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C. R. Acad. Sci. Paris, Ser. I 337 (2003) 185–188
COMPTES RENDUS MATHEMATIQUE

Algebraic Geometry/Group Theory

Real cubic surfaces and real hyperbolic geometry

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Received 25 April 2003; accepted 20 May 2003

Presented by Mikhaël Gromov

Abstract

The moduli space of stable real cubic surfaces is the quotient of real hyperbolic four-space by a discrete, nonarithmetic group. The volume of the moduli space is $37\pi^2/1080$ in the metric of constant curvature -1. Each of the five connected components of the moduli space can be described as the quotient of real hyperbolic four-space by a specific arithmetic group. We compute the volumes of these components. ***To cite this article: D. Allcock et al., C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

Résumé

Surfaces cubiques réelles et géométrie hyperbolique réelle. L'espace des modules des surfaces cubiques stables et réelles est le quotient de l'espace hyperbolique réel de dimension quatre par un groupe non-arithmétique discret. Le volume de l'espace des modules est $37\pi^2/1080$ dans la métrique de courbure constante -1. Chacune des composantes connexes de l'espace des modules peut être décrite comme le quotient de l'espace hyperbolique réel de dimension quatre par un groupe arithmétique spécifique. Nous calculons le volume des composantes. ***Pour citer cet article : D. Allcock et al., C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

1. Results

In [2] we showed that the moduli space of stable cubic surfaces is the quotient of complex hyperbolic four-space by a certain arithmetic group which we described explicitly. The purpose of this Note is to announce a corresponding result for real cubic surfaces: the moduli space is a quotient of real hyperbolic four-space by an explicit discrete group. The group, however, is not arithmetic. We also compute the volume of the moduli space in its metric of curvature -1. It is $37\pi^2/1080 = (4\pi^2/3)(37/1440)$. (The $4\pi^2/3$ is the ratio of the volume of the unit 4-sphere to its Euler characteristic, which appears in the Gauss–Bonnet theorem.)

By the moduli space $\mathcal{M}_0^{\mathbf{R}}$ (resp. $\mathcal{M}_s^{\mathbf{R}}$) we mean the set $\mathcal{C}_0^{\mathbf{R}}$ (resp. $\mathcal{C}_s^{\mathbf{R}}$) of cubic forms with real coefficients that define smooth (resp. stable) surfaces, modulo the action of $\mathrm{GL}(4, \mathbf{R})$. By smooth we mean that the set of complex points is smooth, and by stable we mean stable in the sense of geometric invariant theory. In this case, stable

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1631-073X/$ – see front matter
doi:10.1016/S1631-073X(03)00287-5

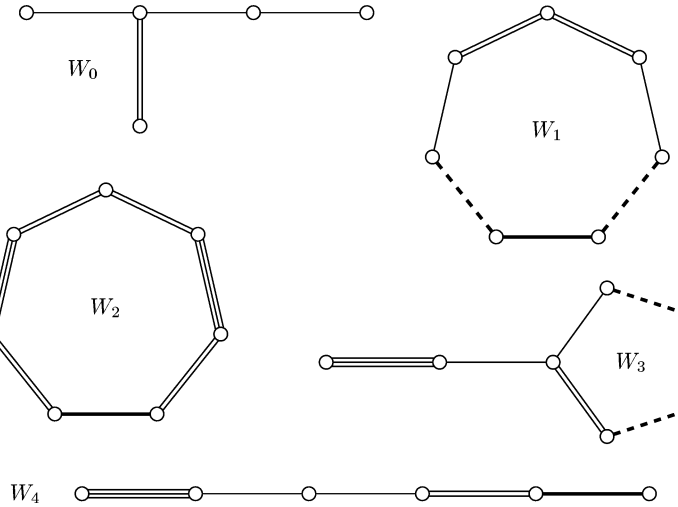

Fig. 1. Coxeter diagrams for the reflection subgroups W_j of the $P\Gamma_j$. Each describes a polyhedron C_j with one facet per atom of the diagram. The bonds indicate if/how pairs of facets meet: an absent (resp. single, double, triple) bond represents an angle of $\frac{\pi}{2}$ (resp. $\frac{\pi}{3}$, $\frac{\pi}{4}$, $\frac{\pi}{6}$), and a dashed (resp. heavy) bond represents ultraparallelism (resp. parallelism at ∞). W_j is the group generated by reflections in the facets of C_j, and C_j is a fundamental domain for W_j.

