Available online at www.sciencedirect.com
SCIENCE DIRECT
ELSEVIER
C. R. Acad. Sci. Paris, Ser. I 337 (2003) 223–226
Comptes Rendus Mathematique

Number Theory

On Euler products and multi-variate Gaussians

Dennis A. Hejhal[a,b]

[a] *Department of Mathematics, Uppsala University, Box 480, 75106 Uppsala, Sweden*
[b] *School of Mathematics, University of Minnesota, Minneapolis, MN 55455, USA*

Received 18 June 2003; accepted after revision 9 July 2003

Presented by Alain Connes

Abstract

In this Note, we extend a recent result of A. Selberg concerning the asymptotic value distribution of Euler products to a multi-dimensional setting. Under certain conditions, an asymptotic development of Edgeworth type is found. ***To cite this article: D.A. Hejhal, C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

Résumé

Sur les produits eulériens et les gaussiennes multidimensionnelles. Nous généralisons à plusieurs variables un résultat récent de A. Selberg concernant la distribution asymptotique de valeurs des produits Eulériens. Sous certaines hypothèses un développement asymptotique de type Edgeworth est établi. ***Pour citer cet article : D.A. Hejhal, C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

1. Preliminaries

Let $L_1, \ldots, L_J$ be a family of J Euler products of degree d satisfying the following hypotheses.

(I) Each $L_j(s)$ is expressible as $\prod_p \prod_{k=1}^{d}(1-\alpha_{kpj}\,p^{-s})^{-1}$ for $\mathrm{Re}(s)>1$, with "root numbers" α_{kpj} having modulus at most 1.

(II) Each $L_j(s)$ admits an analytic continuation to all of $\mathbb{C}$ as a meromorphic function of finite order having a finite number of poles, all situated along $\mathrm{Re}(s)=1$.

(III) Each continued function $L_j(s)$ satisfies a functional equation of type

$$G(s)L_j(s)=\exp(\mathrm{i}\alpha)\overline{G(1-\bar{s})}\;\overline{L_j(1-\bar{s})}$$

with $G(s)=Q^s\prod_{\iota=1}^{h}\Gamma(\lambda_\iota s+\mu_\iota)$ and certain choices of $\alpha\in\mathbb{R}$, $Q>0$, $h\geqslant 1$, $\lambda_\iota>0$, and $\mathrm{Re}(\mu_\iota)\geqslant 0$ (these choices are allowed to depend on j).

E-mail address: hejhal@math.uu.se (D.A. Hejhal).

doi:10.1016/S1631-073X(03)00344-3

(IV) The logarithms of the L_j are "independent" in the sense that one has

$$\sum_{p\leqslant X} p^{-1}c_j(p)\overline{c_k(p)} = \delta_{jk}\aleph_j \log\log X + c_{jk} + \mathrm{O}\big[(\log X)^{-\nu}\big]$$

for $X \geqslant 2$ and certain $\aleph_j > 0,\ c_{jk} \in \mathbb{C},\ \nu \in (0, 1]$, the coefficients $c_j(n)$ being defined by

$$\log L_j(s) = \sum_{n=2}^{\infty} c_j(n)\frac{\Lambda(n)}{\log n} n^{-s}.$$

In addition to (I)–(IV), we shall assume either:

(V_a) that GRH holds for all L_j; or,

(V_b) that, for some $\omega \in (\frac{1}{2}, 1]$ and $\beta > 0$, each L_j satisfies a Selberg-type density condition $N(\sigma, T, T + H) = \mathrm{O}[H(H/\sqrt{T})^{\beta(1/2-\sigma)}\log T]$ for $\frac{1}{2} \leqslant \sigma \leqslant 1$ and $T^\omega \leqslant H \leqslant T$ (the same ω, β being utilized for all L_j).

Consult [3,7,10,11] for further information à propos (I)–(V). Hypothesis (V_b) is known to hold for Dirichlet L-series [10,11,5] as well as Euler products associated with Hecke-normalized $\mathrm{GL}(2)$ modular forms [8].

Elementary use of (IV) shows that one has

$$\psi_j(\sigma, t) \equiv \sum_{p\leqslant t} \big|c_j(p)\big|^2 p^{-2\sigma} = \aleph_j \log\left[\min\left(\log t, \left(\sigma - \frac{1}{2}\right)^{-1}\right)\right] + \mathrm{O}(1) \tag{1}$$

whenever $\frac{1}{2} \leqslant \sigma \leqslant \frac{3}{2}$ and $t \geqslant 2$.

