\frac{\|\mathbf{x}_{s,t}\|}{\varphi(\delta)} = 0$ is separable).

As in Section 4, we let y_t be the solution of the Stratonovich differential equation

$$dy_t = f_0(y_t)\,dt + f(y_t)\circ dB_t,$$

where f_0 is $(1+\varepsilon)$-Lipschitz and f is $(2+\varepsilon)$-Lipschitz. For $h \in W^{1,2}$, we denote by $F(h)$ the solution of the ordinary differential equation

$$dF(h)_t = f_0\big(F(h)_t\big)\,dt + f\big(F(h)_t\big)\,dh_t. \tag{18}$$

We also denote by $\mathbf{F}$ the extension of the Itô map F to the space of $G(\mathbb{R}^d)$-valued path of finite p-variation. From the continuity of $\mathbf{F}$ in the d_φ topology (Corollary 9), we instantaneously obtain the following:

Corollary 32. *The support of the law of $\Gamma(y)$ (the Stratonovich extension of y to a p-rough path) is the closure of $\mathbf{F}(S(W^{1,2}))$ in the topology induced by d_φ, where $\varphi \in \Xi_p$ is such that* $\lim_{x\to 0} \frac{\sqrt{-x\ln(x)}}{\varphi(x)} = 0$.

Projecting on the first level, we improve Stroock–Varadhan's result [37], its extension to Hölder norm [28,6, 7,35] as well as the p-variation result [25]. Our approach allows us to use more refined topologies than the one induced by Hölder distances. Let $d_{1,\varphi}$ be the distance defined by the following formula:

$$d_{1,\varphi}(x, y) = \sup_{0\leqslant s<t\leqslant 1} \frac{|y_{s,t} - x_{s,t}|_{\mathbb{R}^d}}{\varphi(t-s)}.$$

Corollary 33. *The support of the law of y is the closure of $F(W^{1,2})$ in the topology induced by $d_{1,\varphi}$, where $\varphi \in \Xi_p$ is such that* $\lim_{x\to 0} \frac{\sqrt{-x\ln(x)}}{\varphi(x)} = 0$.

6. Large deviations results

6.1. Some preliminary results

For $n \in \mathbb{N}$, we define the map $\Upsilon_n : C_0([0,1], G(\mathbb{R}^d)) \to C_0([0,1], G(\mathbb{R}^d))$ where $\Upsilon_n(\mathbf{x})$ is defined by:

(i) $\forall k \in \{0, \ldots, 2^n\}$, $\Upsilon_n(\mathbf{x})_{\frac{k}{2^n}} = \mathbf{x}_{\frac{k}{2^n}}$,
(ii) $\forall k \in \{0, \ldots, 2^n\}$ and $\forall t \in [0, 2^{-n}]$, $\Upsilon_n(\mathbf{x})_{\frac{k}{2^n}, \frac{k}{2^n}+t} = \delta_{t2^n}(\mathbf{x}_{\frac{k}{2^n}, \frac{k+1}{2^n}})$.

$\Upsilon_n(\mathbf{x})$ is a piecewise linear approximation of $\mathbf{x}$ to which we assign a non-canonical area!

Lemma 34. *For all $n \in \mathbb{N}$,*

$$\left\|\Upsilon_n(\mathbf{x})\right\|_{\phi_p} \leqslant C\|\mathbf{x}\|_{\phi_p}.$$

Proof. One can show, with similar techniques than in Lemma 18, that for all $\alpha, y \in (0, 1]$,

$$\phi_p(\alpha y) \geqslant C\sqrt{\alpha}\phi_p(y). \tag{19}$$

Whenever $0 \leqslant s \leqslant t \leqslant 2^{-n}$, $\Upsilon_n(\mathbf{x})_{\frac{k}{2^n}+s, \frac{k}{2^n}+t}$ is equal to

$$\exp\left(2^n(t-s)\pi_1(\mathbf{x}_{\frac{k}{2^n}, \frac{k+1}{2^n}}) + 2^{2n}(t^2 - s^2)\pi_2(\mathbf{x}_{\frac{k}{2^n}, \frac{k+1}{2^n}})\right).$$

