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Numerical Analysis

The topological asymptotic with respect to a singular boundary perturbation

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Abstract

The aim of the topological sensitivity analysis is to obtain an asymptotic expansion of a design functional with respect to the insertion of a small hole in the domain. The question that we address here is what happens if the hole is located at the boundary of the domain and what happens if the boundary is not regular. The adjoint method and the domain truncation technique are proposed to solve this problem. As a model example, we consider the Laplace equation in a domain with a corner. ***To cite this article: B. Samet, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Résumé

L'asymptotique topologique par rapport à une perturbation singulière du bord. Le but de la sensibilité topologique est d'obtenir une expression asymptotique d'une fonctionnelle de forme par rapport à l'insertion d'un petit trou dans le domaine. Dans cette Note, nous considérons le cas d'un petit trou situé sur un coin du domaine. La méthode de l'état adjoint et la technique de troncature de domaine sont proposées pour résoudre ce probléme. Nous considérons comme exemple modèle, l'équation de Laplace posée dans un domaine avec un coin. ***Pour citer cet article : B. Samet, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Version française abrégée

La méthode de la sensibilité topologique est basée sur l'idée suivante. Soit une fonction coût $\mathcal{J}(\Omega) = J(\Omega, u_\Omega)$, où Ω est un ouvert de $\mathbb{R}^d$, $d \geqslant 2$, et u_Ω est la solution d'un problème aux dérivées partielles posé dans le domaine Ω. Si nous créons un trou $B(x,\varepsilon)$ dans le domaine Ω, nous pouvons montrer (dans la plupart des cas) que la variation de la fonction coût admet l'expression asymptotique suivante :

$$\mathcal{J}\big(\Omega\backslash\overline{B(x,\varepsilon)}\big) - \mathcal{J}(\Omega) = f(\varepsilon)G(x) + \mathrm{o}\big(f(\varepsilon)\big). \tag{1}$$

La fonction $f(\varepsilon)$ est strictement positive et tend vers zéro avec ε. L'expression (1) est appelée « asymptotique topologique ». La fonction G définie dans (1) est appelée « gradient topologique ». Pour minimiser notre critère,

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nous devons créer des trous où le gradient est négatif. L'asymptotique topologique a été obtenue pour des problèmes divers [1–10]. Tous ces problèmes ont en commun le fait que le trou est assez loin du bord du domaine. Dans cette Note, nous considérons le problème suivant. Soit Ω un ouvert borné du plan. Une partie Γ_0 du bord est définie par deux segments formant un angle $\lambda\pi$, $0<\lambda<2$ (voir Fig. 1). Nous notons u_Ω la solution du problème de Laplace posé dans le domaine Ω, vérifiant $u=0$ sur Γ_0 et une condition aux limites sur $\Gamma_1=\partial\Omega\backslash\overline{\Gamma_0}$. Pour $\varepsilon>0$ (assez petit), nous considérons le domaine perturbé $\Omega_\varepsilon=\Omega\backslash\overline{S_\varepsilon}$, où S_ε est le secteur (voir Fig. 1) défini par $S_\varepsilon=\{(r,\theta)\,;\ 0\leqslant r<\varepsilon,\ 0\leqslant\theta\leqslant\lambda\pi\}$. Notre but est de donner une expression asymptotique de la variation $J(u_{\Omega_\varepsilon})-J(u_\Omega)$, où u_{Ω_ε} est la solution du problème de Laplace dans le domaine perturbé avec une condition de Dirichlet imposée sur l'arc de cercle joignant les deux segments du secteur S_ε (problème (8)).

Dans cette Note, nous utilisons la méthode de l'état adjoint et une technique de troncature de domaine [4] pour déterminer une formule générale de l'asymptotique topologique (Theorem 2.5). Ensuite, nous étudions le cas particulier où $\lambda^{-1}\in\mathbb{N}^*$ (Corollary 2.6).

