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C. R. Acad. Sci. Paris, Ser. I 336 (2003) 585–588
COMPTES RENDUS MATHEMATIQUE

Algebraic Geometry

On the Łojasiewicz numbers
Sur les nombres de Łojasiewicz

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Received and accepted 19 February 2003

Presented by Bernard Malgrange

Abstract

Let f be a holomorphic function of two complex variables with an isolated critical point at $0 \in \mathbb{C}^2$. We give some necessary conditions for a rational number to be the smallest $\theta > 0$ in the Łojasiewicz inequality $|\operatorname{grad} f(z)| \geqslant C|z|^\theta$ for z near $0 \in \mathbb{C}^2$. ***To cite this article: E. García Barroso, A. Płoski, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Résumé

Soit f une fonction holomorphe de deux variables complexes ayant un point critique isolé à l'origine. Nous donnons des conditions nécessaires pour qu'un nombre rationnel soit égal au plus petit $\theta > 0$ tel que l'on ait l'inégalité de Łojasiewicz $|\operatorname{grad} f(z)| \geqslant C|z|^\theta$ dans un voisinage de 0 dans $\mathbb{C}^2$. ***Pour citer cet article : E. García Barroso, A. Płoski, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

1. Introduction

Let F be a holomorphic mapping of an open neibourhood of $0 \in \mathbb{C}^2$ into $\mathbb{C}^2$ with an isolated zero at $0 \in \mathbb{C}^2$. The *Łojasiewicz exponent* $\mathcal{L}_0(F)$ of F at 0 is defined to be the smallest $\theta > 0$ such that

$$\left|F(z)\right| \geqslant C|z|^\theta \quad \text{in a neighbourhood of } 0 \in \mathbb{C}^2 \text{ with a constant } C > 0.$$

It is well-known that the exponent $\mathcal{L}_0(F)$ is a rational number (see [3] and [5]). Moreover in [5] (see p. 359) the following is proved

Proposition 1.1. *A rational number is equal to the Łojasiewicz exponent of a holomorphic mapping of $\mathbb{C}^2$ if and only if it appears in the sequence*

$$1, 2, 3, 3\frac{1}{2}, 4, 4\frac{1}{3}, 4\frac{1}{2}, 4\frac{2}{3}, 5, \ldots. \tag{1.1}$$

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1631-073X/03/$ – see front matter
doi:10.1016/S1631-073X(03)00105-5

The fractional parts of the numbers appearing in (1.1) form Farey's sequences

$$F_2=\left\{0,\frac{1}{2},1\right\},\ F_3=\left\{0,\frac{1}{3},\frac{1}{2},\frac{2}{3},1\right\},\ldots.$$

In all this Note a, b, N are integers such that $0<b<a<N$. Thus every term of the sequence (1.1) is a rational number of the form $N+\frac{b}{a}$ or an integer.

Let us consider the germ $(C,0)$ of a singular plane curve with local equation $f=0$. Let $\operatorname{grad} f:=(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y})$ be the gradient of f. The Łojasiewicz exponent $\mathcal{L}_0(\operatorname{grad} f)$ will be called the *Łojasiewicz number* of the germ $(C,0)$ and denoted $\mathcal{L}_0(f)$.

For example, if $(C,0)$ is defined by a homogeneous form of degree $n+1$ then the Łojasiewicz number of $(C,0)$ is equal to n.

The quasihomogeneous singularities provide examples of the Łojasiewicz numbers which are not integers. More precisely we have

Proposition 1.2. *The number $N+\frac{b}{a}$ is the Łojasiewicz number of a quasihomogeneous singularity if and only if $N\equiv b-1$ or $b \bmod (a+1)$.*

It is natural to ask the following question: does there exist for every rational number r of the sequence (1.1) a germ $(C,0)$ with local equation $f=0$ such that $\mathcal{L}_0(f)=r$?

