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Partial Differential Equations

Higher order energy expansions for some singularly perturbed Neumann problems

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Abstract

We consider the following singularly perturbed semilinear elliptic problem:

$$\varepsilon^2\Delta u-u+u^p=0\quad\text{in }\Omega,\quad u>0\quad\text{in }\Omega\quad\text{and}\quad\frac{\partial u}{\partial\nu}=0\quad\text{on }\partial\Omega,$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$, $\varepsilon>0$ is a small constant and p is a subcritical exponent. Let $J_\varepsilon[u]:=\int_\Omega(\frac{\varepsilon^2}{2}|\nabla u|^2+\frac{1}{2}u^2-\frac{1}{p+1}u^{p+1})\,\mathrm{d}x$ be its energy functional, where $u\in H^1(\Omega)$. Ni and Takagi proved that for a single boundary spike solution u_ε, the following asymptotic expansion holds

$$J_\varepsilon[u_\varepsilon]=\varepsilon^N\big[\tfrac{1}{2}I[w]-c_1\varepsilon H(P_\varepsilon)+\mathrm{o}(\varepsilon)\big],$$

where $c_1>0$ is a generic constant, P_ε is the unique local maximum point of u_ε and $H(P_\varepsilon)$ is the boundary mean curvature function. In this Note, we obtain the following higher order expansion of $J_\varepsilon[u_\varepsilon]$:

$$J_\varepsilon[u_\varepsilon]=\varepsilon^N\big[\tfrac{1}{2}I[w]-c_1\varepsilon H(P_\varepsilon)+\varepsilon^2[c_2(H(P_\varepsilon))^2+c_3R(P_\varepsilon)]+\mathrm{o}(\varepsilon^2)\big],$$

where c_2, c_3 are generic constants and $R(P_\varepsilon)$ is the Ricci scalar curvature at P_ε. In particular $c_3>0$. Applications of this expansion will be given. ***To cite this article: J. Wei, M. Winter, C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

Résumé

Développement asymptotique de l'énergie des solutions des problèmes de perturbations singulières. Nous étudions le problème suivant de perturbations singulières :

$$\varepsilon^2\Delta u-u+u^p=0\quad\text{dans }\Omega,\quad u>0\quad\text{dans }\Omega\quad\text{et}\quad\frac{\partial u}{\partial\nu}=0\quad\text{sur }\partial\Omega,$$

où Ω est un domaine ouvert dans $\mathbb{R}^N$, $\varepsilon>0$ est une constante petite et p est un exposant souscritique. L'énergie s'écrit alors $J_\varepsilon[u]:=\int_\Omega(\frac{\varepsilon^2}{2}|\nabla u|^2+\frac{1}{2}u^2-\frac{1}{p+1}u^{p+1})\,\mathrm{d}x$, où $u\in H^1(\Omega)$. Ni et Takagi montrent que pour une solution u_ε avec une pic sur la frontière du domaine, on a le développement asymptotique suivant :

$$J_\varepsilon[u_\varepsilon]=\varepsilon^N\big[\tfrac{1}{2}I[w]-c_1\varepsilon H(P_\varepsilon)+\mathrm{o}(\varepsilon)\big],$$

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où $c_1 > 0$ est une constante générique, P_ε est le point unique de maximum local de u_ε et $H(P_\varepsilon)$ est la fonction de la courbure moyenne sur la frontière. On établit que :

$$J_\varepsilon[u_\varepsilon] = \varepsilon^N\big[\tfrac{1}{2}I[w] - c_1\varepsilon H(P_\varepsilon) + \varepsilon^2[c_2(H(P_\varepsilon))^2 + c_3 R(P_\varepsilon)] + \mathrm{o}(\varepsilon^2)\big],$$

où c_2, c_3 sont les constantes génériques et $R(P_\varepsilon)$ est la courbure scalaire de Ricci en P_ε. En particulier $c_3 > 0$. Nous présentons des applications de ce développement asymptotique. ***Pour citer cet article : J. Wei, M. Winter, C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

