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Examples of wandering domains in p-adic polynomial dynamics

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Abstract For any prime $p > 0$, we contruct p-adic polynomial functions in $\mathbb{C}_p[z]$ whose Fatou sets have wandering domains. ***To cite this article: R.L. Benedetto, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 615–620.***

Exemples des domaines errants dans la dynamique polynôme p-adique

Résumé Soit $p > 0$ un nombre premier. Nous construisons des polynômes p-adiques dans $\mathbb{C}_p[z]$ dont les ensembles de Fatou ont des domaines errants. ***Pour citer cet article : R.L. Benedetto, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 615–620.***

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Soit $p > 0$ un nombre premier fixé, soit $\overline{\mathbb{Q}}_p$ une clôture algébrique du corps $\mathbb{Q}_p$ des nombres rationels p-adiques, et soit $\mathbb{C}_p$ le complété de $\overline{\mathbb{Q}}_p$ pour la valeur absolue p-adique, notée $|\cdot|$. Pour une fraction rationelle $\phi(z) \in \mathbb{C}_p(z)$, la dynamique de ϕ opérant sur $\mathbb{P}^1(\mathbb{C}_p) = \mathbb{C}_p \cup \{\infty\}$ est analogue à la dynamique des fractions rationelles complexes sur la sphère de Riemann ; voir [1,2,4,6,7,9–11], par exemple. En particulier, on peut définir les ensembles p-adiques de Julia, les ensembles de Fatou, et les composantes des ensembles de Fatou, qui se comportent de façon semblable à leurs contre-parties complexes. Bien que quelques résultats partiels suggèrent que l'ensemble de Fatou de $\phi \in \overline{\mathbb{Q}}_p(z)$ ne puisse pas avoir de domaine errant, nous démontrons dans cet article qu'il y a des polynômes dans $\mathbb{C}_p[z]$ avec des domaines errants. Plus précisément, nous démontrons qu'il existe $a \in \mathbb{C}_p$ tel que la fonction ϕ_a définie par l'équation (1) a un domaine errant.

Pour $x \in \mathbb{C}_p$ et $r > 0$, on note le disque ouvert $D_r(x) = \{y \in \mathbb{C}_p : |y-x| < r\}$ et le disque fermé $\overline{D}_r(x) = \{y \in \mathbb{C}_p : |y-x| \leqslant r\}$. Nous considérons $a \in \mathbb{C}_p$ tel que $|a| = |p|^{-(p-1)} > 1$. Dans ce cas, ϕ_a augmente des distances dans $D_1(1)$ par un facteur de $|a|$; voir équation (6). Nous observons avec l'équation (5) que

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ϕ_a^m contracte localement des distances dans $\overline{D}_r(0)$, où $r = |p|^{1-p^{-m}} < 1$, mais $\phi_a^m(\overline{D}_r(0)) = \overline{D}_1(0)$. De plus, on peut étudier la variation de la famille en employant l'équation (4).

Avec ces outils, nous pouvons construire une valeur $a \in \mathbb{C}_p$ et un point $x \in D_1(0)$ telle que l'orbite $\{\phi_a^j(x)\}_{j \geqslant 0}$ suit le modèle dans l'équation (7). Dans cette équation, un 0 en la j-ème position indique que $\phi_a^{j-1}(x) \in D_1(0)$, et un 1 indique que $\phi_a^{j-1}(x) \in D_1(1)$; et pour $i \geqslant 0$, on a $M_i = 2i$ et $m_i = 2p + 2(p-1)i$. Comme la contraction de $\phi_a^{m_i}$ dans $D_1(0)$ est supérieure à l'expansion de $\phi_a^{M_{i+1}}$ dans $D_1(1)$, on voit que x est contenu dans un disque errant de ϕ_a.

1. Introduction

Fix a prime number $p > 0$, and let $\mathbb{Q}_p$ denote the field of p-adic rationals, formed by completing $\mathbb{Q}$ with respect to the unique absolute value satisfying $|p| = 1/p$. Let $\overline{\mathbb{Q}}_p$ be an algebraic closure of $\mathbb{Q}_p$, and let $\mathbb{C}_p$ denote the completion of $\overline{\mathbb{Q}}_p$. The absolute value $|\cdot|$, which extends canonically to $\mathbb{C}_p$, is non-Archimedean, meaning that it satisfies the ultrametric triangle inequality $|x + y| \leqslant \max\{|x|, |y|\}$. Both $\mathbb{Q}_p$ and $\mathbb{C}_p$ are complete with respect to $|\cdot|$, though $\overline{\mathbb{Q}}_p$ is not. Note that $\mathbb{Z} \subset \mathbb{Q}_p \subset \mathbb{C}_p$; every $n \in \mathbb{Z}$ satisfies $|n| \leqslant 1$, with $|p| < 1$. See [5,8,12] for more general background on p-adic fields.