Hence, by inequality (6),

$$\left\|\Upsilon_n(\mathbf{x})_{\frac{k}{2^n}+s, \frac{k}{2^n}+t}\right\| \leqslant C\|\mathbf{x}_{\frac{k}{2^n}, \frac{k+1}{2^n}}\|\left(2^n(t-s) + 2^{n/2}\sqrt{t-s}\right) \leqslant C\|\mathbf{x}_{\frac{k}{2^n}, \frac{k+1}{2^n}}\|2^{n/2}\sqrt{t-s}. \tag{20}$$

Hence, for $\frac{k}{2^n} \leqslant s \leqslant t \leqslant \frac{k+1}{2^n}$,

$$\left\|\Upsilon_n(\mathbf{x})_{s,t}\right\| \leqslant C\|\mathbf{x}\|_{\phi_p}\phi_p(t-s)\frac{\phi_p(2^{-n})}{\phi_p(t-s)}\sqrt{\frac{t-s}{2^{-n}}} \leqslant C\|\mathbf{x}\|_{\phi_p}\phi_p(t-s),$$

applying inequality (19) with $\alpha = \frac{t-s}{2^{-n}}$ and $y = 2^{-n}$.

For general $s \leqslant \frac{j}{2^n} \leqslant \frac{k}{2^n} \leqslant t$, as

$$\begin{aligned}\Upsilon_n(\mathbf{x})_{s,t} &= \Upsilon_n(\mathbf{x})_{s, \frac{j}{2^n}} \otimes \Upsilon_n(\mathbf{x})_{\frac{j}{2^n}, \frac{k}{2^n}} \otimes \Upsilon_n(\mathbf{x})_{\frac{k}{2^n}, t} \\ &= \Upsilon_n(\mathbf{x})_{s, \frac{j}{2^n}} \otimes \mathbf{x}_{\frac{j}{2^n}, \frac{k}{2^n}} \otimes \Upsilon_n(\mathbf{x})_{\frac{k}{2^n}, t},\end{aligned} \tag{21}$$

$$\left\|\Upsilon_n(\mathbf{x})_{s,t}\right\| \leqslant C\|\mathbf{x}\|_{\phi_p}\left(\phi_p\left(\frac{j}{2^n}-s\right)+\phi_p\left(\frac{k-j}{2^n}\right)+\phi_p\left(t-\frac{k}{2^n}\right)\right) \leqslant C\|\mathbf{x}\|_{\phi_p}\phi_p(t-s). \qquad \square$$

Lemma 35. *For all $n \in \mathbb{N}$,*

$$\frac{d_\infty(\mathbf{x}, \Upsilon_n(\mathbf{x}))}{\|\mathbf{x}\|_{\phi_p}} \leqslant C\sqrt{\phi_p(2^{-n})}.$$

Proof. For $s \leqslant \frac{j}{2^n} \leqslant \frac{k}{2^n} \leqslant t$, using Eq. (21), we see that the distance between $\mathbf{x}_{s,t}$ and $\Upsilon_n(\mathbf{x})_{s,t}$ is equal to

$$d\left(\Upsilon_n(\mathbf{x})_{s,\frac{j}{2^n}} \otimes \mathbf{x}_{\frac{j}{2^n},\frac{k}{2^n}} \otimes \Upsilon_n(\mathbf{x})_{\frac{k}{2^n},t}, \mathbf{x}_{s,\frac{j}{2^n}} \otimes \mathbf{x}_{\frac{j}{2^n},\frac{k}{2^n}} \otimes \mathbf{x}_{\frac{k}{2^n},t}\right).$$

We then use inequality (8):

$$d\left(\mathbf{x}_{s,t}, \Upsilon_n(\mathbf{x})_{s,t}\right) \leqslant d\left(\Upsilon_n(\mathbf{x})_{s,\frac{j}{2^n}}, \mathbf{x}_{s,\frac{j}{2^n}}\right) + d\left(\Upsilon_n(\mathbf{x})_{\frac{k}{2^n},t}, \mathbf{x}_{\frac{k}{2^n},t}\right) + \sqrt{d\left(\Upsilon_n(\mathbf{x})_{s,\frac{j}{2^n}}, \mathbf{x}_{s,\frac{j}{2^n}}\right)\|\mathbf{x}_{\frac{j}{2^n},t}\|}.$$