Version française abrégée

Nous étudions le problème suivant de perturbations singulières :

$$\varepsilon^2\Delta u - u + u^p = 0 \quad \text{dans } \Omega, \quad u > 0 \quad \text{dans } \Omega \quad \text{et} \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{sur } \partial\Omega,$$

où Ω est un domaine ouvert dans $\mathbb{R}^N$ de frontière régulière, $\varepsilon > 0$ est une constante petite, Δ est l'opérateur de Laplace dans $\mathbb{R}^N$, ν est la normale extérieure sur $\partial\Omega$ et p vérifie $1 < p < (\frac{N+2}{N-2})_+$ ($=\frac{N+2}{N-2}$ si $N \geqslant 3$; $= +\infty$ si $N = 1, 2$).

L'énergie s'écrit alors $J_\varepsilon[u] := \int_\Omega(\frac{\varepsilon^2}{2}|\nabla u|^2 + \frac{1}{2}u^2 - \frac{1}{p+1}u^{p+1})\,\mathrm{d}x$, où $u \in H^1(\Omega)$. Ni et Takagi [15,16] ont montré que pour une solution u_ε avec une pic sur la frontière du domaine, on a le développement asymptotique suivant : $J_\varepsilon[u_\varepsilon] = \varepsilon^N[\frac{1}{2}I[w] - c_1\varepsilon H(P_\varepsilon) + \mathrm{o}(\varepsilon)]$, où $c_1 > 0$ est une constante générique, $P_\varepsilon \in \partial\Omega$ est le point unique de maximum local de u_ε, $H(P_\varepsilon)$ est la fonction de la courbure moyenne sur la frontière et $I[w]$ est l'énergie de l'état fondamental dans $\mathbb{R}^N$.

Dans ce travail on établit le développement suivant, à un ordre plus élevé de $J_\varepsilon[u_\varepsilon]$:

Théorème 1. *Pour une solution u_ε de* (I) *avec une pic sur la frontière du domaine et avec un point unique du maximum local de u_ε nous avons, pour ε suffisamment petit* :

$$J_\varepsilon = \varepsilon^N\left[\frac{1}{2}I[w] - c_1\varepsilon H(P_\varepsilon) + \varepsilon^2\big[c_2\big(H(P_\varepsilon)\big)^2 + c_3 R(P_\varepsilon)\big] + \mathrm{o}\big(\varepsilon^2\big)\right],$$

òu c_1, c_2, c_3 sont les constantes génériques. De plus $c_1 > 0$, $c_3 > 0$.

Le corollaire suivant donne un raffinement des résultats de [15] et [16].

Corollaire 2. *Pour une solution u_ε de l'énergie minimale de* (I) *et pour ε suffisamment petit nous avons*

$$H(P_\varepsilon) \to \max_{P\in\partial\Omega} H(P), \qquad R(P_\varepsilon) \to \min_{Q\in\partial\Omega,\ H(Q)=\max_{P\in\partial\Omega} H(P)} R(Q).$$

Il y a deux étapes essentielles dans la démonstration du Théorème 1. Dans l'étape 1 nous trouvons une fonction approximativement bonne $w_{\varepsilon,P}$ avec $\varepsilon^2\Delta\widetilde{w}_{\varepsilon,P} - \widetilde{w}_{\varepsilon,P} + w^p_{\varepsilon,P} = \mathrm{O}(\varepsilon^2)$. Dans l'étape nous montrons que $u_\varepsilon = \widetilde{w}_{\varepsilon,P_\varepsilon} + \mathrm{O}(\varepsilon^\tau)$, pour un $\tau > 1$.

