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Hypercyclic semigroups and somewhere dense orbits

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Abstract We study hypercyclicity of linear strongly continuous semigroups. In the case of iterations of a single operator Bourdon and Feldman have recently proved that the existence of somewhere dense orbits implies hypercyclicity. We show the corresponding result for semigroups. As a consequence, a conjecture of Herrero concerning iterations of a single operator also holds for strongly continuous semigroups. ***To cite this article: G. Costakis, A. Peris, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 895–898.***

Semigroupes hypercycliques et orbites quelque part denses

Résumé Nous étudions l'hypercyclicité des semigroupes linéaires et fortement continus. En ce qui concerne l'iteration d'un opérateur, Bourdon et Feldman ont montré que l'existence des orbites quelque part denses implique hypercyclicité. Nous démontrons le resultat correspondant pour des semigroupes. Une conséquence est la generalisation d'une conjecture de Herrero à des semigroupes. ***Pour citer cet article : G. Costakis, A. Peris, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 895–898.***

Version française abrégée

Soit X un espace de Banach et $T = \{T_t : X \to X;\ t \geqslant 0\}$ un semigroupe d'opérateurs bornés. Définissons $F(x) := \overline{\mathrm{Orb}(T, x)} = \overline{\{T_t x;\ t \geqslant 0\}}$ et $U(x)$ l'intérieur de $F(x)$. Nous démontrons

THÉORÈME 5. – *Si $U(x) \neq \emptyset$, alors $F(x) = U(x) = X$.*

Donnons le schéma de la démonstration.

LEMME 1. –
(1) *$F(x)$ est T-invariant ($T_t(F(x)) \subset F(x)$ pour tout $t \geqslant 0$).*
(2) *$U(x) \cap U(y) \neq \emptyset$ implique $U(x) = U(y)$.*
(3) *Si X est complex (respectivement, réel), alors $U(x) \neq \emptyset$ implique $U(x) = U(\lambda x) = \lambda U(x)$, pour tout $\lambda \in \mathbb{C} \setminus \{0\}$ ($\lambda > 0$). En conséquence 0 est un point d'adhérence de l'orbite de x.*

LEMME 2. – *Si $U(x) \neq \emptyset$, alors l'ensemble $\{p(T_t)T_s x;\ t > 0,\ p \neq 0$ polynôme$\}$ est connexe et dense dans X pour tout $s \geqslant 0$.*

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LEMME 3. – *L'ensemble* $X \setminus U(x)$ *est* T*-invariant.*

LEMME 4. – *Si l'intérieur de* $\overline{\mathrm{Orb}(T,x)}$ *est non vide, alors* $p(T_t)$ *est d'image dense pour tout* $t > 0$ *et pour tout polynôme* $p \neq 0$.

Une conséquence de le notre théorème est la suivant

COROLLAIRE 6. – *Si on a l'existence d'une famille finie de vecteurs* $\{x_1, \ldots, x_n\} \subset X$ *telles que* $X = \bigcup_{i=1}^{n} \overline{\mathrm{Orb}(T, x_i)}$, *alors il existe* $i \in \{1, \ldots, n\}$ *tel que* $X = \overline{\mathrm{Orb}(T, x_i)}$.

In [3] and [7] the authors independently showed a conjecture of Herrero [5] and proved that every multi-hypercyclic operator is hypercyclic. More precisely, if $T : E \to E$ is an operator on a locally convex space E that admits a finite collection of orbits whose union is dense in E, then there is a single element whose orbit is dense. Motivated by this result, the second author asked in [7] if every operator T that admits a somewhere dense orbit should be hypercyclic. This has been shown to be true very recently by Bourdon and Feldman [2].

Our purpose is to obtain the corresponding results for a strongly continuous semigroup of bounded linear operators $T = \{T_t;\ t \geqslant 0\}$ on a Banach space X. The investigation of hypercyclicity of semigroups was initiated in [4]. Keeping the notation of this paper, we say that T is *hypercyclic* if there exists $x \in X$ such that $\{T_t x;\ t \geqslant 0\}$ is dense in X.

