Available online at www.sciencedirect.com
SCIENCE DIRECT®

C. R. Acad. Sci. Paris, Ser. I 336 (2003) 985–990

COMPTES RENDUS
MATHEMATIQUE

Partial Differential Equations/Mathematical Problems in Mechanics

Existence of weak solutions for the motion of an elastic structure in an incompressible viscous fluid

Muriel Boulakia

Laboratoire de mathématiques appliquées, Université de Versailles-St-Quentin, 45, avenue des Etats Unis, 78035 Versailles cedex, France

Received 1 April 2003; accepted after revision 6 May 2003

Presented by Pierre-Louis Lions

Abstract

We study here the two dimensional motion of an elastic body immersed in an incompressible viscous fluid. The body and the fluid are contained in a fixed bounded set Ω. We show the existence of a weak solution for regularized elastic deformations as long as elastic deformations are not too important and no collisions occur. A complete proof is given by Boulakia in *existence d'une solution faible pour un problème d'interaction fluide visqueux incompressible-solide élastique* (prepublication 104, UVSQ, 2003). ***To cite this article: M. Boulakia, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Résumé

Existence d'une solution faible pour un problème d'interaction fluide visqueux incompressible-solide élastique. Nous étudions ici le mouvement d'un solide élastique immergé dans un fluide visqueux incompressible en dimension 2. L'ensemble fluide-structure évolue dans une cavité fixe bornée Ω. Nous montrons un résultat d'existence de solution faible de notre problème pour des déformations élastiques régularisées sous réserve qu'il n'y ait pas de chocs et que le solide n'ait pas de trop grosses déformations élastiques. Une preuve complète est donnée par Boulakia dans *existence d'une solution faible pour un problème d'interaction fluide visqueux incompressible-solide élastique* (prépublication 104, UVSQ, 2003). ***Pour citer cet article : M. Boulakia, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Version française abrégée

Sur la partie solide, on cherche le déplacement sous la forme d'un mouvement rigide accompagné d'un petit mouvement élastique : le flot lagrangien X_S s'écrit : $X_S(t,0,y) = a(t) + Q(t)(y - g_0) + Q(t)\xi(t,y)$, $\forall y \in \Omega_S(0)$, $\forall t \in [0,T]$, où $\Omega_S(0)$ est la position du solide à l'instant initial, g_0 représente le centre de gravité du solide à l'instant initial, a la translation, $Q \in SO_2(\mathbb{R})$ la rotation et ξ la déformation élastique du solide. Ce flot détermine la position du solide et donc aussi du fluide à l'instant t : $\Omega_S(t) = X_S(t,0,\Omega_S(0))$ et $\Omega_F(t) = \Omega \setminus \Omega_S(t)$. On suppose ce flot inversible de $\Omega_S(0)$ sur $\Omega_S(t)$ (ce qui sera vérifié par notre solution) et on note $X_S(0,t,\cdot)$ l'inverse. Ceci permet de passer d'un point de vue lagrangien à un point de vue eulérien : la vitesse eulérienne est définie par (1).

E-mail address: boulakia@math.uvsq.fr (M. Boulakia).

1631-073X/03/$ – see front matter
doi:10.1016/S1631-073X(03)00235-8

D'autre part, sur le domaine fluide, la donnée qui intervient naturellement est u_F, la vitesse eulérienne définie sur $\Omega_F(t)$. On note u la vitesse eulérienne globale et X le flot associé.

Sur la partie solide, les équations sont, d'une part, les équations du mouvement rigide, d'autre part, l'équation de l'élasticité linéarisée. Dans notre cas, on a besoin de régulariser cette équation afin d'avoir ξ dans $W^{1,\infty}(0,T;H^3(\Omega_S(0)))$. Quant au mouvement du fluide, il est représenté par les équations de Navier–Stokes incompressibles. Le système complet d'équations est donné par (2)–(10).