1. Introduction

We consider the following singularly perturbed semilinear elliptic problem

$$\varepsilon^2\Delta u - u + f(u) = 0, \quad u > 0 \quad \text{in } \Omega \quad \text{and} \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $\varepsilon > 0$ is a small constant, $\Delta := \sum_{j=1}^N \frac{\partial^2}{\partial x_j \partial x_j}$ denotes the Laplace operator in $\mathbb{R}^N$, ν stands for the unit outer normal to $\partial\Omega$, $f(u) = u^p$ and p satisfies $1 < p < (\frac{N+2}{N-2})_+$ ($=\frac{N+2}{N-2}$ when $N \geqslant 3$; $= +\infty$ when $N = 1, 2$).

Eq. (1) arises in many branches of the applied sciences. For example, it can be viewed as a steady-state equation for the shadow system of the Gierer–Meinhardt model in biological pattern formation [7,18] or of parabolic equations in chemotaxis, population dynamics and phase transitions. Associated with (1) is the energy functional J_ε defined by:

$$J_\varepsilon[u] := \int_\Omega \left(\frac{\varepsilon^2}{2}|\nabla u|^2 + \frac{1}{2}u^2 - F(u)\right) dx \text{ for } u \in H^1(\Omega), \text{ where } F(u) = \int_0^u f(s)\, ds.$$

In the pioneering papers [14,15] and [16], Lin, Ni and Takagi established the existence of least-energy solutions and showed that for ε sufficiently small the least-energy solution has only one local maximum point P_ε with $P_\varepsilon \in \partial\Omega$. Moreover, $H(P_\varepsilon) \to \max_{P\in\partial\Omega} H(P)$ as $\varepsilon \to 0$, where $H(P)$ is the mean curvature of $\partial\Omega$ at P. Since then, many works have been devoted to finding solutions with multiple spikes for the Neumann problem as well as the Dirichlet problem. See [1–6,8–13,15–17,19–21], and the review article [18] and the references therein.

A common tool for proving the existence of spike solutions is by energy expansion: in [15] and [16], Ni and Takagi proved, among others, that for a single boundary spike solution u_ε the following asymptotic expansion for $J_\varepsilon[u_\varepsilon]$ holds true:

$$J_\varepsilon[u_\varepsilon] = \varepsilon^N\left[\frac{1}{2}I[w] - c_1\varepsilon H(P_\varepsilon) + o(\varepsilon)\right], \tag{2}$$

where $c_1 > 0$ is a generic constant, P_ε is the unique local maximum point of u_ε, $H(P_\varepsilon)$ is the mean curvature function at $P_\varepsilon \in \partial\Omega$, w is the unique solution of the following ground-state problem:

$$\Delta w - w + f(w) = 0, \quad w > 0 \quad \text{in } \mathbb{R}^N, \quad w(0) = \max_{y\in\mathbb{R}^N} w(y), \quad \lim_{|y|\to+\infty} w(y) = 0, \tag{3}$$

and $I[w]$ is the ground-state energy $I[w] = \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla w|^2 + \frac{1}{2}w^2 - F(w))\, dy$. Based on (2), Ni and Takagi [16] concluded that the least energy solution must concentrate at a maximum point of the mean curvature function. However, if $H(P)$ has more than one maximum point on $\partial\Omega$, the asymptotic expansion (2) has to be refined to prove such a statement and the next order term in (2) becomes important. This is exactly the purpose of this paper.

We now state our main theorem. First, we introduce boundary deformations. Let $P \in \partial\Omega$. After rotation and translation of the coordinate system we may assume that the inward normal to $\partial\Omega$ at P points in the direction of the positive x_N-axis, that $P = 0$, and that there exists a constant $\delta > 0$ and a smooth function ρ such that $\Omega \cap B_\delta(P) = \{(x', x_N) \mid x_N > \rho(x')\}$. Moreover, we may assume that $\rho(x') = \frac{1}{2}\sum_{i=1}^{N-1} k_i x_i^2 + O(|x'|^3)$, $x' = (x_1, \ldots, x_{N-1})$, where k_i, $i = 1, \ldots, N-1$, are the principal curvatures at P. (Note that $H(P) = \frac{1}{N-1}\sum_{i=1}^{N-1} k_i$ is the mean curvature.) For $N \geqslant 3$, we also need to define $R(P) = \sum_{i\neq j} k_i k_j$, which is called Ricci scalar curvature at P. When $N = 2$, we let $R(P) = 0$.

