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COMPTES RENDUS MATHEMATIQUE

Dynamical Systems

# Unique normal forms for Hopf-zero vector fields

# Formes normales uniques des champs de vecteurs de type Hopf-zéro

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**Abstract**

We consider normal forms of Hopf-zero vector fields in $\mathbf{R}^3$. Unique normal forms under conjugacy and orbital equivalence for the generic case are given. ***To cite this article: G. Chen et al., C. R. Acad. Sci. Paris, Ser. I 336 (2003).***


**Résumé**

Nous étudions l'unicité des formes normales de champs de vecteurs de type Hopf-zéro dans $\mathbf{R}^3$. Des formes normales uniques dans le cas générique sont données par rapport aux changements de coordonnées et pour l'équivalence orbitale. ***Pour citer cet article : G. Chen et al., C. R. Acad. Sci. Paris, Ser. I 336 (2003).***


## 1. Introduction and main results

Denote by $X$ a germ of smooth ($C^\infty$) vector field in $\mathbf{R}^3$ whose linear part has eigenvalues zero and $\pm \mathrm{i}$ and whose nonlinear part is generic. Then in formal conjugacy category we can always normalize the linear part of $X$ to the form $X_1 = x_3\partial_2 - x_2\partial_3$. Denote by $H_k$ $(k \geqslant 2)$ the set of homogeneous vector fields of resonant terms of degree $k$. One has, for $m \geqslant 1$,

$$H_{2m} = \mathrm{Span}\bigl\{r^{2m}\partial_1, x_1^{2m-2i}r^{2i}\partial_1, x_1^{2m-2i-1}r^{2i}V_1, x_1^{2m-2i-1}r^{2i}X_1,\ 0 \leqslant i \leqslant m-1\bigr\},$$
$$H_{2m+1} = \mathrm{Span}\bigl\{x_1^{2m-2i+1}r^{2i}\partial_1, x_1^{2m-2i}r^{2i}V_1, x_1^{2m-2i}r^{2i}X_1,\ 0 \leqslant i \leqslant m\bigr\},$$

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where $V_1 = x_2\partial_2 + x_3\partial_3$ and $r^2 = x_2^2 + x_3^2$. Then the Poincaré–Dulac normal form can be written as

$$X = X_1 + \sum_{k\geqslant 2} X_k, \quad \text{with } X_k \in H_k. \tag{1}$$

With a linear transformation one can obtain, in the generic case, a new vector field in which the quadratic terms take the form:

$$X_2 = \left(x_1^2 \pm r^2\right)\partial_1 + x_1(b_0V_1 + c_0X_1), \tag{2}$$

where $b_0$ and $c_0$ are real parameters. To simplify notations and calculations, throughout the Note we shall suppose that this step has already been done.

We remark that the signs $\pm$ and the values of $b_0$ and $c_0$ are invariants of $X$ in the conjugacy context and therefore they do not depend on the choice of individual coordinates. In the orbital equivalence context, however, it does not make any further sense to specify the value of $c_0$. Indeed one can always eliminate the term $c_0x_1X_1$ in (2) by reparameterizing time (see shortly). Thus the quadratic terms can be reduced, under orbital equivalence, to

$$\widetilde{X}_2 = \left(x_1^2 \pm r^2\right)\partial_1 + b_0x_1V_1.$$

Let $H_1 = \mathrm{Span}\{X_1\}$. We have $[H_1, H_j] = \{0\}$ for any $j \geqslant 1$. One also has $[H_i, H_j] \subset H_{i+j-1}$ for any $i, j$. Let $\mathcal{L}_j = H_{j+1}$ then $\sum_{k\geqslant 0}\mathcal{L}_k$ is a graded Lie algebra.To obtain a unique normal form we need to find a vector field $\Phi$ such that $\exp(\mathrm{ad}(\Phi))(X)$ is in a normal form (see [1] or [4]).

We have obtained the following unique normal forms of $X$ with respect to both conjugacy and orbital equivalence, under nondegeneracy conditions. In terms of (2), the nondegeneracy conditions can be expressed as follows, for any integer $m \geqslant 1$,

$$2m + 2jb_0 \neq 2j, 2j+1, 2j+2, \quad \text{for } 0 < j \leqslant m. \tag{3}$$

We state our results in the following. We refer to [2] for more details.

