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COMPTES RENDUS MATHEMATIQUE

Combinatorics

Counting moves in knight's tours
Décompte des mouvements dans les tours de cavalier

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Abstract

A knight's tour contains eight types of elementary moves. We prove that the only asymptotic constraints on the numbers of moves of each type are the trivial ones: for all proportions compatible with these constraints, there exists a sequence of tours asymptotically achieving these proportions. We deduce a positive answer to the question asked by A. Grigis in C. R. Acad. Sci. Paris, Ser. I 335 989–992 about the existence of tours with an arbitrarily large index. ***To cite this article: P. Dehornoy, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

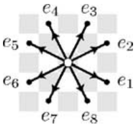
Résumé

Un tour de cavalier contient huit types de mouvements élémentaires. Nous montrons que les seules restrictions sur le nombre asymptotique de ces mouvements sont les bornes triviales existant pour toute boucle : pour tout choix de proportions compatible avec ces bornes, il existe une suite de tours réalisant asymptotiquement ce choix. On déduit l'existence de tours d'indice arbitrairement grand, ce qui répond positivement à la question de A. Grigis, C. R. Acad. Sci. Paris, Ser. I 335 989–992. ***Pour citer cet article : P. Dehornoy, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Version française abrégée

1. La composition d'un tour de cavalier

Un cavalier sur un échiquier peut effectuer les huit mouvements $e_1, \ldots, e_8$ représentés ci-contre. Un chemin de cavalier est une suite finie de mouvements $e_1, \ldots, e_8$; une boucle de cavalier est un chemin fermé, et un *tour* (ou tour ré-entrant) est une boucle qui visite une et une seule fois chaque case d'un échiquier [3,4]. Nous étudions ici les proportions respectives de chacun des huit mouvements possibles dans un tour de cavalier.



Définition 1. Pour tout chemin de cavalier Γ, on note $\varepsilon_i(\Gamma)$ la proportion de mouvements e_i dans Γ ; la suite $(\varepsilon_1(\Gamma), \ldots, \varepsilon_8(\Gamma))$ est appelée la *composition* de Γ.

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La composition vérifie les contraintes $\sum \varepsilon_i = 1$ et (1). Dans cette Note, nous montrons :

Proposition 2. *Pour toute suite* $(\varepsilon_1, \ldots, \varepsilon_8)$ *dans* $\mathbb{R}_+^8$ *vérifiant* $\sum \varepsilon_i = 1$ *et* (1)*, il existe une suite infinie de tours de cavalier* $n \times n$ *dont la composition est* $(\varepsilon_1, \ldots, \varepsilon_8) + \mathrm{O}(\frac{1}{n})$.

Lemme 3. *Soit* $\mathbf{c}_1 = (\frac{1}{2}, 0, 0, 0, \frac{1}{2}, 0, 0, 0)$, $\mathbf{c}_5 = (\frac{5}{12}, 0, 0, \frac{1}{3}, 0, \frac{1}{4}, 0, 0)$, $\mathbf{c}_6 = (0, \frac{5}{12}, 0, 0, \frac{1}{4}, 0, \frac{1}{3}, 0)$. *Pour* $k = 2, \ldots, 4$ (*resp.* $7, \ldots, 12$)*, soit* $\mathbf{c}_k$ *obtenue en décalant les indices de* 1 (*resp. de* 2) *dans* $\mathbf{c}_{k-1}$ (*resp.* $\mathbf{c}_{k-2}$)*. Alors l'ensemble* Δ *des suites* $(\varepsilon_1, \ldots, \varepsilon_8)$ *in* $\mathbb{R}_+^8$ *satisfaisant* (1) *et* $\sum \varepsilon_i = 1$ *est l'enveloppe convexe de* $\mathbf{c}_1, \ldots, \mathbf{c}_{12}$.

La stratégie pour démontrer la Proposition 2 consiste à montrer que les compositions $\mathbf{c}_1$ et $\mathbf{c}_5$ sont réalisables, puis que tous les $\mathbf{c}_k$ le sont, et, enfin, que toutes les compositions dans Δ le sont.

2. Première construction

Réaliser la composition $\mathbf{c}_1$ consiste à construire des tours contenant un nombre quadratique de mouvements e_1 et e_5, et un nombre au plus linéaire des autres mouvements. Ceci est réalisé dans la suite de tours Γ_n représentée dans la Fig. 1.