Then we simply bound $d(\Upsilon_n(\mathbf{x})_{s,\frac{j}{2^n}}, \mathbf{x}_{s,\frac{j}{2^n}})$ by $\|\Upsilon_n(\mathbf{x})_{s,\frac{j}{2^n}}\| + \|\mathbf{x}_{s,\frac{j}{2^n}}\| \leqslant C\|\mathbf{x}\|_{\phi_p}\phi_p(2^{-n})$, and similarly for $d(\Upsilon_n(\mathbf{x})_{\frac{k}{2^n},t}, \mathbf{x}_{\frac{k}{2^n},t})$. Hence,

$$\frac{d(\mathbf{x}_{s,t}, \Upsilon_n(\mathbf{x})_{s,t})}{\|\mathbf{x}\|_{\phi_p}} \leqslant C\phi_p(2^{-n}) + C\sqrt{\phi_p(2^{-n})\phi_p(1)}. \qquad \square$$

Corollary 36. *For all $n \in \mathbb{N}$,*

$$d_\varphi\left(\mathbf{x}, \Upsilon_n(\mathbf{x})\right) \leqslant C\|\mathbf{x}\|_{\phi_p}\left(\sup_{0\leqslant x\leqslant C'\sqrt{\phi_p(2^{-n})}} \frac{\varphi^{-1}}{\phi_p^{-1}}(x)\right)^{1/p}.$$

In particular, if $\|\mathbf{x}\|_{\phi_p} < \infty$, $d_\varphi(\mathbf{x}, \Upsilon_n(\mathbf{x}))$ converges to 0 when n tends to infinity.

Proof. We apply Proposition 23, with $A = C\|\mathbf{x}\|_{\phi_p} \geqslant d_{\phi_p}(\Upsilon_n(\mathbf{x}), \mathbf{x})$. It gives

$$d_\varphi\left(\mathbf{x}, \Upsilon_n(\mathbf{x})\right) \leqslant C\|\mathbf{x}\|_{\phi_p}\left(\frac{\varphi^{-1}}{\phi_p^{-1}}\left(\frac{d_\infty(\mathbf{x}, \Upsilon_n(\mathbf{x}))}{C\|\mathbf{x}\|_{\phi_p}}\right)\right)^{1/p}.$$

The result is then given by Lemma 35. $\square$

Remark 37. A feature of this approximation is that it does not rely on dyadic (or nested) approximations which are fundamental for our earlier martingale approach. Indeed, the 2^{-n} appearing on the right-hand side of above estimate is readily replaced by the mesh of any dissection upon which $\Upsilon_n(\mathbf{x})$ is constructed. Introducing the right area in our approximation improves its convergence properties.

6.2. Schilder and Freidlin–Wentzell theorem with refined norms

We extend Schilder theorem [8,9,4] to the enhanced Brownian motion in our refined topology. First, we need the following lemma

Lemma 38. *Let $\varphi \in \Xi_p$, such that $\lim_{x\to 0} \frac{\sqrt{x}}{\varphi(x)} = 0$. The maps*

$$\Upsilon_n : \left(C_0\left([0,1], G(\mathbb{R}^d)\right), d_\infty\right) \to \left(C_0\left([0,1], G(\mathbb{R}^d)\right), d_\varphi\right)$$

are continuous.

Proof. The map Υ_n from $(C_0([0,1], G(\mathbb{R}^d)), d_\infty)$ into itself is clearly continuous as easily seen using (8) as before. Let $\mathbf{x} \in C_0([0,1], G(\mathbb{R}^d))$ and $s \leqslant \frac{j}{2^n} \leqslant \frac{k}{2^n} \leqslant t$,

$$\begin{aligned}\|\Upsilon_n(\mathbf{x})_{s,t}\| &\leqslant \|\Upsilon_n(\mathbf{x})_{s,\frac{j}{2^n}}\| + \|\mathbf{x}_{\frac{j}{2^n},\frac{k}{2^n}}\| + \|\Upsilon_n(\mathbf{x})_{\frac{k}{2^n},t}\| \\ &\leqslant C\|\mathbf{x}_{\frac{j-1}{2^n},\frac{j}{2^n}}\|2^{n/2}\sqrt{\frac{j}{2^n}-s} + C\|\mathbf{x}_{\frac{k-1}{2^n},\frac{k}{2^n}}\|2^{n/2}\sqrt{t-\frac{k}{2^n}} + 1_{j<k}2^{n/2}\|\mathbf{x}_{\frac{j}{2^n},\frac{k}{2^n}}\|\sqrt{\frac{k}{2^n}-\frac{j}{2^n}} \\ &\leqslant C2^{n/2}\|\mathbf{x}\|_\infty\sqrt{t-s}.\end{aligned}$$

