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Asymptotic behavior for doubly degenerate parabolic equations

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Abstract

We use mass transportation inequalities to study the asymptotic behavior for a class of doubly degenerate parabolic equations of the form

$$\frac{\partial\rho}{\partial t} = \operatorname{div}\bigl\{\rho\nabla c^*\bigl[\nabla\bigl(F'(\rho)+V\bigr)\bigr]\bigr\} \quad \text{in } (0,\infty)\times\Omega, \quad \text{and} \quad \rho(t=0)=\rho_0 \quad \text{in } \{0\}\times\Omega, \tag{1}$$

where Ω is $\mathbb{R}^n$, or a bounded domain of $\mathbb{R}^n$ in which case $\rho\nabla c^*[\nabla(F'(\rho)+V)]\cdot\nu = 0$ on $(0,\infty)\times\partial\Omega$. We investigate the case where the potential V is *uniformly c-convex*, and the degenerate case where $V=0$. In both cases, we establish an exponential decay in relative entropy and in the c-Wasserstein distance of solutions – or self-similar solutions – of (1) to equilibrium, and we give the explicit rates of convergence. In particular, we generalize to all $p>1$, the HWI inequalities obtained by Otto and Villani (J. Funct. Anal. 173 (2) (2000) 361–400) when $p=2$. This class of PDEs includes the Fokker–Planck, the porous medium, fast diffusion and the parabolic p-Laplacian equations. ***To cite this article: M. Agueh, C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

Résumé

Comportement asymptotique des équations paraboliques doublement dégénérées. Nous utilisons des inégalités de transport de masse pour étudier le comportement asymptotique des équations paraboliques doublement dégénérées de la forme (1), où Ω est soit $\mathbb{R}^n$, ou un domaine borné de $\mathbb{R}^n$ auquel cas $\rho\nabla c^*[\nabla(F'(\rho)+V)]\cdot\nu = 0$ sur $(0,\infty)\times\partial\Omega$. Nous examinons le cas où le potentiel V est *uniformément c-convexe*, et le cas dégénéré où $V=0$. Dans ces deux cas, nous montrons une décroissance exponentielle de la différence d'entropies et de la distance de Wasserstein – suivant le coût c – des solutions de l'équation et de sa solution stationnaire, et nous précisons les taux de convergence. En particulier, nous généralisons à tous les $p>1$ les inégalités HWI obtenues dans Otto et Villani (J. Funct. Anal. 173 (2) (2000) 361–400) lorsque $p=2$. Cette classe d'équations contient les équations de Fokker–Planck, des milieux poreux et du p-Laplacien. ***Pour citer cet article : M. Agueh, C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

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Nous considérons les équations aux dérivées partielles de la forme (1), où ρ_0 est une densité de probabilité sur Ω, et $c:\mathbb{R}^n \to \mathbb{R}$, $F:[0,\infty) \to \mathbb{R}$ et $V:\mathbb{R}^n \to \mathbb{R}$ vérifient les hypothèses (HC), (HF) et (HV) ci-dessous. Nous nous intéressons au comportement asymptotique des solutions de (1). Rappelons que si $c(x) = \frac{|x|^2}{2}$ et $D^2V \geqslant \lambda I$ avec $\lambda > 0$, la différence d'entropies et la distance de Wasserstein de la solution de (1) et de sa solution stationnaire décroissent exponentiellement avec des taux de convergence de 2λ et λ respectivement (voir [7] et [3]). Mais quand $c(x) = \frac{|x|^q}{q}$ où $q \neq 2$, les seuls résultats connus sont apparemment les résultats de Kamin et Vázquez [6] et de Del Pino et Dolbeault [5]. Kamin et Vázquez [6] ont prouvé que la solution du p-Laplacien converge suivant les normes L^1 et L^∞ vers sa solution stationnaire, mais sans aucune precision du taux de convergence, tandis que Del Pino et Dolbeault [5] ont établi une décroissance exponentielle de cette solution vers la solution stationnaire, mais seulement pour les p appartenant à l'interval $\frac{2n+1}{n+1} \leqslant p < n$. Apparemment, il n'y avait pas de résultats sur le taux de convergence de la solution du p-Laplacien pour les p vérifiant $2 \neq p \geqslant n$. Dans cet article, nous généralisons à tous les $p > 1$ les résultats précédents, et nous améliorons les taux de convergence obtenus dans [5] lorsque $p > 2$ (voir Théorèmes 2.3 et 3.2). En particulier, nous généralisons à tous les $p > 1$, les inégalités HWI établies dans [7] et [8] quand $p = 2$ (voir Théorème 2.2).

