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Up to isometries, a deformation is a continuous function of its metric tensor

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Abstract If the Riemann–Christoffel tensor associated with a field of class $\mathcal{C}^2$ of positive definite symmetric matrices of order three vanishes in a connected and simply connected open subset $\Omega \subset \mathbb{R}^3$, then this field is the metric tensor field associated with a deformation of class $\mathcal{C}^3$ of the set Ω, uniquely determined up to isometries of $\mathbb{R}^3$. We establish here that the mapping defined in this fashion is continuous, for *ad hoc* metrizable topologies. ***To cite this article: P.G. Ciarlet, F. Laurent, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 489–493.***

Aux isométries près, une déformation est une fonction continue de son tenseur métrique

Résumé Si le tenseur de Riemann–Christoffel associé à un champ de classe $\mathcal{C}^2$ de matrices symétriques définies positives d'ordre trois s'annule sur un ouvert connexe et simplement connexe $\Omega \subset \mathbb{R}^3$, alors ce champ est celui du tenseur métrique associé à une déformation de classe $\mathcal{C}^3$ de l'ensemble Ω, déterminée de façon unique à une isométrie de $\mathbb{R}^3$ près. On établit ici la continuité de l'application ainsi définie, pour des topologies métrisables convenables. ***Pour citer cet article : P.G. Ciarlet, F. Laurent, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 489–493.***

1. Formulation of the problem

All spaces, matrices, etc., considered are real. The notations $\mathbb{M}^3$, $\mathbb{O}^3$, $\mathbb{S}^3$, and $\mathbb{S}^3_>$, respectively designate the sets of all square matrices of order three, of all orthogonal matrices of order three, of all symmetric matrices of order three, and of all symmetric and positive definite matrices of order three.

Latin indices and exponents vary in the set $\{1, 2, 3\}$, except when they are used for indexing sequences or when otherwise indicated, and the summation convention with respect to repeated indices or exponents is used in conjunction with this rule. Kronecker's symbols are designated by δ_{ij} or δ_i^j according to the context.

Let $\boldsymbol{E}^3$ denote a three-dimensional Euclidean space, let $\boldsymbol{a} \cdot \boldsymbol{b}$ denote the Euclidean inner product of $\boldsymbol{a}, \boldsymbol{b} \in \boldsymbol{E}^3$, and let $|\boldsymbol{a}| = \sqrt{\boldsymbol{a} \cdot \boldsymbol{a}}$ denote the Euclidean norm of $\boldsymbol{a} \in \boldsymbol{E}^3$. Let $\rho(\boldsymbol{A})$ denote the spectral radius

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and let $|\boldsymbol{A}| := \{\rho(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A})\}^{1/2}$ denote the spectral norm of a matrix $\boldsymbol{A} \in \mathbb{M}^3$. Finally, $\boldsymbol{id}$ denotes the identity mapping of $\boldsymbol{E}^3$.

Let there be also given a three-dimensional vector space, identified with $\mathbb{R}^3$. Let x_i denote the coordinates of a point $x \in \mathbb{R}^3$ and let $\partial_i := \partial/\partial x_i$, $\partial_{ij} := \partial^2/\partial x_i \partial x_j$, and $\partial_{ijk} := \partial^3/\partial x_i \partial x_j \partial x_k$. Let Ω be an open subset of $\mathbb{R}^3$. The notation $K \Subset \Omega$ means that K is a compact subset of Ω. If $g \in \mathcal{C}^\ell(\Omega;\mathbb{R})$, $\ell \geqslant 0$, and $K \Subset \Omega$, we let

$$|g|_{\ell,K} = \sup_{\substack{x \in K \\ |\alpha| = \ell}} \left|\partial^\alpha g(x)\right| \quad \text{and} \quad \|g\|_{\ell,K} = \sup_{\substack{x \in K \\ |\alpha| \leqslant \ell}} \left|\partial^\alpha g(x)\right|.$$

If $\boldsymbol{\Theta} \in \mathcal{C}^\ell(\Omega; \boldsymbol{E}^3)$ or $\boldsymbol{A} \in \mathcal{C}^\ell(\Omega; \mathbb{M}^3)$, we use similar notations, where $|\cdot|$ denotes the Euclidean vector norm or the matrix spectral norm, respectively.