The answer to this question is negative. In this Note we will prove

Theorem 1.3. *The rational numbers $a+1+\frac{b}{a}$ where a,b are integers such that $1<b<a$ and a,b are coprime, are not the Łojasiewicz numbers of plane curve germs.*

Obviously all the numbers $a+1+\frac{b}{a}$ where $1<b<a$ and a, b coprime appear in the sequence (1.1). The proof of our theorem is given in the third section of this Note. It is based on Teissier's formula for the Łojasiewicz number and some properties of polar invariants.

Example 1. Using Proposition 1.2 and Theorem 1.3 it is easy to check that the Łojasiewicz numbers less than 6 are

$$1,2,3,3\frac{1}{2},4,4\frac{1}{3},4\frac{1}{2},5,5\frac{1}{4},5\frac{1}{3},5\frac{1}{2},5\frac{2}{3}.$$

Indeed all numbers above different from $5\frac{1}{2}$ are the Łojasiewicz numbers of quasihomogeneous singularities calculated above. Moreover for $f=y^8+xy^5+x^3y^2+x^6$ we get $\mathcal{L}_0(f)=5\frac{1}{2}$ (see [4], p. 311). The Łojasiewicz number of the branches of semi-group $\langle 4,6,13\rangle$ is $5\frac{1}{2}$ too. There is no quasihomogeneous singularity with the Łojasiewicz number equal to $5\frac{1}{2}$.

2. Polar invariants

Let $(C,0)$ be a plane singular germ and let $(L,0)$ be a smooth branch which is not tangent to $(C,0)$. Consider the local reduced equations $f=0$ and $l=0$ of $(C,0)$ and $(L,0)$ and put $j(f,l):=\frac{\partial f}{\partial x}\frac{\partial l}{\partial y}-\frac{\partial f}{\partial y}\frac{\partial l}{\partial x}$.

If $j(f,l)=h_1\cdots h_s$ is the decomposition of $j(f,l)$ into irreducible factors in the ring $\mathcal{O}_{\mathbb{C}^2,0}$ then we set $\mathcal{Q}(C,0)=\{\frac{(f,h_i)_0}{\operatorname{ord} h_i}\colon i\in\{1,\ldots,s\}\}$ where $(f,h)_0$ is the intersection number of f and h at the origin.

We call the elements of $\mathcal{Q}(C,0)$ polar invariants of $(C,0)$. They do not depend on the choice of $(L,0)$ and are topological invariants of the germ $(C,0)$ (see [7], Théorème 6, p. 275). For every $q\in\mathcal{Q}(C,0)$ we put $A_q:=\{i\colon \frac{(f,h_i)_0}{\operatorname{ord} h_i}=q\}$, $j_q:=\prod_{i\in A_q}h_i$ and $m_q=\operatorname{ord} j_q$. We call m_q the multiplicity of q. Clearly $qm_q=(f,j_q)_0$ is an integer.

Let us observe the inequality $m_q \leqslant \operatorname{ord} f - 1$, with equality if and only if $\mathcal{Q}(C,0) = \{q\}$. In the sequel we put $\eta_0(f) = \max\mathcal{Q}(C,0)$.

Teissier's formula for the Łojasiewicz number is $\mathcal{L}_0(f) = \eta_0(f) - 1$. Teissier also proved (see [7], Corollaire 3.4, p. 281) that the degree of C^0-sufficiency $\mathrm{Suff}_0(f)$ of the function germ f is $\mathrm{Suff}_0(f) = [\mathcal{L}_0(f)] + 1 = [\eta_0(f)]$ (see [7], Théorème 8, p. 280).

We call $(C,0)$ an Eggers singularity if $\mathcal{Q}(C,0)$ has exactly one element. We need

Classification of Eggers' singularities (see [2], Korollar 3, p. 16)). If $\mathcal{Q}(C,0) = \{\eta\}$ with $\eta \in \mathbb{Q}$ then $(C,0)$ is topologically equivalent to a plane curve singularity of type $y^n - x^m = 0$ or of type $y^n - yx^m = 0$ with $n \leqslant m$. Moreover $\eta = m$ for the first type and $\eta = \frac{mn}{n-1}$ for the second type of singularity.