1. Introduction

We consider equations of the form (1), where Ω is either $\mathbb{R}^n$, or a bounded domain of $\mathbb{R}^n$ in which case we impose the Neumann condition $\rho\nabla c^\star[\nabla(F'(\rho)+V)]\cdot\nu = 0$ on the boundary $(0,\infty)\times\partial\Omega$. Here, ρ_0 is a probability density on Ω, and $c:\mathbb{R}^n \to \mathbb{R}$, $F:[0,\infty) \to \mathbb{R}$ and $V:\mathbb{R}^n \to \mathbb{R}$ satisfy:

- (HC) $c \in C^1(\mathbb{R}^n)$, nonnegative, strictly convex and satisfies $c(0) = 0$, and for all $x \in \mathbb{R}^n$, there exist $q > 1$ and $\alpha, \beta > 0$, such that $\beta|x|^q \leqslant c(x) \leqslant \alpha(|x|^q + 1)$.
- (HF) $F \in C^2(0,\infty)$, strictly convex and satisfies $F(0) = 0$, $(0,\infty) \ni x \mapsto x^n F(x^{-n})$ is convex, and, either $\lim_{x\to\infty} \frac{F(x)}{x} = \infty$ or $\lim_{x\to\infty} \frac{F(x)}{x} = 0$ and $F'(x) < 0$ for $x \in (0,\infty)$.
- (HV) $V \in C^1(\mathbb{R}^n)$, nonnegative and convex.

The existence and uniqueness of solutions to (1) is proved in [1] when Ω is bounded. When $\Omega = \mathbb{R}^n$, existence of solutions to (1) is known for particular examples of c, F and V. In this paper, we study the long time behavior of the solutions to (1). In [7] and [3], it was shown that when $c(x) = \frac{|x|^2}{2}$ and $D^2V \geqslant \lambda I$ for some $\lambda > 0$, solutions to (1) decay exponentially fast in relative entropy and in the 2-Wasserstein distance at the rates 2λ and λ respectively. But, when $c(x) = \frac{|x|^q}{q}$ with $q \neq 2$, the only results known so far seem to be the results of Kamin and Vázquez [6] and Del Pino and Dolbeault [5]. In [6], the authors proved a convergence in L^1 and L^∞ norm of self-similar solutions of the p-Laplacian equation to equilibrium, with no rates. This result was improved in [5], where it was established an exponential decay in relative entropy at the rate $q(1 - \frac{1}{p}(p-1)^{1/q})$ – where q is the conjugate of p – but only when p is restricted to the interval $\frac{2n+1}{n+1} \leqslant p < n$. No results seemed to be known so far when $2 \neq p \geqslant n$. In this work, we extend to all $p > 1$ the results obtained by the previous authors, and we also improve the rates of convergence in [5] when $p > 2$. Indeed, let us first recall the notion of *uniform c-convexity* introduced in [4]: $V:\mathbb{R}^n \to \mathbb{R}$ *is uniformly c-convex with* $\mathrm{Hess}_c V \geqslant \lambda I$ *for some* $\lambda \in \mathbb{R}$, *if for all* $a, b \in \mathbb{R}^n$,

