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Propagation of chaos for pressureless gas equations with viscosity

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Abstract We use A.S. Sznitman ideas of probabilistic phenomenon of propagation of chaos for Burgers equation, and we derive the existence and uniqueness of a weak solution of the following system of pressureless gas equations with viscosity:

$$(\mathcal{S})\quad \begin{cases} \frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x}(u\rho) = \frac{1}{2}\frac{\partial^2}{\partial^2 x}\rho, \\ \frac{\partial}{\partial t}(u\rho) + \frac{\partial}{\partial x}(u^2\rho) = \frac{1}{2}\frac{\partial^2}{\partial^2 x}(u\rho), \\ \rho(\mathrm{d}x,t) \to \rho(\mathrm{d}x,0),\, u(x,t)\rho(\mathrm{d}x,t) \to u_0(x)\rho(\mathrm{d}x,0) \quad \text{weakly as } t \to 0^+. \end{cases}$$

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Propagation du chaos pour un système de gaz sans pression avec viscosité

Résumé Dans cette Note on utilise les idées de A.S. Sznitman dans son étude de la propagation du chaos probabiliste pour l'équation de Burgers, et on obtient l'existence et l'unicité d'une solution faible au système ($\mathcal{S}$) de gaz sans pression avec viscosité cité dans l'abstract. ***Pour citer cet article : A. Dermoune, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 935–940.***

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On considère un système de N particules qui satisfait :

$$\mathrm{d}X_t^i = \mathrm{d}B_t^i + \frac{\sum_{j\neq i} u_0(X_0^j)V(N(X_t^i - X_t^j))}{\sum_{j\neq i} V(N(X_t^i - X_t^j))}\,\mathrm{d}t, \quad i = 1,\ldots,N, \tag{1}$$

où $(B^i : 1 \leqslant i \leqslant N)$ sont N-mouvement browniens standards indépendants, V est une densité de probabilité continue, positive et paire. La fonction u_0 est donnée et elle est continue et bornée. Les positions initiales $(X_0^1,\ldots,X_0^N)$ des N-particules sont des variables aléatoires indépendantes et équidistribuées suivant une

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loi de probabilité $\rho(\mathrm{d}x,0)$ et indépendantes du N-mouvement browniens $B^1,\dots,B^N$. En imitant la propagation du chaos probabiliste pour l'équation de Burgers [6], on montre l'existence de solutions de la diffusion nonlinéaire

$$X_t = X_0 + B_t + \int_0^t \mathbf{E}\big[u_0(X_0) \mid X_s\big]\,\mathrm{d}s, \quad \mathcal{L}(X_0) = \rho(\mathrm{d}x,0). \tag{2}$$

Nous obtenons un résultat d'existence et d'unicité d'une solution faible au système $(\mathcal{S})$, et nous en déduisons l'unicité de la loi (notée par P) des solutions à (2). Il en découle que la suite des lois P_N (sur $C(\mathbf{R}_+,\mathbf{R})^N$ des processus $(X^i, 1 \leqslant i \leqslant N)$ solution de (1)) est P-chaotique.

1. Introduction

In Dermoune and Djehiche [2,3], the local existence and uniqueness, and the global existence of weak solution of $(\mathcal{S})$ were established via (2). In this Note we mimic a work of Sznitman [6] for the Burgers equation in order to get the solutions of (2) by the propagation of chaos phenomenon. For the definition and more details concerning the propagation of chaos see [7] and the references herein.

Let us first explain some notation, which is needed later. We denote by

$$H = \overline{H} \times \overline{H} \times C_0^+(\mathbf{R}_+,\mathbf{R}),$$

where $C_0^+(\mathbf{R}_+,\mathbf{R})$ is the set of continuous increasing functions on $\mathbf{R}_+$ with values 0 at time 0, and

$$\overline{H} = \left\{ (X,B) \in C(\mathbf{R}_+,\mathbf{R}) \times C(\mathbf{R}_+,\mathbf{R}),\ \frac{\mathrm{d}(X-B)}{\mathrm{d}t} \in C(\mathbf{R}_+,\mathbf{R}) \right\}.$$

The space $\overline{H}$ is endowed with the topology defined by: $(X_n,B_n) \to (X,B)$ iff for all $T>0$,

$$\sup_{t\leqslant T}\big|X_n(t)-X(t)\big|, \quad \sup_{t\leqslant T}\big|B_n(t)-B(t)\big| \quad \text{and} \quad \sup_{t\leqslant T}\left|\frac{\mathrm{d}(X_n-B_n)}{\mathrm{d}t} - \frac{\mathrm{d}(X-B)}{\mathrm{d}t}\right|$$

go to 0 as $n \to +\infty$. We will denote the canonical coordinates on H by (X^1,B^1,X^2,B^2,A). The space H endowed with the product topology, is a complete metric space. We denote by $(G_t : t \geqslant 0)$ the canonical filtration on H, and $M(H)$ is the space of probability measures on H, endowed with the topology of weak convergence.