When $\frac{j}{2^n} \leqslant s \leqslant t \leqslant \frac{j+1}{2^n}$, we also have

$$\|\Upsilon_n(\mathbf{x})_{s,t}\| \leqslant C\|\mathbf{x}_{\frac{k}{2^n},\frac{k+1}{2^n}}\|2^{n/2}\sqrt{t-s},$$

as already notice in Eq. (20). Hence,

$$\sup_{s<t}\frac{\|\Upsilon_n(\mathbf{x})_{s,t}\|}{\sqrt{t-s}} \leqslant C_n\|\mathbf{x}\|_\infty.$$

The proof is then finished applying a slight modification (replacing ϕ_p by $\sqrt{\cdot}$) of Proposition 23. □

Theorem 39. *Let $\varphi \in \Xi_p$, such that $\lim_{x\to 0}\frac{\sqrt{-x\ln(x)}}{\varphi(x)} = 0$. The random variables $\delta_\varepsilon\mathbf{B}$ satisfies a large deviation principle in the topology induced by d_φ with good rate function*

$$I(\mathbf{x}) = \begin{cases}\frac{1}{2}\int_0^1 |x'_u|^2\,du, & \text{if } S(x) = \mathbf{x} \text{ for some } x \in W^{1,2}, \\ +\infty & \text{otherwise.}\end{cases}$$

Proof. The large deviation result in [25] tells us that $\delta_\varepsilon\mathbf{B}$ satisfies a large deviation principle with good rate function I using the topology induced by d_∞. By Lemma 38, if we prove that $\Upsilon_n(\delta_\varepsilon\mathbf{B})$ is an exponentially good approximation of $\delta_\varepsilon\mathbf{B}$, i.e.

$$\lim_{n\to\infty}\overline{\lim_{\varepsilon\to 0}}\,\varepsilon^2\log\mathbb{P}\big(d_\varphi\big(\Upsilon_n(\delta_\varepsilon\mathbf{B}), \delta_\varepsilon\mathbf{B}\big) > \delta\big) = -\infty \tag{22}$$

and that for all α,

$$\lim_{n\to\infty}\sup_{\mathbf{x}, I(\mathbf{x})\leqslant\alpha} d_\varphi\big(\Upsilon_n(\mathbf{x}), \mathbf{x}\big) = 0, \tag{23}$$

we will have shown our theorem, by applying Theorem 4.2.23 in [9].

Let us first prove Eq. (23). First observe that if $I(\mathbf{x}) \leqslant \alpha$, letting $x \in W^{1,2}$ be such that $S(x) = \mathbf{x}$, we have, by definition of our homogeneous norm and by Cauchy–Schwartz, $\|\mathbf{x}_{s,t}\| \leqslant \int_s^t |x'_u|\,du \leqslant 2\sqrt{t-s}I(\mathbf{x})$. Hence, $\|\mathbf{x}\|_{\phi_p} \leqslant 2\sup_{0\leqslant s<t\leqslant 1}\frac{\sqrt{t-s}}{\phi_p(t-s)}\sqrt{\alpha} = C\sqrt{\alpha}$ and hence by Corollary 36,

$$\sup_{\mathbf{x}, I(\mathbf{x})\leqslant\alpha} d_\varphi\big(\Upsilon_n(\mathbf{x}), \mathbf{x}\big) \leqslant C\sqrt{\alpha}\left(\sup_{0\leqslant x\leqslant C'\sqrt{\phi_p(2^{-n})}}\frac{\varphi^{-1}}{\phi_p^{-1}}(x)\right)^{1/p} \to_{n\to\infty} 0.$$