$$V(b) - V(a) \geqslant \nabla V(a)\cdot(b-a) + \lambda c(b-a). \tag{2}$$

Note that when $c(x) = \frac{|x|^2}{2}$ and V is twice differentiable, then (2) means that $D^2V \geqslant \lambda I$. We show in Section 2 that, if $c(x) = \frac{|x|^q}{q}$ with $q > 1$, and if $\mathrm{Hess}_c V \geqslant \lambda I$ for some $\lambda > 0$, then solutions to (1) decay exponentially fast in relative entropy and in the q-Wasserstein distance at the rates $p\lambda^{p-1}$ and $(p-1)\lambda^{p-1}$ respectively, where p is the conjugate of q (Theorem 2.3). There, we use the generalized Log-Sobolev and transport inequalities (Proposition 2.1) established in [4]. Note that our result extends previous results obtained in [7] and [3] for $p = q = 2$. As a by-product, we generalize to all $p > 1$, the HWI inequalities obtained in [7] and [8] for $p = 2$ (Theorem 2.2). In Section 3, we show that if $c(x) = \frac{|x|^q}{q}$ with $2 \neq q > 1$, $V = 0$ and $\Omega = \mathbb{R}^n$, then solutions to (1) decay exponentially fast in relative entropy, and – for $q > 2$ – in the q-Wasserstein distance at the rates 1 and $\frac{1}{q}$ respectively (Theorem 3.2). For that, we establish another Log-Sobolev type inequality (Proposition 3.1) using an argument in [2]. Note that this result extends to all $p \geqslant n$ results obtained in [5] for $p < n$, and the rates are sharper when $p > 2$. In the sequel, the set of probability densities over Ω is denoted by $\mathbf{P}_a(\Omega)$, and $\mathrm{H}_V^F(\rho) := \int_{\mathbb{R}^n} (F(\rho) + \rho V)\,\mathrm{d}x$ is the *free energy* of $\rho \in \mathbf{P}_a(\Omega)$. For $\rho_0, \rho_1 \in \mathbf{P}_a(\Omega)$, $\mathrm{H}_V^F(\rho_0|\rho_1) := \mathrm{H}_V^F(\rho_0) - \mathrm{H}_V^F(\rho_1)$ denotes the *relative energy of* ρ_0 *with respect to* ρ_1, and

$$I_{c^*}(\rho_0|\rho_\infty) := \int_\Omega \rho_0 \nabla\bigl(F'(\rho_0) + V\bigr) \cdot \nabla c^\star\bigl(\nabla(F'(\rho_0) + V)\bigr)\,\mathrm{d}x, \tag{3}$$

is the *generalized relative Fisher information of* ρ_0 *with respect to* ρ_∞ *measured against* c^* (see [4]), where $\rho_\infty \in \mathbf{P}_a(\Omega)$ satisfies $\rho_\infty \nabla(F'(\rho_\infty) + V) = 0$ a.e., and $c^*(y) := \sup_{x \in \mathbb{R}^n}\{x \cdot y - c(x)\}$ is the Legendre transform of c. When $c(x) = \frac{|x|^q}{q}$ and p is the conjugate of q, $\frac{1}{p} + \frac{1}{q} = 1$, we denote I_{c^*} by I_p. The *c-Wasserstein work* between ρ_0 and ρ_1 is defined by

$$W_c(\rho_0, \rho_1) := \inf\left\{ \int_{\mathbb{R}^n} c(x - Tx)\rho_0(x)\,\mathrm{d}x;\ T_\#\rho_0 = \rho_1 \right\}, \tag{4}$$

where $T_\#\rho_0 = \rho_1$ means that $\rho_1(B) = \rho_0(T^{-1}(B))$ for all Borel sets $B \subset \mathbb{R}^n$. When $c(x) = \frac{|x|^q}{q}$, $W_c = \frac{1}{q} W_q^q$, where W_q is the q-Wasserstein distance.

The following *energy inequality* will be needed in our analysis (for its proof, we refer to [1] and [4]): *if* c, F *and* V *satisfy* (HC), (HF) *and* (HV), *and if* $\mathrm{Hess}_c\, V \geqslant \lambda I$ *for some* $\lambda \in \mathbb{R}$, *then for all* $\rho_0 \in \mathbf{P}_a(\Omega) \cap W^{1,\infty}(\Omega)$ *and* $\rho_1 \in \mathbf{P}_a(\Omega)$ *with support of* ρ_0 *in* Ω, *we have for* $T_\#\rho_0 = \rho_1$ *optimal in* (4),

$$\mathrm{H}_V^F(\rho_0|\rho_1) + \lambda W_c(\rho_0, \rho_1) \leqslant \int_\Omega (x - Tx) \cdot \nabla\bigl(F'(\rho_0) + V\bigr)\rho_0\,\mathrm{d}x. \tag{5}$$

2. Doubly degenerate PDEs with uniformly c-convex confinement potentials

We study the asymptotic behavior of (1) assuming that V is uniformly c-convex (2) with $\mathrm{Hess}_c\, V \geqslant \lambda I$ for some $\lambda > 0$. Here Ω is either $\mathbb{R}^n$, or an open bounded convex subset of $\mathbb{R}^n$ in which case we impose the Neumman condition $\rho \nabla c^\star[\nabla(F'(\rho) + V)] \cdot \nu = 0$ on the boundary $(0, \infty) \times \partial\Omega$.