Définition. On dira que (ρ_F, u) est une solution faible du problème (2)–(10) si les conditions (13)–(16) sont satisfaites et si la formulation variationnelle (18) est vérifiée avec l'espace des fonctions test défini par (17).

Théorème 1. *Soient* $\xi^1 \in H^3(\Omega_S(0))$, $u_F^0 \in H^1(\Omega_F(0))$, $\rho_F^0 \in L^\infty(\Omega)$, $a^0, a^1 \in \mathbb{R}^2$ *vérifiant les conditions* (11), (12). *On définit* : $d(t) = d(\partial\Omega_S(t), \partial\Omega)$ *et* $\gamma(t) = \inf_{y\in\Omega_S(0)} |\det \nabla X_S(t,0,y)|$ *et on suppose que* $d(0) > 0$. *Alors, il existe au moins un couple* (ρ_F, u) *solution faible du problème* (2)–(10) *définie sur* $(0,T)$ *où* :
$T = \sup\{t > 0 \mid d(t) > 0,\ \gamma(t) > 0$ *et* $X_S(t,0,\cdot)$ *injective*$\}$ (*on a une minoration explicite de* T *par une constante strictement positive dépendant des données et de* ε).

De plus, cette solution vérifie l'estimation d'énergie (*E.E.*).

Pour montrer ce théorème, on prouve d'abord l'existence de solutions en dimension finie (méthode de Galerkin) pour le problème linéarisé. On a besoin de ramener les vitesses aux domaines de référence, ceci nécessite la construction de vitesses admissibles (c'est-à-dire tenant compte du couplage fluide-structure) à partir de vitesses découplées définies sur $\Omega_S(0)$ et $\Omega_F(0)$: on reprend ici la démarche de l'article [2]. La linéarisation permet de traîter le problème en dimension finie N comme un simple système d'équations différentielles ordinaires. On passe ensuite facilement à l'existence de solution approchée du problème non linéaire complet en dimension finie par un théorème de point fixe sur un intervalle $[0, T_0]$ avec T_0 indépendant de N.

Pour passer au problème continu, on utilise un résultat de compacité sur les densités donné par [4] et on montre la compacité des vitesses en obtenant une inégalité de la forme :
$\int_0^{T_0-h}\int_\Omega \left|\sqrt{\rho_F^N(t+h)}u^N(t+h) - \sqrt{\rho_F^N(t)}u^N(t)\right|^2 \mathrm{d}x\,\mathrm{d}t \leqslant \delta(h)$ avec $\lim_{h\mapsto 0}\delta(h) = 0$. Cela donne une convergence forte de $(\sqrt{\rho_F^N}u^N)_{N\in\mathbb{N}}$ dans $L^2((0,T_0)\times\Omega)$ ce qui est suffisant pour passer à la limite dans la formulation variationnelle. On prolonge alors la solution en partant de la nouvelle configuration de référence à l'instant T_0, puis on réitère ce raisonnement jusqu'au temps T.

1. Introduction and equation of motion

On an elastic structure, we have a rigid motion combined with an elastic motion with small deformations: the Lagrangian flow X_S is defined by: $X_S(t,0,y) = a(t) + Q(t)(y - g_0) + Q(t)\xi(t,y)$, $\forall y \in \Omega_S(0)$, $\forall t \in [0,T]$, where $\Omega_S(0)$ is the initial domain occupied by the structure, g_0 the center of mass of the solid, a the translation, $Q \in SO_2(\mathbb{R})$ the rotation and ξ the elastic deformation of the structure. The vector $X_S(t,0,y)$ gives the position at time t of the particle located in y at initial time. We can define U_S the Lagrangian velocity and: $\Omega_S(t) = X_S(t,0,\Omega_S(0))$ and $\Omega_F(t) = \Omega \setminus \Omega_S(t)$ that denote respectively the domain occupied by the body at time t and the domain occupied by the fluid at time t. Moreover, the flow X_S is supposed invertible (this hypothesis will be satisfied by our solution). Thus, we define u_S the Eulerian velocity by:

$$u_S(t,x) = U_S\big(t, X_S(0,t,x)\big) = \frac{\partial X_S}{\partial t}\big(t,0,X_S(0,t,x)\big), \quad \forall x \in \Omega_S(t),\ \forall t \in [0,T] \tag{1}$$

with: $X_S(0,t,x) = X_S(t,0,\cdot)^{-1}(x)$, $\forall x \in \Omega_S(t), \forall t \in [0,T]$.

