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Linear statistics for zeros of Riemann's zeta function

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Abstract We consider a smooth counting function of the scaled zeros of the Riemann zeta function, around height T. We show that the first few moments tend to the Gaussian moments, with the exact number depending on the statistic considered. ***To cite this article: C.P. Hughes, Z. Rudnick, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 667–670.***

Statisiques linéaires pour les zéros de la fonction zêta de Riemann

Résumé Nous considérons une fonction de comptage lisse des zéros de la fonction zêta de Riemann, normalisés au voisinage de la hauteur T. Nous montrons que les premiers moments sont Gaussiens, le nombre exact de tels moments dépendant de la moyenne choisie et de la fonction de comptage des zéros. ***Pour citer cet article : C.P. Hughes, Z. Rudnick, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 667–670.***

1. Introduction

In this paper we will examine linear statistics of zeros of the Riemann zeta function. Denote its nontrivial zeros by $1/2 + \mathrm{i}\gamma_j$, $j = \pm 1, \pm 2, \ldots$ with $\gamma_{-j} = -\gamma_j$ and $\mathfrak{Re}(\gamma_1) \leqslant \mathfrak{Re}(\gamma_2) \leqslant \cdots$. Let $N(T)$ denote the number of zeros in the strip $0 < \mathfrak{Re}(\gamma) \leqslant T$, then $N(T) = \overline{N}(T) + S(T)$ where

$$\overline{N}(T) = 1 + \frac{1}{\pi}\mathfrak{Im}\log\left(\pi^{-\mathrm{i}T/2}\Gamma\left(\frac{1}{4} + \frac{1}{2}\mathrm{i}T\right)\right) = \frac{T}{2\pi}\log\frac{T}{2\pi\mathrm{e}} + \frac{7}{8} + \mathcal{O}\left(\frac{1}{T}\right).$$

Selberg [3] investigated the remainder term $S(t)$ and showed that it has a Gaussian value distribution, in the sense that if $T^a \leqslant H \leqslant T$ with $1/2 < a \leqslant 1$, then for $k \geqslant 1$ an integer, as $T \to \infty$,

$$\frac{1}{H}\int_T^{T+H}\left|\frac{S(t)}{\sqrt{(\log\log t)/2\pi^2}}\right|^{2k}\mathrm{d}t \sim \frac{(2k)!}{k!2^k}.$$

Fujii [1] has similar results for the remainder term in counting the number of zeros in intervals of size $h \leqslant t$ around t, showing that so long as $h\log t \to \infty$ they have Gaussian moments too.

Rather than study $S(t)$ itself, instead we will investigate the distribution of a smooth version of the counting function in intervals of size comparable to the mean spacing, $2\pi/\log T$. In particular, for a real-

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valued even function f, and real numbers τ and $T > 1$, set

$$N_f(\tau) := \sum_{j=\pm 1,\pm,2,\dots} f\left(\frac{\log T}{2\pi}(\gamma_j - \tau)\right).$$

If f is the characteristic function of an interval $[-1,1]$ and if all the γ_j are real, then $N_f(\tau)$ counts the number of zeros in the interval $[\tau - 2\pi/\log T, \tau + 2\pi/\log T]$. However, we will take f so that its Fourier transform, $\widehat{f}(u) := \int_{-\infty}^{\infty} f(x)\,\mathrm{e}^{-2\pi \mathrm{i} x u}\,\mathrm{d}x$, is smooth and of compact support (that is $\widehat{f} \in \mathrm{C}_c^\infty(\mathbb{R})$), and will not assume the Riemann hypothesis.

