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Algebraic Geometry

# Virtual Betti numbers of real algebraic varieties

Clint McCrory<sup>a</sup>, Adam Parusiński<sup>b</sup>

<sup>a</sup> *Mathematics Department, University of Georgia, Athens, GA 30602, USA*

<sup>b</sup> *Département de mathématiques, Université d'Angers, 2, bd Lavoisier, 49045 Angers cedex, France*

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## Abstract

We show that for all  $i \geq 0$  the  $i$ -th mod 2 Betti number of compact nonsingular real algebraic varieties has a unique extension to a *virtual Betti number*  $\beta_i$  defined for all real algebraic varieties, such that if  $Y$  is a closed subvariety of  $X$  then  $\beta_i(X) = \beta_i(X \setminus Y) + \beta_i(Y)$ . We show by example that there is no natural weight filtration on the  $\mathbb{Z}_2$ -cohomology of real algebraic varieties with compact supports such that the virtual Betti numbers are the weighted Euler characteristics. **To cite this article:** C. McCrory, A. Parusiński, *C. R. Acad. Sci. Paris, Ser. I 336 (2003)*.

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## Résumé

**Nombres de Betti virtuels des variétés algébriques réelles.** On montre que pour tout entier positif  $i$  le  $i$ -ième nombre de Betti de la cohomologie à coefficients dans  $\mathbb{Z}_2$  des variétés algébriques réelles compactes nonsingulières admet une unique extension en un *nombre de Betti virtuel*  $\beta_i$ , défini pour toute variété algébrique réelle, telle que pour une sous-variété fermée  $Y \subset X$ ,  $\beta_i(X) = \beta_i(X \setminus Y) + \beta_i(Y)$ . On donne un exemple qui montre qu'il n'existe pas de filtration par le poids naturelle sur la cohomologie à coefficients dans  $\mathbb{Z}_2$  des variétés algébriques réelles telle que les nombres de Betti virtuels soient les caractéristiques d'Euler par le poids associées à cette filtration. **Pour citer cet article :** C. McCrory, A. Parusiński, *C. R. Acad. Sci. Paris, Ser. I 336 (2003)*.

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## Version française abrégée

Soit  $K_0(\mathcal{V}_{\mathbb{R}}(\mathbb{R}))$  l'anneau de Grothendieck des variétés algébriques réelles. Nous définissons *le polynôme de Poincaré virtuel* qui est un homomorphisme d'anneaux  $\beta(\cdot, t) : K_0(\mathcal{V}_{\mathbb{R}}(\mathbb{R})) \rightarrow \mathbb{Z}[t]$ . Pour  $X$  lisse et compacte  $\beta(X, t) = \sum_i \dim_{\mathbb{Z}_2} H^i(X; \mathbb{Z}_2) t^i$ . On appelle *nombres de Betti virtuels de  $X$*  les coefficients de  $\beta(X, t)$ . L'existence des nombres de Betti virtuels a été aussi annoncée dans [12]. On démontre que des variétés équivalentes dans  $K_0(\mathcal{V}_{\mathbb{R}}(\mathbb{R}))$  ont la même dimension. Ceci implique que les mesures motiviques de [9,10,5], qui prennent leurs valeurs dans un localisé de l'anneau de Grothendieck complété par rapport à la dimension, sont bien définies dans le cadre réel. Nos définitions d'une variété algébrique réelle et d'un isomorphisme birégulier sont celles de [3].

*E-mail addresses:* [clint@math.uga.edu](mailto:clint@math.uga.edu) (C. McCrory), [parus@tonton.univ-angers.fr](mailto:parus@tonton.univ-angers.fr) (A. Parusiński).

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### 0.1. Le polynôme de Poincaré virtuel

Le résultat suivant, dû à Bittner [2], est une conséquence du théorème de factorisation faible [1].