1. Introduction

Classical shape optimization methods are based on the perturbation of the boundary of the initial shape. The initial and the final shape have the same topology. The aim of topological optimization is to find an optimal shape without any *a priori* assumption about the topology of the structure. Unlike the case of classical shape optimization, the topology of the structure may change during the optimization process, as, for example, through the inclusion of holes. Recently, the notion of topological sensitivity brings a new approach for topological optimization. It provides an asymptotic expansion of a shape function with respect to the creation of a small hole in the domain. To present the basic idea, we consider Ω a domain of $\mathbb{R}^d$, $d\geqslant 2$, and $\mathcal{J}(\Omega)=J(u_\Omega)$ a cost function to be minimized, where u_Ω is solution to a given PDE problem defined in Ω. For $\varepsilon>0$, let $\Omega\backslash\overline{B(x,\varepsilon)}$ be the perturbed domain. Then, an asymptotic expansion of the function $\mathcal{J}$ can be obtained in the following form:

$$\mathcal{J}\big(\Omega\backslash\overline{B(x,\varepsilon)}\big)-\mathcal{J}(\Omega)=f(\varepsilon)G(x)+\mathrm{o}\big(f(\varepsilon)\big). \tag{2}$$

Here, $f(\varepsilon)$ is an explicit positive function going to zero with ε. Hence, to minimize the criterion $\mathcal{J}$ we just have to create infinitely small holes at some points $\tilde{x}$ where the function G (called the topological gradient) is negative. The expression (2) is called "topological asymptotic". The topological asymptotic has been obtained for various problems [1–10]. In all these publications, the hole is located far enough from the boundary of the domain. In this work, we consider an initial domain $\Omega\subset\mathbb{R}^2$ with a corner. The perturbed domain is defined by $\Omega_\varepsilon=\Omega\backslash\overline{S_\varepsilon}$, where S_ε is given by $S_\varepsilon=\{(r,\theta);\ 0\leqslant r<\varepsilon,\ 0\leqslant\theta\leqslant\lambda\pi\}$, $0<\lambda<2$ (see Fig. 1). Our aim is to obtain the topological

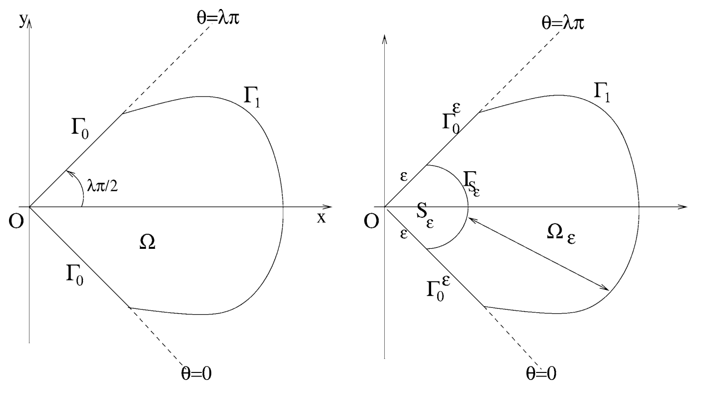

Fig. 1. The initial domain and the same domain after perturbation.

Fig. 1. Le domaine initial et le domain perturbé.

asymptotic for the Laplace equation in the corner domain Ω. We use the adjoint method and a domain truncation technique [4] to obtain a general formula of the topological asymptotic ($0 < \lambda < 2$). In the case of $\lambda^{-1} \in \mathbb{N}^*$, we obtain a simplified formula.

2. The Laplace problem in a domain with a corner

2.1. The adjoint method

In this subsection, we recall the adjoint method introduced in [1]. Let $\mathcal{V}$ be a Hilbert space. For all $\varepsilon \geqslant 0$, let a_ε be a bilinear and continuous form on $\mathcal{V}$ and ℓ be a linear and continuous form on $\mathcal{V}$. We assume that for all $\varepsilon \geqslant 0$, the bilinear form a_ε is coercive. Using the Lax–Milgram theorem, the following problem: find $u_\varepsilon \in \mathcal{V}$ such that

$$a_\varepsilon(u_\varepsilon, v) = \ell(v) \quad \forall v \in \mathcal{V} \tag{3}$$

has one and only one solution. We consider a cost function: $j(\varepsilon) = J(u_\varepsilon)$, where $J \in \mathcal{C}^1(\mathcal{V}, \mathbb{R})$. Let $v_0 \in \mathcal{V}$ the solution to the adjoint problem:

$$a_0(v, v_0) = -DJ(u_0) \cdot v \quad \forall v \in \mathcal{V}. \tag{4}$$

We call v_0 the adjoint state. We assume that

$$\left\| a_\varepsilon - a_0 - f(\varepsilon)\delta_a \right\|_{\mathcal{L}_2(\mathcal{V})} = \mathrm{o}\big(f(\varepsilon)\big), \tag{5}$$

where $f(\varepsilon) > 0$, $\lim_{\varepsilon \to 0^+} f(\varepsilon) = 0$ and δ_a is a bilinear and continuous form on $\mathcal{V}$. We have the following theorem.