means that the complex surface has no singularities besides nodes. The space $\mathcal{M}_0^{\mathbf{R}}$ has five connected components (see [5]), which we denote by $\mathcal{M}_{0,j}^{\mathbf{R}}$ for $j = 0, 1, \ldots, 4$.

For each component $\mathcal{M}_{0,j}^{\mathbf{R}}$ of the moduli space we exhibit an arithmetic lattice $P\Gamma_j \subset \mathrm{PO}(4,1)$, a union Δ_j of two- and three-dimensional real hyperbolic subspaces of $\mathbf{RH}^4$, and an isomorphism

$$\mathcal{M}_{0,j}^{\mathbf{R}} \cong P\Gamma_j \backslash \big(\mathbf{RH}^4 - \Delta_j\big) \tag{1}$$

of real analytic orbifolds. We give two concrete descriptions of the $P\Gamma_j$, one arithmetic and one geometric. First, $P\Gamma_j$ is the projective orthogonal group of the integer quadratic form $-x_0^2 + m_1x_1^2 + \cdots + m_4x_4^2$, where j of the m_i are 3's and the rest are 1's. Second, $P\Gamma_j$ is, up to a group of order at most two, the Coxeter group W_j defined in Fig. 1. More precisely, $P\Gamma_j$ is the semidirect product of W_j by the group of diagram automorphisms, which is either trivial or of order two. Yoshida has treated the case $j = 0$ in [7].

The points of Δ_j represent nodal surfaces which are limits of smooth surfaces of type j. Since a surface with a real node is a limit of two different topological types of real surface, it is natural to glue various pairs $\mathcal{M}_{0,j}^{\mathbf{R}}$ and $\mathcal{M}_{0,j'}^{\mathbf{R}}$ together by identifying part of Δ_j with part of $\Delta_{j'}$. Carrying this out in practice means gluing certain faces of the polyhedra C_j to each other and taking care to deal with the diagram automorphisms. A miracle occurs and the result of these gluings turns out to be a quotient of $\mathbf{RH}^4$ in its own right:

Theorem 1. *There is a nonarithmetic lattice $P\Gamma^{\mathbf{R}} \subset \mathrm{PO}(4,1)$, a union Δ of two- and three-dimensional hyperbolic subspaces of $\mathbf{RH}^4$, and an isomorphism*

$$\mathcal{M}_0^{\mathbf{R}} \cong P\Gamma^{\mathbf{R}} \backslash \big(\mathbf{RH}^4 - \Delta\big)$$

of real analytic orbifolds. This identification extends to a homeomorphism $\mathcal{M}_s^{\mathbf{R}} \cong P\Gamma^{\mathbf{R}} \backslash \mathbf{RH}^4$.

The clue to the nonarithmeticity is that $\mathcal{M}_s^{\mathbf{R}}$ is obtained by gluing together arithmetic orbifolds whose groups fall into two commensurability classes. In the spirit of Gromov-Piatetski and Shapiro [4], one expects the resulting group to be non-arithmetic. To prove the nonarithmeticity we use the Galois-conjugation criterion in [3]. Namely,