Finally, we denote: $X_S(t,s,x) = X_S(t,0,X_S(0,s,x))$, $\forall x \in \Omega_S(s)$, $\forall t, s \in [0,T]$.

On the fluid domain, we have an Eulerian point of view: let u_F be the Eulerian velocity, u the global Eulerian velocity on Ω and X the associated Lagrangian flow.

The unknowns are a, Q or ω the corresponding rotation velocity, ξ, u_F, ρ_F the density of the fluid and p the pressure of the fluid.

The motion of the fluid is described by the incompressible viscous Navier–Stokes equations and the motion of the solid is described by the equations of the rigid motion and by the linearized elasticity equation. In our case, we need a regularization on this equation: we add a regularizing term in order to get ξ in $W^{1,\infty}(0,T;H^3(\Omega_S(0)))$. At last, we assume that the fluid adheres to the boundary.

The equations of the motion are given by the following system of equations:

$$m\frac{\mathrm{d}^2a}{\mathrm{d}t^2}(t)=\int\limits_{\partial\Omega_S(t)}\sigma_F n_x\,\mathrm{d}\gamma_x,\qquad J\frac{\mathrm{d}\omega}{\mathrm{d}t}(t)=\int\limits_{\partial\Omega_S(t)}(\sigma_F n_x)\cdot\big(x-a(t)\big)^{\perp}\,\mathrm{d}\gamma_x,\tag{2}$$

$$\rho_S^0(y)\frac{\partial^2\xi}{\partial t^2}(t,y)+\varepsilon A_3\frac{\partial^2\xi}{\partial t^2}(t,y)-\operatorname{div}\Sigma_E^2\big(\xi(t,y)\big)=0,\quad\forall y\in\Omega_S(0),\tag{3}$$

$$\rho_F(t,x)\left(\frac{\partial u_F}{\partial t}+(u_F\cdot\nabla)u_F\right)(t,x)-\operatorname{div}\sigma_F(t,x)=0,\quad\forall x\in\Omega_F(t),\tag{4}$$

$$\frac{\partial\rho_F}{\partial t}(t,x)+\operatorname{div}(\rho_F u)(t,x)=0,\quad\forall x\in\Omega,\tag{5}$$

$$\operatorname{div}u_F(t,x)=0,\quad\forall x\in\Omega_F(t),\tag{6}$$

$$u_F(t,x)=\frac{\mathrm{d}a}{\mathrm{d}t}(t)+\omega(t)\big(x-a(t)\big)^{\perp}+Q(t)\frac{\partial\xi}{\partial t}\big(t,X(0,t,x)\big),\quad\forall x\in\partial\Omega_S(t),\tag{7}$$

$$u_F(t,x)=0,\quad\forall x\in\partial\Omega,\tag{8}$$

with the initial conditions:

$$a(0)=a^0,\quad\frac{\mathrm{d}a}{\mathrm{d}t}(0)=a^1,\ \omega(0)=\omega^0,\quad Q(0)=\mathrm{Id},\quad\xi(0,\cdot)=0\quad\text{on }\Omega_S(0),\tag{9}$$

$$\frac{\partial\xi}{\partial t}(0,\cdot)=\xi^1\quad\text{on }\Omega_S(0),\quad u_F(0,\cdot)=u_F^0\quad\text{on }\Omega_F(0),\quad\rho_F(0,\cdot)=\rho_F^0\chi_{\Omega_F(0)}\quad\text{on }\Omega,\tag{10}$$