**Théorème 0.1.** Soit  $X \mapsto e(X)$  une fonction définie sur les variétés algébriques réelles compactes lisses à valeurs dans un groupe abélien  $G$ , telle que  $e(X) = e(Y)$  si  $X$  et  $Y$  sont isomorphes, et  $e(\emptyset) = 0$ . Supposons que pour tout éclatement  $\text{Bl}_C X \rightarrow X$  d'une variété compacte lisse le long d'un centre lisse  $C$  et de diviseur exceptionnel  $E$ ,

$$e(\text{Bl}_C X) - e(E) = e(X) - e(C).$$

Alors  $e$  s'étend de manière unique en un homomorphisme de groupes  $K_0(\mathcal{V}_{\mathbb{R}}(\mathbb{R})) \rightarrow G$ .

Si  $e$  est à valeurs dans un anneau commutatif  $R$  et  $e(X \times Y) = e(X) \cdot e(Y)$  pour toutes variétés  $X, Y$  compactes lisses alors cette extension  $K_0(\mathcal{V}_{\mathbb{R}}(\mathbb{R})) \rightarrow R$  est un homomorphisme d'anneaux.

Pour  $i \geq 0$  on écrit  $b_i(X) = \dim_{\mathbb{Z}_2} H^i(X; \mathbb{Z}_2)$ . Soit  $\tilde{X} = \text{Bl}_C X$  comme dans l'énoncé du Théorème 0.1. On peut démontrer, par un argument de suites exactes, que

$$b_i(\text{Bl}_C X) - b_i(E) = b_i(X) - b_i(C).$$

Alors on obtient le résultat suivant. (Navarro nous a informé que Théorème 0.2 résulte aussi du Théorème 2.2.2 de [8].)

**Théorème 0.2.** Pour tout entier positif  $i$  il existe un unique homomorphisme de groupes  $\beta_i : K_0(\mathcal{V}_{\mathbb{R}}(\mathbb{R})) \rightarrow \mathbb{Z}$  tel que  $\beta_i(X) = b_i(X)$  pour toute variété  $X$  compacte lisse.

Il existe un unique homomorphisme d'anneau  $\beta(\cdot, t) : K_0(\mathcal{V}_{\mathbb{R}}(\mathbb{R})) \rightarrow \mathbb{Z}[t]$  tel que  $\beta(X, t) = \sum_i b_i(X) t^i$  pour tout  $X$  compacte nonsingulière.

On appelle  $\beta_i(X)$ , resp.  $\beta(X, t) = \sum_i \beta_i(X) t^i$ , le  $i$ -ème nombre de Betti virtuel de  $X$ , resp. le polynôme de Poincaré virtuel de  $X$ .

**Corollaire 0.3.** Le polynôme de Poincaré virtuel  $\beta(X, t)$  est de degré  $n = \dim X$  et  $\beta_n(X) > 0$ . En particulier,  $[X] = [Y]$  implique  $\dim X = \dim Y$ , et  $[X] \neq 0$  si  $X \neq \emptyset$ .

Pour toute variété algébrique réelle  $\beta(X, -1)$  est égale à la caractéristique d'Euler à support compact  $\chi_c(X) = \sum_i (-1)^i \dim H_c^i(X; \mathbb{Z}_2)$ . Mais, en général,  $\beta_i(X) \neq \dim H_c^i(X; \mathbb{Z}_2)$ , et les nombres de Betti virtuels ne sont pas des invariants topologiques.

### 0.2. Filtrations par le poids

La version complexe du Théorème 0.1 appliquée au polynôme de Poincaré  $\sum_i \dim_{\mathbb{C}} H^i(X; \mathbb{C}) t^i$  donne un invariant des variétés algébriques complexes  $\beta^{\mathbb{C}}(X, t)$  lié à la filtration par le poids de Deligne [4] par la formule  $\beta^{\mathbb{C}}(X, t) = \sum_{i,j} (-1)^{i+j} w_j^i t^j$ , où  $w_j^i(X) = \dim_{\mathbb{C}} W_j^i(X) / W_{j-1}^i(X)$ . Totaro [12] a annoncé l'existence d'une filtration par le poids naturelle sur  $H_c^i(X; \mathbb{Z}_2)$  pour les variétés algébriques réelles. Mais cette filtration ne donne pas le polynôme de Poincaré virtuel. En plus, on donne un exemple qui montre qu'il n'existe pas de filtration par le poids naturelle qui définisse les nombres de Betti virtuels.