Proposition 4. *Pour* $n \geqslant 16$ *avec* $n = 0 \pmod 4$*, le diagramme* Γ_n *est un tour de cavalier* $n \times n$ *de composition* $\mathbf{c}_1 + \mathrm{O}(\frac{1}{n})$ *quand* n *tend vers l'infini.*

3. Seconde construction

Pour réaliser la composition $\mathbf{c}_5$, on utilise de même un motif élémentaire de composition $\mathbf{c}_5$, à savoir l'union des pièces X : , et Y : . Ceci est réalisé dans la suite de tours $\widetilde{\Gamma}_n$ représentée dans la Fig. 2.

Proposition 5. *Pour* $n \geqslant 20$ *avec* $n = 8 \pmod{12}$*, le diagramme* $\widetilde{\Gamma}_n$ *est un tour de cavalier* $n \times n$ *de composition* $\mathbf{c}_5 + \mathrm{O}(\frac{1}{n})$ *quand* n *tend vers l'infini.*

4. Somme de deux tours

En utilisant une rotation et une symétrie, on déduit de Γ_n et $\widetilde{\Gamma}_n$ une série de tours $\Gamma_n^{(k)}$ de composition asymptotique $\mathbf{c}_k$ pour $k = 1, \ldots, 12$. Il reste alors à construire des tours de composition $\sum \lambda_k \mathbf{c}_k$ lorsque les λ_k sont des rationnels positifs de somme 1. Pour cela, on utilise une version rectangulaire $\Gamma_{p,q}^{(k)}$ de $\Gamma_n^{(k)}$ de taille $p \times q$, on colle les uns au-dessus des autres les diagrammes $\Gamma_{\lambda_k n, n}^{(k)}$, puis on les connecte grâce à l'opération # de la Fig. 3. Le passage à λ_k réel est facile.

5. Application à l'indice d'un tour

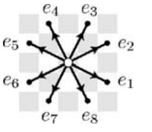
Les valeurs de tout paramètre obtenu par combinaison linéaire des $\varepsilon_i(\Gamma)$ s'obtiennent par projection du polytope convexe Δ, et leur ensemble est donc l'enveloppe convexe des projections des $\mathbf{c}_k$. Dans [2], A. Grigis associe à chaque tour de cavalier Γ une paire d'entiers $(\ell(\Gamma), \ell'(\Gamma))$, appelée *indice*, déterminée par (2). En projetant les douze sommets de Δ suivant (2), on déduit :

Corollaire 6. *Soit* Ω *l'enveloppe convexe des huit points* $(\pm\frac{1}{6}, \pm\frac{1}{12})$ *et* $(\pm\frac{1}{12}, \pm\frac{1}{6})$*. Alors l'indice d'un tour* $n \times n$ *est de la forme* $(\alpha n^2, \alpha' n^2)$ *avec* (α, α') *dans* Ω*. Inversement, pour tout* (α, α') *dans* Ω*, il existe une suite infinie de tours* $n \times n$ *d'indice* $(\alpha n^2, \alpha' n^2) + \mathrm{O}(n)$.

En particulier, l'image de $\mathbf{c}_5$ par les formules (2) est le couple $(\frac{n^2}{6}, \frac{n^2}{12})$, et donc les tours $\widetilde{\Gamma}_n$, dont la composition asymptotique est $\mathbf{c}_5$, ont un index asymptotiquement équivalent à $(\frac{n^2}{6}, \frac{n^2}{12})$.

1. The composition of a knight's tour

On a chessboard, a knight may move to one of the eight vertices of an octagon, as shown in the margin. A *knight's walk* is a sequence of knight's moves. A knight's *loop* is a knight's walk where the last move returns to the initial square. An $m \times n$ knight's *tour* – or *re-entrant tour*, or *circuit* – is a knight's loop that visits every square in an $m \times n$ chessboard once. There exist $m \times n$ tours provided m, n are large enough and at least one of them is even – see [3,4] for an introduction. Here we analyse knight's tours by counting how many moves of each type they contain.



Definition 1. For Γ a knight's walk, we denote by $\varepsilon_i(\Gamma)$ the proportion of e_i's in Γ, i.e., the number of e_i's divided by the total number of moves. The sequence $(\varepsilon_1(\Gamma), \ldots, \varepsilon_8(\Gamma))$ is called the *composition* of Γ.