Although $\mathbb{Q}_p$ is locally compect, $\overline{\mathbb{Q}}_p$ and $\mathbb{C}_p$ are not. Still, $\mathbb{C}_p$ is algebraically closed and complete, analogous to $\mathbb{C}$; the projective line $\mathbb{P}^1(\mathbb{C}_p) = \mathbb{C}_p \cup \{\infty\}$ is a non-Archimedean version of the Riemann sphere. The dynamics of rational functions $\phi(z) \in \mathbb{C}_p(z)$ acting on $\mathbb{P}^1(\mathbb{C}_p)$ have exhibited many parallels with the existing theory of complex dynamics; see [1,2,4,6,7,9–11], for example. The failure of local compactness, and hence of the Arzelà–Ascoli theorem, means that p-adic Fatou and Julia sets should be defined in terms of equicontinuity, rather than normality.

Topologically, $\mathbb{P}^1(\mathbb{C}_p)$ and its subsets are totally disconnected. Nevertheless, the author [1,2] and Rivera-Letelier [9,11] have developed several related definitions of components of p-adic Fatou sets which behave as useful analogs of connected components of complex Fatou sets.

The author has proven [1] that if $\phi \in \overline{\mathbb{Q}}_p(z)$ (acting on the full $\mathbb{P}^1(\mathbb{C}_p)$) has nonempty Julia set $\mathcal{J}$ with no recurrent critical points of order divisible by p in $\mathcal{J}$, then the Fatou set of ϕ has no wandering domains. (In fact, the proof in [1] applies equally well to any of the definitions of components even if $\mathcal{J}$ is empty.) The main result of this paper implies that the first hypothesis (that the coefficients lie in $\overline{\mathbb{Q}}_p$) cannot be removed.

THEOREM 1.1. – *There exists $a \in \mathbb{C}_p$ such that the Julia set $\mathcal{J}$ of*

$$\phi_a(z) = (1-a)z^{p+1} + az^p \tag{1}$$

is nonempty, the Fatou set $\mathcal{F}$ of ϕ_a has a wandering domain, and all critical points of ϕ_a lie in $\mathcal{F}$.

Compared to Sullivan's complex No Wandering Domains Theorem [13], Theorem 1.1 gives a sharp contrast between non-Archimedean and complex dynamics. Moreover, our result also provides a counterexample disproving Rivera-Letelier's Conjecture de Non-Errance and his related statement on Structure Conjecturale de l'Ensemble de Fatou in [9, Section 4.3]. However, both of those conjectures may still be true if the hypothesis that all coefficients lie in $\overline{\mathbb{Q}}_p$ is added; see the conjecture in [1, Section 1].

A generalization of the method of this paper can actually be used to prove the density of parameters for which ϕ_a has a wandering domain in the set $\{a \in \mathbb{C}_p : |a| > 1\}$. The argument works for any algebraically closed complete non-Archimedean field with the property that $|p| < 1$. However, in the interest of clarity, we restrict our attention here to announcing the existence of p-adic wandering domains, and we leave the generalizations to a forthcoming paper [3].

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2. Disks

We will denote the closed disk of radius $r > 0$ about a point $a \in \mathbb{C}_p$ by $\overline{D}_r(a)$, and the open disk by $D_r(a)$. We recall some basic properties of non-Archimedean disks. Every disk is both open and closed as a topological set. Any point in a disk U is a center, but the radius of U is a well-defined real number, being the same as the diameter of U. If two disks in $\mathbb{C}_p$ intersect, then one contains the other. If $f \in \mathbb{C}_p[z]$ is a non-constant polynomial, and if $U \subset \mathbb{C}_p$ is a disk, then $f(U)$ is also a disk. If $a, b \in \mathbb{C}_p$, $r, s > 0$, and $f \in \mathbb{C}_p[z]$ with $f(a) = b$, then f maps $\overline{D}_r(a)$ bijectively onto $\overline{D}_s(b)$ if and only if for every $x \in \overline{D}_r(a)$,

$$\bigl|f(x) - f(a)\bigr| = \frac{s}{r} \cdot |x - a|.$$

We also recall Hsia's criterion [7] for equicontinuity, which is a non-Archimedean analogue of the Montel–Carathéodory theorem. Hsia stated his result for arbitrary meromorphic functions on more general non-Archimedean fields, but for simplicity, we rephrase it for our special case.