where $\xi^1\in H^3(\Omega_S(0))$, $u_F^0\in H^1(\Omega_F(0))$, $\rho_F^0\in L^\infty(\Omega)$ and:

$$\int\limits_{\Omega_S(0)}\xi^1(y)\cdot n_y\,\mathrm{d}y=0,\quad 0<M_1\leqslant\rho_F^0\leqslant M_2\quad\text{on }\Omega_F(0),\quad u_F^0(y)=0\quad\text{on }\partial\Omega,\tag{11}$$

$$u_F^0(y)=a^1+\omega^0\big(y-a^0\big)^{\perp}+\xi^1(y)\quad\text{on }\partial\Omega_S(0),\quad\operatorname{div}u_F^0=0\quad\text{on }\Omega_F(0).\tag{12}$$

Here m is the mass of the solid, J is the inertia moment related to the mass center, n_x is the outwards unit nomal to $\partial\Omega_S(t)$. The Cauchy stress tensor in the fluid and the second Piola–Kirchhoff tensor in the solid are respectively given by: $\sigma_F(t,x)=2\nu\varepsilon(u)(t,x)-p(t,x)\,\mathrm{Id}$, and $\Sigma_E^2(\xi)=\lambda\,\mathrm{tr}(\varepsilon(\xi))\,\mathrm{Id}+2\mu\varepsilon(\xi)$ with $\nu>0$ the viscosity of the fluid, $\varepsilon(u)=\frac{1}{2}(\nabla u+\nabla u^t)$ and λ, μ the Lamé constants of the elastic media such that $\lambda+2\mu>0$.

Moreover ε is a fixed positive real and A_3 is a regularizing differential operator such that:

$$\langle A_3u,v\rangle_{H^{-3}(\Omega_S(0))\times H^3(\Omega_S(0))}=(u,v)_{H^3(\Omega_S(0))}+\langle u,v\rangle_{3,\partial\Omega_S(0)},\quad\forall u,v\in H^3\big(\Omega_S(0)\big)^2.$$

Denoting Σ_F^1 the first Piola–Kirchhoff tensor corresponding to σ_F, we also suppose that:

$$\int\limits_{\partial\Omega_S(0)}\big(\Sigma_E^2(t,y)n_y\big)\cdot\frac{\partial\eta}{\partial t}(t,y)\,\mathrm{d}\gamma(y)+\varepsilon\left\langle\frac{\partial^2\xi}{\partial t^2}(t,\cdot),\frac{\partial\eta}{\partial t}(t,\cdot)\right\rangle_{3,\partial\Omega_S(0)}$$
$$=\int\limits_{\partial\Omega_S(0)}\big(\Sigma_F^1(t,y)n_y\big)\cdot\frac{\partial\eta}{\partial t}(t,y)\,\mathrm{d}\gamma(y),\quad\text{for }\eta\text{ regular enough.}$$

Remark 1. We can also regularize ξ by adding a viscosity term (for instance $\varepsilon A_3 \frac{\partial \xi}{\partial t}$) in the elasticity equation in order to have $\xi \in H^1(0,T;H^3(\Omega_S(0)))$.

2. Variational formulation

We look for a solution (ρ_F, u) such that:

(i) $u \in L^\infty\big(0,T;L^2(\Omega)\big)^2 \cap L^2(0,T;H_0^1(\Omega))^2,\ \rho_F \in L^\infty((0,T)\times\Omega).$ (13)

(ii) The flow corresponding to u is defined on Ω and satisfies:

$$X(t,0,y) = a(t) + Q(t)(y-g_0) + Q(t)\xi(t,y) \quad \text{on } \Omega_S(0), \quad \text{with}$$

$$a \in W^{1,\infty}(0,T)^2, \quad Q \in W^{1,\infty}\big(0,T;\mathrm{SO}_2(\mathbb{R})\big), \quad \xi \in W^{1,\infty}\big(0,T;H^3\big(\Omega_S(0)\big)\big)^2, \tag{14}$$