Now we can state the main result of this paper.

Theorem 1.1. *Let u_ε be a single boundary spike solution of* (1) *with a unique local maximum point $P_\varepsilon \in \partial\Omega$. Then, for ε sufficiently small, we have*:

$$J_\varepsilon = \varepsilon^N\left[\frac{1}{2}I[w] - c_1\varepsilon H(P_\varepsilon) + \varepsilon^2\left[c_2\big(H(P_\varepsilon)\big)^2 + c_3 R(P_\varepsilon)\right] + o\big(\varepsilon^2\big)\right], \tag{4}$$

where $c_1 = \frac{N-1}{N+1}\int_{R_+^N}(w')^2 y_N\, dy > 0$, and c_2, c_3 are generic constants. Moreover, we have $c_3 > 0$. Here $R_+^N = \{(y', y_N) \mid y_N > 0\}$.

As a corollary, we give a refinement of the results of [15] and [16].

Corollary 1.2. *Let u_ε be a least energy solution of* (1). *Then, for ε sufficiently small, we have*:

$$H(P_\varepsilon) \to \max_{P\in\partial\Omega} H(P), \qquad R(P_\varepsilon) \to \min_{Q\in\partial\Omega,\ H(Q)=\max_{P\in\partial\Omega} H(P)} R(Q). \tag{5}$$

Remark 1. The result (5) shows that the least energy solution will concentrate at a global maximum mean curvature point with smallest scalar curvature. For example, for $N = 3$, and suppose that the mean curvature function $H(P)$ has two global maximum points P_1 and P_2. Let the principal curvatures at P_i be given by $k_{i,j}$, $i = 1, 2$, $j = 1, 2$. Then $R(P_i) = k_{i,1}k_{i,2}$, $i = 1, 2$. The spike will approach the point with smaller R. However, if $N = 2$, (5) yields no new results. In that case, we have to expand $J_\varepsilon[u_\varepsilon]$ up to the order $\mathrm{O}(\varepsilon^3)$ to obtain more information on the spike locations.

Remark 2. Theorem 1.1 holds true if we replace $-u + u^p$ with more general nonlinearities; see [22].

2. Two important lemmas

In this section we present two main lemmas needed to prove Theorem 1.1. We begin with the following on good approximate functions:

Lemma 2.1. *For each $P \in \partial\Omega$, there exists a smooth function $\widetilde{w}_{\varepsilon,P}$ such that*

$$\varepsilon^2 \Delta \widetilde{w}_{\varepsilon,P} - \widetilde{w}_{\varepsilon,P} + f(\widetilde{w}_{\varepsilon,P}) = \mathrm{O}\big(\varepsilon^{1+\sigma}\big), \tag{6}$$

$$J_\varepsilon[\widetilde{w}_{\varepsilon,P}] = \varepsilon^N \left[\frac{1}{2} I[w] - c_1 \varepsilon H(P) + \varepsilon^2 \big[c_2\big(H(P)\big)^2 + c_3 R(P)\big] + \mathrm{o}\big(\varepsilon^2\big)\right], \tag{7}$$

where $\sigma = \min(1, p-1)$ and c_1, c_2, c_3 are generic constants. In particular, $c_3 = \frac{1}{16}\int_{\mathbb{R}^N_+} [|\nabla \Psi_0|^2 + |\Psi_0|^2 - f'(w)\Psi_0^2]\,\mathrm{d}y > 0$, where Ψ_0 satisfies $\Delta\Psi_0 - \Psi_0 + f'(w)\Psi_0 = 0$ in $\mathbb{R}^N_+$, $\frac{\partial \Psi_0}{\partial y_N} = \frac{w'}{|y|}(y_1^2 - y_2^2)$ on $\partial\mathbb{R}^N_+$.