THEOREM 2.1 (Hsia). – *Let F be a family of rational functions on a disk $U \subset \mathbb{C}_p$, and suppose that there are two distinct points $a_1, a_2 \in \mathbb{P}^1(\mathbb{C}_p)$ such that for all $f \in F$, $x \in U$, and $i = 1, 2$, we have $f(x) \neq a_i$. Then F is an equicontinuous family.*

3. The family

We consider the family $\{\phi_a\}$ defined in equation (1), with $|a| = |p|^{-(p-1)} > 1$. For any such a, ϕ_a has a superattracting (hence Fatou) fixed point at $z = 0$, and a repelling (hence Julia) fixed point at $z = 1$. Furthermore, it is not difficult to see that the filled Julia set $\mathcal{K}$ (that is, the set of points not attracted to ∞) is completely contained in $D_1(0) \cup D_1(1)$. The only critical points of ϕ_a besides ∞ lie in $\overline{D}_{|p|}(0)$, which is a bounded open set that maps into itself. Hence, all critical points are Fatou.

Fix $a \in \mathbb{C}_p$ with $|a| = |p|^{-(p-1)}$. If $y_0 \in D_1(1)$ and $|y_1 - y_0| < 1$, then it is immediate from the definition of ϕ_a and the ultrametric triangle inequality that

$$\bigl|\phi_a(y_1) - \phi_a(y_0)\bigr| = |a| \cdot |y_1 - y_0|. \tag{2}$$

If $|p| < |y_0| < 1$ and $|y_1 - y_0| \leqslant |p|^2$, then it is only slightly more difficult to show that

$$\bigl|\phi_a(y_1) - \phi_a(y_0)\bigr| = |a| \cdot |y_0|^p \cdot |y_1 - y_0|. \tag{3}$$

On the other hand, if we fix $y_0 \in D_1(0)$ and $y_1 \in D_1(1)$, and if we choose two parameters $a, b \in \mathbb{C}_p$, then

$$\bigl|\phi_b(y_0) - \phi_a(y_0)\bigr| = |y_0|^p \cdot |b - a| \quad \text{and} \quad \bigl|\phi_b(y_1) - \phi_a(y_1)\bigr| = |y_1 - 1| \cdot |b - a|. \tag{4}$$

4. Local mapping properties of ϕ_a^n

Let $S = |p|^2$. If $x \in D_1(0)$ or $x \in D_1(1)$, then using induction and Eqs. (2) and (3), we can easily prove the following statements concerning the next few iterates of x.

LEMMA 4.1. – *Let $a \in \mathbb{C}_p$ with $|a| = |p|^{-(p-1)}$. Let $m \geqslant 1$ and $x \in \mathbb{C}_p$ with $|x| \leqslant |p|^{1-p^{-m}}$. Then for all $0 \leqslant i \leqslant m$,*

$$\bigl|\phi_a^i(x)\bigr| = |p|^{1-p^i}|x|^{p^i}, \quad \textit{and for all } r \in (0, S], \quad \phi_a^i\bigl(\overline{D}_r(x)\bigr) \subset \overline{D}_{r\cdot|p|^{i-e_i}}\bigl(\phi_a^i(x)\bigr),$$

where $e_i = p^{1-m} + p^{2-m} + \cdots + p^{i-m} < 2$. In particular, if $|x| = |p|^{1-p^{-m}}$, then $|\phi_a^m(x)| = 1$,

$$\phi_a^m\bigl(\overline{D}_r(x)\bigr) \subset \overline{D}_{r\cdot|p|^{m-2}}\bigl(\phi_a^m(x)\bigr) \quad \textit{for all } r \in (0, S], \tag{5}$$

and $\phi^i(x) \in D_1(0)$ for all $0 \leqslant i \leqslant m - 1$, Thus, the iterates of x are pushed away from 0, but the function ϕ_a^m is locally contracting.

On the other hand, all distances within $D_1(1)$ are stretched by a factor of exactly $|a|$, giving us the following simpler statement for that disk.