Table 1

Type	Topology	Real lines	$\pi_1^{\text{orb}}(\mathcal{M}_{0,j}^{\mathbf{R}})$	Euler char.	Volume	Fraction (%)
0	$\mathbf{RP}^2 + 3$ handles	27	S_5	$1/1920$	0.00685	2.03
1	$\mathbf{RP}^2 + 2$ handles	15	$(S_3 \times S_3) \rtimes \mathbf{Z}/2$	$1/288$	0.04569	13.51
2	$\mathbf{RP}^2 + 1$ handle	7	$(D_\infty \times D_\infty) \rtimes \mathbf{Z}/2$	$5/576$	0.11423	33.78
3	$\mathbf{RP}^2$	3	$\circ\overset{\infty}{—}\circ—\circ—\circ$	$1/96$	0.13708	40.54
4	$\mathbf{RP}^2 \cup S^2$	3		$1/384$	0.03427	10.14
				$37/1440$	0.33813	100.00

it happens that $P\Gamma^{\mathbf{R}}$ preserves an integral quadratic form over $\mathbf{Z}[\sqrt{3}]$ which has signature $(4,1)$ and whose Galois conjugate also has signature $(4,1)$. We note that $P\Gamma^{\mathbf{R}}$ is not a Coxeter group, even up to finite index, but it contains an index two subgroup whose fundamental domain is a union of copies of the C_j and happens to be a Coxeter polyhedron.

The homeomorphism $\mathcal{M}_s^{\mathbf{R}} \cong P\Gamma^{\mathbf{R}}\backslash\mathbf{RH}^4$ is not an orbifold isomorphism, but it becomes one if the orbifold structure on $P\Gamma^{\mathbf{R}}\backslash\mathbf{RH}^4$ is suitably changed. This can be done explicitly enough to compute the orbifold fundamental group $\pi_1^{\text{orb}}(\mathcal{M}_s^{\mathbf{R}}) \cong \mathbf{Z}/2 \times (\mathbf{Z} * \mathbf{Z}/2)$, and to see that $\mathcal{M}_s^{\mathbf{R}}$ is a bad orbifold in the sense of Thurston.

The theory of Coxeter groups makes it easy to compute the orbifold Euler characteristic of $W_j\backslash\mathbf{RH}^4$, and hence the volume of this quotient. Dividing by a factor of two if necessary, we obtain the volume of $P\Gamma_j\backslash\mathbf{RH}^4$, which is the volume of $\mathcal{M}_{0,j}^{\mathbf{R}}$. It follows that the hyperbolic volume of $P\Gamma^{\mathbf{R}}\backslash\mathbf{RH}^4$ is the sum of these volumes. The results are displayed in Table 1. For each j we give the topology of that type of real cubic surface, the number of its real lines, the orbifold fundamental group of $\mathcal{M}_{0,j}^{\mathbf{R}}$, and the orbifold Euler characteristic and volume of $P\Gamma_j\backslash\mathbf{RH}^4$. S_n and D_∞ denote symmetric and infinite dihedral groups. Note that the component corresponding to the simplest topology has the greatest volume, just over 40% of the total, and the component corresponding to surfaces with the most real lines has the smallest volume.

2. About the proof

The identification of the components of the moduli space with quotients of real hyperbolic space depends on the construction of [1,2]. Given a smooth complex cubic surface S, let T be the triple cover of projective 3-space branched along S, and let $(H^3(T),\sigma)$ denote the resulting special Hodge structure, where σ is the symmetry coming from the branched covering transformation. The period map which assigns to S the class of $(H^3(T),\sigma)$ defines an isomorphism between the moduli space of stable cubic surfaces and $P\Gamma\backslash\mathbf{CH}^4$. Here $P\Gamma$ is the projective automorphism group of the Hermitian form $h(x,y) = -x_0\bar{y}_0 + x_1\bar{y}_1 + \cdots + x_4\bar{y}_4$ on the lattice $\Lambda = \mathcal{E}^{4,1}$, where $\mathcal{E} = \mathbf{Z}[\sqrt[3]{1}]$. The locus $\mathcal{H}$ of $\mathbf{CH}^4$ representing singular surfaces is the union of the orthogonal complements of the norm 1 vectors of Λ. In more detail, the Hodge structure on $H^3(T)$, together with a choice of isomorphism $i : H^3(T,\mathbf{Z}) \to \Lambda$ of Hermitian $\mathcal{E}$-modules determines a complex line in $\Lambda_{\mathbf{C}} = \Lambda \otimes_{\mathcal{E}} \mathbf{C} \cong \mathbf{C}^{4,1}$ which is negative for h. Thus L is a point of $\mathbf{CH}^4$, well defined up to the action of $P\Gamma$.