Proof of Proposition 1.2. Let n, m be integers such that $1 < n \leqslant m$. We set $l_{n,m} = y^n + x^m$, $f_{n,m} = y^n + yx^m$ and $g_{n,m} = xy^m + yx^n$. All polynomials listed above are quasihomogeneous and $\mathcal{L}_0(l_{n,m}) = m$, $\mathcal{L}_0(f_{n,m}) = \frac{mn}{n-1} - 1$, $\mathcal{L}_0(g_{n,m}) = \frac{mn-1}{n-1} - 1$ (see [4], Corollary 1.4). On the other hand every quasihomogeneous singularity is topologically equivalent to one of the singularities $l_{n,m} = 0$ or $f_{n,m} = 0$ or $g_{n,m} = 0$. Since the Łojasiewicz number is a topological invariant we see that the Łojasiewicz numbers of quasihomogeneous singularities are integer or rationals of the form $\frac{mn}{n-1} - 1$ or $\frac{mn-1}{n-1} - 1$. It suffices to observe that $\frac{mn}{n-1} = N + \frac{b}{a}$ with $N \equiv b \bmod (a+1)$ (if $n = a+1$, $m = qa + b$) and $\frac{mn-1}{n-1} = N + \frac{b}{a}$ with $N \equiv b - 1 \bmod (a+1)$ (if $n = a+1$, $m = qa + b + 1$).

3. Proof of the result

Let $\mu_0(f) = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})_0$ be the Milnor number of the germ $f = 0$; it is a topological invariant of the germ $f = 0$ (see [7], Théorème 5, p. 274).

Let $\mathrm{in} f$ be the initial form of the power series from $\mathbb{C}\{x,y\}$ representing the germ f. Recall that f is an ordinary singularity if and only if the form $\mathrm{in} f$ has no multiple factor, i.e., if the germ $f = 0$ has exactly $\operatorname{ord} f$ tangents. The lemma below is the two-dimensional case of a result due to Teissier (see [6], Propositions 2.1 and 2.7). It can be also obtained from the well-known inequality for the intersection multiplicity of two curves (see [1], Proposition 2.6.7).

Lemma 3.1. *Let $f = 0$ be a germ. Then $\mu_0(f) \geqslant (\operatorname{ord} f - 1)^2$ with equality if and only if f is an ordinary singularity.*

Lemma 3.2. *If $\eta_0(f) \neq \operatorname{ord} f$ then $\operatorname{ord} f + 1 \leqslant [\eta_0(f)]$.*

Proof. Since $\mathrm{Suff}_0(f) = [\eta_0(f)]$ we get the equalities $\mu_0(f) = \mu_0(\tilde{f})$ and $\eta_0(f) = \eta_0(\tilde{f})$, where $\tilde{f}$ is the Taylor polynomial of f of degree $\leqslant [\eta_0(f)]$. Using Bezout's theorem we get $\mu_0(f) = \mu_0(\tilde{f}) \leqslant (\deg \tilde{f} - 1)^2 \leqslant ([\eta_0(f)] - 1)^2$. On the other hand if $\eta_0(f) \neq \operatorname{ord} f$ then $f = 0$ is not an ordinary singularity (see Lemma 3.3 of [4]) and $\mu_0(f) > (\operatorname{ord} f - 1)^2$ by Lemma 3.1. Consequently we get $\operatorname{ord} f < [\eta_0(f)]$ and the lemma follows.