Because of the energy inequality (5), the density function $\rho_\infty \in \mathbf{P}_a(\Omega)$ satisfying

$$\rho_\infty \nabla\bigl(F'(\rho_\infty) + V\bigr) = 0 \quad \text{a.e.}, \tag{6}$$

minimizes $\{\mathrm{H}_V^F(\rho),\ \rho \in \mathbf{P}_a(\Omega)\}$, and if $\mathrm{Hess}_c\, V \geqslant \lambda I$ for some $\lambda > 0$, it is the unique minimizer. ρ_∞ is the stationary solution to (1). The following generalized transport and logarithmic Sobolev inequalities of [4], will be used in Theorem 2.3 below, to obtain the rates of convergence of solutions to (1):

Proposition 2.1 (Generalized transport and Log-Sobolev inequalities). *In addition to* (HC), (HF) *and* (HV), *assume that* c *is even and that* $\mathrm{Hess}_c\, V \geqslant \lambda I$ *for some* $\lambda > 0$. *If* $\rho_\infty \in \mathbf{P}_a(\Omega) \cap W^{1,\infty}(\Omega)$ *satisfies* (6), *then*

(i) *For all probability densities* $\rho \in \mathbf{P}_a(\Omega)$, *the following transport inequality holds*:

$$W_c(\rho, \rho_\infty) \leqslant \frac{1}{\lambda} \mathrm{H}_V^F(\rho|\rho_\infty). \tag{7}$$

(ii) *For all* $\mu > 0$ *and all probability densities* $\rho_0 \in \mathbf{P}_a(\Omega) \cap W^{1,\infty}(\Omega)$ *and* $\rho_1 \in \mathbf{P}_a(\Omega)$, *we have*

$$\mathrm{H}_V^F(\rho_0|\rho_1) + (\lambda - \mu) W_c(\rho_0, \rho_1) \leqslant \mu \int_\Omega c^* \left(\frac{\nabla(F'(\rho_0) + V)}{\mu} \right) \rho_0 \, \mathrm{d}x. \tag{8}$$

In particular, if $c(x) = \frac{|x|^q}{q}$ *for some* $q > 1$, *we have the generalized Log-Sobolev inequality*:

$$\mathrm{H}_V^F(\rho_0|\rho_1) \leqslant \frac{1}{p\lambda^{p-1}} I_p(\rho_0|\rho_\infty). \tag{9}$$

Proof. (7) follows from (5), and (8) follows from (5) and Young inequality applied with $c_\mu := \mu c$: $\nabla(F'(\rho_0) + V) \cdot (I - T) \leqslant c_\mu(I - T) + c_\mu^*(\nabla(F'(\rho_0) + V))$. If $c(x) = \frac{|x|^q}{q}$, choose $\mu = \lambda$ in (8) to get (9).

As by-product of (8), we obtain the following generalization to all $p, q > 1$ of the HWI inequalities:

Theorem 2.2 (Generalized p-HWI inequalities). *In addition to the hypotheses* (HC), (HF) *and* (HV), *assume that* c *is even and* q*-homogeneous, and that* $\mathrm{Hess}_c\, V \geqslant \lambda I$ *for some* $\lambda > 0$. *If* ρ_∞ *satisfies* (6), *then, for all probability densities* $\rho_0 \in \mathbf{P}_a(\Omega) \cap W^{1,\infty}(\Omega)$ *and* $\rho_1 \in \mathbf{P}_a(\Omega)$, *we have*

$$\mathrm{H}_V^F(\rho_0|\rho_1) \leqslant \frac{p}{(p-1)^{1/q}} \hat{I}_{c^*}(\rho_0|\rho_\infty)^{1/p} W_c(\rho_0, \rho_1)^{1/q} - \lambda W_c(\rho_0, \rho_1), \quad \textit{where} \tag{10}$$

$\hat{I}_{c^*}(\rho_0|\rho_\infty) := \int_\Omega c^*(\nabla(F'(\rho_0) + V)) \rho_0 \, \mathrm{d}x$. *In particular, if* $c(x) = \frac{|x|^q}{q}$, *then*

$$\mathrm{H}_V^F(\rho_0|\rho_1) \leqslant I_p(\rho_0|\rho_\infty)^{1/p} W_q(\rho_0, \rho_1) - \frac{\lambda}{q} W_q(\rho_0, \rho_1)^q. \tag{11}$$