(iii) $\operatorname{div} u = 0 \quad \text{on } \Omega_F(t),$ (15)

(iv)
$$\begin{cases} \dfrac{\partial \rho_F}{\partial t} + \operatorname{div}(\rho_F u) = 0 \quad \text{on } \Omega, \\ \rho_F(t=0) = \rho_F^0 \chi_{\Omega_F(0)}. \end{cases} \tag{16}$$

Let V be the test-function space (notice that its definition depends on the solution itself):

$$\begin{aligned} V = \bigg\{ & v \in H^1\big((0,T)\times\Omega\big)^2 \;\Big|\; v(t,x) = \frac{\mathrm{d}b}{\mathrm{d}t}(t) + r(t)\big(x-a(t)\big)^\perp + Q(t)\frac{\partial \eta}{\partial t}\big(t, X(0,t,x)\big) \\ & \text{on } \Omega_S(t),\ \text{with } b \in H^2(0,T)^2,\ r \in H^1(0,T),\ \eta \in H^2\big(0,T;H^3\big(\Omega_S(0)\big)\big)^2, \\ & \text{and } \operatorname{div} v(t,\cdot) = 0 \text{ on } \Omega_F(t),\ v(t,\cdot) \in H_0^1(\Omega),\ \forall t \in [0,T],\ v(T) = 0 \bigg\}. \end{aligned} \tag{17}$$

Definition 2.1. We will say that (ρ_F, u) is a weak solution of (2)–(10) if the conditions (13)–(16) are satisfied and if the following equality holds, $\forall v \in V$:

$$\begin{aligned} & m\int_0^T \frac{\mathrm{d}a}{\mathrm{d}t}(t)\cdot\frac{\mathrm{d}^2 b}{\mathrm{d}t^2}(t)\,\mathrm{d}t + J\int_0^T \omega(t)\frac{\mathrm{d}r}{\mathrm{d}t}(t)\,\mathrm{d}t + \int_0^T\int_{\Omega_S(0)} \rho_S^0(y)\frac{\partial \xi}{\partial t}(t,y)\cdot\frac{\partial^2 \eta}{\partial t^2}(t,y)\,\mathrm{d}y\,\mathrm{d}t \\ & + \int_0^T\int_\Omega \rho_F(t,x)u(t,x)\cdot\frac{\partial v}{\partial t}(t,x)\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Omega \rho_F(t,x)u(t,x)\otimes u(t,x):\nabla v(t,x)\,\mathrm{d}x\,\mathrm{d}t \\ & + \varepsilon\int_0^T\int_{\Omega_S(0)} a_3\bigg(\frac{\partial \xi}{\partial t}(t,y), \frac{\partial^2 \eta}{\partial t^2}(t,y)\bigg)\,\mathrm{d}y\,\mathrm{d}t - 2\nu\int_0^T\int_{\Omega_F(t)} \varepsilon(u):\varepsilon(v)\,\mathrm{d}x\,\mathrm{d}t \\ & - \lambda\int_0^T\int_{\Omega_S(0)} \mathrm{tr}\big(\varepsilon(\xi)\big)\,\mathrm{tr}\bigg(\varepsilon\bigg(\frac{\partial \eta}{\partial t}\bigg)\bigg)\,\mathrm{d}y\,\mathrm{d}t - 2\mu\int_0^T\int_{\Omega_S(0)} \varepsilon(\xi):\varepsilon\bigg(\frac{\partial \eta}{\partial t}\bigg)\,\mathrm{d}y\,\mathrm{d}t = -ma^1\cdot\frac{\mathrm{d}b}{\mathrm{d}t}(0) \\ & - J\omega^0 r(0) - \int_{\Omega_S(0)} \rho_S^0(y)\xi^1(y)\cdot\frac{\partial \eta}{\partial t}(0,y)\,\mathrm{d}y - \int_{\Omega_F(0)} \rho_F^0(y)u_F^0(y)\cdot v(0,y)\,\mathrm{d}y \\ & - \varepsilon\int_{\Omega_S(0)} a_3\bigg(\xi_1(y), \frac{\partial \eta}{\partial t}(0,y)\bigg)\,\mathrm{d}y. \end{aligned} \tag{18}$$