Now we come back to the system (1), its drift is continuous and bounded. It follows that the latter system of stochastic differential equation has a unique weak solution (see [5]).

Let us consider the law $\overline{P}_N$ of the empirical process

$$\overline{X}_N = \frac{1}{N(N-1)} \sum_{i\neq j} \delta_{(X^i,B^i,X^j,B^j,\int_0^\cdot NV(N(X_s^i-X_s^j))\,\mathrm{d}s)},$$

which takes its values in $M(H)$. Now we can announce our main results.

THEOREM 1. – *The laws $(\overline{P}_N)$ are tight. Let $\overline{P}_\infty$ be a limit of a subsequence of $(\overline{P}_N)$. We have $\overline{P}_\infty$ a.e. $m \in M(H)$, (X^1,B^1) and (X^2,B^2) are m-independent with the same law, (B^1,B^2) is a two-dimensional G_t-Brownian motion. Moreover for $i=1,2$,*

$$X_t^i = X_0^i + B_t^i + \int_0^t \mathbf{E}_m\big[u_0\big(X_0^i\big) \mid X_s^i\big]\,\mathrm{d}s, \quad \mathcal{L}\big(X_0^i\big) = \rho(\mathrm{d}x,0).$$

Let $S(\rho(\mathrm{d}x,0),u_0)$ be the set of laws of processes satisfying (2) in some probability space.

COROLLARY 2. – *The system* $(\mathcal{S})$ *has a unique weak solution* (u,ρ) *in the class* $\mathcal{U}_{||u_0||_\infty} \times C(\mathbf{R}_+, M(\mathbf{R}))$, *of measurable velocity* u *bounded by* $||u_0||_\infty$ *and* $(t \in \mathbf{R}_+ \to \rho(\mathrm{d}x,t)) \in C(\mathbf{R}_+, M(\mathbf{R}))$. *The set* $S(\rho(\mathrm{d}x,0),u_0)$ *is a singleton* $\{P\}$. *There exists* $m \in M(H)$ *such that* $\{P_\infty : P_\infty$ *is a limit of subsequence of* $(\overline{P}_N)\} = \{\delta_m\}$, *and under* m *the distribution of* (X^1) *(or* X^2*) is equal to* P.

COROLLARY 3. – *The laws* P_N *of* $(X^1,\dots,X^N)$ *solution of* (1) *is* P*-chaotic.*

Proof of Corollary 2. – The existence of a weak solution is a consequence of Theorem 1 and the fact that if (X,B) is a solution of (2), then by Itô's formula we show that $(\rho(\mathrm{d}x,t) = \mathcal{L}(X_t),\ u(x,t) = \mathbf{E}[u_0(X_0) \mid X_t = x])$ is a weak solution of the system $(\mathcal{S})$.

Let us prove the unicity. First of all if u is measurable and bounded, then any weak solution $\rho \in C(\mathbf{R}_+, M(\mathbf{R}))$ of the first equation of $(\mathcal{S})$ has for any $t>0$ a continuous bounded density $x \to \rho(x,t)$ with respect to Lebesgue measure (see [4]). Now we write the system $(\mathcal{S})$ as the following parabolic systems

$$\partial_t(\rho,q) + \partial_x\bigl(f(\rho,q)\bigr) = \frac{1}{2}\partial^2_{xx}(\rho,q), \quad t>0,\ x \in \mathbf{R}, \tag{3}$$

where $f(\rho,q) = (q, q^2/\rho)$. Let us consider the domain

$$D = \bigl\{(\rho,q) \in \mathbf{R}_+ \times \mathbf{R} : |q| \leqslant ||u_0||_\infty \rho\bigr\}.$$

The map f from D to $\mathbf{R}^2$ is Lipschitz continuous, because f' is bounded on D. We are interested in the set $\mathbf{S}$ of solutions (ρ,q) of (3) such that $(\rho(x,t), q(x,t)) \in D$ for all $t>0, x \in \mathbf{R}$, with initial values $(\rho(\mathrm{d}x,0), q(\mathrm{d}x,0)) = (\rho(\mathrm{d}x,0), u_0(x)\rho(\mathrm{d}x,0)) := (\rho_0(\mathrm{d}x), q_0(\mathrm{d}x))$. We are going to show that $\mathbf{S}$ is a singleton.