The proof of Lemma 2.1 is technical and we refer to Sections 2 and 3 of [22].

Our next lemma is about the expansion of u_ε which is a single boundary spike solution of (1). Let P_ε be its local maximum point. The key observation is that by using $\widetilde{w}_{\varepsilon,P_\varepsilon}$ as our approximating function, we just need to expand u_ε up to $\mathrm{O}(\varepsilon^\tau)$ for some $\tau > 1$. In fact, we do not even need to know the exact asymptotic expansion in $\mathrm{O}(\varepsilon^\tau)$. We now choose $\tau = 1 + \frac{\sigma}{2}$. Thus we get:

Lemma 2.2. *For ε sufficiently small, we have $u_\varepsilon = \widetilde{w}_{\varepsilon,P_\varepsilon} + \varepsilon^\tau \phi_\varepsilon$, where ϕ_ε satisfies*:

$$\|\phi_\varepsilon\|_{L^\infty(\overline{\Omega})} \leqslant C, \tag{8}$$

$$\varepsilon^{-N} \int_\Omega \big(\varepsilon^2 |\nabla \phi_\varepsilon|^2 + |\phi_\varepsilon|^2\big)\,\mathrm{d}x \leqslant C. \tag{9}$$

Proof. We sketch the main ideas of the proof. For details, see Section 5 of [22]. Substituting $u_\varepsilon = \widetilde{w}_{\varepsilon,P_\varepsilon} + \varepsilon^\tau \phi_\varepsilon$ into (1), we see from (6) that ϕ_ε satisfies:

$$\varepsilon^2 \Delta \phi_\varepsilon - \phi_\varepsilon + f'(\widetilde{w}_{\varepsilon,P_\varepsilon})\phi_\varepsilon = \mathrm{O}\big(\varepsilon^{\sigma/2}\big) + N_\varepsilon[\phi_\varepsilon] \quad \text{in } \Omega, \qquad \frac{\partial \phi_\varepsilon}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \tag{10}$$

where $N_\varepsilon[\phi_\varepsilon] = -\varepsilon^{-\tau}[f(\widetilde{w}_{\varepsilon,P_\varepsilon} + \varepsilon^\tau \phi_\varepsilon) - f(\widetilde{w}_{\varepsilon,P_\varepsilon}) - \varepsilon^\tau f'(\tilde{w}_{\varepsilon,P_\varepsilon})\phi_\varepsilon] = \mathrm{o}(1)|\phi_\varepsilon|$, by the mean value theorem.

Now we can prove (8). Suppose not, then there exists a sequence $\varepsilon_k \to 0$ such that $M_{\varepsilon_k} := \|\phi_{\varepsilon_k}\|_{L^\infty(\overline{\Omega})} \to +\infty$. For simplicity of notation, we still denote ε_k by ε. Let $M_\varepsilon = |\phi_\varepsilon(x_\varepsilon)|$, where $x_\varepsilon \in \overline{\Omega}$. Without loss of generality, we may assume that x_ε is a maximum point of ϕ_ε. We proceed by proving two claims.

Claim 2.3. $|x_\varepsilon - P_\varepsilon|/\varepsilon \leqslant C$.