3. Main result

Theorem 3.1. *Let* $\xi^1 \in H^3(\Omega_S(0))$, $u_F^0 \in H^1(\Omega_F(0))$, $\rho_F^0 \in L^\infty(\Omega)$ *satisfying* (11), (12). *We define*: $d(t) = d(\partial\Omega_S(t), \partial\Omega)$ *and* $\gamma(t) = \inf_{y\in\Omega_S(0)} |\det\nabla X_S(t,0,y)|$ *and we suppose that* $d(0) > 0$. *Then, there exists at least one weak solution of* (2)–(10) *defined on* $(0,T)$ *where*: $T = \sup\{t > 0 \mid d(t) > 0,\ \gamma(t) > 0$ *and* $X_S(t,0,\cdot)$ *one-to-one*$\}$ (*with* T *greater than an explicit constant* > 0 *depending on the data and* ε).

Moreover, this solution satisfies the energy estimate (*with* E_0 *the initial energy*):

$$\frac{1}{2}m\left|\frac{\mathrm{d}a}{\mathrm{d}t}(t)\right|^2 + \frac{1}{2}J\left|\omega(t)\right|^2 + \frac{1}{2}\int_{\Omega_S(0)} \rho_S^0(y)\left|\frac{\partial\xi}{\partial t}(t,y)\right|^2 \mathrm{d}y + \frac{1}{2}\int_{\Omega_F(t)} \rho_F(t,x)\left|u_F(t,x)\right|^2 \mathrm{d}x$$
$$+ \frac{1}{2}\varepsilon\left\|\frac{\partial\xi}{\partial t}(t,\cdot)\right\|^2_{H^3(\Omega_S(0))} + \frac{\lambda}{2}\int_{\Omega_S(0)} \left|\mathrm{tr}\big(\varepsilon\big(\xi(t,y)\big)\big)\right|^2 \mathrm{d}y + \mu\int_{\Omega_S(0)} \left|\varepsilon\big(\xi(t,y)\big)\right|^2 \mathrm{d}y$$
$$+ 2\nu\int_0^t\int_{\Omega_F(s)} \left|\varepsilon\big(u_F(s,x)\big)\right|^2 \mathrm{d}x\,\mathrm{d}s \leqslant E_0. \tag{E.E.}$$

Up to now, we consider a time $T > 0$ such that: $d(t) > \alpha$, $\gamma(t) > \beta$, $\forall t \in [0,T]$, with $\alpha > 0$, $\beta > 0$. This is a priori possible, thanks to (E.E.).

4. Representation of velocities

The goal is to represent any u satisfying (13)–(16) by velocities defined on fixed reference domains. Here, we use the same method as in the paper of Desjardins, Esteban, Grandmont and Le Tallec [2].

Suppose that we have (w_F, a, Q, ξ) such that: (1) $w_F \in Y_0$ with Y_0 defined by: $Y_0 = \{w_F \in L^\infty(0,T; L^2(\Omega_F(0)))^2 \cap L^2(0,T; H_0^1(\Omega_F(0)))^2 \mid \operatorname{div} w_F = 0 \text{ on } \Omega_F(0)\}$ (2) $(a, Q, \xi) \in Y_1$ with Y_1 defined by: $Y_1 = \{(a, Q, \xi) \in W^{1,\infty}(0,T)^2 \times W^{1,\infty}(0,T; \mathrm{SO}_2(\mathbb{R})) \times W^{1,\infty}(0,T; H^3(\Omega_S(0)))^2 \mid \|\xi\|_{L^\infty(0,T;H^3(\Omega_S(0)))^2} \leqslant \kappa\}$ where κ will be defined later. With (a, Q, ξ) given in Y_1, we can define a flow. This flow is not compatible with an incompressible fluid velocity: we have to add a term of dilatation or compression of the solid volume that will balance the volume variations due to the elastic deformations. Let $\eta \in H^3(\Omega_S(0))$ be a lifting of the unit outward normal on $\partial\Omega_S(0)$. We define: $X_S(t,0,y) = a(t) + Q(t)(y - g_0) + Q(t)(\xi(t,y) + \lambda(t)\eta(y))$, $\forall y \in \Omega_S(0)$ where $\lambda(t)$ is such that:

$$\left|\Omega_S(t)\right| = \left|\Omega_S(0)\right| \quad\Longleftrightarrow\quad \int_{\Omega_S(0)} \det\nabla X_S(t,0,y)\,\mathrm{d}y = \left|\Omega_S(0)\right|, \quad \forall t \in [0,T]. \tag{19}$$

By applying the local inversion theorem, we prove easily that this is always possible for κ small enough. We can also suppose that κ is small enough for having $X_S(t,0,\cdot)$ invertible from $\Omega_S(0)$ on $\Omega_S(t)$. This allows us to define u_S the corresponding Eulerian velocity on $\Omega_S(t)$. Then, we extend u_S by $u_{S,p}$ defined on Ω by a Stokes problem:

$$\begin{cases} -\Delta u_{S,p} + \nabla q = 0, \quad \Omega_F(t), \qquad \operatorname{div} u_{S,p} = 0, \quad \Omega_F(t),\\ u_{S,p} = u_S, \quad \partial\Omega_S(t), \qquad u_{S,p} = 0, \quad \partial\Omega \cap \partial\Omega_F(t) \end{cases}$$

with $\Omega_F(t) = \Omega \setminus \Omega_S(t)$. This is possible thanks to (19). Moreover, strictly following the arguments of [1], we can obtain a regularity result for Stokes problem that gives $u_{S,p} \in L^\infty(0,T; W^{2,4}(\Omega))^2$. At last, we denote by $X_{S,p} \in W^{1,\infty}((0,T) \times \Omega)^2$ the corresponding flow.

On the other hand, we can extend w_F on Ω by 0 and we denote by X_F the corresponding flow. Then, we define: $\forall t \in [0,T]$, $\forall y \in \Omega$, $X(t,0,y) = X_{S,p}(t,0,X_F(t,0,y))$. Let u be the associated Eulerian velocity. We can now easily check that u satisfies (13)–(15). By this way, we have represented a velocity compatible with solid and fluid motions. Finally, a result of di Perna and Lions [3] gives the existence of ρ_F solution of (16). This ends the representation of velocities. We denote Θ the map that sends (w_F, a, Q, ξ) on u. We remark that we can also do the reciprocal construction that sends u on (w_F, a, Q, ξ).

5. Finite dimensional problem

We will follow the previous paragraph to construct approximate solutions in finite dimension. First, we will solve a linearized finite dimensional problem and then we will get a solution of the non linear finite dimensional problem thanks to a fixed point theorem.

Let $(\varphi_i)_{i\geqslant 1}$ be a basis of $\{\varphi \in H_0^1(\Omega_F(0))^2 \mid \operatorname{div}\varphi = 0 \text{ sur } \Omega_F(0)\}$ and $(\psi_i)_{i\geqslant 0}$ be a basis of $H^3(\Omega_S(0))^2$. Suppose that we have $(\widetilde{w}_F^N, \tilde{a}^N, \widetilde{Q}^N, \tilde{\xi}^N) \in Y_0 \times Y_1$ defined by: $\widetilde{w}_F^N(t,\cdot) = \sum_{i=1}^N \tilde{\gamma}_i(t)\varphi_i$, $\tilde{\xi}^N(t,\cdot) = \sum_{i=1}^N \tilde{\alpha}_i(t)\psi_i$ with $\sup_{0\leqslant t\leqslant T}(\sum_{i=1}^N \tilde{\alpha}_i^2(t))^{1/2} \leqslant \kappa$. Then, we can construct $\tilde{u}^N = \Theta(\widetilde{w}_F^N, \tilde{a}^N, \widetilde{Q}^N, \tilde{\xi}^N)$ and $\tilde{\rho}_F^N$. By this way, we linearize the variational formulation in finite dimension by replacing a, Q and X by $\tilde{a}^N$, $\widetilde{Q}^N$ and $\widetilde{X}_{S,p}^N$ in the definition of the test-functions space and by replacing ρ_F, $\Omega_F(t)$ and $u\otimes u$ by $\tilde{\rho}_F^N$, $\widetilde{\Omega}_F^N(t)$ and $\tilde{u}^N \otimes u^N$ in the weak formulation (18).