Proof. Rewrite (8) as $\mathrm{H}_V^F(\rho_0|\rho_1) + \lambda W_c(\rho_0, \rho_1) \leqslant \mu W_c(\rho_0, \rho_1) + \frac{1}{\mu^{p-1}} \hat{I}_{c^*}(\rho_0|\rho_\infty)$, and show that the minimum over μ is attained at $\bar{\mu} = [\frac{(p-1)\hat{I}_{c^*}(\rho_0|\rho_\infty)}{W_c(\rho_0, \rho_1)}]^{1/p}$. If $c(x) = \frac{|x|^q}{q}$, then $W_c = \frac{1}{q} W_q^q$ and $\hat{I}_{c^*} = \frac{1}{p} I_p$.

Theorem 2.3. *In addition to* (HF) *and* (HV), *assume that* $c(x) = \frac{|x|^q}{q}$, *and* $\mathrm{Hess}_c\, V \geqslant \lambda I$ *for some* $\lambda > 0$. *If* $\rho_0 \in \mathbf{P}_a(\Omega)$ *is such that* $\mathrm{H}_V^F(\rho_0) < \infty$, *then for any solution* ρ *of* (1) *with* $\mathrm{H}_V^F(\rho(t)) < \infty$,

$$\mathrm{H}_V^F\big(\rho(t)|\rho_\infty\big) \leqslant \mathrm{e}^{-p\lambda^{p-1}t} \mathrm{H}_V^F(\rho_0|\rho_\infty) \quad \textit{and} \quad W_q\big(\rho(t), \rho_\infty\big) \leqslant \mathrm{e}^{-(p-1)\lambda^{p-1}t} \left[\frac{q\mathrm{H}_V^F(\rho_0|\rho_\infty)}{\lambda} \right]^{1/q}. \tag{12}$$

Proof. For a solution ρ of (1), we have $\frac{\mathrm{d}\mathrm{H}_V^F(\rho(t)|\rho_\infty)}{\mathrm{d}t} = -I_{c^*}(\rho(t)|\rho_\infty)$. Combine the subsequent equality and (9), to obtain the first inequality in (12). Then combine this inequality and (7) to deduce the second inequality in (12).

Example. If $c(x) = \frac{|x|^2}{2}$, F satisfies (HF), and $D^2 V \geqslant \lambda I$ for some $\lambda > 0$, in which case (1) is the generalized Fokker–Planck equation (see [7] and [3]) $\frac{\partial\rho}{\partial t} = \mathrm{div}[\rho \nabla(F'(\rho) + V)]$, Theorem 2.3 gives an exponential decay in relative entropy and in the 2-Wasserstein distance of the solutions of this equation to the equilibrium solution ρ_∞ (6) at the rates 2λ and λ respectively.

3. Doubly degenerate PDE without confinement potentials

In this section, we study the asymptotic behavior for

$$\begin{cases} \dfrac{\partial\rho}{\partial t} = \operatorname{div}\{\rho\nabla c^*[\nabla(F'(\rho))]\} & \text{in } (0,\infty)\times\mathbb{R}^n, \\ \rho(t=0)=\rho_0 & \text{in } \{0\}\times\mathbb{R}^n. \end{cases} \tag{13}$$

It is known – at least for $c(x)=\frac{|x|^q}{q}$ where $q>1$, and $F(x)=x\ln x$ or $F(x)=\frac{x^\gamma}{\gamma-1}$ – that, after rescaling in time and space:

$$\tau=\beta(t),\quad y=\frac{x}{R(t)},\quad \text{and}\quad \hat\rho(\tau,y)=R(t)^n\rho(t,x), \tag{14}$$

where $\beta(0)=0$, $\lim_{t\to\infty}\beta(t)=\infty$ and $R(0)=1$, ρ solves (13) if and only if $\hat\rho$ solves:

$$\begin{cases} \dfrac{\partial\hat\rho}{\partial\tau} = \operatorname{div}\{\hat\rho(\nabla c^*[\nabla(F'(\hat\rho))]+\nabla c^\star(\nabla c))\} & \text{in } (0,\infty)\times\mathbb{R}^n, \\ \hat\rho(\tau=0)=\rho_0 & \text{in } \{0\}\times\mathbb{R}^n, \end{cases} \tag{15}$$