Let $\boldsymbol{\Theta} \in \mathcal{C}^1(\Omega; \boldsymbol{E}^3)$ be an *immersion*. Then the *metric tensor field* $(g_{ij}) \in \mathcal{C}^0(\Omega; \mathbb{S}^3_>)$ of the open set $\boldsymbol{\Theta}(\Omega)$ is defined by means of its *covariant components*

$$g_{ij}(x) := \partial_i \boldsymbol{\Theta}(x) \cdot \partial_j \boldsymbol{\Theta}(x), \quad x \in \Omega.$$

When $\mathbb{R}^3$ is identified with $\boldsymbol{E}^3$, immersions such as $\boldsymbol{\Theta} = (\Theta_i) \in \mathcal{C}^1(\Omega; \boldsymbol{E}^3)$ may be thought of as *deformations* of the set Ω, viewed as a *reference configuration*, in the sense of *geometrically exact three-dimensional elasticity* (although they should then be in addition injective and orientation-preserving in order to qualify for this definition; for details, see, e.g., [3, Sect. 1.4] or [1, Chap. XII, Sect. 1]). In this context, the matrix $(g_{ij}(x))$ is usually denoted $\boldsymbol{C}(x) := (g_{ij}(x))$, and is called the (right) *Cauchy–Green tensor at* x. Note that one also has

$$\big(g_{ij}(x)\big) = \boldsymbol{\nabla\Theta}(x)^{\mathrm{T}} \boldsymbol{\nabla\Theta}(x).$$

We now recall two classical results from differential geometry, which are essential to the ensuing analysis. Theorem 1 provides sufficient conditions guaranteeing that, given a smooth enough matrix field $\boldsymbol{C} = (g_{ij}) : \Omega \to \mathbb{S}^3_>$ there exists an immersion $\boldsymbol{\Theta} : \Omega \to \boldsymbol{E}^3$ such that $g_{ij} = \partial_i \boldsymbol{\Theta} \cdot \partial_j \boldsymbol{\Theta}$ in Ω, i.e., such that $\boldsymbol{C}$ is the metric tensor field of the set $\boldsymbol{\Theta}(\Omega)$, while Theorem 2 specifies how two such immersions differ (a self-contained, complete, and essentially elementary, proof of these well-known results, whose outline follows with some modifications and simplifications that of [2], is found in [4]).

THEOREM 1 (Global Existence Theorem). – *Let Ω be a connected and simply connected open subset of $\mathbb{R}^3$ and let $\boldsymbol{C} = (g_{ij}) \in \mathcal{C}^2(\Omega; \mathbb{S}^3_>)$ be a matrix field that satisfies*

$$R_{qijk} := \partial_j \Gamma_{ikq} - \partial_k \Gamma_{ijq} + \Gamma^p_{ij} \Gamma_{kqp} - \Gamma^p_{ik} \Gamma_{jqp} = 0 \quad \textit{in } \Omega,$$

where

$$\Gamma_{ijq} := \tfrac{1}{2}(\partial_j g_{iq} + \partial_i g_{jq} - \partial_q g_{ij}) \quad \textit{and} \quad \Gamma^p_{ij} := g^{pq} \Gamma_{ijq}, \quad \textit{where } (g^{pq}) := (g_{ij})^{-1}.$$

Then there exists an immersion $\boldsymbol{\Theta} \in \mathcal{C}^3(\Omega; \boldsymbol{E}^3)$ such that

$$\boldsymbol{C} = \boldsymbol{\nabla\Theta}^{\mathrm{T}} \boldsymbol{\nabla\Theta} \quad \textit{in } \Omega.$$