Suppose not, that is $|x_\varepsilon - P_\varepsilon|/\varepsilon \to +\infty$. Then $-1 + f'(\widetilde{w}_{\varepsilon,P_\varepsilon}(x_\varepsilon)) \leqslant -\frac{1}{4}$ for ε small. Since $\partial\phi_\varepsilon/\partial\nu = 0$, by the Hopf boundary Lemma, $x_\varepsilon \notin \partial\Omega$. So $x_\varepsilon \in \Omega$, which implies $\Delta\phi_\varepsilon(x_\varepsilon) \leqslant 0$. From (10) we then deduce that

$$\bigl(1 - f'\bigl(\widetilde{w}_{\varepsilon,P_\varepsilon}(x_\varepsilon)\bigr)\bigr)M_\varepsilon + \mathrm{o}(1)M_\varepsilon + \mathrm{O}\bigl(\varepsilon^{\tau-1}\bigr) \leqslant 0$$

and hence M_ε is bounded, a contradiction. Let $\hat{\phi}_\varepsilon(y) = \phi_\varepsilon(x)/M_\varepsilon$, where $\varepsilon y = x - P$.

Claim 2.4. $\hat{\phi}_\varepsilon(y) \to 0$ *in* $C^1_{\mathrm{loc}}(\mathbb{R}^N_+)$, *as* $\varepsilon \to 0$.

In fact, from the equation for $\hat{\phi}_\varepsilon$, we see that as $\varepsilon \to 0$, $\hat{\phi}_\varepsilon \to \hat{\phi}_0$, where $\Delta\hat{\phi}_0 - \hat{\phi}_0 + f'(w)\hat{\phi}_0 = 0$, $|\hat{\phi}_0| \leqslant 1$, in $\mathbb{R}^N_+$, $\frac{\partial\hat{\phi}_0}{\partial y_N} = 0$ on $\partial\mathbb{R}^N_+$. By the nondegeneracy of w, there exist $N-1$ constants $a_1, \dots, a_{N-1}$ such that $\hat{\phi}_0 = \sum_{j=1}^{N-1} a_j \frac{\partial w}{\partial y_j}$. On the other hand, we know that $\nabla_{x_k} u_\varepsilon(P_\varepsilon) = 0$, $k = 1, \dots, N-1$, and hence

$$0 = \nabla_{x_k}\bigl(\widetilde{w}_{\varepsilon,P_\varepsilon}(P_\varepsilon) + \varepsilon^\tau\phi_\varepsilon(P_\varepsilon)\bigr) = \mathrm{O}(\varepsilon) + \varepsilon^{\tau-1}M_\varepsilon\nabla_{y_k}\hat{\phi}_\varepsilon(0).$$

Thus we have $\nabla_{y_k}\hat{\phi}_\varepsilon(0) \to 0$ which shows that $\nabla_{y_k}\hat{\phi}_0(0) = 0$. This implies $\nabla_{y_k}(\sum_{j=1}^{N-1} a_j \frac{\partial w}{\partial y_j})_{y=0} = 0$, $k = 1, \dots, N-1$. Thus $a_1 = \dots = a_{N-1} = 0$. This proves Claim 2.4.

Eq. (8) now follows from Claim 2.3 and Claim 2.4: let $y_\varepsilon = (x_\varepsilon - P_\varepsilon)/\varepsilon$. Then by Claim 2.3, $|y_\varepsilon| \leqslant C$. So we may assume that $y_\varepsilon \to y_0$ as $\varepsilon \to 0$. Since $\hat{\phi}_\varepsilon(y_\varepsilon) = 1$, we have $\hat{\phi}_0(y_0) = 1$, which contradicts Claim 2.4.

Multiplying (10) by ϕ_ε, integrating over Ω and using (8), we obtain (9). □

3. Proofs of Theorem 1.1 and Corollary 1.2

We prove Theorem 1.1 by using Lemmas 2.1 and 2.2.