For convenience, set $\psi_0(\sigma, t) = \sum_{p\leqslant t} p^{-2\sigma}$. Also select any numbers $0 < c_1,\ \Theta,\ \delta < 1,\ 1 < \kappa,\ c_2 < \infty$, and let $\chi_{ab}(u)$ denote the indicator function of $[a, b]$. If GRH holds, let ω be any number in $(0, 1]$; otherwise, take ω as in (V_b).

Selberg has shown that, under these conditions,

$$\int_T^{T+H} \Big|\log L_j(\sigma + \mathrm{i}t) - \sum_{p\leqslant x} c_j(p)p^{-\sigma-\mathrm{i}t}\Big|^{2k}\,\mathrm{d}t = \mathrm{O}\big[H(Ak)^{4k}\big] \tag{2}$$

holds with $x = T^{\Theta\omega/k}$ anytime $T^\omega \leqslant H \leqslant T,\ \frac{1}{2} \leqslant \sigma \leqslant 1,\ 1 \leqslant k \leqslant (\log T)^{9/10}$. Cf. [10,6,13]. The constant A will depend solely on $\Theta,\ \omega,\ L_1, \ldots, L_J$.[1]

In the case $J = 1$, by combining (2) with certain Fourier integral approximations to $\chi_{ab}(u)$ (cf. [1,12,14]) and standard moment properties of Dirichlet polynomials (as, for instance, in [9] or [7, Eqs. (4.4), (4.5)]), Selberg was able to show further that

$$\int_T^{T+H} \chi_{ab}\big[\mathrm{Re}(\mathrm{or\,Im})\log L_1(\sigma + \mathrm{i}t)\big]\,\mathrm{d}t = H\int_{a/\sqrt{\pi\psi_1}}^{b/\sqrt{\pi\psi_1}} \exp\big(-\pi v^2\big)\,\mathrm{d}v + \mathrm{O}(H)\frac{\log^2\psi_1}{\sqrt{\psi_1}} \tag{3}$$

holds with an implied constant independent of $[a, b]$ whenever $\frac{1}{2} \leqslant \sigma \leqslant \frac{1}{2} + (\log T)^{-\delta}$ and $c_1T^\omega \leqslant H \leqslant c_2T$. See [10,13] and [7, §4]. It is understood here that $\psi_1 = \psi_1(\sigma, T)$ and that T is kept bigger than some suitable $T_0(\omega, c_1, c_2, \delta, L_1)$; of course, by (1), $\psi_1 \approx \log\log T$.

Relation (3) can be viewed as a partial refinement of the pointwise limit assertion

$$\lim_{T\to\infty}\frac{1}{H} m\big\{t \in [T, T + H]\colon\ (\pi\psi_1)^{-1/2}\log L_1(\sigma + \mathrm{i}t) \in [a, b] \times [c, d]\big\} = \int_a^b\!\!\int_c^d \mathrm{e}^{-\pi(u^2+v^2)}\,\mathrm{d}v\,\mathrm{d}u \tag{4}$$

[1] Likewise for the *implied* constant associated with the "big O".

which follows from the (relatively easily proved) moment estimate

$$\int_T^{T+H} \bigl(\log L_1(\sigma+\mathrm{i}t)\bigr)^k \overline{\bigl(\log L_1(\sigma+\mathrm{i}t)\bigr)}^\ell \,\mathrm{d}t = \delta_{k\ell} k! H \psi_1^k + \mathrm{O}_{k\ell}(H)\psi_1^{(k+\ell-1)/2} \tag{5}$$

by means of some basic probability theory (cf. [2, Problem 30.6]).

It would naturally be of interest to *extend* relation (3) to a full-fledged multi-variate setting. The counterpart of (4) for arbitrary J has been known for some time and is due to Selberg (unpublished); see [3, §5] for an exposition of this when $\sigma = \frac{1}{2}$. The case of general $\sigma \in [\frac{1}{2}, \frac{1}{2} + (\log T)^{-\delta}]$ is similar.

In light of the fact (see [7]) that (2) can be improved to read

$$\int_T^{T+H} \left|\log L_j(\sigma+\mathrm{i}t) - \sum_{n\leqslant x} c_j(n)\frac{\Lambda(n)}{\log n} n^{-\sigma-\mathrm{i}t}\right|^{2k} \mathrm{d}t = \mathrm{O}\bigl[H(Ak)^{4k} x^{k(1/2-\sigma)}\bigr] \tag{6}$$

at least for $\frac{1}{2} \leqslant \sigma \leqslant \frac{1}{2} + (\log T)^{-\delta}$ and sufficiently small Θ, there arises a suspicion that – in revamping (3) – keeping σ slightly bigger than $\frac{1}{2}$ may make it feasible for matters to ultimately take the form of an *asymptotic development* (in powers of $\sqrt{\psi_\ell}$) akin to an Edgeworth expansion. Concerning the latter topic, cf., e.g., [4, Chapter 7].