THEOREM 2 (Rigidity Theorem). – *Let Ω be a connected open subset of $\mathbb{R}^3$ and let $\boldsymbol{\Theta} \in \mathcal{C}^1(\Omega; \boldsymbol{E}^3)$ and $\widetilde{\boldsymbol{\Theta}} \in \mathcal{C}^1(\Omega; \boldsymbol{E}^3)$ be two immersions whose associated metric tensors $\boldsymbol{C} = \boldsymbol{\nabla\Theta}^{\mathrm{T}} \boldsymbol{\nabla\Theta}$ and $\widetilde{\boldsymbol{C}} = \boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}^{\mathrm{T}} \boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}$ satisfy*

$$\boldsymbol{C} = \widetilde{\boldsymbol{C}} \quad \textit{in } \Omega.$$

Then there exist a vector $\boldsymbol{a} \in \boldsymbol{E}^3$ and a matrix $\boldsymbol{Q} \in \mathbb{O}^3$ such that

$$\boldsymbol{\Theta}(x) = \boldsymbol{a} + \boldsymbol{Q}\widetilde{\boldsymbol{\Theta}}(x) \quad \textit{for all } x \in \Omega.$$

Together, Theorems 1 and 2 establish the existence of a mapping $\mathcal{F}$ that associates to any matrix field $\boldsymbol{C} = (g_{ij}) \in \mathcal{C}^2(\Omega; \mathbb{S}^3_>)$ satisfying $R_{qijk} = 0$ in Ω (the functions R_{qijk} being defined in terms of the functions g_{ij} as in Theorem 1) a well-defined element $\mathcal{F}(\boldsymbol{C})$ in the quotient set $\mathcal{C}^3(\Omega; \boldsymbol{E}^3)/\mathcal{R}$, where

$(\boldsymbol{\Theta}, \widetilde{\boldsymbol{\Theta}}) \in \mathcal{R}$ means that there exists a vector $\boldsymbol{a} \in \boldsymbol{E}^3$ and a matrix $\boldsymbol{Q} \in \mathbb{O}^3$ such that $\boldsymbol{\Theta}(x) = \boldsymbol{a} + \boldsymbol{Q}\widetilde{\boldsymbol{\Theta}}(x)$ for all $x \in \Omega$.

A natural question thus arises as to whether there exist *ad hoc* topologies on the space $\mathcal{C}^2(\Omega; \mathbb{S}^3)$ and on the quotient set $\mathcal{C}^3(\Omega; \boldsymbol{E}^3)/\mathcal{R}$ such that the mapping $\mathcal{F}$ defined in this fashion is *continuous*.

2. A key preliminary result

The next theorem constitutes the key step toward establishing the continuity of the mapping $\mathcal{F}$ (see Theorem 4 in Section 3). Complete proofs will be found in [5].

THEOREM 3. – *Let Ω be a connected and simply connected open subset of $\mathbb{R}^3$. Let $\boldsymbol{C} = (g_{ij}) \in \mathcal{C}^2(\Omega; \mathbb{S}^3_>)$, and $\boldsymbol{C}^n = (g^n_{ij}) \in \mathcal{C}^2(\Omega, \mathbb{S}^3_>)$, $n \geqslant 0$, be matrix fields respectively satisfying $R_{qijk} = 0$ in Ω and $R^n_{qijk} = 0$ in Ω, $n \geqslant 0$ (with self-explanatory notations), such that*

$$\lim_{n\to\infty} \|\boldsymbol{C}^n - \boldsymbol{C}\|_{2,K} = 0 \quad \textit{for all } K \Subset \Omega.$$