Theorem 2.1. *We have that*

$$j(\varepsilon) - j(0) = f(\varepsilon)\delta_a(u_0, v_0) + \mathrm{o}\big(f(\varepsilon)\big).$$

2.2. Problem formulation

Let Ω be a bounded domain of $\mathbb{R}^2$. The boundary of Ω, denoted by $\partial\Omega$, is assumed to be smooth except at a point O, in the vicinity of which $\partial\Omega$ is defined by two straight line segments Σ_1, Σ_2 forming an angle $\lambda\pi$, $0 < \lambda < 2$ (see Fig. 1). The boundary $\partial\Omega$ is split into parts Γ_0, Γ_1 such that $\Gamma_0 = \Sigma_1 \cup \Sigma_2$ and $\Gamma_1 = \partial\Omega \backslash \overline{\Gamma_0}$. We consider the Laplace problem: find $u_\Omega \in \mathcal{V}(\Omega)$ such that

$$\begin{cases} -\Delta u_\Omega = 0 & \text{in } \Omega, \\ u_\Omega = 0 & \text{on } \Gamma_0, \\ \dfrac{\partial u_\Omega}{\partial n} = h & \text{on } \Gamma_1, \end{cases} \tag{6}$$

where $h \in H_{00}^{1/2}(\Gamma_1)'$ and the functional space $\mathcal{V}(\Omega)$ is defined by: $\mathcal{V}(\Omega) = \{v \in H^1(\Omega);\ v_{|\Gamma_0} = 0\}$. It is clear that problem (6) has one and only one solution. We consider now a cost function $\mathcal{J}(\Omega) = J(u_\Omega)$. We assume (for simplicity) that the function J is defined in a neighbor part of Γ_1. The adjoint problem is: find $v_\Omega \in \mathcal{V}(\Omega)$ such that:

$$a(v, v_\Omega) = -DJ(u_\Omega) \cdot v \quad \forall v \in \mathcal{V}(\Omega), \tag{7}$$

where $a(u, v) = \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x$ $\forall u, v \in \mathcal{V}(\Omega)$. Let $u_{\Omega\varepsilon}$ be the solution to the perturbed problem

$$\begin{cases} -\Delta u_{\Omega\varepsilon} = 0 & \text{in } \Omega_\varepsilon, \\ u_{\Omega\varepsilon} = 0 & \text{on } \Gamma_{S_\varepsilon} \cup \Gamma_0^\varepsilon, \\ \dfrac{\partial u_{\Omega\varepsilon}}{\partial n} = h & \text{on } \Gamma_1, \end{cases} \tag{8}$$

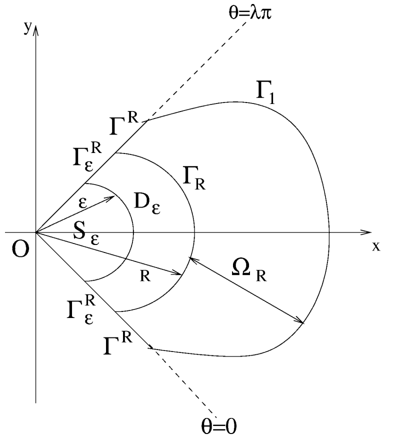

Fig. 2. The truncated domain.

Fig. 2. Le domaine tronqué.

where $\Gamma_0^\varepsilon = \{(r,\theta);\, r > \varepsilon,\, \theta \in \{0, \lambda\pi\}\} \cap \Gamma_0$ and $\Gamma_{S_\varepsilon} = \{(r,\theta);\, r = \varepsilon,\, 0 \leqslant \theta \leqslant \lambda\pi\}$.