Supposons que  $W$  est une filtration par le poids  $0 \subset W_0^i(X) \subset W_1^i(X) \subset \dots \subset W_j^i(X) = H_c^i(X; \mathbb{Z}_2)$  et notons  $w_j^i(X) = \dim_{\mathbb{Z}_2} W_j^i(X) / W_{j-1}^i(X)$ .

Dans le cas complexe la filtration par le poids satisfait les conditions suivantes :

- (1)  $f^* W_j^i(Y) \subset W_j^i(X)$  pour tout morphisme algébrique  $f : X \rightarrow Y$ .
- (2) Pour un tel morphisme,  $f^* W_j^i(Y) = W_j^i(X) \cap \text{Im } f^*$ .

- (3)  $W_{i-1}^i(X) = 0$  pour  $X$  compacte et lisse.  
 (4)  $W_{i-1}^i(X) = \text{Ker } p^*$  si  $p: \tilde{X} \rightarrow X$  est une résolution d'une variété compacte  $X$ .  
 (5)  $\beta_j(X) = (-1)^j \sum_i (-1)^i w_j^i(X)$ .

La filtration par le poids de Totaro, qui est construite à partir d'une hyperrésolution cubique, cf. [8,6], satisfait les conditions (1) et (3) mais pas les autres.

Il n'est pas difficile de trouver des exemples qui montrent qu'il n'existe pas de filtration réelle par le poids qui satisfait (2) et (4). De même, il n'en existe pas qui satisfait (1) et (5); voir Exemple 3.3 de la version anglaise.

## 1. Introduction

The *Grothendieck group* of real algebraic varieties is the Abelian group generated by symbols  $[X]$ , where  $X$  is a real algebraic variety, with relations

- (1)  $[X] = [Y]$  if  $X$  and  $Y$  are isomorphic,  
 (2)  $[X] = [X \setminus Y] + [Y]$  if  $Y$  is a closed subvariety of  $X$ .

The product of varieties induces a ring structure

- (3)  $[X] \cdot [Y] = [X \times Y]$ ,

and the resulting ring, denoted by  $K_0(\mathcal{V}_{\mathbb{R}}(\mathbb{R}))$ , is called the *Grothendieck ring* of real algebraic varieties.

We define a ring homomorphism  $\beta(\cdot, t): K_0(\mathcal{V}_{\mathbb{R}}(\mathbb{R})) \rightarrow \mathbb{Z}[t]$ , the *virtual Poincaré polynomial*. For  $X$  nonsingular and compact  $\beta(X, t)$  is the classical Poincaré polynomial for cohomology with  $\mathbb{Z}_2$  coefficients. The coefficients of the virtual Poincaré polynomial are the *virtual Betti numbers* of  $X$ . The existence of the virtual Betti numbers for certain real analytic spaces, including real algebraic varieties, has also been announced by Totaro [12].

Kontsevich's motivic measure on the arc space of a complex algebraic variety takes values in the completion of the localized Grothendieck ring (cf. [9,10,5]). Completion with respect to virtual dimension is possible because complex varieties which are equivalent in the Grothendieck ring have the same dimension. Using the virtual Poincaré polynomial we prove that dimension is an invariant of the Grothendieck ring of real algebraic varieties. It follows that motivic measures can be defined on arc spaces of real varieties.

In [11] Quarez studies the Grothendieck ring of semialgebraic homeomorphism classes of semialgebraic sets, and he observes that the Euler characteristic with compact supports is an isomorphism from this ring to  $\mathbb{Z}$ . In particular, in this ring the class of a non-empty semialgebraic set can be zero, and two semialgebraic sets of different dimensions may represent the same class. This makes the construction of motivic measures – more precisely, completion with respect to virtual dimension – impossible for semialgebraic sets.

Our definitions of real algebraic variety and (biregular) isomorphism are those of Bochnak, Coste and Roy [3]. Thus a real algebraic variety may be realized – in many different ways – as the set of real points of a scheme (reduced, finite type) over the field of real numbers. By the cohomology of a real algebraic variety we mean classical singular cohomology with respect to the standard (Euclidean) topology. For background on real algebraic varieties we refer the reader to [3].