LEMMA 4.2. – *Let $a \in \mathbb{C}_p$ with $|a| = |p|^{-(p-1)}$. Let $M \geqslant 1$ and $x \in \mathbb{C}_p$ with $|x-1| \leqslant |a|^{-M}$. Then for all $0 \leqslant i \leqslant M$,*

$$\left|\phi_a^i(x) - 1\right| = |a|^i \cdot |x-1|, \quad \textit{and for all } r \in \left(0, |a|^{-M}\right], \quad \phi_a^i\left(\overline{D}_r(x)\right) = \overline{D}_{r\cdot|a|^i}\left(\phi_a^i(x)\right).$$

In particular, if $|x-1| = |a|^{-M}$, then $|\phi_a^M(x) - 1| = 1$,

$$\phi_a^M\left(\overline{D}_r(x)\right) = \overline{D}_{r\cdot|a|^M}\left(\phi_a^M(x)\right) \quad \textit{for all } r \in \left(0, |a|^{-M}\right], \tag{6}$$

and $\phi^i(x) \in D_1(1)$ for all $0 \leqslant i \leqslant M-1$.

5. Perturbations

Set the notation $\Phi_n(a, z) = \phi_a^n(z)$. For fixed $x \in \mathbb{C}_p$, $\Phi_n(\cdot, x)$ is a polynomial function of the parameter a. The following lemmas show how that function behaves locally in certain circumstances.

LEMMA 5.1. – *Let $a \in \mathbb{C}_p$ with $|a| = |p|^{-(p-1)}$. Let $m \geqslant 2$, let $n \geqslant 0$, and let $x \in \mathbb{C}_p$ satisfying $|\phi_a^n(x)| = |p|^{1-p^{-m}}$. Let $A \geqslant |p|^{p-1}$ be a real number, let $\varepsilon \in (0, A^{-1}S]$, and suppose that*

$$\Phi_n\left(\overline{D}_\varepsilon(a), x\right) \subset \overline{D}_{A\varepsilon}\left(\phi_a^n(x)\right) \quad \textit{and} \quad A \leqslant |p|^{p+1-m}.$$

Then $\Phi_{n+m}(\cdot, x)$ maps $\overline{D}_\varepsilon(a)$ bijectively onto $\overline{D}_{\varepsilon/|a|}(\phi_a^{n+m}(x))$.

Note that the two displayed conditions in Lemma 5.1 say, first, that A is large enough to bound the size of a certain image disk, and second, that m is large enough to make $|p|^{-m+p+1}$ even larger than A.

LEMMA 5.2. – *Let $a \in \mathbb{C}_p$ with $|a| = |p|^{-(p-1)}$. Let $M \geqslant 0$, let $n \geqslant 1$, and let $x \in \mathbb{C}_p$ satisfying $|\phi_a^n(x) - 1| \leqslant |a|^{-M}$. Let $\varepsilon \in (0, |a|^{1-M}]$. Suppose that $\Phi_n(\cdot, x)$ maps $\overline{D}_\varepsilon(a)$ bijectively onto $\overline{D}_{\varepsilon/|a|}(\phi_a^n(x))$. Then $\Phi_{n+M}(\cdot, x)$ maps $\overline{D}_\varepsilon(a)$ bijectively onto $\overline{D}_{\varepsilon\cdot|a|^{(M-1)}}(\phi_a^{n+M}(x))$.*

We sketch the proofs as follows. Pick $b \in \overline{D}_\varepsilon(a) \setminus \{a\}$, and for every $i \geqslant 0$, let $\delta_i = |\phi_b^{n+i}(x) - \phi_a^{n+i}(x)|$. By the ultrametric triangle inequality, for $i \geqslant 1$ we have $\delta_i \leqslant \max\{B_i, C_i\}$, with equality if $B_i \neq C_i$, where

$$B_i = \left|\phi_b\left(\phi_b^{n+i-1}(x)\right) - \phi_a\left(\phi_b^{n+i-1}(x)\right)\right|, \quad \text{and} \quad C_i = \left|\phi_a\left(\phi_b^{n+i-1}(x)\right) - \phi_a\left(\phi_a^{n+i-1}(x)\right)\right|.$$

Define

$$s_i = |b-a| \cdot \max\left\{A \cdot |p|^{i-e_i}, |p|^{p-p^{-m+i}}\right\}, \quad \text{and} \quad t_i = |b-a| \cdot |a|^{i-1},$$

where e_i is as in the statement of Lemma 4.1.