The composition of a tour is not arbitrary: by construction we have $\sum \varepsilon_i = 1$, and projecting to the axes gives two linear constraints whenever Γ is a loop, namely

$$\begin{aligned} &2\varepsilon_1 + 2\varepsilon_2 + \varepsilon_3 - \varepsilon_4 - 2\varepsilon_5 - 2\varepsilon_6 - \varepsilon_7 + \varepsilon_8 = 0, \\ &-\varepsilon_1 + \varepsilon_2 + 2\varepsilon_3 + 2\varepsilon_4 + \varepsilon_5 - \varepsilon_6 - 2\varepsilon_7 - 2\varepsilon_8 = 0. \end{aligned} \tag{1}$$

We prove that no other restriction applies to the composition of large knight's tours:

Proposition 2. *For each* 8*-tuple* $(\varepsilon_1, \ldots, \varepsilon_8)$ *in* $\mathbb{R}_+^8$ *satisfying* $\sum \varepsilon_i = 1$ *and* (1)*, there exists an infinite sequence of* $n \times n$ *knight's tours with composition* $(\varepsilon_1, \ldots, \varepsilon_8) + \mathrm{O}(\frac{1}{n})$.

So, for instance, there exist tours containing one half of e_1 moves, one half of e_5 moves, and a negligible number of the six other moves.[1]

Lemma 3. *Let* $\mathbf{c}_1 = (\frac{1}{2}, 0, 0, 0, \frac{1}{2}, 0, 0, 0)$, $\mathbf{c}_5 = (\frac{5}{12}, 0, 0, \frac{1}{3}, 0, \frac{1}{4}, 0, 0)$, $\mathbf{c}_6 = (0, \frac{5}{12}, 0, 0, \frac{1}{4}, 0, \frac{1}{3}, 0)$. *For* $k = 2, \ldots, 4$ (*resp.* $7, \ldots, 12$), *define* $\mathbf{c}_k$ *from* $\mathbf{c}_{k-1}$ (*resp.* $\mathbf{c}_{k-2}$) *by shifting the entries by* 1 (*resp. by* 2)*. Then the set* Δ *of all* $(\varepsilon_1, \ldots, \varepsilon_8)$ *in* $\mathbb{R}_+^8$ *satisfying* $\sum \varepsilon_i = 1$ *and* (1) *is the convex hull of* $\mathbf{c}_1, \ldots, \mathbf{c}_{12}$.

Proof. The set Δ is defined by three linear equalities, plus the eight conditions $\varepsilon_i \geqslant 0$. So, it is a convex polytope in $\mathbb{R}^8$, and, therefore, it is the convex hull of its extremal points. The dimension of Δ is 5, so at least five coordinates of these extremal points are 0, and a systematic search is easy. (The results have been double-checked using Maple and the program `cdd++` [1].) □

Let us say that $(\varepsilon_1, \ldots, \varepsilon_8)$ is *achievable* if there exists an infinite sequence of $n \times n$ tours Γ_n whose composition is $(\varepsilon_1, \ldots, \varepsilon_8) + \mathrm{O}(\frac{1}{n})$ (we only require that Γ_n be defined for infinitely many n's). Our aim is to prove that every point in Δ is achievable. Our strategy is to show that $\mathbf{c}_1$ and $\mathbf{c}_5$ are achievable, then that all $\mathbf{c}_k$'s are achievable, and, finally, that every point in Δ is achievable.

[1] Here we consider rectangular or square chessboards only. A variation is to consider chessboards lying on a torus: then the constraints become equalities *modulo* n, and there exist tours asymptotically containing e_1 moves only.

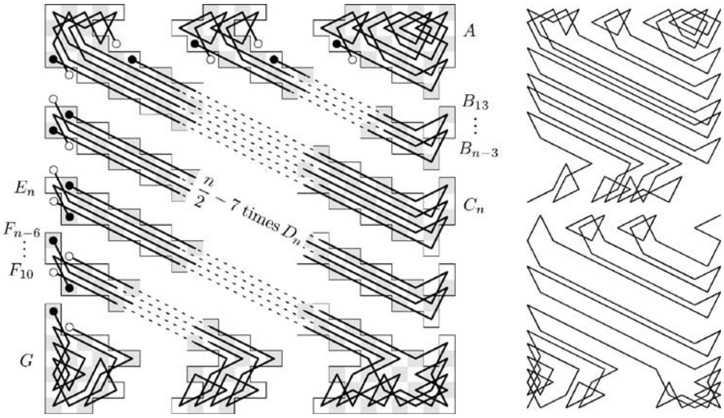


Fig. 1. The tour Γ_n, and the decomposition of the tour Γ_{16} into two diagonal paths.