Let $\gamma(t,x)$ be the standard gaussian density and $\gamma_x(t,x)$ denotes its derivative with respect to x. A solution of (3) with initial value $(\rho_0(\mathrm{d}x), q_0(\mathrm{d}x))$ then satisfies

$$\bigl(\rho(t), q(t)\bigr) = \gamma(t) * (\rho_0, q_0) - \int_0^t \gamma_x(t-s) * f\bigl(\rho(s), q(s)\bigr)\,\mathrm{d}s.$$

Let us consider two solutions (ρ^1,q^1), (ρ^2,q^2) belonging to $\mathbf{S}$. We set $\rho = \rho^1-\rho^2$, $q = q^1-q^2$, and we get

$$\bigl(\rho(t), q(t)\bigr) = -\int_0^t \gamma_x(t-s) * \bigl[f\bigl(\rho^1(s), q^1(s)\bigr) - f\bigl(\rho^2(s), q^2(s)\bigr)\bigr]\,\mathrm{d}s.$$

From the fact that $(\rho^i(x,s), q^i(x,s)) \in D$, $i=1,2$, for all $s>0, x \in \mathbf{R}$, and that f is Lipschitz continuous on D, we have

$$\begin{aligned}
\bigl\|\bigl(\rho(t),q(t)\bigr)\bigr\|_\infty &\leqslant C\int_0^t \bigl\|\gamma_x(t-s)\bigr\|_1 \bigl\|\bigl(\rho(s),q(s)\bigr)\bigr\|_\infty\,\mathrm{d}s \\
&\leqslant C\int_0^t \frac{1}{\sqrt{t-s_1}}\bigl\|\bigl(\rho(s_1),q(s_1)\bigr)\bigr\|_\infty\,\mathrm{d}s_1 \\
&\leqslant C^2\int_0^t\biggl[\int_0^{s_1}\frac{1}{\sqrt{(t-s_1)(s_1-s_2)}}\bigl\|\bigl(\rho(s_2),q(s_2)\bigr)\bigr\|_\infty\,\mathrm{d}s_2\biggr]\mathrm{d}s_1.
\end{aligned}$$

We observe that

$$\int_{s_2}^t \frac{1}{\sqrt{(t-s_1)(s_1-s_2)}}\,\mathrm{d}s_1 = 2\tan^{-1}\sqrt{\frac{s_1-s_2}{t-s_1}}\Bigg|_{s_2}^t = \pi.$$

Hence

$$\bigl\|\bigl(\rho(t),q(t)\bigr)\bigr\|_\infty \leqslant \pi C^2\int_0^t \bigl\|\bigl(\rho(s),q(s)\bigr)\bigr\|_\infty\,\mathrm{d}s,$$

and from Gronwall's inequality, we have

$$\left\|\left(\rho(t), q(t)\right)\right\|_{\infty} = 0 \quad \forall t > 0.$$

The proof of the second part of Corollary 2 (respectively Corollary 3) is the same as in [8] (the end of the proof of Proposition 4.4) (respectively is a consequence of a result of [8], Lemme 3.1). The proof of Theorem 1 is given in many steps in the next section.

2. Proof of Theorem 1

Step 1. The proof of tightness of $(\overline{P}_N)$ is the same as in [6], Proposition 3.8.

Let $\overline{P}_\infty$ be a limit of a subsequence $\{N_k : k \geqslant 1\} \in \mathbf{N}$. Since we shall be concerned with the properties of the possible limits $\overline{P}_\infty$, we may assume for simplicity that $N_k = k$ for all $k \geqslant 1$.

The proof of the following step is the same as in [6], Proposition 4.2.

Step 2. For $\overline{P}_\infty$ a.e. $m \in M(H)$, (X^1, B^1), (X^2, B^2) are m-independent with the same law, (B^1, B^2) is a two-dimensional G_t-Brownian motion and the law of X_0^1 or X_0^2 under m is $\rho(\mathrm{d}x, 0)$.