As $T \to \infty$, we consider the fluctuations of $N_f(\tau)$ as τ varies near T in an interval of size about $H = T^a$, where $0 < a \leqslant 1$. More precisely, given a weight function $w \geqslant 0$, with $\int_{-\infty}^{\infty} w(x)\,\mathrm{d}x = 1$, and $\widehat{w}$ compactly supported, we define an averaging operator

$$\langle W \rangle_{T,H} := \int_{-\infty}^{\infty} W(\tau) w\left(\frac{\tau - T}{H}\right)\frac{\mathrm{d}\tau}{H}.$$

We will show that for $\widehat{f} \in \mathrm{C}_c^\infty(\mathbb{R})$ the first few moments $\langle (N_f)^m \rangle_{T,H}$ of N_f are Gaussian:

THEOREM 1.1. – *Let $H = T^a$ with $0 < a \leqslant 1$, and let $\widehat{f} \in \mathrm{C}_c^\infty(\mathbb{R})$ be such that $\operatorname{supp} \widehat{f} \subset (-2a/m, 2a/m)$ with $m \geqslant 1$ an integer. Then the first m moments of N_f converge as $T \to \infty$ to those of a Gaussian random variable with expectation $\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x$ and variance*

$$\sigma_f^2 = \int_{-\infty}^{\infty} \min\bigl(|u|, 1\bigr)\,\widehat{f}(u)^2\,\mathrm{d}u. \tag{1}$$

A similar result holds in random matrix theory [2]: if U is an $n \times n$ unitary matrix with eigenvalues $\mathrm{e}^{\mathrm{i}\theta_j}$, one can define a version of N_f for the scaled angles $n\theta_j/2\pi$ and show that the first m moments of N_f converge to those of a Gaussian with mean $\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x$ and variance given by (1), provided $\operatorname{supp} \widehat{f} \subseteq [-2/m, 2/m]$. However, the higher moments are not Gaussian. Such mock-Gaussian behaviour is also found for linear statistics of low-lying zeros of Dirichlet L-functions [2].

2. Proofs

Set $\Omega(r) = \frac{1}{2}\Psi(\frac{1}{4} + \frac{1}{2}\mathrm{i}r) + \frac{1}{2}\Psi(\frac{1}{4} - \frac{1}{2}\mathrm{i}r) - \log\pi$, where $\Psi(s) = \frac{\Gamma'}{\Gamma}(s)$ is the polygamma function. We need a smooth version of Riemann's explicit formula:

LEMMA 2.1. – *Let $g \in \mathrm{C}_c^\infty(\mathbb{R})$ have compact support, and let $h(r) = \int_{-\infty}^{\infty} g(u)\,\mathrm{e}^{\mathrm{i}ru}\,\mathrm{d}u$. Then*

$$\sum h(\gamma_j) = h\left(-\frac{\mathrm{i}}{2}\right) + h\left(\frac{\mathrm{i}}{2}\right) + \frac{1}{2\pi}\int_{-\infty}^{\infty} h(r)\Omega(r)\,\mathrm{d}r - \sum_{n=1}^{\infty} \frac{\Lambda(n)}{\sqrt{n}}\bigl(g(\log n) + g(-\log n)\bigr), \tag{2}$$

where $\Lambda(n)$ is the von Mangoldt function.

Setting

$$h(r) = f\left(\frac{\log T}{2\pi}(r - \tau)\right), \qquad g(u) = \frac{\mathrm{e}^{-\mathrm{i}\tau u}}{\log T}\widehat{f}\left(\frac{u}{\log T}\right),$$

where $\widehat{f} \in \mathrm{C}_c^\infty(\mathbb{R})$, we have $N_f(\tau) = \overline{N_f}(\tau) + S_f(\tau)$ where

$$\overline{N_f}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f\left(\frac{\log T}{2\pi}(r - \tau)\right)\Omega(r)\,\mathrm{d}r + f\left(\frac{\log T}{2\pi}\left(\frac{\mathrm{i}}{2} - \tau\right)\right) + f\left(\frac{\log T}{2\pi}\left(-\frac{\mathrm{i}}{2} - \tau\right)\right) \tag{3}$$

and

$$S_f(\tau) = -\frac{1}{\log T}\sum_{n\geqslant 2}\frac{\Lambda(n)}{\sqrt{n}}\widehat{f}\left(\frac{\log n}{\log T}\right)\left(\mathrm{e}^{\mathrm{i}\tau\log n} + \mathrm{e}^{-\mathrm{i}\tau\log n}\right). \tag{4}$$

LEMMA 2.2. – *For all* $\widehat{f}\in \mathrm{C}_c^\infty(\mathbb{R})$,

$$\langle \overline{N_f}\rangle_{T,H} = \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x + \mathcal{O}\left(\frac{1}{\log T}\right), \quad T\to\infty.$$