Fig. 1. Le tour Γ_n, et la décomposition du tour Γ_{16} en deux chemins diagonaux.

2. The first construction

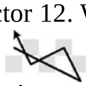
In order to prove that the composition $\mathbf{c}_1$ is achievable, we need to construct a sequence of tours with approximately 50% e_1 moves and 50% e_5 moves. The idea is to tile the chessboard with elementary pieces consisting of one e_1 and one e_5. Covering the central part of the chessboard with alternate lines of e_1's and e_5's is easy, but, as can be expected, problems occur near borders and corners. It is likely that the classical braid method in which several strands travel in parallel [3] cannot be applied, and that a more intricate construction is necessary. Our solution is displayed in Fig. 1. The tour Γ_n is defined for $n \geqslant 16$ with $n = 0 \pmod 4$, and it consists of $n - 9$ pieces of seven types, namely, from top to bottom (the index denotes the overall width of the piece):

$$A, B_{13}, B_{17}, \ldots, B_{n-7}, B_{n-3}, C_n, (n/2 - 7) \text{ times } D_n, E_n, F_{n-6}, F_{n-10}, \ldots, F_{14}, F_{10}, G.$$

Proposition 4. *For $n \geqslant 16$ with $n = 0 \pmod 4$, the diagram Γ_n is an $n \times n$ knight's tour, and its composition has the form $\mathbf{c}_1 + \mathrm{O}(\frac{1}{n})$ when n grows to infinity.*

Proof. Fig. 1 shows that the pieces $A, \ldots, G$ involved in Γ_n tile an $n \times n$ chessboard. To see that Γ_n is a tour, we mark one or two input (resp. output) squares on each piece $A, \ldots, G$ with a white (resp. black) disk. The convention is that the input squares do not belong to the piece. Starting from the white disk of the A piece, one crosses A up to its black disk, then one crosses the B pieces from their right white disk to their left black disk, etc. Finally, we return to the left black disk of the rightmost B piece, which is the square we started from. (The tour Γ_n consists of two complementary walks connecting the top right corner to the bottom left corner, as shown for Γ_{16} on the right of Fig. 1.) It is clear that the numbers of e_1's and e_5's in Γ_n are quadratic in n, while the numbers of all other moves are at most linear in n. Moreover, the numbers of e_1's and e_5's in B_p, D_p, and F_p are equal to p up to a constant, so the composition must be $\mathbf{c}_1 + \mathrm{O}(\frac{1}{n})$. □

3. The second construction

To prove that $\mathbf{c}_5$ is achievable, we have to construct a series of tours whose composition is $(5, 0, 0, 4, 0, 3, 0, 0)$ up to a factor 12. We start with a double pattern containing 5 e_1's, 4 e_4's, and 3 e_6's, namely the union of the two pieces, X: , and Y: . As above, we try to tile the chessboard using this pattern everywhere but on the boundaries.

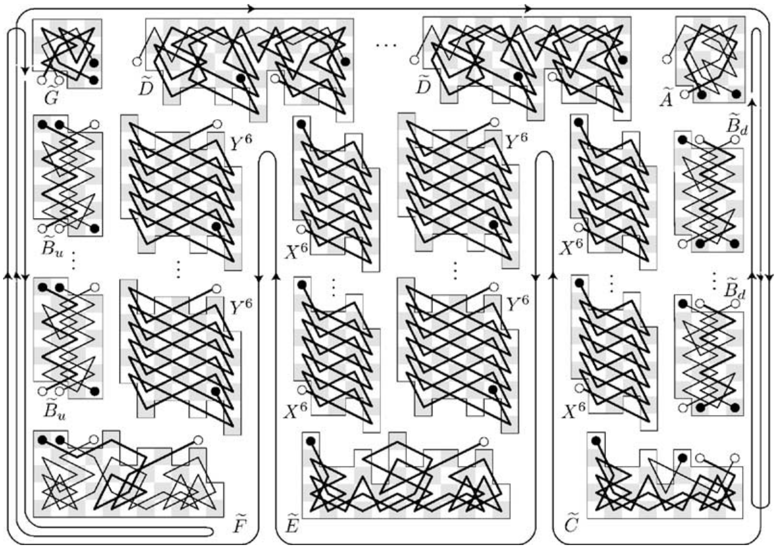


Fig. 2. The tour $\widetilde{\Gamma}_n$.