Let $\boldsymbol{\Theta} \in \mathcal{C}^3(\Omega; \boldsymbol{E}^3)$ be any mapping that satisfies $\nabla\boldsymbol{\Theta}^{\mathrm{T}}\nabla\boldsymbol{\Theta} = \boldsymbol{C}$ in Ω (such mappings exist by Theorem 1). Then there exist mappings $\boldsymbol{\Theta}^n \in \mathcal{C}^3(\Omega; \boldsymbol{E}^3)$ satisfying $(\nabla\boldsymbol{\Theta}^n)^{\mathrm{T}}\nabla\boldsymbol{\Theta}^n = \boldsymbol{C}^n$ in Ω, $n \geqslant 0$, such that

$$\lim_{n\to\infty} \|\boldsymbol{\Theta}^n - \boldsymbol{\Theta}\|_{3,K} = 0 \quad \textit{for all } K \Subset \Omega.$$

For clarity, the proof of Theorem 3 is broken into those of three lemmas. Lemma 1 deals with the special case where $\boldsymbol{C} = \boldsymbol{I}$; Lemma 2 deals with the special case where the mapping $\boldsymbol{\Theta} \in \mathcal{C}^3(\Omega; \boldsymbol{E}^3)$ is injective; finally, Lemma 3 deals with the general case.

LEMMA 1. – *Let Ω be a connected and simply connected open subset of $\mathbb{R}^3$. Let $\boldsymbol{C}^n = (g^n_{ij}) \in \mathcal{C}^2(\Omega; \mathbb{S}^3_>)$, $n \geqslant 0$, be matrix fields satisfying $R^n_{qijk} = 0$ in Ω, $n \geqslant 0$, such that*

$$\lim_{n\to\infty} \|\boldsymbol{C}^n - \boldsymbol{I}\|_{2,K} = 0 \quad \textit{for all } K \Subset \Omega.$$

Then there exist mappings $\boldsymbol{\Theta}^n \in \mathcal{C}^3(\Omega; \boldsymbol{E}^3)$ satisfying $(\nabla\boldsymbol{\Theta}^n)^{\mathrm{T}}\nabla\boldsymbol{\Theta}^n = \boldsymbol{C}^n$ in Ω, $n \geqslant 0$, such that

$$\lim_{n\to\infty} \|\boldsymbol{\Theta}^n - \boldsymbol{id}\|_{3,K} = 0 \quad \textit{for all } K \Subset \Omega,$$

where $\boldsymbol{id}$ denotes the identity mapping of $\mathbb{R}^3$, identified here with $\boldsymbol{E}^3$.

Sketch of proof. – (i) *Let there be given any mappings $\boldsymbol{\Theta}^n \in \mathcal{C}^3(\Omega; \boldsymbol{E}^3)$, $n \geqslant 0$, that satisfy $(\nabla\boldsymbol{\Theta}^n)^{\mathrm{T}}\nabla\boldsymbol{\Theta}^n = \boldsymbol{C}^n$ in Ω* (such mappings exist by Theorem 1). *Then $\lim_{n\to\infty} |\boldsymbol{\Theta}^n - \boldsymbol{id}|_{\ell,K} = \lim_{n\to\infty} |\boldsymbol{\Theta}^n|_{\ell,K} = 0$ for all $K \Subset \Omega$ and for $\ell = 2, 3$.*

Given any immersion $\boldsymbol{\Theta} \in \mathcal{C}^3(\Omega; \boldsymbol{E}^3)$, let $\boldsymbol{g}_i := \partial_i\boldsymbol{\Theta}$ and let the vectors $\boldsymbol{g}^q$ be defined by means of the relations $\boldsymbol{g}_i \cdot \boldsymbol{g}^q = \delta_i^q$. Then proving (i) relies on the relation

$$\partial_{ij}\boldsymbol{\Theta} = \partial_i \boldsymbol{g}_j = (\partial_i \boldsymbol{g}_j \cdot \boldsymbol{g}_q)\boldsymbol{g}^q = \tfrac{1}{2}(\partial_j g_{iq} + \partial_i g_{jq} - \partial_q g_{ij})\boldsymbol{g}^q$$

applied to the mappings $\boldsymbol{\Theta}^n$ and on the uniform boundedness on any $K \Subset \Omega$ of the norms $|(g^n_{ij})^{-1}|_{0,K}$.