Lemma 3.3. *Suppose that $\mathcal{L}_0(f) \notin \mathbb{N}$ and write $\mathcal{L}_0(f) = N + \frac{b}{a}$ with $0 < b < a$ and $\mathrm{g.c.d.}(a,b) = 1$. Let m be the multiplicity of the polar invariant $\eta_0(f)$. Then a is a divisor of m and $a \leqslant m \leqslant \operatorname{ord} f - 1 \leqslant N - 1$.*

Proof. By Teissier's formula $\mathcal{L}_0(f) = \eta_0(f) - 1 = \frac{m\eta_0(f) - m}{m}$. On the other hand $\mathcal{L}_0(f) = \frac{aN+b}{a}$ where $a, aN+b$ are coprime. Hence the integer a divides the multiplicity m. Therefore $a \leqslant m$ and obviously $m \leqslant \operatorname{ord} f - 1$. The inequality $\operatorname{ord} f - 1 \leqslant N - 1$ follows from Lemma 3.2 for $N = [\mathcal{L}_0(f)] = [\eta_0(f)] - 1$ and $\eta_0(f) = \mathcal{L}_0(f) + 1 \notin \mathbb{N}$, in particular $\eta_0(f) \neq \operatorname{ord} f$.

Proof of the theorem. Suppose that $\mathcal{L}_0(f) = a + 1 + \frac{b}{a}$ where $1 < b < a$ and a, b are coprime. Keeping the notation of Lemma 3.3 we see that $a = N - 1$. Consequently $a = m = \operatorname{ord} f - 1 = N - 1$ and $f = 0$ is an Eggers singularity of order N. Since $\mathcal{L}_0(f) \notin \mathbb{N}$ we get by the classification of Eggers singularities that there exists $M \in \mathbb{N}$, $M \geqslant N$ such that

$$\mathcal{L}_0(f) = \frac{MN}{N-1} - 1.$$

Thus $\frac{MN}{N-1} - 1 = a + 1 + \frac{b}{a} = N + \frac{b}{N-1}$ and $b - 1 = NM - N^2$ is divided by N which is a contradiction because $0 < b - 1 < N$.

4. Concluding remarks

Teissier's collection $\{(q, m_q)\colon q \in \mathcal{Q}(C, 0)\}$ is encoded by means of the Jacobian Newton polygon $\mathcal{N}_j(C)$ of the germ $(C, 0)$ (see [8], pp. 195–197). It is the Newton polygon intersecting both axes determined by the conditions:

(1) The slopes of the lines supporting $\mathcal{N}_j(C)$ are $-\frac{1}{q}$, $q \in \mathcal{Q}(C, 0)$.
(2) The length of the projection of the segment of slope $-\frac{1}{q}$ on the vertical axis is equal to the multiplicity of q.

In the sequel by the inclination of a segment of the Newton polygon we mean the negative of the reciprocal of its slope. We say that a Newton polygon $\mathcal{N}$ is *very special* if it possesses the following properties:

(1) There exist integers $\mu, \mu' > 0$ such that $\mathcal{N}$ intersects the axes at the points $(0, \mu')$ and $(\mu + \mu', 0)$,
(2) the inclinations of the segments of $\mathcal{N}$ are greater than or equal to $\mu' + 1$,
(3) if η is the greatest inclination of the segmets of $\mathcal{N}$ then $\eta = \mu' + 1$ or $\eta \geqslant \mu' + 2$.

Let us consider the Jacobian Newton polygon $\mathcal{N}_j(C)$ of the germ $(C, 0)$ with local equation $f = 0$. Then $\mathcal{N}_j(C)$ is very special; it suffices to take $\mu' = \operatorname{ord} f - 1$ and $\mu = \mu_0(f)$ (see [7], Remark 1.4 and Lemma 3.2 of this note). Thus $\eta = \eta_0(f) = \mathcal{L}_0(f) + 1$. Reasoning like that in the proof of Lemma 3.3 it follows that if the Newton polygon $\mathcal{N}$ is very special then $\eta - 1$ appears in the sequence (1.1). Not every very special Newton polygon is a Jacobian Newton polygon. Our theorem shows that the Jacobian Newton polygons are subject to stronger arithmetical restrictions. The polygon having only one segment joining the points $(0, \mu')$ and $((\mu')^2 + \mu' + 2, 0)$ is very special but is not a Jacobian Newton polygon of a plane curve germ.

Acknowledgement

The authors are grateful to Bernard Teissier for his helpful comments and suggestions. This research was partially supported by KBN Grant 2P03A01522.

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