To obtain inequality (22), we just need to apply Corollary 36 to the enhanced Brownian motion. Indeed, letting $\alpha_n = C(\sup_{0\leqslant x\leqslant C'\sqrt{\phi_p(2^{-n})}}\frac{\varphi^{-1}}{\phi_p^{-1}}(x))^{1/p}$, we see that

$$\mathbb{P}\big(d_\varphi\big(\delta_\varepsilon\Upsilon_n(\mathbf{B}), \delta_\varepsilon\mathbf{B}\big) > \delta\big) = \mathbb{P}\left(d_\varphi\big(\Upsilon_n(\mathbf{B}), \mathbf{B}\big) > \frac{\delta}{\varepsilon}\right) \leqslant \mathbb{P}\left(C\|\mathbf{B}\|_{\phi_p}\alpha_n > \frac{\delta}{\varepsilon}\right) \leqslant \mathbb{P}\left(M > \frac{\delta}{\alpha_n\varepsilon}\right),$$

where M is the random variable in Proposition 19. As

$$\mathbb{P}\left(M > \frac{\delta}{\alpha_n\varepsilon}\right) \leqslant E\big(\exp(\lambda M^2)\big)\exp\left(-\lambda\left(\frac{\delta}{\alpha_n\varepsilon}\right)^2\right),$$

$$\overline{\lim_{\varepsilon\to 0}}\varepsilon^2 \log \mathbb{P}\big(d_\varphi\big(\delta_\varepsilon \Upsilon_n(\mathbf{B}), \delta_\varepsilon \mathbf{B}\big) > \delta\big) \leqslant \overline{\lim_{\varepsilon\to 0}}\varepsilon^2 \log E\big(\exp(\lambda M^2)\big) - \lambda\left(\frac{\delta}{\alpha_n}\right)^2 \leqslant -\lambda\left(\frac{\delta}{\alpha_n}\right)^2,$$

which gives inequality (22), since α_n is a deterministic converging to 0. □

By the continuity of the Itô map in the topology induced by d_φ, we obtain the following extension of Freidlin–Wentzell theorem [8,9,5].

Corollary 40. *Let $\varphi \in \Xi_p$, such that $\lim_{x\to 0} \frac{\sqrt{-x\ln(x)}}{\varphi(x)} = 0$. Let y_t^ε be the solution of the Stratonovich differential equation*

$$dy_t^\varepsilon = f_0(y_t^\varepsilon)\,dt + \varepsilon f(y_t^\varepsilon) \circ dB_t,$$

where f_0 is $(1+\alpha)$-Lipschitz and f is $(2+\alpha)$-Lipschitz, $0 < \alpha < 1$. The Stratonovich extension of y^ε to a geometric rough path, i.e. $\Gamma(y_t^\varepsilon) = \mathbf{F}(\delta_\varepsilon \mathbf{B})$ ($\mathbf{F}$ has been defined in Eq. (18)) satisfies a large deviation principle in the topology induced by d_φ with good rate function

$$J(\mathbf{x}) = \inf_{\mathbf{F}(\mathbf{y})=\mathbf{x}} I(\mathbf{y}).$$

Remark that if we only consider the first level of our paths y^ε, we obtain the classical Freidlin–Wentzell theorem in the topology induced by $d_{1,\varphi}$.

6.3. Strassen law

A classical corollary of Schilder is the law of the iterated logarithm [8,9,25]:

Corollary 41. *Let $K = \{\mathbf{x} = S(x),\ x \in W^{1,2}$ and $\int_0^1 |x_u'|^2\,du \leqslant 1\}$ and*

$$\xi_t^n = \delta_{(2n\log\log n)^{-1/2}} \mathbf{B}_{nt}.$$

Then, if $\varphi \in \Xi_p$ is such that $\lim_{x\to 0} \frac{\sqrt{-x\ln(x)}}{\varphi(x)} = 0$,

$$\lim_{n\to\infty} d_\varphi(\xi^n, K) = 0,$$

and the set of limit points of in $C_0([0,1], G(\mathbb{R}^d))$ with the topology induced by d_φ is equal to K.

Acknowledgements

The authors would like to thank G. Ben Arous, T. Lyons and S. Varadhan for related discussions. The first author gratefully acknowledges support from the Austrian Academy of Science.

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