(ii) *There exist mappings $\widetilde{\boldsymbol{\Theta}}^n \in \mathcal{C}^3(\Omega; \boldsymbol{E}^3)$ that satisfy $(\nabla\widetilde{\boldsymbol{\Theta}}^n)^{\mathrm{T}}\nabla\widetilde{\boldsymbol{\Theta}}^n = \boldsymbol{C}^n$ in Ω, $n \geqslant 0$, and $\lim_{n\to\infty} |\widetilde{\boldsymbol{\Theta}}^n - \boldsymbol{id}|_{1,K} = 0$ for all $K \Subset \Omega$.*

Let $\boldsymbol{\psi}^n \in \mathcal{C}^3(\Omega; \boldsymbol{E}^3)$ be mappings that satisfy $(\nabla\boldsymbol{\psi}^n)^{\mathrm{T}}\nabla\boldsymbol{\psi}^n = \boldsymbol{C}^n$ in Ω, $n \geqslant 0$ (such mappings exist by Theorem 1) and let x_0 denote a point in the set Ω. Since $\lim_{n\to\infty} \nabla\boldsymbol{\psi}^n(x_0)^{\mathrm{T}}\nabla\boldsymbol{\psi}^n(x_0) = \boldsymbol{I}$ by assumption, Part (i) implies that there exist orthogonal matrices $\boldsymbol{Q}^n$, $n \geqslant 0$, such that $\lim_{n\to\infty} \boldsymbol{Q}^n\nabla\boldsymbol{\psi}^n(x_0) = \boldsymbol{I}$. Then the mappings

$$\widetilde{\boldsymbol{\Theta}}^n := \boldsymbol{Q}^n\boldsymbol{\psi}^n \in \mathcal{C}^3(\Omega; \boldsymbol{E}^3), \quad n \geqslant 0,$$

satisfy $(\nabla\widetilde{\boldsymbol{\Theta}}^n)^{\mathrm{T}}\nabla\widetilde{\boldsymbol{\Theta}}^n = \boldsymbol{C}^n$ in Ω, so that their gradients $\nabla\widetilde{\boldsymbol{\Theta}}^n \in \mathcal{C}^2(\Omega; \mathbb{M}^3)$ satisfy

$$\lim_{n\to\infty} \left|\partial_i\nabla\widetilde{\boldsymbol{\Theta}}^n\right|_{0,K} = 0 \quad \text{for all } K \Subset \Omega,$$

by part (i). In addition, $\lim_{n\to\infty} \nabla\widetilde{\boldsymbol{\Theta}}^n(x_0) = \lim_{n\to\infty} \boldsymbol{Q}^n\nabla\boldsymbol{\psi}^n(x_0) = \boldsymbol{I}$.

Hence a classical theorem about the differentiability of the limit of a sequence of mappings that are continuously differentiable on a connected open set and that take their values in a Banach space shows that *the mappings* $\nabla\widetilde{\boldsymbol{\Theta}}^n$ *uniformly converge on every compact subset of* Ω *toward a limit* $\boldsymbol{R} \in \mathcal{C}^1(\Omega; \mathbb{M}^3)$ *that satisfies* $\partial_i \boldsymbol{R}(x) = \boldsymbol{0}$ for all $x \in \Omega$. This shows that $\boldsymbol{R}$ is a constant mapping since Ω is connected. Consequently, $\boldsymbol{R} = \boldsymbol{I}$ since in particular $\boldsymbol{R}(x_0) = \lim_{n\to\infty} \nabla\widetilde{\boldsymbol{\Theta}}^n(x_0) = \boldsymbol{I}$.