For Lemma 5.1, we show by induction (using Eqs. (3) and (4) and Lemma 4.1) that $B_i, C_i \leqslant s_i$ for all $1 \leqslant i \leqslant m$. Then we observe that $C_m < B_m = s_m = |b-a|/|a|$, proving that $\delta_m = |b-a|/|a|$, as desired. Similarly, for Lemma 5.2, we show that $B_i < C_i = t_i$, for all $1 \leqslant i \leqslant M$. Thus, $\delta_M = t_M = |a|^{M-1} \cdot |b-a|$.

6. Proof of Theorem 1.1

Let $a_0 = p^{-(p-1)} \in \mathbb{C}_p$. For each $i \geqslant 0$, define $M_i = 2i$ and $m_i = 2p + 2(p-1)i$. Set $r_i = |p|^{1-p^{-m_i}}$ and $\varepsilon_i = |a_0|^{1-M_i} S$.

By Lemma 4.1, any $y \in \mathbb{C}_p$ with $|y| = r_0$ satisfies $|\phi_{a_0}^{m_0}(y)| = 1$. Because $\phi_{a_0}^{m_0}(0) = 0$, it follows that $\phi_{a_0}^{m_0}(\overline{D}_{r_0}(0)) \supset \overline{D}_1(0)$. In particular, there is some $x \in \overline{D}_{r_0}(0)$ with $\phi_{a_0}^{m_0}(x) = 1$. By the same lemma, we must have $|x| = r_0$. We will find $a \in \overline{D}_1(a_0)$ such that the orbit $\{\phi_a^j(x)\}_{j \geqslant 0}$ can be described by

$$\underbrace{0, \ldots, 0}_{m_0}, \underbrace{1, \ldots, 1}_{M_1}, \underbrace{0, \ldots, 0}_{m_1}, \underbrace{1, \ldots, 1}_{M_2}, \underbrace{0, \ldots, 0}_{m_2}, \ldots \tag{7}$$

where a 0 in the j-th position in the sequence indicates that $\phi_a^{j-1}(x) \in D_1(0)$, and a 1 indicates that $\phi_a^{j-1}(x) \in D_1(1)$.

For $i \geqslant 0$, define

$$n_i = \sum_{k=1}^{i}(m_{k-1} + M_k) = 2i + pi(i+1), \quad \text{and} \quad N_i = n_i + m_i = p(i+1)(i+2).$$

That is, n_i is the number of terms in (7) up to but not including the block of m_i 0's, and N_i is the number of terms up to but not including the block of M_{i+1} 1's.

For every $i \geqslant 0$, we will find $a_i \in \overline{D}_{\varepsilon_{i-1}}(a_{i-1})$ so that for every $a \in \overline{D}_{\varepsilon_i}(a_i)$, the orbit $\{\phi_a^j(x)\}$ follows (7) up to the $j = N_i$ iterate, with $\phi_{a_i}^{N_i}(x) = 1$ and such that

$$\Phi_{N_i}(\cdot, x) : \overline{D}_{\varepsilon_i}(a_i) \to \overline{D}_{\varepsilon_i/|a_i|}(1) \quad \text{is bijective.} \tag{8}$$

Note that every a_i will lie in $\overline{D}_{\varepsilon_0}(a_0)$, and therefore $|a_i| = |a_0| = |p|^{-(p-1)}$.

We proceed by induction on i. For $i = 0$, we already have $\phi_{a_0}^{N_0}(x) = 1$, and by Lemma 4.1, the orbit $\{\phi_{a_0}^j(x)\}$ follows (7) up to the $N_0 = m_0$ iterate. By Lemma 5.1 (with $n = n_0 = 0$, $m = m_0$, $a = a_0$, $A = |p|^{(p-1)}$, and $\varepsilon = \varepsilon_0$), condition (8) holds. Also, by Lemma 4.1, the orbit $\{\phi_a^j(x)\}$ is correct up to $j = N_0$ for every $a \in \overline{D}_{\varepsilon_0}(a_0)$. Hence, the $i = 0$ case is already done.