We denote for $i = 1, 2$, by $A_t^i = X_t^i - X_0^i - B_t^i$. From the definition of H we have for all $t \geqslant 0$, $A_t^i = \int_0^t a_s^i \,\mathrm{d}s$, where $s \to a_s^i$ is continuous.

Step 3. For $\overline{P}_\infty$ a.e. $m \in M(H)$, for all $t \geqslant 0$,

(i) $|a_t^1|$ (or a_t^2) is bounded by $\|u_0\|_\infty$, and (ii) for $\overline{P}_\infty$ a.e. $m \in M(H)$,

$$H_t = \left|X_t^1 - X_t^2\right| - \left|X_0^1 - X_0^2\right| - \int_0^t \operatorname{sgn}\left(X_s^1 - X_s^2\right) \mathrm{d}\left(A_s^1 - A_s^2\right) - A_t,$$

is a martingale, and then $A_t = L^0(X^1 - X^2)_t$ is the symmetric local time in 0 of $X^1 - X^2$.

Proof. – (i) For each integer n, let $f_n(x) = 1$ for $x \geqslant 0$, $f_n(x) = 0$ for $x \leqslant -1/n$ and linear in between. The map $m \to \mathbf{E}_m[f_n(\|u_0\|_\infty - |a_t^1|)]$ is bounded continuous on $M(H)$, and it equals to 1 under $\overline{P}_N$, for all N. It follows that $\overline{P}_\infty$ a.e. $m \in M(H)$,

$$\mathbf{E}_m\left[f_n\left(\|u_0\|_\infty - \left|a_t^1\right|\right)\right] = 1, \quad \forall n,$$

and then

$$m\left(\|u_0\|_\infty - \left|a_t^1\right| \geqslant 0\right) = 1.$$

(ii) The proof is based on (i), and it is nearly the same as in [6], Proposition 4.3.

Step 4. For $\overline{P}_\infty$ a.e. $m \in M(H)$, $\int_0^t a_s^1 \,\mathrm{d}A_s - u_0(X_0^2) A_t = 0$, for all $t \geqslant 0$.

Proof. – We mimic [6], Proposition 4.3, as following. We set

$$F(m) = \mathbf{E}_m\left[\left|\int_0^t a_s^1 \,\mathrm{d}A_s - u_0\left(X_0^2\right) A_t\right|\right].$$

We want to show that

$$F(m) = 0, \quad \overline{P}_\infty m \text{ a.e.}$$

Introduce $F_c(m) = \mathbf{E}_m[|\int_0^t a_s^1 \,\mathrm{d}(A_s \wedge c) - u_0(X_0^2)(A_t \wedge c)|]$, where $A_t \wedge c := \min(A_t, c)$. The map $m \to F_c(m)$ is bounded continuous on $M(H)$, and we have for $N \leqslant +\infty$,

$$\mathbf{E}_{\overline{P}_N}\left[\left|F(m) - F_c(m)\right|\right] \leqslant 2\|u_0\|_\infty \mathbf{E}_{\overline{P}_N}\left[\mathbf{E}_m\left[(A_t - c)_+\right]\right].$$

From the density of occupation formula and an inequality of Barlow–Yor [1], we have for all $N \leqslant +\infty$,

$$\mathbf{E}_{\overline{P}_N}\left[\mathbf{E}_m\left[A_t^2\right]\right] \leqslant K(t),$$

where $K(t)$ is some constant which depends only on t and $\|u_0\|_\infty$. We derive that for all $N \leqslant +\infty$,

$$\mathbf{E}_{\overline{P}_N}\left[\mathbf{E}_m\left[(A_t - c)_+\right]\right] \leqslant \frac{K(t)}{c}.$$