where we used that $\nabla c^*\circ\nabla c=I$. Solutions $\hat\rho$ of (15) are known as self-similar solutions of (13). Our goal here is to investigate the asymptotic behavior of $\hat\rho$ to the stationary solution $\hat\rho_\infty$ of (15), or in other words, the intermediate asymptotics of ρ to the solution $\rho_\infty(t,x)=\frac{1}{R(t)^n}\hat\rho_\infty(\frac{x}{R(t)})$ of (13). Note that when $c(x)=\frac{|x|^2}{2}$, Eqs. (1) and (15) are equivalent, where the potential V being here c, but this is not the case when c is not 2-homogeneous. In the sequel, we define $\hat\rho_\infty\in\mathbf{P}_a(\mathbb{R}^n)$ by

$$\hat\rho_\infty=\overline{(F')^{-1}}(K_\infty-c),$$

where K_∞ is the unique constant such that $\int_{\mathbb{R}^n}\hat\rho_\infty\,\mathrm{d}y=1$, and $\overline{(F')^{-1}}$ denotes the generalized inverse of F'. Since $\hat\rho_\infty\nabla(F'(\hat\rho_\infty)+c)=0$, we have, because of (6), that $\hat\rho_\infty$ minimizes $\{\mathrm{H}_V^F(\hat\rho)\colon\ \hat\rho\in\mathbf{P}_a(\mathbb{R}^n)\}$, and for any solution $\hat\rho(\tau)$ of (15), the following energy dissipation equation holds

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\mathrm{H}_V^F(\hat\rho(\tau))=-\int_{\mathbb{R}^n}\hat\rho\nabla(F'(\hat\rho)+c)\cdot[\nabla c^\star(\nabla(F'(\hat\rho)))+\nabla c^\star(\nabla c)]\,\mathrm{d}y:=-\bar I_{c^\star}(\hat\rho|\hat\rho_\infty). \tag{16}$$

The following Log-Sobolev type inequality will be needed in our analysis.

Proposition 3.1. *Assume that F satisfies* (HF). *Then, for any nonnegative strictly convex C^1-function $c:\mathbb{R}^n\to\mathbb{R}$ such that $\lim_{|x|\to\infty}\frac{c(x)}{|x|}=\infty$, and for all $\rho_0\in\mathbf{P}_a(\mathbb{R}^n)\cap W^{1,\infty}(\mathbb{R}^n)$ and $\rho_1\in\mathbf{P}_a(\mathbb{R}^n)$,*

$$H_c^F(\rho_0|\rho_1)\leqslant H_c(\rho_0)+\int\rho_0\nabla(F'(\rho_0))\cdot x\,\mathrm{d}x+\int\rho_0c^*(-\nabla(F'(\rho_0)))\,\mathrm{d}x. \tag{17}$$

In particular, if $c(x)=\frac{|x|^q}{q}$, then, for $q=2$,

$$\mathrm{H}_c^F(\rho_0|\rho_1)\leqslant\frac{1}{2}\bar I_{c^*}(\rho_0|\hat\rho_\infty), \tag{18}$$

and for $q\neq2$,

$$\mathrm{H}_c^F(\rho_0|\rho_1)\leqslant\bar I_{c^*}(\rho_0|\hat\rho_\infty). \tag{19}$$

Proof. (17) follows from (5) and Young inequality applied with c:

$$-\nabla(F'(\rho_0))\cdot T(x)\leqslant c(T(x))+c^*(-\nabla(F'(\rho_0))).$$

If $c(x)=\frac{|x|^2}{2}$, we have that

$$\bar I_4(\rho_0|\hat\rho_\infty):=\int \rho_0\nabla c(x)\cdot\nabla c^*\big(\nabla\big(F'(\rho_0)\big)\big)\,\mathrm{d}x=\int \rho_0 x\cdot\nabla\big(F'(\rho_0)\big)\,\mathrm{d}x:=\bar I_3(\rho_0|\hat\rho_\infty).$$