The function $u_{\Omega\varepsilon}$ is defined on the variable set Ω_ε, thus it belongs to a functional space which depends on ε. Hence, if we want to derive the asymptotic expansion of a function of the form $j(\varepsilon) = J(u_{\Omega\varepsilon})$, we cannot apply directly the tools of Subsection 2.1, which require a fixed functional space. However, a functional space independent of ε can be constructed by using a domain truncation technique.

2.3. The domain truncation

Let $R > \varepsilon$ be such that the sector $\overline{S_R}$ is included in Ω. Here, S_R is defined by: $S_R = \{(r,\theta);\, 0 \leqslant r < R,\, 0 \leqslant \theta \leqslant \lambda\pi\}$. We introduce the following notations: the truncated domain $\Omega \backslash \overline{S_R}$ is denoted by Ω_R, $D_\varepsilon = S_R \backslash \overline{S_\varepsilon}$, $\Gamma_\varepsilon^R = \{(r,\theta);\, \varepsilon \leqslant r \leqslant R,\, \theta \in \{0, \lambda\pi\}\}$, $\Gamma^R = \{(r,\theta);\, r \geqslant R,\, \theta \in \{0, \lambda\pi\}\} \cap \Gamma_0$ and $\Gamma_R = \{(r,\theta);\, r = R,\, 0 \leqslant \theta \leqslant \lambda\pi\}$ (see Fig. 2). For $\varphi \in H_{00}^{1/2}(\Gamma_R)$ and $\varepsilon > 0$, we consider u_ε^φ the solution to the problem

$$\begin{cases} -\Delta u_\varepsilon^\varphi = 0 & \text{in } D_\varepsilon, \\ u_\varepsilon^\varphi = 0 & \text{on } \Gamma_{S_\varepsilon} \cup \Gamma_\varepsilon^R, \\ u_\varepsilon^\varphi = \varphi & \text{on } \Gamma_R. \end{cases} \tag{9}$$

For $\varepsilon = 0$, u_0^φ is the solution to the problem

$$\begin{cases} -\Delta u_0^\varphi = 0 & \text{in } D_0, \\ u_0^\varphi = 0 & \text{on } \partial D_0 \backslash \overline{\Gamma_R}, \\ u_0^\varphi = \varphi & \text{on } \Gamma_R, \end{cases} \tag{10}$$

where $D_0 = S_R$. For $\varepsilon \geqslant 0$, let T_ε be the Dirichlet-to-Neumann operator defined by: $T_\varepsilon \varphi = \nabla u_\varepsilon^\varphi \cdot n_{|\Gamma_R}$, where $n_{|\Gamma_R}$ is the outward normal to the boundary Γ_R. The truncated problem is: find $u_\varepsilon \in \mathcal{V}(\Omega_R)$ such that

$$\begin{cases} -\Delta u_\varepsilon = 0 & \text{in } \Omega_R, \\ u_\varepsilon = 0 & \text{on } \Gamma^R, \\ \dfrac{\partial u_\varepsilon}{\partial n} = h & \text{on } \Gamma_1, \\ \dfrac{\partial u_\varepsilon}{\partial n} + T_\varepsilon u_{\varepsilon|\Gamma_R} = 0 & \text{on } \Gamma_R, \end{cases} \tag{11}$$

where the functional space $\mathcal{V}(\Omega_R)$ is defined by

$$\mathcal{V}(\Omega_R) = \{v \in H^1(\Omega_R);\ v_{|\Gamma^R} = 0\}. \tag{12}$$

The variational formulation associated to problem (11) is the following: find $u_\varepsilon \in \mathcal{V}(\Omega_R)$ such that

$$a_\varepsilon(u_\varepsilon, v) = \ell(v) \quad \forall v \in \mathcal{V}(\Omega_R), \tag{13}$$

where the bilinear form a_ε and the linear form ℓ are defined by: $a_\varepsilon(u, v) = \int_{\Omega_R} \nabla u \cdot \nabla v \,\mathrm{d}x + \int_{\Gamma_R} T_\varepsilon u \cdot v \,\mathrm{d}\gamma$ and $\ell(v) = \int_{\Gamma_1} h \cdot v \,\mathrm{d}\gamma(x)$. The following result is standard in PDE theory.