Consider $\varepsilon > 0$, we can take c large enough such that the last quantity is less than ε. So,

$$\begin{aligned}\mathbf{E}_{\overline{P}_\infty}\left[\left|F(m)\right|\right] &\leqslant \varepsilon + \mathbf{E}_{\overline{P}_\infty}\left[\left|F_c(m)\right|\right] = \varepsilon + \lim_{N\to+\infty} \mathbf{E}_{\overline{P}_N}\left[\left|F_c(m)\right|\right] \\ &\leqslant 2\varepsilon + \limsup_{N\to+\infty} \mathbf{E}_{\overline{P}_N}\left[\left|F(m)\right|\right],\end{aligned}$$

and

$$\begin{aligned}F(\overline{X}_N) &= \left|\frac{1}{N(N-1)}\sum_{i\neq j}\left(\int_0^t a_s^i N V\left(N\left(X_s^i - X_s^j\right)\right)\mathrm{d}s - u_0\left(X_0^j\right)\int_0^t NV\left(N\left(X_s^i - X_s^j\right)\right)\mathrm{d}s\right)\right| \\ &= \left|\frac{1}{N(N-1)}\sum_{i=1}^N\left(\int_0^t a_s^i N\sum_{j\neq i} V\left(N\left(X_s^i - X_s^j\right)\right)\mathrm{d}s - N\int_0^t \sum_{j\neq i} u_0\left(X_0^j\right)V\left(N\left(X_s^i - X_s^j\right)\right)\mathrm{d}s\right)\right|.\end{aligned}$$

Now from (1) we have

$$a_s^i = \frac{\sum_{j\neq i} u_0(X_0^j)V(N(X_s^i - X_s^j))}{\sum_{j\neq i} V(N(X_s^i - X_s^j))},$$

and then

$$a_s^i \sum_{j\neq i} V\left(N\left(X_s^i - X_s^j\right)\right) - \sum_{j\neq i} u_0\left(X_0^j\right)V\left(N\left(X_s^i - X_s^j\right)\right) = 0, \quad \forall s, i.$$

It follows that for all $N < +\infty$, $F(\overline{X}_N) = 0$ and then $\mathbf{E}_{\overline{P}_\infty}[|F(m)|] \leqslant 2\varepsilon$ for all $\varepsilon > 0$, which achieves the proof.

Step 5. Let $X_t = X_0 + B_t + \int_0^t a_s\,\mathrm{d}s$ be a continuous semi-martingale, where $s \to a_s$ is bounded continuous, and B is a Brownian motion. Let Y be an independent copy of X. We have

$$\mathbf{E}_Y\left[u_0(Y_0)L^0(X-Y)_t\right] = 2\int_0^t \mathbf{E}\left[u_0(X_0) \mid X_s\right]\rho(X_s, s)\,\mathrm{d}s,$$

and

$$\mathbf{E}_Y\left[\int_0^t a_s\,\mathrm{d}L^0(X-Y)_s\right] = 2\int_0^t a_s\rho(X_s, s)\,\mathrm{d}s,$$

where $\rho(x, s)$ is the density of X_s.

Proof. – Similar to [6], Propositions 2.1, 2.3.

The following step achieves the proof of Theorem 1.

Step 6. For $\overline{P}_\infty$ a.e. $m \in M(H)$,

$$a_t^1 = \mathbf{E}_m\left[u_0\left(X_0^1\right) \mid X_t^1\right]\mathrm{d}t \quad \text{a.e.}$$

Proof. – In fact from Steps 2 and 5, we have for all $t > 0$,

$$\mathbf{E}_m\left[\int_0^t a_s^1\,\mathrm{d}L^0\left(X^1 - X^2\right)_s \mid X^1\right] = \int_0^t a_s^1\rho\left(X_s^1, s\right)\mathrm{d}s,$$

and

$$\mathbf{E}_m\left[u_0\left(X_0^2\right)L^0\left(X^1-X^2\right)_t \mid X^1\right]=\int_0^t \mathbf{E}_m\left[u_0\left(X_0^1\right)\mid X_s^1\right]\rho\left(X_s^1,s\right)\mathrm{d}s.$$

Now from Steps 3 and 4, we get for all $t \geqslant 0$,

$$\int_0^t a_s^1\rho\left(X_s^1,s\right)\mathrm{d}s=\int_0^t \mathbf{E}_m\left[u_0\left(X_0^1\right)\mid X_s^1\right]\rho\left(X_s^1,s\right)\mathrm{d}s.$$

We derive that

$$a_s^1\rho\left(X_s^1,s\right)=\mathbf{E}_m\left[u_0\left(X_0^1\right)\mid X_s^1\right]\rho\left(X_s^1,s\right)\mathrm{d}s \quad \text{a.e.}$$

Now the property $\rho(x,s)>0$, for all $x\in\mathbf{R}$, $s>0$, completes the proof.

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