Given $x \in X$, we will denote by $F(x) := \overline{\mathrm{Orb}(T,x)} = \overline{\{T_t x;\ t \geqslant 0\}}$, the closure of the orbit of x. We are interested in the case of existence of somewhere dense orbits. In other words, when there are $x \in X$ such that the interior $U(x)$ of $F(x)$ is not empty. In such a case we observe that $U(x)$ coincides with the interior of the set of accumulation points of the orbit of x

$$U(x) = \mathrm{int}\Big\{ y \in X;\ \exists (t_n) \uparrow \infty : \lim_n T_{t_n} x = y \Big\}.$$

This is a consequence of the fact that $\overline{\{T_s x;\ s \in [0,t]\}}$ is a compact set (therefore has empty interior), and the easy observation that $\mathrm{int}(\overline{A}) = \mathrm{int}(\overline{A \setminus G})$ for all G closed with empty interior. The following basic properties of $F(x)$ are stated without proof

LEMMA 1. –
(1) $F(x)$ *is* T*-invariant* ($T_t(F(x)) \subset F(x)$ *for all* $t \geqslant 0$).
(2) $U(x) \cap U(y) \neq \emptyset$ *implies* $U(x) = U(y)$.
(3) *If* X *is complex* (*respectively, real*), *then* $U(x) \neq \emptyset$ *implies that* $U(x) = U(\lambda x) = \lambda U(x)$, *for all* $\lambda \in \mathbb{C} \setminus \{0\}$ ($\lambda > 0$). *In particular, zero is an accumulation point of the orbit of* x.

Another elementary result for single operators R is that, if R admits a somewhere dense orbit $\mathrm{Orb}(R, x)$, then any iteration of x is cyclic (i.e., the linear span of its orbit is dense in X) [6]. The corresponding result that we will need for semigroups is as follows

LEMMA 2. – *If* $U(x) \neq \emptyset$, *then the set* $\{p(T_t)T_s x;\ t > 0,\ p \neq 0$ *polynomial*$\}$ *is connected and dense in* X *for all* $s \geqslant 0$.

Proof. – Since $U(x) \neq \emptyset$, then $\mathrm{span}(U(x)) = X$. Given $y \in X \setminus \{0\}$, we find $y_i \in U(x)$ and $\lambda_i \in \mathbb{C} \setminus \{0\}$ such that $y = \sum \lambda_i y_i$. Given $s \geqslant 0$ and $\varepsilon > 0$, we fix values $t_i > s + 1$ such that

$$\Big\| y - \sum \lambda_i T_{t_i} x \Big\| < \frac{\varepsilon}{2}.$$

Now, since $\lim_{t \to 0} T_t = I$ pointwise, there is $0 < t' < 1$ such that

$$\Big\| \sum \lambda_i (T_{t_i} - T_{s_i} T_{t_i}) x \Big\| < \frac{\varepsilon}{2}, \quad \forall s_i \in [0, t'].$$

We write $t_i = m_i t' + s - s_i$ for some $m_i \in \mathbb{N}$, $s_i \in [0, t']$, and then

$$\left\| y - \sum \lambda_i T_{t'}^{m_i} T_s x \right\| < \frac{\varepsilon}{2} + \left\| \sum \lambda_i (T_{t_i} x - T_{m_i t' + s} x) \right\| = \frac{\varepsilon}{2} + \left\| \sum \lambda_i (T_{t_i} - T_{s_i} T_{t_i}) x \right\| < \varepsilon,$$

which implies $\| y - p(T_{t'}) T_s x \| < \varepsilon$, for $p(z) := \sum \lambda_i z^{m_i}$.

Concerning the connectedness of our set we observe that, since for fixed t and s the subset $\{p(T_t)T_s x;\ p \neq 0$ polynomial$\}$ is a subspace, there is an arc contained in this subspace except zero connecting arbitrary $p(T_t)T_s x$ and $q(T_t)T_s x$, $p, q \neq 0$. We finally have that the trivial arc connecting $q(T_t)T_s x$ with $q(T_{t'})T_s x$ completes an arc contained in our set. □

The discrete analog of next result was one of the key points that allowed Bourdon and Feldman [2, Lemma 2.3] to show that operators with somewhere dense orbit are hypercyclic.