Proposition 2.2. *Problem* (11) *has one and only one solution which is the restriction to Ω_R of the solution to* (8).

We have now at our disposal the fixed Hilbert space $\mathcal{V}(\Omega_R)$ required by the adjoint method. The variation of the bilinear form $a_\varepsilon - a_0$ can be written:

$$(a_\varepsilon - a_0)(u, v) = \int_{\Gamma_R} (T_\varepsilon - T_0)u \cdot v \,\mathrm{d}\gamma(x). \tag{14}$$

Hence, our problem reduces to the computation of $(T_\varepsilon - T_0)\varphi$ for $\varphi = u_{|\Gamma_R}$.

2.4. The asymptotic expansion

We have the following result.

Proposition 2.3. *The operator T_ε is given by the explicit expression*:

$$T_\varepsilon \varphi = \sum_{n \in \mathbb{N}^*} \left(\frac{n}{\lambda}\right) \frac{\varepsilon^{n/\lambda} R^{(-n/\lambda)-1} + \varepsilon^{-n/\lambda} R^{(n/\lambda)-1}}{\varepsilon^{-n/\lambda} R^{n/\lambda} - \varepsilon^{n/\lambda} R^{-n/\lambda}} \varphi_n \sin\left(n\frac{\theta}{\lambda}\right), \quad \forall \varphi \in H_{00}^{1/2}(\Gamma_R),$$

where $\varphi_n = \int_0^{\lambda\pi} \varphi(R, \theta) \sin(n\theta/\lambda) \,\mathrm{d}\theta$.

We introduce the operator δ_T defined by: $\delta_T \varphi = 2/(\lambda R^{(2/\lambda)+1})\varphi_1 \sin(\theta/\lambda)$. Using Proposition 2.3, we obtain the following result.

Proposition 2.4. *We have that*

$$\left\| T_\varepsilon - T_0 - \varepsilon^{2/\lambda}\delta_T \right\|_{\mathcal{L}(H_{00}^{1/2}(\Gamma_R), H_{00}^{1/2}(\Gamma_R)')} = \mathrm{o}\left(\varepsilon^{2/\lambda}\right).$$

It follows from Proposition 2.4, (14) and Theorem 2.1 that the following theorem holds.

Theorem 2.5. *The function j has the following asymptotic expansion*:

$$j(\varepsilon) - j(0) = \varepsilon^{2/\lambda}\delta_a(u_\Omega, v_\Omega) + \mathrm{o}\left(\varepsilon^{2/\lambda}\right),$$

where u_Ω is the solution to (6), *v_Ω is the solution to* (7) *and δ_a is defined by*:

$$\delta_a(u, v) = \pi \frac{u_1}{R^{1/\lambda}} \frac{v_1}{R^{1/\lambda}}, \quad \forall u, v \in \mathcal{V}(\Omega_R). \tag{15}$$

Here, $X_1 = 2(\lambda\pi)^{-1} \int_0^{\lambda\pi} X(R, \theta) \sin(\frac{\theta}{\lambda}) \,\mathrm{d}\theta$, *$X = u$ or v.*

As j is usually independent of R and $\delta_a(u_\Omega, v_\Omega)$ is independent of ε, it follows from the uniqueness of an asymptotic expansion that $\delta_a(u_\Omega, v_\Omega)$ is also independent of R. Using (15) leads to the following result.

Corollary 2.6. *If* $\lambda^{-1} \in \mathbb{N}^*$*, then*

$$j(\varepsilon) - j(0) = \pi \left[\left(\frac{1}{\lambda}\right)!\right]^{-2} \varepsilon^{2/\lambda} \frac{\partial^{1/\lambda} u_\Omega}{\partial x^{1/\lambda}}(O) \frac{\partial^{1/\lambda} v_\Omega}{\partial x^{1/\lambda}}(O) + \mathrm{o}\big(\varepsilon^{2/\lambda}\big). \tag{16}$$

Remark. In our situation, we computed the expansion of the topological asymptotic by the use of the adjoint method and the domain truncation technique. However, other interesting cases seem worthy of study. For example, what happen if the initial angle is rounded? or if one cut it by a straight line? These questions are open.

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