**Theorem 1.1** (Conjugacy equivalence normal form). *Let a vector field be given with a linear part $X_1$ as above and a generic quadratic part $X_2$, i.e., verifying the conditions* (3)*. Then it can be formally conjugated to the following unique normal form*

$$X = X_1 + X_2 + \left(a_1x_1^3 + \sum_{m\geqslant 2} a_m r^{2m}\right)\partial_1 + \sum_{m\geqslant 1}\left(b_m r^{2m}V_1 + c_m r^{2m}X_1\right). \tag{4}$$

**Theorem 1.2** (Orbital equivalence normal form). *Let a generic vector field be given as in the above theorem, then it is formally orbitally equivalent to the following unique normal form*

$$\widetilde{X} = X_1 + \widetilde{X}_2 + \widetilde{X}_3 + \sum_{m\geqslant 2}\left(\tilde{a}_m r^{2m}\partial_1 + \tilde{b}_m r^{2m}V_1\right), \tag{5}$$

*where $\widetilde{X}_2 = (x_1^2 \pm r^2)\partial_1 + b_0x_1V_1$ and $\widetilde{X}_3 = \tilde{a}_1x_1^3\partial_1 + \tilde{b}_1r^2V_1$.*

## 2. Main steps of the proofs

Having put the vector field to a resonant normal form (1), we need to consider the adjoint map $[X_2, \cdot]$ restricted to each $H_k$ for $k \geqslant 2$, and find a complementary subspace $\mathcal{C}_{k+1}$ of its range in $H_{k+1}$. We can prove the following.

**Lemma 2.1.** *Let $m \geqslant 1$ be an integer. Suppose that generic conditions* (3) *are satisfied.*

(a) *If $m = 1$, then $\ker[X_2, \cdot]|_{H_2} = \mathrm{Span}\{X_2\}$ and*

$$H_3 = [X_2, H_2] \oplus \mathcal{C}_3, \quad \textit{with } \mathcal{C}_3 = \mathrm{Span}\left\{x_1^3\partial_1, r^2V_1, r^2X_1\right\}.$$

(b) *If* $m \geqslant 2$, *then* $\ker[X_2, \cdot]|_{H_{2m}} = \{0\}$ *and*

$$H_{2m+1} = [X_2, H_{2m}] \oplus \mathcal{C}_{2m+1}, \quad \text{with } \mathcal{C}_{2m+1} = \mathrm{Span}\{r^{2m} V_1, r^{2m} X_1\}.$$

(c) *If* $m \geqslant 1$ *then* $\ker[X_2, \cdot]|_{H_{2m+1}} = \{0\}$ *and*

$$H_{2m+2} = [X_2, H_{2m+1}] \oplus \mathcal{C}_{2m+2}, \quad \text{with } \mathcal{C}_{2m+2} = \mathrm{Span}\{r^{2(m+1)} \partial_1\}.$$

Let $k \geqslant 2$ be an integer. Assume that we have obtained a conjugacy normal form up to order $k$. To compute a normal form of order $k+1$, we consider the linear map $[X_2, \Phi_k] + \cdots + [X_k, \Phi_2]$ for $\Phi_i \in H_i$, subjected to the conditions

$$\sum_{i=0}^{j-2} [X_{2+i}, \Phi_{j-i}] = 0 \quad \text{for } 2 \leqslant j \leqslant k-1. \tag{6}$$

We compute its range in $H_{k+1}$. Using Lemma 2.1 and by induction, we obtain $\Phi_j = \alpha X_j$ for $j = 2, \ldots, k-1$. Hence

$$\sum_{i=0}^{k-2} [X_{2+i}, \Phi_{k-i}] = [X_2, \Phi_k] + \sum_{i=1}^{k-2} [X_{2+i}, \alpha X_{k-i}] = [X_2, \Phi_k - \alpha X_k].$$

Its range coincides with that of the adjoint operator $[X_2, \cdot]$. This proves the uniqueness of the conjugacy normal form of Theorem 1.1. We notice that a similar unique conjugacy normal form is given in [6] by using a different method.

Orbital equivalence consists of using coordinate transformations and reparametrization of the time, i.e., multiplication by a function. So the problem is to find a formal series $\widetilde{G} = 1 + \sum_{j \geqslant 1} G_j$, where $G_j$ is homogeneous of degree $j$, and a vector field $\phi$ so that $\exp(\mathrm{ad}(\phi))(\widetilde{G} X)$ is as simple as possible.

Let $X$ be a vector field in a conjugacy normal form as described in (4). Let $g_0 = 1 - c_0 x_1$. Then

$$\widehat{X} = g_0 X = X - c_0 x_1 X = X_1 + X_2 - c_0 x_1 X_1 + \cdots = X_1 + \widetilde{X}_2 + \cdots,$$

where the dots represent terms of degree higher than 2. One can then consider terms of degree 3 by renormalizing again by coordinate transformations. The conjugacy normal form obtained for $\widehat{X}$ is in the same form as before. More generally for $m \geqslant 1$, let $g_m = 1 - c_m r^{2m}$. Then

$$g_m \widehat{X} = \widehat{X} - c_m r^{2m} \widehat{X} = X_1 + \widetilde{X}_2 + \cdots + X_m - c_m r^{2m} X_1 + \cdots,$$

It is clear now that one can use a sequence of multiplications and coordinate transformations to convert any generic Hopf-zero vector field to a new one in the form (5). It remains to prove the uniqueness of this normal form $\widetilde{X}$.