We have now a classical problem with the unknowns a^N, Q^N, γ_i and α_i. We can easily solve this linear ordinary differential system. Then, in order to apply Schauder's fixed point theorem, we need to keep a small elastic deformation, i.e.: $\sup_{0\leqslant t\leqslant T}(\sum_{i=1}^N \alpha_i^2(t))^{1/2} \leqslant \kappa$. Thanks to the energy estimate, this is checked if we take $T = T_0 = \sqrt{\varepsilon/(2E_0)}\kappa$. Now, we can apply Schauder's theorem, the compactness resulting immediately from the ordinary differential system. The fixed point satisfies: $u^N = \Theta(w_F^N, a^N, Q^N, \xi^N)$ with $w_F^N(t,\cdot) = \sum_{i=1}^N \gamma_i(t)\varphi_i$, $\xi^N(t,\cdot) = \sum_{i=1}^N \alpha_i(t)\psi_i$ and it is a solution of the finite dimensional approximation of our problem.

6. Proof of Theorem 3.1

In order to pass to the limit when N goes to infinity, we have to show results of compactness. The strong convergence of $(\rho_F^N)_{N\in\mathbb{N}}$ is obtained directly by using a result of di Perna and Lions [4]. For the compactness of the velocity, we prove the following estimate on $(\sqrt{\rho_F^N}u^N)_{N\in\mathbb{N}}$:

$$\int_0^{T_0-h}\int_\Omega \left|\sqrt{\rho_F^N(t+h)}u^N(t+h) - \sqrt{\rho_F^N(t)}u^N(t)\right|^2 \mathrm{d}x\,\mathrm{d}t \leqslant \delta(h) \tag{20}$$

with $\lim_{h\mapsto 0}\delta(h) = 0$. To show this inequality, the idea is to use admissible test-functions close to $u^N(t+h) - u^N(t)$. We will not explain the details of the proof: this is quite technical. This gives the compactness of $(\sqrt{\rho_F^N}u^N)_{N\in\mathbb{N}}$ in $L^2((0,T_0)\times\Omega)$. Now, we can pass to the limit in the variational problem and we obtain a weak solution of (2)–(10) defined on $[0,T_0]$ satisfying the energy inequality (E.E.).

Now, transporting the solution on the new reference domains $\Omega_S(T_0)$ and $\Omega_F(T_0)$, we can repeat the same proof and extend the solution to the intervall $[0, T_0 + A]$ for a fixed $A > 0$. Reitering this process, we can reach the time T wanted.

Acknowledgements

I want to thank deeply my advisor Pr. Jean-Pierre Puel for his help and all his advices that allowed me to bring this work to its end.

References

[1] J.A. Bello, L^r regularity for the Stokes and Navier–Stokes problems, Ann. Math. Pura Appl. 170 (1996) 187–206.
[2] B. Desjardins, M.J. Esteban, C. Grandmont, P. Le Tallec, Weak solutions for a fluid-elastic structure interaction model, Rev. Mat. Complut. 14 (2001) 523–538.
[3] R.J. Di Perna, P.-L. Lions, Ordinary differential equations, transport theory and Sobolev spaces, Invent. Math. 98 (1989) 511–547.
[4] P.-L. Lions, Mathematical Topics in Fluid Mechanics, Oxford Science Publications, 1996.