We call an antilinear involution ("anti-involution") of $\mathbf{CH}^4$ integral if it arises from an anti-involution of Λ. We write K_0 for the set of all pairs (L,χ) where $L \in \mathbf{CH}^4 - \mathcal{H}$ and χ is an integral anti-involution that preserves L. If the surface S is defined by an equation with real coefficients, then complex conjugation $\kappa(X_0,\ldots,X_4) = (\overline{X}_0,\ldots,\overline{X}_4)$ acts on $H^3(T,\mathbf{Z})$ as an anti-involution with respect to the $\mathcal{E}$-module structure. Let χ be the corresponding integral anti-involution $i \circ \kappa^* \circ i^{-1}$ of $\mathbf{CH}^4$. This associates to S and a choice of i a pair $(L,\chi) \in K_0$, and defines a period map

$$\mathcal{M}_0^{\mathbf{R}} \to P\Gamma\backslash K_0, \tag{2}$$

which we show is an isomorphism of real analytic orbifolds.

Another way to look at K_0 is as a disjoint union of incomplete real hyperbolic manifolds. To see this, let $\mathbf{RH}^4_\chi$ be the set of fixed points in $\mathbf{CH}^4$ of χ. Then

$$K_0 = \coprod_\chi \bigl(\mathbf{RH}^4_\chi - \mathcal{H}\bigr),$$

where χ varies over the integral anti-involutions of $\mathbf{CH}^4$. Now let C be a set of of representatives for the conjugacy classes of integral anti-involutions of $\mathbf{CH}^4$ under the action of $P\Gamma$. Let $P\Gamma_\chi$ be the centralizer of χ in $P\Gamma$. Then the quotient of K_0 by $P\Gamma$ is

$$P\Gamma \backslash K_0 = \coprod_{\chi \in C} P\Gamma_\chi \backslash \bigl(\mathbf{RH}^4_\chi - \mathcal{H}\bigr).$$

To understand this quotient in detail, we need to classify the integral anti-involutions χ of $\mathbf{CH}^4$, modulo the action of $P\Gamma$. One shows that there are just five classes, given by

$$\chi_j(z_0, \ldots, z_4) = (\bar{z}_0, \varepsilon_1 \bar{z}_1, \varepsilon_2 \bar{z}_2, \varepsilon_3 \bar{z}_3, \varepsilon_4 \bar{z}_4), \tag{3}$$

where j of the ε_i are -1 and the rest are $+1$. It is clear that each $P\Gamma_{\chi_j}$ is a subgroup of the projective automorphism group of the $\mathbf{Z}$-lattice Λ^{χ_j} fixed by χ_j, and one can check that it is the full projective isometry group. Computing the quadratic forms on the Λ^{χ_j} leads to the quadratic forms used to describe the $P\Gamma_j$ in (1), so $P\Gamma_{\chi_j} = P\Gamma_j$. This yields (1), where $\Delta_j = \mathbf{RH}^4_{\chi_j} \cap \mathcal{H}$. We found the Coxeter diagrams by using Vinberg's algorithm [6].

In order to carry out the gluing process leading to Theorem 1, we computed which points of the Weyl chambers C_j lie in $\mathcal{H}$; it turns out that $C_j \cap H$ is a union of faces of C_j. Then we had to figure out which faces of the C_j and $C_{j'}$ to glue to each other and how; for this we studied how the various $\mathbf{RH}^4_\chi$ meet in $\mathbf{CH}^4$. Finally we worked out the result of the gluing by explicitly manipulating polyhedra in $\mathbf{RH}^4$.

Acknowledgements

First author partly supported by NSF grants DMS-0070930 and DMS-0231585; second and third authors partly supported by NSF grants DMS-9900543 and DMS-0200877.

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