We now consider some special cases for the function $G$. Let $m \geqslant 2$ and $G = \sum_{j \geqslant 2m} G_j$. Then

$$\widehat{X} = \widetilde{G} \widetilde{X} = \widetilde{X} + G \widetilde{X} = \widetilde{X} + \sum_{j \geqslant 2m} \sum_{i=1}^{j-1} G_i \widetilde{X}_{j-i}.$$

Now we renormalize $\widehat{X}$ under conjugacy equivalence. Let $\phi = \Phi_{2m} + \Phi_{2m+1} + \cdots$ where $\Phi_j \in H_j$. It turns out that, to keep $\widehat{X}$ in a normal form up to order $2m+1$, $G_{2m}, \Phi_{2m}$ should be solutions of the equation

$$[\Phi_{2m}, \widetilde{X}_2] + G_{2m} X_1 = 0.$$

One finds that $G_{2m} = \widetilde{X}_2(h_{2m-1})$ and $\Phi_{2m} = \sum_{j=0}^{m-1} \gamma_j^{(2m)} x_1^{2(m-j)-1} r^{2j} X_1 = h_{2m-1} X_1$, where $\gamma_j^{(2m)}$ are arbitrary real parameters and $h_{2m-1} = \sum_{j=0}^{m-1} \gamma_j^{(2m)} x_1^{2(m-j)-1} r^{2j}$. Here $\widetilde{X}_2(f)$ denotes the directional derivative of $f$ along $\widetilde{X}_2$.

Hence terms of order $2m+1$ in the orbital normal form of $\widetilde{X}$ are unchanged with arbitrary $\gamma_j^{(2m)}$.

In order to determine $\Phi_{2m+1}$ we need to substitute $\Phi_{2m}$ and $G_{2m}$, and to solve similar equations as above. In fact we obtain $\Phi_{2m+1}$ and $G_{2m}$ which are solutions of the following equation

$$\left[\Phi_{2m+1}, \widetilde{X}_2\right] + \left[\Phi_{2m}, \widetilde{X}_3\right] + G_{2m}\widetilde{X}_2 + G_{2m+1}X_1 = 0.$$

One can prove that

$$\Phi_{2m+1} = h_{2m-1}\widetilde{X}_2 + h_{2m}X_1 \quad \text{and} \quad G_{2m+1} = \widetilde{X}_3(h_{2m-1}) + \widetilde{X}_2(h_{2m}),$$

where $h_{2m} = \sum_{j=0}^{m} \gamma_j^{(2m+1)} x_1^{2(m-j)} r^{2j}$ and $\gamma_j^{(2m+1)}$ are arbitrary real parameters.

The key step in the proof of the uniqueness of the orbital normal form is to prove that for any $m \geqslant 1$, the parameters $\gamma_j^{(2m+1)}$ keep the normal form of any order invariant. This is stated in the following lemma.We refer to [2] for more details and proofs.

**Lemma 2.2.** *Let notations be as above. Let* $h = \alpha_0 + \sum_{k\geqslant 1} h_k$, $\alpha_0 \in \mathbf{R}$ *and*

$$\phi = \alpha_0\left(\widetilde{X} - X_1\right) + \sum_{k\geqslant 1} h_k\widetilde{X} = h\widetilde{X} - \alpha_0 X_1.$$

*Then there exists a formal series* $G = \sum_{k\geqslant 2} G_k$ *which is uniquely determined from* $h$ *and* $\widetilde{X}$ *such that*

$$\exp\left(\mathrm{ad}(\phi)\right)\left(\widetilde{X} + G\widetilde{X}\right) = \widetilde{X}.$$

According to [3], where necessary and sufficient conditions are given for formal finite determinacy of germs of vector fields, Hopf-zero vector fields are not finitely determined in the formal conjugacy category. In other words, it cannot be reduced to a polynomial normal form. The above result shows that they are not finitely determined in the orbital equivalence case either. On the other hand, in [5], it is proved that vector fields having two pairs of purely imaginary eigenvalues generically are formally orbitally finitely determined, though they are not finitely determined under the conjugacy equivalence. Thus the consequence of the present Note and the result of [5] bring out an interesting problem in normal form theory: in terms of algebraic structure between eigenvalues of a vector field, is it possible to give necessary or sufficient conditions for a vector field to be finitely determined in the orbital equivalence category?

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