Proof. – Since $\widehat{f}\in \mathrm{C}_c^\infty(\mathbb{R})$ we have that $f(x)$ decreases faster than any power of $1/|x|$, as $x\to\pm\infty$. Furthermore, in the bulk of the integral, when r is close to τ,

$$\Omega\left(\tau + \frac{2\pi x}{\log T}\right) = \Omega(\tau) + \mathcal{O}\left(\frac{1}{1+|\tau|}\frac{|x|}{\log T}\right),$$

and since Stirling's formula yields $\Omega(r) = \log(1+|r|) + \mathcal{O}(1)$ for all $r\in\mathbb{R}$ an asymptotic analysis gives

$$\int_{-\infty}^{\infty} f\left(\frac{\log T}{2\pi}(r-\tau)\right)\Omega(r)\frac{\mathrm{d}r}{2\pi} = \frac{\Omega(\tau)}{\log T}\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x + \mathcal{O}\left(\frac{1}{1+|\tau|}\frac{1}{(\log T)^2}\right) + \mathcal{O}\left(\frac{\log(1+|\tau|)}{(\log T)^A}\right) \tag{5}$$

for any $A>1$. Therefore,

$$\left\langle \frac{1}{2\pi}\int_{-\infty}^{\infty} f\left(\frac{\log T}{2\pi}(r-\tau)\right)\Omega(r)\,\mathrm{d}r\right\rangle_{T,H} = \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x + \mathcal{O}\left(\frac{1}{\log T}\right).$$

The averages of the polar terms $f\left(\frac{\log T}{2\pi}(\frac{\mathrm{i}}{2}-\tau)\right) + f\left(\frac{\log T}{2\pi}(-\frac{\mathrm{i}}{2}-\tau)\right)$ are bounded by $\mathcal{O}\left(\frac{1}{H\log T}\right)$ since by Parseval

$$\int_{-\infty}^{\infty} f\left(\frac{\log T}{2\pi}\left(\frac{\mathrm{i}}{2}-\tau\right)\right) w\left(\frac{\tau - T}{H}\right)\frac{\mathrm{d}\tau}{H} = \int_{-\infty}^{\infty}\frac{2\pi}{\log T}\widehat{f}\left(-\frac{2\pi y}{\log T}\right)\mathrm{e}^{\pi y}\widehat{w}(Hy)\,\mathrm{e}^{-2\pi\mathrm{i}Ty}\,\mathrm{d}y$$

and since $\widehat{w}$ has compact support, the integral is over $|y|\ll 1/H$ and is bounded by $\mathcal{O}(1/H\log T)$. □

PROPOSITION 2.3. – *For* f *with* $\widehat{f}\in \mathrm{C}_c^\infty(\mathbb{R})$, *if* $H\to\infty$ *then the mean value of* N_f *is given by*

$$\langle N_f\rangle_{T,H} = \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x + \mathcal{O}\left(\frac{1}{\log T}\right), \quad T\to\infty. \tag{6}$$

Proof. – In view of Lemma 2.2, it suffices to show that the mean value of S_f is zero as $H\to\infty$. Indeed, we have

$$\langle S_f\rangle_{T,H} = \frac{-1}{\log T}\sum_n \frac{\Lambda(n)}{\sqrt{n}}\widehat{f}\left(\frac{\log n}{\log T}\right)\left(\widehat{w}\left(\frac{H}{2\pi}\log n\right)\mathrm{e}^{-\mathrm{i}T\log n} + \widehat{w}\left(-\frac{H}{2\pi}\log n\right)\mathrm{e}^{\mathrm{i}T\log n}\right).$$

Since $\widehat{w}$ has compact support, and the prime powers n are at least 2, the summands vanish once $H\gg 1$. □