Proof of Theorem 1.1. Since both $\widetilde{w}_{\varepsilon,P_\varepsilon}$ and ϕ_ε satisfy the Neumann boundary condition, we get:

$$\begin{aligned} J_\varepsilon[u_\varepsilon] = {} & J_\varepsilon[\widetilde{w}_{\varepsilon,P}] + \varepsilon^\tau \int_\Omega \bigl(\varepsilon^2\nabla\widetilde{w}_{\varepsilon,P}\nabla\phi_\varepsilon + \widetilde{w}_{\varepsilon,P}\phi_\varepsilon - f(\widetilde{w}_{\varepsilon,P})\phi_\varepsilon\bigr)\,\mathrm{d}x \\ & + \frac{\varepsilon^{2\tau}}{2}\biggl(\int_\Omega \bigl(\varepsilon^2|\nabla\phi_\varepsilon|^2 + |\phi_\varepsilon|^2\bigr)\,\mathrm{d}x - \int_\Omega f'(\widetilde{w}_{\varepsilon,P_\varepsilon})\phi_\varepsilon^2\,\mathrm{d}x\biggr) \\ & - \int_\Omega \biggl[F\bigl(\widetilde{w}_{\varepsilon,P_\varepsilon} + \varepsilon^\tau\phi_\varepsilon\bigr) - F(\widetilde{w}_{\varepsilon,P_\varepsilon}) - \varepsilon^\tau f(\widetilde{w}_{\varepsilon,P_\varepsilon})\phi_\varepsilon - \frac{\varepsilon^{2\tau}}{2}f'(\widetilde{w}_{\varepsilon,P_\varepsilon})\phi_\varepsilon^2\biggr]\,\mathrm{d}x. \end{aligned}$$

By Lemma 2.2, the last two terms are $\mathrm{o}(\varepsilon^{N+2})$. Now integrating by parts and using (6) we obtain

$$\varepsilon^\tau \int_\Omega \bigl(\varepsilon^2\nabla\widetilde{w}_{\varepsilon,P}\nabla\phi_\varepsilon + \widetilde{w}_{\varepsilon,P}\phi_\varepsilon - f(\widetilde{w}_{\varepsilon,P})\phi_\varepsilon\bigr)\,\mathrm{d}x = \varepsilon^\tau \int_\Omega S_\varepsilon[\widetilde{w}_{\varepsilon,P_\varepsilon}]\phi_\varepsilon\,\mathrm{d}x = \mathrm{O}\bigl(\varepsilon^{N+1+\tau+\sigma}\bigr).$$

Hence $J_\varepsilon[u_\varepsilon] = J_\varepsilon[\widetilde{w}_{\varepsilon,P}] + \mathrm{o}(\varepsilon^{N+2})$ which, by Lemma 2.2, finishes the proof of Theorem 1.1. □

Proof of Corollary 1.2. Let u_ε be a least energy solution of (1). By Theorem 1.1, we have:

$$c_\varepsilon := J_\varepsilon[u_\varepsilon] = \varepsilon^N\biggl[\frac{1}{2}I[w] - c_1\varepsilon H(P_\varepsilon) + \varepsilon^2\bigl(c_2\bigl(H(P_\varepsilon)\bigr)^2 + c_3R(P_\varepsilon)\bigr) + \mathrm{o}\bigl(\varepsilon^2\bigr)\biggr]. \tag{11}$$

On the other hand, by using $\widetilde{w}_{\varepsilon,Q}$ as test function, we see that

$$c_\varepsilon \leqslant \varepsilon^N \left[\frac{1}{2} I[w] - c_1 \varepsilon H(Q) + \varepsilon^2 \big(c_2 \big(H(Q)\big)^2 + c_3 R(Q)\big) + \mathrm{o}\big(\varepsilon^2\big)\right], \tag{12}$$

where we take Q such that $H(Q) = \max_{P \in \partial\Omega} H(P)$. Comparing (11) with (12), we arrive at

$$c_1 H(Q) - \varepsilon\big(c_2 \big(H(Q)\big)^2 + c_3 R(Q)\big) + \mathrm{o}(\varepsilon) \leqslant c_1 H(P_\varepsilon) - \varepsilon\big(c_2 \big(H(P_\varepsilon)\big)^2 + c_3 R(P_\varepsilon)\big) + \mathrm{o}(\varepsilon).$$