Then, the right-hand side of (17) reads as $\frac{1}{2}\bar I_{c^*}(\rho_0|\hat\rho_\infty)$. This proves (18). If $c(x)=\frac{|x|^q}{q}$ with $q\neq 2$, we use Young inequality with c: $\pm\nabla c(x)\cdot\nabla c^*(\nabla(F'(\rho_0)))\leqslant c^*(\pm\nabla c(x))+c(\nabla c^*(\nabla(F'(\rho_0))))$, to have that

$$\big|\bar I_4(\rho_0|\hat\rho_\infty)\big|\leqslant\frac{1}{q}\bar I_1(\rho_0|\hat\rho_\infty)+\frac{1}{p}\bar I_2(\rho_0|\hat\rho_\infty),\tag{20}$$

where $\bar I_1(\rho_0|\hat\rho_\infty):=\int\rho_0\nabla(F'(\rho_0))\cdot\nabla c^*(\nabla(F'(\rho_0)))\,\mathrm{d}x$ and $\bar I_2(\rho_0|\hat\rho_\infty):=\int\rho_0 x\cdot\nabla c(x)\,\mathrm{d}x$.

Then, we deduce from (17) and (20) that

$$\mathrm{H}_c^F(\rho_0|\rho_1)\leqslant\bar I_1(\rho_0|\hat\rho_\infty)+\bar I_2(\rho_0|\hat\rho_\infty)+\bar I_3(\rho_0|\hat\rho_\infty)-\left(\frac{1}{q}\bar I_1(\rho_0|\hat\rho_\infty)+\frac{1}{p}\bar I_2(\rho_0|\hat\rho_\infty)\right)\leqslant\bar I_{c^*}(\rho_0|\hat\rho_\infty).$$

Theorem 3.2 (Trend to equilibrium for (15)). *Assume that F satisfies* (HF), $c(x)=\frac{|x|^q}{q}$ *and* $\mathrm{H}_c^F(\rho_0)<\infty$. *Then, for any solution $\hat\rho$ of* (15) *with* $\mathrm{H}_c^F(\hat\rho(\tau))<\infty$, *we have, if $q=2$, then*

$$\mathrm{H}_V^F\big(\hat\rho(\tau)|\hat\rho_\infty\big)\leqslant \mathrm{e}^{-2\tau}\mathrm{H}_V^F(\rho_0|\hat\rho_\infty)\quad\textit{and}\quad W_2\big(\hat\rho(\tau)|\hat\rho_\infty\big)\leqslant\mathrm{e}^{-\tau}\sqrt{2\mathrm{H}_V^F(\rho_0|\hat\rho_\infty)};\tag{21}$$

if $q\neq 2$, then

$$\mathrm{H}_V^F\big(\hat\rho(\tau)|\hat\rho_\infty\big)\leqslant \mathrm{e}^{-\tau}\mathrm{H}_V^F(\rho_0|\hat\rho_\infty),\quad\textit{and for } q>2,\quad W_q\big(\hat\rho(\tau)|\hat\rho_\infty\big)\leqslant\mathrm{e}^{-\tau/q}\left[\frac{q\mathrm{H}_V^F(\rho_0|\hat\rho_\infty)}{\lambda_q}\right]^{1/q},\tag{22}$$

where $\lambda_q>0$ is such that $\mathrm{Hess}_c\, c\geqslant\lambda_q I$.

Proof. If $q=2$, combine (16) and (18) to obtain the first inequality in (21). Then combine this inequality with (7) to deduce the second inequality in (21). The proof of (22) is similar.

Example. If $c(x)=\frac{|x|^q}{q}$ ($\frac{1}{p}+\frac{1}{q}=1$ and $2\neq p>1$), and $F(x)=\frac{mx^\gamma}{\gamma(\gamma-1)}$ where $\gamma=m+\frac{p-2}{p-1}$, $\frac{1}{p-1}\neq m\geqslant\frac{n-(p-1)}{n(p-1)}$ (resp. $F(x)=\frac{1}{p-1}x\ln x$), then (13) reads as $\frac{\partial\rho}{\partial t}=\mathrm{div}(|\nabla\rho^m|^{p-2}\nabla\rho^m)$ (resp. $m=\frac{1}{p-1}$), and $\hat\rho_\infty=(K_\infty+\frac{1-\gamma}{qm}|x|^q)_+^{1/(\gamma-1)}$ (resp. $\hat\rho_\infty=\mathrm{e}^{-(p-1)|x|^q/q}/\sigma$, $\sigma=\int_{\mathbb{R}^n}\mathrm{e}^{-(p-1)|x|^q/q}\,\mathrm{d}x$). Then Theorem 3.2 gives the decay rates in (22).

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