LEMMA 3. – *The set $X \setminus U(x)$ is T-invariant.*

Proof. – If $U(x) \neq \emptyset$, we choose $s > 0$ such that $T_s x \in U(x)$ and set $x_s = T_s x$.

Suppose that there exist $r > 0$ and $y \in X \setminus U(x)$ such that $T_r y \in U(x)$. Continuity of T_r allows us to assume that $y \in X \setminus F(x)$. Lemma 2 implies the existence of $t > 0$ and a polynomial $p \neq 0$ such that $p(T_t)x_s$ is close enough to y to ensure (1) $p(T_t)x_s \in X \setminus F(x)$, and (2) $T_r(p(T_t)x_s) \in F(x)$. Finally, the coincidence of $U(x)$ with the interior of the set of accumulation points of the orbit of x, the continuity of $p(T_t)$, and the T-invariance of $F(x)$ imply, in this order, that

$$p(T_t)x_s \in p(T_t)\bigl(U(x)\bigr) = p(T_t)\bigl(U(T_r x_s)\bigr) \subset F\bigl(T_r\bigl(p(T_t)x_s\bigr)\bigr) \subset F(x),$$

which contradicts (1). □

Our final lemma concerns the density of the range of $p(T_t)$ for every $t > 0$ and for all polynomials $p \neq 0$, whenever T admits a somewhere dense orbit. The corresponding discrete version was a consequence of $\sigma_p(R^*) = \emptyset$ if R admits a somewhere dense orbit (see e.g., [6]), which was inspired in a result of Bourdon [1]. In [4, Theorem 3.3] the authors observe that the adjoint of the infinitesimal generator of a hypercyclic semigroup should have empty point spectrum. It is easy to generalize this result for semigroups that admit a somewhere dense orbit. However, this is not enough for our purposes. A much finer result is needed.

LEMMA 4. – *If T admits a somewhere dense orbit, then the point spectrum $\sigma_p(T_t^*)$ of the adjoint T_t^* of T_t is empty for all $t > 0$. As a consequence $p(T_t)$ has a dense range for all $t > 0$ and for every polynomial $p \neq 0$.*

Proof. – Let us first suppose that X is a complex space. Proceeding by contradiction, if $U(x) \neq \emptyset$ for some $x \in X$ and there are $s > 0$, $\lambda \in \mathbb{C}$, and $x^* \in X^* \setminus \{0\}$ such that $T_t^* x^* = \lambda x^*$, we fix $y \in U(x)$ such that $\langle y, x^* \rangle \neq 0$. We will distinguish two cases:

(1) $|\lambda| < 1$.

We find then $(t_n) \uparrow \infty$ such that $\lim_n T_{t_n} x = y$, and we write $t_n = m_n s + s_n$, where $m_n \in \mathbb{N}$, $(m_n) \uparrow \infty$, and $s_n \in [0, s]$. Thus

$$\langle y, x^* \rangle = \lim_n \langle T_{t_n} x, x^* \rangle = \lim_n \bigl\langle T_{s_n} x, (T_s^*)^{m_n} x^* \bigr\rangle = \lim_n \lambda^{m_n} \langle T_{s_n} x, x^* \rangle = 0,$$

since the first factor tends to 0 and the second one is uniformly bounded. This is a contradiction.

(2) $|\lambda| \geqslant 1$.