## 2. The virtual Poincaré polynomial

A *generalized Euler characteristic* of real algebraic varieties, with values in a ring  $R$ , is a ring homomorphism  $e: K_0(\mathcal{V}_{\mathbb{R}}(\mathbb{R})) \rightarrow R$ . The Euler characteristic for cohomology with compact supports,  $\chi_c(X) =$

$\sum_i (-1)^i \dim H_c^i(X; \mathbb{Z}_2)$ , is an example of such a homomorphism. In fact any generalized Euler characteristic which is a homeomorphism invariant factors through  $\chi_c$ . (The standard Euler characteristic  $\chi(X) = \sum_i (-1)^i \dim H^i(X; \mathbb{Z}_2)$  does not respect the sum relation (2).)

We will prove the existence of the virtual Poincaré polynomial using the following result of Bittner [2], which is proved using the weak factorization theorem of Abramovich et al. [1]. Navarro has pointed out that our Theorem 2.2 can also be obtained from Theorem 2.2.2 of [8].

**Theorem 2.1.** *Let  $X \mapsto e(X)$  be a function defined on compact nonsingular real algebraic varieties, with values in a Abelian group  $G$ , such that  $e(X) = e(Y)$  if  $X$  and  $Y$  are isomorphic, and  $e(\emptyset) = 0$ . Suppose that for every compact nonsingular variety  $X$  and closed nonsingular subvariety  $C$ , the blowup  $\text{Bl}_C X \rightarrow X$  with center  $C$  and exceptional divisor  $E$  satisfies*

$$e(\text{Bl}_C X) - e(E) = e(X) - e(C).$$

Then  $e$  extends uniquely to a group homomorphism  $K_0(\mathcal{V}_{\mathbb{R}}(\mathbb{R})) \rightarrow G$ .

If  $e$  takes values in a ring  $R$  and  $e(X \times Y) = e(X) \cdot e(Y)$  for all  $X, Y$  compact nonsingular then this extension of  $e$  is a ring homomorphism  $K_0(\mathcal{V}_{\mathbb{R}}(\mathbb{R})) \rightarrow R$ .

For  $i \geq 0$  we let  $b_i(X) = \dim_{\mathbb{Z}_2} H^i(X; \mathbb{Z}_2)$ , the  $i$ -th Betti number of  $X$  with  $\mathbb{Z}_2$  coefficients.

**Theorem 2.2.** *For each nonnegative integer  $i$  there exists a unique group homomorphism  $\beta_i : K_0(\mathcal{V}_{\mathbb{R}}(\mathbb{R})) \rightarrow \mathbb{Z}$  such that  $\beta_i(X) = b_i(X)$  for  $X$  compact nonsingular.*

There exists a unique ring homomorphism  $\beta(\cdot, t) : K_0(\mathcal{V}_{\mathbb{R}}(\mathbb{R})) \rightarrow \mathbb{Z}[t]$  such that  $\beta(X, t) = \sum_i b_i(X)t^i$  for  $X$  compact nonsingular.

**Proof.** Let  $\tilde{X} = \text{Bl}_C X$ , with notation as in Theorem 2.1. By Poincaré duality we have short exact sequences

$$0 \rightarrow H^i(X; \mathbb{Z}_2) \rightarrow H^i(C; \mathbb{Z}_2) \oplus H^i(\tilde{X}; \mathbb{Z}_2) \rightarrow H^i(E; \mathbb{Z}_2) \rightarrow 0,$$

so  $b_i(\text{Bl}_C X) - b_i(E) = b_i(X) - b_i(C)$ , and the first part of the theorem follows from Theorem 2.1. The second part of the theorem follows from Theorem 2.1 and the Künneth formula for compact nonsingular varieties.  $\square$

**Definition 2.3.** The integer  $\beta_i(X)$  is the  $i$ -th virtual Betti number of the real algebraic variety  $X$ , and the polynomial  $\beta(X, t) = \sum_i \beta_i(X)t^i$  is the virtual Poincaré polynomial of  $X$ .