2. Statement of results

Set $\Phi(x) = \int_0^x \exp(-\pi u^2)\,\mathrm{d}u$ and write $\psi_j = \psi_{j-J}$, $L_j = L_{j-J}$ whenever $J+1 \leqslant j \leqslant 2J$.

Theorem 2.1. *Given the situation of Section* 1. *Keep* $\sigma \in [\frac{1}{2}, \frac{1}{2} + (\log T)^{-\delta}]$, $H \in [c_1 T^\omega, c_2 T]$, *and* T *bigger than some suitable* $T_0(L_1,\ldots,L_J,\omega,c_1,c_2,\delta,\kappa)$. *Let* $N = [\![\psi_0(\sigma,T)^\kappa]\!]$, $y = T^{\omega/2JN}$, *and* $\mathcal{L}_j(t) = L_j(\sigma+\mathrm{i}t)$. *Then, for any numbers* $a_j < b_j$, *one has*

$$\int_T^{T+H} \prod_{j=1}^{2J} \chi_{a_j b_j}\bigl[\mathrm{Re}(\mathrm{Im})\log \mathcal{L}_j(t)\bigr]\,\mathrm{d}t = H\prod_{j=1}^{2J}\left[\Phi\left(\frac{b_j}{\sqrt{\pi\psi_j}}\right) - \Phi\left(\frac{a_j}{\sqrt{\pi\psi_j}}\right)\right] + \mathrm{O}(H)\frac{\log^2\psi_0}{\sqrt{\psi_0}},$$

wherein "Re" *refers to* $j \leqslant J$ *and* "Im" *to* $j > J$. *When* σ *exceeds* $\frac{1}{2} + (\log y)^{-1}$, *the remainder term can be replaced by*

$$\mathrm{O}(H) y^{(1-2\sigma)/3} + \mathrm{O}(H)\psi_0^{-\kappa/2} + H \sum_{2\leqslant|\mathbf{n}|\leqslant 1+[\![\kappa]\!]} A(\mathbf{n}) \prod_{j=1}^{2J} \frac{1}{(\sqrt{\pi\psi_j})^{n_j}}\left[\Phi^{(n_j)}\left(\frac{b_j}{\sqrt{\pi\psi_j}}\right) - \Phi^{(n_j)}\left(\frac{a_j}{\sqrt{\pi\psi_j}}\right)\right],$$

where the coefficients $A(\mathbf{n})$ ($\mathbf{n} \in \mathbb{N}^{2J}$) *are certain numbers depending solely on* $\{L_1,\ldots,L_J\}$. *The implied constants associated with the various "big* O*" terms are understood here to depend on at most* $\{L_1,\ldots,L_J,\omega,c_1,c_2,\delta,\kappa\}$. (*In particular*: *they are independent of* a_j *and* b_j.)

3. About the proof

The proof is basically a multi-variable adaptation of the ideas in [7, §4]; cf. also [13]. Taking $y = T^{\omega/2JN}$ (as above) and

$$\Sigma_{yj}(\sigma,t) = \sum_{n\leqslant y} c_j(n)\frac{\Lambda(n)}{\log n} n^{-\sigma-\mathrm{i}t},$$

one first seeks to develop – for general $\sigma \in [\frac{1}{2}, \frac{1}{2} + (\log T)^{-\delta}]$ – a prototypical Edgeworth expansion (minus the $O(H)y^{(1-2\sigma)/3}$ term) for $(\Sigma_{y1}(\sigma,t), \ldots, \Sigma_{yJ}(\sigma,t))$. To accomplish this, one approximates the various functions $\chi_{a_j b_j}(u)$ by Beurling–Selberg type functions of bandwidth $\Omega = (\text{const.})\psi_0(\sigma,T)^{(\kappa-1)/2}$ as in [7] and then patiently pushes through the resultant bookkeeping utilizing two main tricks; viz.,

(A) use of [9] to express each $\mathbb{C}^J$ multimoment $\int_T^{T+H} \Sigma_y(\sigma,t)^{\mathbf{k}}\,\overline{\Sigma_y(\sigma,t)}^{\mathbf{h}}\,\mathrm{d}t$ as

$$H\int_0^1 \ldots \int_0^1 \Sigma_y(\sigma,\theta)^{\mathbf{k}}\,\overline{\Sigma_y(\sigma,\theta)}^{\mathbf{h}} \prod \mathrm{d}\theta_p + (\text{good error term}), \tag{7}$$

where $\Sigma_y = (\Sigma_{y1}, \ldots, \Sigma_{yJ})$ and

$$\Sigma_{yj}(\sigma,\theta) = \sum_{n\leqslant y} c_j(n)\frac{\Lambda(n)}{\log n} n^{-\sigma}\exp\bigl(2\pi \mathrm{i}\theta(n)\bigr), \quad \theta(n) = \sum_{p^f \| n} f\theta_p;$$

(B) exploitation of complex-variable techniques to systematically express differences of numerous (J_0 Bessel function-like) θ_p-integrals as Cauchy-type multiple integrals in the other ("v-type" Fourier transform space) variables; cf. here [7, Eq. (4.7) and the first line of the subsequent paragraph].