(iii) There exist mappings $\boldsymbol{\Theta}^n \in \mathcal{C}^3(\Omega; \boldsymbol{E}^3)$ satisfying $(\nabla\boldsymbol{\Theta}^n)^{\mathrm{T}}\nabla\boldsymbol{\Theta}^n = \boldsymbol{C}^n$ in Ω, $n \geqslant 0$, and

$$\lim_{n\to\infty} |\boldsymbol{\Theta}^n - \boldsymbol{id}|_{\ell,K} = 0 \quad \text{for all } K \Subset \Omega \text{ and for } \ell = 0, 1.$$

To see this, apply again the theorem about the differentiability of the limit of a sequence of mappings used in part (ii) to the mappings $\boldsymbol{\Theta}^n := (\widetilde{\boldsymbol{\Theta}}^n - \{\widetilde{\boldsymbol{\Theta}}^n(x_0) - x_0\}) \in \mathcal{C}^3(\Omega; \boldsymbol{E}^3)$, $n \geqslant 0$.

LEMMA 2. – *Let* Ω *be a connected and simply connected open subset of* $\mathbb{R}^3$. *Let* $\boldsymbol{C} = (g_{ij}) \in \mathcal{C}^2(\Omega; \mathbb{S}^3_>)$ *and* $\boldsymbol{C}^n = (g^n_{ij}) \in \mathcal{C}^2(\Omega; \mathbb{S}^3_>)$, $n \geqslant 0$, *be matrix fields satisfying respectively* $R_{qijk} = 0$ *in* Ω *and* $R^n_{qijk} = 0$ *in* Ω, $n \geqslant 0$, *such that*

$$\lim_{n\to\infty} \|\boldsymbol{C}^n - \boldsymbol{C}\|_{2,K} = 0 \quad \textit{for all } K \Subset \Omega.$$

Assume that there exists an injective mapping $\boldsymbol{\Theta} \in \mathcal{C}^3(\Omega; \boldsymbol{E}^3)$ *such that* $\nabla\boldsymbol{\Theta}^{\mathrm{T}}\nabla\boldsymbol{\Theta} = \boldsymbol{C}$ *in* Ω. *Then there exist mappings* $\boldsymbol{\Theta}^n \in \mathcal{C}^3(\Omega; \boldsymbol{E}^3)$ *satisfying* $(\nabla\boldsymbol{\Theta}^n)^{\mathrm{T}}\nabla\boldsymbol{\Theta}^n = \boldsymbol{C}^n$ *in* Ω, $n \geqslant 0$, *such that*

$$\lim_{n\to\infty} \|\boldsymbol{\Theta}^n - \boldsymbol{\Theta}\|_{3,K} = 0 \quad \textit{for all } K \Subset \Omega.$$

Sketch of proof. – Let $\widehat{\Omega} := \boldsymbol{\Theta}(\Omega)$ and define the matrix fields $(\hat{g}^n_{ij}) \in \mathcal{C}^2(\widehat{\Omega}; \mathbb{S}^3_>)$, $n \geqslant 0$, by letting

$$\left(\hat{g}^n_{ij}(\hat{x})\right) := \nabla\boldsymbol{\Theta}(x)^{-\mathrm{T}}\left(g^n_{ij}(x)\right)\nabla\boldsymbol{\Theta}(x)^{-1} \quad \text{for all } \hat{x} = \boldsymbol{\Theta}(x) \in \widehat{\Omega}.$$

Then the assumptions of Lemma 2 imply that $\lim_{n\to\infty} \|\hat{g}^n_{ij} - \delta_{ij}\|_{2,\widehat{K}} = 0$.