The following result is proved by induction on the dimension of  $X$ .

**Corollary 2.4.** *The virtual Poincaré polynomial  $\beta(X, t)$  is of degree  $n = \dim X$ , and  $\beta_n(X) > 0$ . In particular,  $[X] = [Y]$  implies  $\dim X = \dim Y$ , and  $[X] \neq 0$  if  $X \neq \emptyset$ .*

The generalized Euler characteristics  $\chi_c(X)$  and  $\beta(X, -1) = \sum_i (-1)^i \beta_i(X)$  are equal for  $X$  compact and nonsingular. It follows from Theorem 2.1 that  $\beta(X, -1) = \chi_c(X)$  for all real algebraic varieties  $X$ . But  $\beta_i(X) \neq \dim H_c^i(X; \mathbb{Z}_2)$  in general; in fact the virtual Betti numbers  $\beta_i(X)$  are not topological invariants.

For example, the “figure eight” curve  $X = \{(x, y) \mid y^2 = x^2 - x^4\}$  has a resolution homeomorphic to a circle, and the preimage of the singular point of  $X$  is two points. It follows that  $\beta_1(X) = 1$ . On the other hand, if  $Y$  is the union of two tangent circles then  $Y$  is homeomorphic to  $X$ , but  $\beta_1(Y) = 2$ .

### 3. Complex and real weight filtrations

We obtain a virtual Poincaré polynomial  $\beta^{\mathbb{C}}(X, t) = \sum_j \beta_j^{\mathbb{C}}(X)t^j$  for complex algebraic varieties if we apply the complex version of Theorem 2.1 to the Poincaré polynomial  $P_{\mathbb{C}}(X, t) = \sum_i \dim_{\mathbb{C}} H^i(X; \mathbb{C})t^i$ . The virtual Poincaré polynomial is related to the Hodge–Deligne polynomial (cf. [10]) by

$$\beta^{\mathbb{C}}(X, t) = \sum_{i,j} (-1)^{i+j} w_j^i(X)t^j,$$

with coefficients  $w_j^i(X) = \dim_{\mathbb{C}} W_j^i(X)/W_{j-1}^i(X)$ , where

$$0 \subset W_0^i(X) \subset W_1^i(X) \subset \dots \subset W_i^i(X) = H_c^i(X; \mathbb{C})$$

is the *weight filtration* of cohomology with compact supports [4].

Applying the complex version of Theorem 2.1 to the Poincaré polynomial with coefficients in  $\mathbb{Z}_p$ ,  $p$  prime, we get further examples of generalized Euler characteristics of complex algebraic varieties. The existence of these invariants also follows from the work of Gillet and Soulé [6].

It is natural to ask whether the virtual Betti numbers of real algebraic varieties are associated to a weight filtration on mod 2 cohomology with compact supports. Totaro [12] has announced that there is a natural weight filtration on  $H_c^i(X; \mathbb{Z}_2)$  for a class of varieties  $X$  including real algebraic varieties. However, from Example 3.3 below it can be seen that there is no natural real weight filtration such that the virtual Betti numbers are the weighted Euler characteristics.

**Definition 3.1.** A *real weight filtration*  $W$  assigns to every real algebraic variety  $X$ , and to every  $i \geq 0$ , a filtration

$$0 \subset W_0^i(X) \subset W_1^i(X) \subset \dots \subset W_i^i(X) = H_c^i(X; \mathbb{Z}_2).$$

If  $W$  is a real weight filtration, for all  $i, j$  we let  $w_j^i(X) = \dim_{\mathbb{Z}_2} W_j^i(X)/W_{j-1}^i(X)$ .

We will consider the following properties of a real weight filtration. The classical complex weight filtration considered above has all of these properties.