The leading term of (7) effectively "morphs" each $p^{-\mathrm{i}t}$ (with $p \leqslant y$) into an independent random variable $\exp(2\pi\mathrm{i}\theta_p)$. The upshot of this is that $\int_T^{T+H} \prod \chi_{a_j b_j}[\mathrm{Re}\ (\text{or}\,\mathrm{Im})\,\Sigma_{yj}(\sigma,t)]\,\mathrm{d}t$ ultimately takes the form of an Edgeworth expansion in powers of $\sqrt{\psi_\ell(\sigma,y)}$ having coefficients $A(\sigma;\mathbf{n})$ which are built up out of constants like c_{jk} (cf. Section 1) and certain *absolutely convergent* Dirichlet series on $\{\sigma > \frac{1}{3}\}$ whose entries ξ_n are polynomial expressions in $\{\mathrm{Re}\, c_j(m), \mathrm{Im}\, c_j(m)\colon\ j \in [1,J], m \in [2,n]\}$. (Note the n.)

Since, however, $\sigma \in [\frac{1}{2}, \frac{1}{2} + (\log T)^{-\delta}]$, there is no harm in replacing each of the aforementioned Dirichlet series by its value at $\sigma = 1/2$. This gives $A(\mathbf{n})$. Passage to $\log L_j(\sigma + \mathrm{i}t)$ can then be carried out (utilizing (6)) in much the same way as in [7];[2] the final result is that of Section 2.[3]

Complete details of this proof will be published elsewhere.

References

[1] A. Beurling, On functions with a spectral gap, in: The Collected Works of Arne Beurling, Vol. 2, Birkhäuser, 1989, pp. 370–372.
[2] P. Billingsley, Probability and Measure, 2nd edition, Wiley, 1986.
[3] E. Bombieri, D. Hejhal, On the distribution of zeros of linear combinations of Euler products, Duke Math. J. 80 (1995) 821–862.
[4] H. Cramér, Random Variables and Probability Distributions, 2nd edition, Cambridge University Press, 1962.
[5] A. Fujii, On the zeros of Dirichlet L-functions I, Trans. Amer. Math. Soc. 196 (1974) 225–235.
[6] A. Ghosh, On Riemann's zeta function – sign changes of $S(T)$, in: H. Halberstam (Ed.), in: Recent Progress in Analytic Number Theory, Vol. 1, Academic Press, 1981, pp. 25–46.
[7] D. Hejhal, On a result of Selberg concerning zeros of linear combinations of L-functions, Intern. Math. Res. Notices (2000) 551–577.
[8] W. Luo, Zeros of Hecke L-functions associated with cusp forms, Acta Arith. 71 (1995) 139–158.
[9] H. Montgomery, R. Vaughan, Hilbert's inequality, J. London Math. Soc. 8 (2) (1974) 73–82, especially p. 75.
[10] A. Selberg, Old and new conjectures and results about a class of Dirichlet series, in: Collected Papers, Vol. 2, Springer-Verlag, 1991, pp. 47–63.
[11] A. Selberg, Contributions to the theory of the Riemann zeta function, Archiv for Math. og Naturv. B 48 (1946) 89–155, especially §§ 2, 3.
[12] A. Selberg, Lectures on sieves, in: Collected Papers, Vol. 2, Springer-Verlag, 1991, pp. 65–247, especially §§20, 21.
[13] K.-M. Tsang, The distribution of the values of the Riemann zeta function, Ph.D. dissertation, Princeton University, 1984.
[14] J. Vaaler, Some extremal functions in Fourier analysis, Bull. Amer. Math. Soc. (N.S.) 12 (1985) 183–216.

[2] In this connection, note especially the $O(y^{1-2\sigma})$ assertion in [7, Lemma 3].

[3] Taking $\nu = 0$ in Section 1 (IV) leads to a similar but weaker result (since the "constants like c_{jk}" which enter into the coefficients $A(\mathbf{n})$ will now contain "fuzz" of size $O(1)$).