Since $c_1 > 0$, $c_3 > 0$, we obtain (5). □

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References

[1] P.F. Adimurthi, S.L. Yadava, Interaction between the geometry of the boundary and positive solutions of a semilinear Neumann problem with critical nonlinearity, J. Funct. Anal. 113 (2) (1993) 318–350.
[2] N. Alikakos, M. Kowalczyk, Critical points of a singular perturbation problem via reduced energy and local linking, J. Differential Equations 159 (1999) 403–426.
[3] P. Bates, E.N. Dancer, J. Shi, Multi-spike stationary solutions of the Cahn–Hilliard equation in higher-dimension and instability, Adv. Differential Equations 4 (1999) 1–69.
[4] P. Bates, G. Fusco, Equilibria with many nuclei for the Cahn–Hilliard equation, J. Differential Equations 4 (1999) 1–69.
[5] E.N. Dancer, S. Yan, Multipeak solutions for a singular perturbed Neumann problem, Pacific J. Math. 189 (1999) 241–262.
[6] M. del Pino, P. Felmer, Spike-layered solutions of singularly perturbed elliptic problems in a degenerate setting, Indiana Univ. Math. J. 48 (1999) 883–898.
[7] A. Gierer, H. Meinhardt, A theory of biological pattern formation, Kybernetik (Berlin) 12 (1972) 30–39.
[8] M. Grossi, A. Pistoia, J. Wei, Existence of multipeak solutions for a semilinear Neumann problem via nonsmooth critical point theory, Calc. Var. Partial Differential Equations 11 (2000) 143–175.
[9] C. Gui, J. Wei, Multiple interior peak solutions for some singular perturbation problems, J. Differential Equations 158 (1999) 1–27.
[10] C. Gui, J. Wei, M. Winter, Multiple boundary peak solutions for some singularly perturbed Neumann problems, Ann. Inst. H. Poincaré, Anal. Non Linéaire 17 (2000) 47–82.
[11] M. Kowalczyk, Multiple spike layers in the shadow Gierer–Meinhardt system: existence of equilibria and approximate invariant manifold, Duke Math. J. 98 (1999) 59–111.
[12] Y.-Y. Li, On a singularly perturbed equation with Neumann boundary condition, Comm. Partial Differential Equations 23 (1998) 487–545.
[13] Y.-Y. Li, L. Nirenberg, The Dirichlet problem for singularly perturbed elliptic equations, Comm. Pure Appl. Math. 51 (1998) 1445–1490.
[14] C.-S. Lin, W.-M. Ni, I. Takagi, Large amplitude stationary solutions to a chemotaxis systems, J. Differential Equations 72 (1988) 1–27.
[15] W.-M. Ni, I. Takagi, On the shape of least energy solution to a semilinear Neumann problem, Comm. Pure Appl. Math. 41 (1991) 819–851.
[16] W.-M. Ni, I. Takagi, Locating the peaks of least energy solutions to a semilinear Neumann problem, Duke Math. J. 70 (1993) 247–281.
[17] W.-M. Ni, J. Wei, On the location and profile of spike-layer solutions to singularly perturbed semilinear Dirichlet problems, Comm. Pure Appl. Math. 48 (1995) 731–768.
[18] W.-M. Ni, Diffusion, cross-diffusion, and their spike-layer steady states, Notices Amer. Math. Soc. 45 (1998) 9–18.
[19] J. Wei, On the construction of single-peaked solutions to a singularly perturbed semilinear Dirichlet problem, J. Differential Equations 129 (1996) 315–333.
[20] J. Wei, On the boundary spike layer solutions of singularly perturbed semilinear Neumann problem, J. Diffential Equations 134 (1997) 104–133.
[21] J. Wei, M. Winter, Stationary solutions for the Cahn–Hilliard equation, Ann. Inst. H. Poincaré Anal. Non Linéaire 15 (1998) 459–492.
[22] J. Wei, M. Winter, Higher order energy expansions and location of spikes for some singularly perturbed Neumann problems, Submitted.