- (1) *Naturality.* If  $f : X \rightarrow Y$  is an algebraic morphism, then  $f^*W_j^i(Y) \subset W_j^i(X)$  for all  $i, j$ .
- (2) *Strict naturality.* For such a morphism,  $f^*W_j^i(Y) = W_j^i(X) \cap \text{Im } f^*$  for all  $i, j$ .
- (3) *Manifold.* If  $X$  is compact and nonsingular, then  $W_{i-1}^i(X) = 0$  for all  $i$ .
- (4) *Resolution.* If  $X$  is compact and  $p : \tilde{X} \rightarrow X$  is a resolution of singularities, then  $W_{i-1}^i(X) = \text{Ker } p^*$  for all  $i$ .
- (5) *Euler characteristic.* For all  $X$  and all  $j$ , the virtual Betti number  $\beta_j$  is given by

$$\beta_j(X) = (-1)^j \sum_i (-1)^i w_j^i(X).$$

Since the virtual Betti numbers of a compact nonsingular variety equal the classical Betti numbers, the Euler characteristic condition (5) implies the manifold condition (3).

Totaro’s real weight filtration is constructed using cubical hyperresolutions of cohomological descent, cf. [8,6]. This filtration satisfies conditions (1) and (3) but not the other conditions.

**Example 3.2.** Let  $X \subset \mathbb{R}^3$  be the intersection of the circular cylinder  $x^2 + y^2 = 1$  with the union of the two parabolic cylinders  $x = z^2$  and  $x = -z^2$ . Let  $p : \tilde{X} \rightarrow X$  be the resolution of singularities given by the inclusions of the two irreducible components. Let  $C$  be the unit circle in the  $(x, y)$ -plane, and let  $q : X \rightarrow C$  be the projection. Now  $\dim H_c^i(X; \mathbb{Z}_2) = 3$ , and the sequence

$$0 \longrightarrow H_c^i(C; \mathbb{Z}_2) \xrightarrow{q^*} H_c^i(X; \mathbb{Z}_2) \xrightarrow{p^*} H_c^i(\tilde{X}; \mathbb{Z}_2) \longrightarrow 0$$

is exact. If the weight filtration  $0 \subset W_0^1(X) \subset W_1^1(X) = H^1(X)$  has property (4), then  $W_0^1(X) = \text{Ker } p^* = \text{Im } q^*$ . Strict naturality (2) implies that  $q^*W_0^1(C) = W_0^1(X)$ . But property (4) implies that  $W_0^1(C) = 0$ , which is a contradiction. *Therefore a real weight filtration cannot satisfy both strict naturality (2) and resolution (4).*

**Example 3.3.** Let  $X$  be the divisor with normal crossings in 3-space with the following three components:  $X_1$  is the sphere of radius  $\sqrt{2}$  with center  $(1, 0, 0)$ ,  $X_2$  is the sphere of radius  $\sqrt{2}$  with center  $(-1, 0, 0)$ , and  $X_3$  is the torus of revolution which is a tube of radius  $\sqrt{2}/2$  with core the circle of radius 4 in the  $(y, z)$ -plane with center  $(0, 4, 0)$ . The Mayer–Vietoris spectral sequence of  $X = X_1 \cup X_2 \cup X_3$  (cf. [7], Chapter 2, §5) does not degenerate at  $E_2$ , and  $E_3 = E_\infty$ . The alternating sum of the ranks of the entries of the  $j$ -th row of  $E_1$  (and therefore also of  $E_2$ ) equals the virtual Betti number  $\beta_j$ . But the weight filtration on the cohomology of  $X$  defined by the Mayer–Vietoris spectral sequence, which coincides with Totaro’s weight filtration, satisfies  $w_j^i = \dim E_\infty^{i-j, j}$ . Thus, since  $E_2 \neq E_\infty$ , this filtration does not satisfy condition (5). Now suppose  $W$  is a real weight filtration with the naturality and Euler characteristic properties. The latter property and naturality with respect to the inclusion  $X_3 \rightarrow X$  imply that  $w_1^2(X) = 2$ . On the other hand, condition (5) and naturality with respect to the inclusions  $X_i \cap X_j \rightarrow X$ ,  $1 \leq i < j \leq 3$ , imply that  $w_1^2(X) \geq 3$ . *Thus there does not exist a real weight filtration satisfying both naturality (1) and the Euler characteristic condition (5).*

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