Given $\hat{x} = (\hat{x}_i) \in \widehat{\Omega}$, let $\widehat{\partial}_i = \partial/\partial\hat{x}_i$. By Lemma 1 applied over the set $\widehat{\Omega}$, there exist mappings $\widehat{\boldsymbol{\Theta}}^n \in \mathcal{C}^3(\widehat{\Omega}; \boldsymbol{E}^3)$ satisfying $\widehat{\partial}_i\widehat{\boldsymbol{\Theta}}^n \cdot \widehat{\partial}_j\widehat{\boldsymbol{\Theta}}^n = \hat{g}^n_{ij}$ in $\widehat{\Omega}$, $n \geqslant 0$, such that $\lim_{n\to\infty} \|\widehat{\boldsymbol{\Theta}}^n - \widehat{\boldsymbol{id}}\|_{3,\widehat{K}} = 0$ for all $\widehat{K} \Subset \widehat{\Omega}$. Then the mappings $\boldsymbol{\Theta}^n \in \mathcal{C}^3(\Omega; \mathbb{S}^3_>)$, $n \geqslant 0$, defined by letting $\boldsymbol{\Theta}^n(x) = \widehat{\boldsymbol{\Theta}}^n(\hat{x})$ for all $x = \widehat{\boldsymbol{\Theta}}(\hat{x}) \in \Omega$, satisfy $\lim_{n\to\infty} \|\boldsymbol{\Theta}^n - \boldsymbol{\Theta}\|_{3,K} = 0$.

LEMMA 3. – *The assumption that the mapping* $\boldsymbol{\Theta} : \Omega \subset \mathbb{R}^3 \to \boldsymbol{E}^3$ *is injective is superfluous in Lemma* 2, *all its other assumptions holding verbatim. In other words, Theorem* 3 *holds.*

Sketch of proof. – (i) *Let* $\boldsymbol{\Theta} \in \mathcal{C}^3(\Omega; \boldsymbol{E}^3)$ *be any mapping that satisfies* $\nabla\boldsymbol{\Theta}^{\mathrm{T}}\nabla\boldsymbol{\Theta} = \boldsymbol{C}$ *in* Ω. *Then there exists a countable number of open balls* $B_r \subset \Omega$, $r \geqslant 1$, *such that* $\Omega = \bigcup_{r=1}^{\infty} B_r$ *and such that, for each* $r \geqslant 1$, *the set* $\bigcup_{s=1}^{r} B_s$ *is connected and the restriction of* $\boldsymbol{\Theta}$ *to* B_r *is injective.*

These assertions, which essentially rely on the assumed connectedness of the set Ω, are established by an iterative procedure.

(ii) *By Lemma* 2, *there exist mappings* $\boldsymbol{\Theta}^n_1 \in \mathcal{C}^3(B_1; \boldsymbol{E}^3)$ *and* $\widetilde{\boldsymbol{\Theta}}^n_2 \in \mathcal{C}^3(B_2; \boldsymbol{E}^3)$, $n \geqslant 0$, *that satisfy*

$$\left(\nabla\boldsymbol{\Theta}^n_1\right)^{\mathrm{T}}\nabla\boldsymbol{\Theta}^n_1 = \boldsymbol{C}^n \text{ in } B_1 \quad \text{and} \quad \lim_{n\to\infty} \left\|\boldsymbol{\Theta}^n_1 - \boldsymbol{\Theta}\right\|_{3,K} = 0 \quad \text{for all } K \Subset B_1,$$

$$\left(\nabla\widetilde{\boldsymbol{\Theta}}^n_2\right)^{\mathrm{T}}\nabla\widetilde{\boldsymbol{\Theta}}^n_2 = \boldsymbol{C}^n \text{ in } B_2 \quad \text{and} \quad \lim_{n\to\infty} \left\|\widetilde{\boldsymbol{\Theta}}^n_2 - \boldsymbol{\Theta}\right\|_{3,K} = 0 \quad \text{for all } K \Subset B_2,$$

and by Theorem 2, *there exist vectors* $\boldsymbol{a}^n \in \boldsymbol{E}^3$ *and matrices* $\boldsymbol{Q}^n \in \mathbb{O}^3$, $n \geqslant 0$, *such that* $\widetilde{\boldsymbol{\Theta}}^n_2(x) = \boldsymbol{a}^n + \boldsymbol{Q}^n\boldsymbol{\Theta}^n_1(x)$ *for all* $x \in B_1 \cap B_2$. *Then* $\lim_{n\to\infty} \boldsymbol{a}^n = \boldsymbol{0}$ *and* $\lim_{n\to\infty} \boldsymbol{Q}^n = \boldsymbol{I}$.