Proof of Theorem 1.1. – Assume supp $\widehat{f}\subseteq[-\rho,\rho]$, with $\rho < 2a/m$. From (5) and Proposition 2.3 it follows that

$$\left\langle\left(N_f - \langle N_f\rangle_{T,H}\right)^m\right\rangle_{T,H} = \left\langle S_f^m\right\rangle_{T,H}\left(1 + \mathcal{O}\left(\frac{1}{\log T}\right)\right)$$

and so it is sufficient to show that the m-th moment of S_f is the same as that of a centered normal random variable with variance given by (1), that is the mth moment vanishes for m odd and if $m=2k$ is even equals $\frac{(2k)!}{2^k k!}\sigma_f^{2k}$. Using Eq. (4), multiplying out $(S_f)^m$ and integrating we find

$$\left\langle (S_f)^m \right\rangle_{T,H} = \left(-\frac{1}{\log T}\right)^m \sum_{\varepsilon_1,\dots,\varepsilon_m=\pm 1} \sum_{n_1,\dots,n_m} \prod_{j=1}^m \frac{\Lambda(n_j)}{\sqrt{n_j}} \widehat{f}\left(\frac{\log n_j}{\log T}\right)$$
$$\times \widehat{w}\left(\frac{H}{2\pi} \sum_{j=1}^m \varepsilon_j \log n_j\right) \mathrm{e}^{-\mathrm{i}T \sum_{j=1}^m \varepsilon_j \log n_j}.$$

Since $\widehat{w}$ has compact support, in order to get a nonzero contribution we need

$$\left|\sum_{j=1}^m \varepsilon_j \log n_j\right| \ll \frac{1}{H}.$$

Set $M = \prod_{\varepsilon_j=+1} n_j$ and $N = \prod_{\varepsilon_j=-1} n_j$. If $M \neq N$ then assume w.l.o.g. that $M > N$, say $M = N + u$ with $u \geqslant 1$. Thus for a non-zero contribution we need

$$\frac{1}{H} \gg \log \frac{M}{N} = \log\left(1 + \frac{u}{N}\right) \gg \frac{1}{N}$$

and hence $T^a = H \ll N \leqslant \sqrt{MN} \leqslant T^{m\rho/2}$ since $n_j \leqslant T^\rho$ by assumption on the support of $\widehat{f}$. Since $\rho < 2a/m$, this is a contradiction. Therefore $M = N$, and $\sum \varepsilon_j \log n_j = 0$.

Thus for $T \gg 1$, we find (taking into account that $\widehat{w}(0) = \int_{-\infty}^{\infty} w(x)\,\mathrm{d}x = 1$)

$$\left\langle (S_f)^m \right\rangle_{T,H} = \left(-\frac{1}{\log T}\right)^m \sum_{\varepsilon_1,\dots,\varepsilon_m=\pm 1} \sum_{\substack{n_1,\dots,n_m \geqslant 2 \\ \Sigma_{j=1}^m \varepsilon_j \log n_j = 0}} \prod_{j=1}^m \frac{\Lambda(n_j)}{\sqrt{n_j}} \widehat{f}\left(\frac{\log n_j}{\log T}\right).$$

A standard argument in this subject (given in detail in [2]) now shows that the only terms which do not vanish as $T \to \infty$ are those where $m = 2k$ is even, and there is a partition $\{1, \dots, 2k\} = S \cup S'$ into disjoint subsets and a bijection $\sigma : S \to S'$ such that $n_j = n_{\sigma(j)}$ and $\varepsilon_j = -\varepsilon_{\sigma(j)}$. There are $k!\binom{2k}{k}$ such terms, and so

$$\left\langle (S_f)^{2k} \right\rangle_{T,H} = \frac{(2k)!}{k!} \left(\frac{1}{\log^2 T} \sum_n \frac{\Lambda(n)^2}{n} \widehat{f}\left(\frac{\log n}{\log T}\right)^2\right)^k + \mathcal{O}\left(\frac{1}{\log T}\right).$$

We note that by the Prime Number Theorem, as $T \to \infty$

$$\frac{1}{(\log T)^2} \sum_n \frac{\Lambda(n)^2}{n} \widehat{f}\left(\frac{\log n}{\log T}\right)^2 \sim \int_0^\infty u \widehat{f}(u)^2\,\mathrm{d}u + \mathcal{O}\left(\frac{1}{\log T}\right).$$

Since supp $\widehat{f} \subset (-1, 1)$, the integral coincides with $\sigma_f^2/2$ in (1) as required. □

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