For $i \geqslant 1$, assume that we are given a_{i-1} with the desired properties. Let $\rho = |a_0|^{1-M_i} \leqslant \varepsilon_{i-1}$; then for every $a \in \overline{D}_\rho(a_{i-1})$, the orbit $\{\phi_a^j(x)\}$ agrees with (7) up to $j = N_{i-1}$. By Lemma 5.2 (with $a = a_{i-1}$, $M = M_i$, $n = N_{i-1}$, and $\varepsilon = \rho$), there exists $c_1 \in \mathbb{C}_p$ such that $|c_1 - a_{i-1}| = \rho$ and

$$\Phi_{n_i}(c_1, x) = 0 \quad \text{and} \quad \Phi_{n_i}(\cdot, x) : \overline{D}_\sigma(c_1) \to \overline{D}_{r_i}(0) \quad \text{is bijective,} \tag{9}$$

where $\sigma = r_i \cdot |a_0|^{1-M_i} \in (0, \rho)$. By Lemma 4.2, the orbit $\{\phi_a^j(x)\}$ is correct up to $j = n_i$ for every $a \in \overline{D}_\sigma(c_1)$.

Choose $c_2 \in \overline{D}_\sigma(c_1)$ so that $|\Phi_{n_i}(c_2, x)| = r_i$. By Lemma 4.1, $|\Phi_{N_i}(c_2, x)| = 1$. Furthermore it is clear that $\Phi_{N_i}(c_1, x) = 0$. Because the polynomial image of a disk is a disk, it follows that $\Phi_{N_i}(\overline{D}_\sigma(c_1), x) \supset \overline{D}_1(0)$. We may therefore choose $a_i \in \overline{D}_\sigma(c_1)$ so that $\Phi_{N_i}(a_i, x) = 1$.

By Eq. (9), the radius of $\Phi_{n_i}(\overline{D}_{\varepsilon_i}(a_i))$ must be $\varepsilon_i \cdot |a_0|^{M_i-1} = S$. Therefore, by Lemma 5.1 (with $n = n_i$, $m = m_i$, $a = a_i$, $A = |a_i|^{M_i-1}$, and $\varepsilon = \varepsilon_i$), condition (8) holds on $\overline{D}_{\varepsilon_i}(a_i)$. By Lemma 4.1, the orbit $\{\phi_a^j(x)\}$ is correct up to $j = N_i$ for every $a \in \overline{D}_{\varepsilon_i}(a_i)$. Our construction of a_i is complete.

The sequence $\{a_i\}_{i\geqslant 0}$ is a Cauchy sequence, because for any $0 \leqslant i \leqslant j$, we have $|a_i - a_j| \leqslant \varepsilon_i$, and $\varepsilon_i \to 0$. Therefore, the sequence has a limit $a \in \mathbb{C}_p$, with $|a - a_0| \leqslant \varepsilon_0$. By construction, $a \in \overline{D}_{\varepsilon_i}(a_i)$ for every $i \geqslant 0$; hence, the orbit $\{\phi_a^j(x)\}$ follows (7) exactly. In light of Lemmas 4.1 and 4.2, we must have $|\phi_a^{n_i}(x)| = |p|^{1-p^{-m_i}}$, and $|\phi_a^{N_i}(x) - 1| = |a|^{-M_{i+1}}$, for any $i \geqslant 0$. We only need to verify that ϕ_a has a wandering domain containing x.

Let $U = \overline{D}_S(x)$; we will show that U is contained in a wandering domain of the Fatou set $\mathcal{F}$ of ϕ_a. Every iterate $U_n = \phi_a^n(U)$ is a disk; we claim that for any $i \geqslant 0$, the radius of U_{n_i} is at most $S = |p|^2$, and the radius of U_{N_i} is at most $|a|^{-M_{i+1}}S$. The claim is easily proven by induction, as follows. For $i = 0$, $U_{n_0} = U_0 = U$, and by Eq. (5), U_{N_0} has radius at most $|p|^{m_0-2}S = |a|^{-M_1}S$. For $i \geqslant 1$, we assume the radius of $U_{N_{i-1}}$ is at most $|a|^{-M_i}S$. By Eq. (6), the radius of U_{n_i} is at most S; and by Eq. (5), the radius of U_{N_i} is at most $|p|^{m_i-2}S = |a|^{-M_{i+1}}S$.

In particular, no U_{n_i} contains the point 1; and because 1 is fixed, it follows that no U_n contains 1. Clearly, no U_n contains ∞ either. By Hsia's theorem, then, the family $\{\phi_a^n\}$ is equicontinuous on U, and therefore $U \subset \mathcal{F}$.

By any of the definitions of components in [1,2,9,11], the component V of $\mathcal{F}$ containing U must be a disk (see, for example, [2, Theorem 5.4.d]). Again, no iterate of V can contain 1, and therefore the symbolic dynamics of any point in V are also described by Eq. (7). Because those dynamics are not preperiodic, it follows that V must be wandering.

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