Fig. 2. Le tour $\widetilde{\Gamma}_n$.

Our solution, called $\widetilde{\Gamma}_n$, is described in Fig. 2. It is defined for $n \geqslant 20$ with $n = 8 \pmod{12}$. Assuming $n = 12p + 8$, starting from the top right corner, and following the thick path, the tour $\widetilde{\Gamma}_n$ is described by the sequence $\widetilde{A}\widetilde{B}_d^{2p}\widetilde{C}X^{12p}\widetilde{D}(Y^{12p}\widetilde{E}X^{12p}\widetilde{D})^{p-1}Y^{12p}\widetilde{F}\widetilde{B}_u^{2p}\widetilde{G}$.

Proposition 5. *For $n \geqslant 20$ with $n = 8 \pmod{12}$, the diagram $\widetilde{\Gamma}_n$ is an $n \times n$ knight's tour, and its composition has the form $\mathbf{c}_5 + \mathrm{O}(\frac{1}{n})$ when n grows to infinity.*

Proof. As shown in the figure, the pieces make a partition of an $n \times n$ chessboard. The tour $\widetilde{\Gamma}_n$ consists of the thick path which zigzags through the central part of the chessboard from the top right corner to the top left corner, followed by the thin path, which completes the tour along the left side (back and forth), the top side, and the right side (back and forth). The composition is asymptotically $\mathbf{c}_5$ because the X and Y pieces appear an equal quadratic number of times, while the other pieces appear a constant or linear number of times. □

4. Connected sum of two tours

The horizontal symmetry s transforms a tour of composition $(\varepsilon_1, \ldots, \varepsilon_8)$ into one of composition $(\varepsilon_2, \varepsilon_1, \varepsilon_8, \varepsilon_7, \ldots, \varepsilon_3)$, while the $\frac{\pi}{2}$ rotation r transforms it into one of composition $(\varepsilon_7, \varepsilon_8, \varepsilon_1, \ldots, \varepsilon_6)$. Define $\Gamma_n^{(1)}, \ldots, \Gamma_n^{(12)}$ to be Γ_n, $s(\Gamma_n)$, $r(\Gamma_n)$, $sr(\Gamma_n)$, $\widetilde{\Gamma}_n$, $s(\widetilde{\Gamma}_n)$, $r(\widetilde{\Gamma}_n)$, $rs(\widetilde{\Gamma}_n)$, $r^2(\widetilde{\Gamma}_n)$, $r^2s(\widetilde{\Gamma}_n)$, $r^3(\widetilde{\Gamma}_n)$, and $r^3s(\widetilde{\Gamma}_n)$, respectively. Then, for each k, the tours $\Gamma_n^{(k)}$ witness that $\mathbf{c}_k$ is achievable.

It remains to show that any composition $\sum \lambda_k \mathbf{c}_k$ with $\lambda_1, \ldots, \lambda_{12} \geqslant 0$ and $\sum \lambda_k = 1$ is achievable. We begin with the case of rational λ_k's. For convenient values of $p, q \pmod{12}$, there exists a rectangular $p \times q$ version $\Gamma_{p,q}^{(k)}$ of $\Gamma_n^{(k)}$ with composition $\mathbf{c}_k$. In order to achieve $\sum \lambda_k \mathbf{c}_k$, we consider the tours $\Gamma_{n_k,n}^{(k)}$ with $n_k = \lambda_k n$. Let Γ be the diagram obtained by putting the tours $\Gamma_{n_k,n}^{(k)}$ one above the other. Then Γ has the needed average composition, and it suffices to show that we can arrange it into a tour (without changing the composition too much).