Since $y \in F(x)$, there is $r \geqslant 0$ such that $|\langle T_r x, x^* \rangle| > \alpha := |\langle y, x^* \rangle|/2$. Equicontinuity of $\{T_t;\ t \in [0, s]\}$ gives the existence of an $\varepsilon > 0$ such that $|\langle T_t u, x^* \rangle| < \alpha$ if $\|u\| < \varepsilon$ and $t \in [0, s]$. By Lemma 1 zero is an accumulation point of the orbit of x, so let $t' > r$ be such that $\|T_{t'} x\| < \varepsilon$. We write $t' = ms - t + r$ for some $m \in \mathbb{N}$ and $t \in [0, s]$. We thus have

$$\alpha > \bigl|\bigl\langle T_t(T_{t'} x), x^* \bigr\rangle\bigr| = \bigl|\bigl\langle T_r x, (T_s^*)^m x^* \bigr\rangle\bigr| = |\lambda|^m \bigl|\langle T_r x, x^* \rangle\bigr| > \alpha,$$

which is a contradiction.

Therefore T_s^* has no eigenvalues. This means that $T_t - \lambda I$ has a dense range for all $\lambda \in \mathbb{C}$ and $t > 0$. Decomposing $p(T_t)$ as a product of monomials on T_t we get the density of the range of $p(T_t)$.

If X is a real space, we consider its complexification $\widetilde{X} := X + \mathrm{i}X$, and the complexification $\widetilde{T_t} := T_t + \mathrm{i}T_t$ of T_t. An easy adaptation of the above argument shows that $\widetilde{T_t}^*$ has no eigenvalues, and we conclude that $p(T_t)$ has a dense range (see, e.g., [7, Lemma 2]). $\square$

Bourdon observed that our proof actually shows that if T admits a vector x whose orbit has zero as accumulation point, then any eigenvector of T_t^* is orthogonal to all accumulation points of $\mathrm{Orb}(T, x)$.

We finally establish the main result of the paper. It essentially mimics the argument of Bourdon and Feldman [2, Theorem 2.4] for iterations of a single operator.

THEOREM 5. – *If a strongly continuous semigroup $T = \{T_t : X \to X;\ t \geqslant 0\}$ admits a somewhere dense orbit* $\mathrm{Orb}(T, x)$; *then it is hypercyclic and* $\mathrm{Orb}(T, x)$ *is dense in* X.

Proof. – Suppose $U(x) \neq \emptyset$ but $F(x) \neq X$. Our goal is to show that $p(T_t)x \in U(x) \cup (X \setminus F(x))$ for all $t > 0$ and for each polynomial $p \neq 0$. To do this we observe that Lemma 4 implies that the image of the dense set $G := U(x) \cup (X \setminus F(x))$ under $p(T_t)$ is dense in X. If $p(T_t)x$ belongs to $F(x) \cap (X \setminus U(x))$, the T-invariance lemma gives $p(T_t)F(x) \subset X \setminus U(x)$. On the other hand, if there were $y \in X \setminus F(x)$ such $p(P_t)y \in U(x)$, Lemma 2 and the continuity of $p(P_t)$ would lead to the existence of some $q(T_s)x \in X \setminus F(x)$ with $p(T_t)q(T_s)x \in U(x)$. However,

$$p(T_t)q(T_s)x = q(T_s)p(T_t)x \in q(T_s)F(x) = F\big(q(T_s)x\big) \subset X \setminus U(x),$$

by the T-invariance lemma. Therefore $p(T_t)(G) \subset X \setminus U(x)$, contradicting the density of the set.

This implies that $p(T_t)x$ belongs either to $U(x)$ or to $X \setminus F(x)$. Thus the connected set $H := \{p(T_t)x;\ t > 0,\ p \neq 0\}$ is contained in a disjoint union of open sets. Since $x \in H \cap U(x)$ we have $H \subset U(x)$. Lemma 2 concludes that $U(x)$ is dense in X, so $X = F(x)$. $\square$

An analogous result to Herrero's conjecture [5] holds in the context of strongly continuous semigroups. A semigroup T is *multi-hypercyclic* if there is a finite collection of vectors $\{x_1, \ldots, x_n\} \subset X$ such that

$$X = \bigcup_{i=1}^{n} \overline{\mathrm{Orb}(T, x_i)}.$$

This obviously implies the existence of somewhere dense orbits, so the following result is immediate from our main theorem.

COROLLARY 6. – *Any multi-hypercyclic strongly continuous semigroup of bounded operators on a Banach space is hypercyclic.*

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