This can be done by using the classical connecting operation displayed in Fig. 3, which can be traced back to [5]. Assume that Γ contains one e_1 and Γ' contains one e_5 in the positions shown in Fig. 3 (left). We denote by $\Gamma \# \Gamma'$ the result of gluing Γ above Γ' and replacing the distinguished e_1 and e_5 by e_7 and e_3, as shown in Fig. 3 (right)

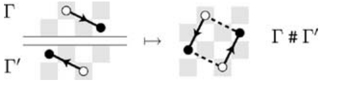


Fig. 3. Connecting Γ and Γ' into $\Gamma\#\Gamma'$.

Fig. 3. Connection de Γ et Γ' en $\Gamma\#\Gamma'$.

– to make $\Gamma\#\Gamma'$ well defined, we require # to be applied to the leftmost possible position. It is easy to check that $\Gamma\#\Gamma'$ is a tour. It is easy to see that the tours $\Gamma_n^{(k)}$ are eligible for the #-operation.

(Actually, the gluing argument is not readily correct, because the tours $\Gamma_{p,q}^{(k)}$ are not defined for $p < q$ in the cases $k = 1, 2$: "high" versions of Γ_n exist, but "broad" versions do not. The obvious solution is to use the vertical counterpart of the # operation and to append the needed versions of Γ_n and $s(\Gamma_n)$ on the left, instead of on the top.)

In order to complete the proof of Proposition 2, it remains to extend the result to compositions $\sum \lambda_k \mathbf{c}_k$ with λ_k in $\mathbb{R}$. This is easily done using a continuity argument, because the convergence rate in the above construction does not depend on the choice of the λ_k's in $\mathbb{Q}$.

5. Application to the index

Once we know that the range of the composition $(\varepsilon_1(\Gamma), \ldots, \varepsilon_8(\Gamma))$ is the convex domain Δ, we deduce that the range of any parameter defined by linear combinations of the $\varepsilon_i(\Gamma)$'s is the projection of Δ, hence it is the convex hull of the projections of the twelve points $\mathbf{c}_1, \ldots, \mathbf{c}_{12}$.

In [2], A. Grigis associates with every tour Γ a pair of integers $(\ell(\Gamma), \ell'(\Gamma))$ called the *index* of Γ. This index describes the monodromy of the lifting of Γ in some lattice of $\mathbb{Z}^4$. A. Grigis asks whether there exist tours with an arbitrarily large index. Now, the index is defined from the ε_i's by

$$\begin{aligned} 10\ell &= (2\varepsilon_1 - 2\varepsilon_2 + \varepsilon_3 + \varepsilon_4 - 2\varepsilon_5 + 2\varepsilon_6 - \varepsilon_7 - \varepsilon_8)n^2, \\ 10\ell' &= (\varepsilon_1 + \varepsilon_2 - 2\varepsilon_3 + 2\varepsilon_4 - \varepsilon_5 - \varepsilon_6 + 2\varepsilon_7 - 2\varepsilon_8)n^2. \end{aligned} \tag{2}$$

Then Proposition 2 immediately enables us to answer the question in the positive:

Corollary 6. *Let Ω be the convex hull of the eight points $(\pm\frac{1}{6}, \pm\frac{1}{12})$ and $(\pm\frac{1}{12}, \pm\frac{1}{6})$ in $\mathbb{R}^2$. Then the index of an $n \times n$ knight's tour must be $(\alpha n^2, \alpha' n^2)$ for some (α, α') in Ω. Conversely, for each (α, α') in Ω, there exists a sequence of $n \times n$ knight's tours with index $(\alpha n^2, \alpha' n^2) + \mathrm{O}(n)$.*

In particular, the image of $\mathbf{c}_5$ under (2) is the pair $(\frac{n^2}{6}, \frac{n^2}{12})$. So the tours $\widetilde{\Gamma}_n$, whose asymptotic composition is $\mathbf{c}_5$, have asymptotic index $(\frac{n^2}{6}, \frac{n^2}{12})$.

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References

[1] K. Fukuda, http://www.cs.mcgill.ca/fukuda/download/cdd/README.cdd+.
[2] A. Grigis, L'indice d'un tour de cavalier, C. R. Acad. Sci. Paris, Ser. I 335 (12) 989–992.
[3] G. Jelliss, Knight's tour notes, http://www.ktn.freeuk.com.
[4] W.W. Rouse, H.S.M. Coxeter, Mathematical Recreations and Essays, 13th edition, Dover, New York, 1987.
[5] A.T. Vandermonde, Remarques sur les problèmes de situation, Mém. Acad. Sci. Paris (1774) 566–574.