The proof hinges on the relations $\boldsymbol{\Theta}(x) = \lim_{p\to\infty} \widetilde{\boldsymbol{\Theta}}^p_2(x) = \lim_{p\to\infty}(\boldsymbol{a}^p + \boldsymbol{Q}^p\boldsymbol{\Theta}^p_1(x))$ for all $x \in B_1 \cap B_2$.

(iii) *Let the mappings* $\boldsymbol{\Theta}^n_2 \in \mathcal{C}^3(B_1 \cup B_2; \boldsymbol{E}^3)$, $n \geqslant 0$, *be defined by* $\boldsymbol{\Theta}^n_2(x) := \boldsymbol{\Theta}^n_1(x)$ *for all* $x \in B_1$, *and* $\boldsymbol{\Theta}^n_2(x) := (\boldsymbol{Q}^n)^{\mathrm{T}}(\widetilde{\boldsymbol{\Theta}}^n_2(x) - \boldsymbol{a}^n)$ *for all* $x \in B_2$. *Then* $\lim_{n\to\infty} \|\boldsymbol{\Theta}^n_2 - \boldsymbol{\Theta}\|_{3,K} = 0$ *for all* $K \Subset B_1 \cup B_2$.

The plane containing the intersection of the boundaries of the open balls B_1 and B_2 is the common boundary of two closed half-spaces in $\mathbb{R}^3$, H_1 containing the center of B_1, and H_2 containing that of B_2 (by construction, the set $B_1 \cup B_2$ is connected; see part (i)). Any compact subset K of $B_1 \cup B_2$ may thus be written as $K = K_1 \cup K_2$, where $K_1 := (K \cap H_1) \subset B_1$ and $K_2 := (K \cap H_2) \subset B_2$. Hence $\lim_{n\to\infty} \|\boldsymbol{\Theta}_2^n - \boldsymbol{\Theta}\|_{3,K_1} = 0$ and $\lim_{n\to\infty} \|\boldsymbol{\Theta}_2^n - \boldsymbol{\Theta}\|_{3,K_2} = 0$, the second relation following from the definition of the mapping $\boldsymbol{\Theta}_2^n$ on $B_2 \supset K_2$ and on the relations $\lim_{n\to\infty} \|\widetilde{\boldsymbol{\Theta}}_2^n - \boldsymbol{\Theta}\|_{3,K_2} = 0$ (part (ii)), and $\lim_{n\to\infty} \boldsymbol{Q}^n = \boldsymbol{I}$ and $\lim_{n\to\infty} \boldsymbol{a}^n = \boldsymbol{0}$ (part (iii)).

(iv) *It remains to iterate the procedure described in* parts (ii) *and* (iii). □

3. Continuity in metric spaces

Let $(K_i)_{i\geqslant 0}$ be any sequence of subsets of Ω that satisfy $K_i \Subset \Omega$ and $K_i \subset \operatorname{int} K_{i+1}$ for all $i \geqslant 0$, and $\Omega = \bigcup_{i=0}^{\infty} K_i$, and let

$$d_\ell(\boldsymbol{\psi}, \boldsymbol{\Theta}) := \sum_{i=0}^{\infty} \frac{1}{2^i} \frac{\|\boldsymbol{\psi} - \boldsymbol{\Theta}\|_{\ell,K_i}}{1 + \|\boldsymbol{\psi} - \boldsymbol{\Theta}\|_{\ell,K_i}}.$$

Let model $\dot{\mathcal{C}}^3(\Omega; \boldsymbol{E}^3) := \mathcal{C}^3(\Omega; \boldsymbol{E}^3)/\mathcal{R}$ denote the quotient set of $\mathcal{C}^3(\Omega; \boldsymbol{E}^3)$ by the equivalence relation $\mathcal{R}$, where $(\boldsymbol{\Theta}, \widetilde{\boldsymbol{\Theta}}) \in \mathcal{R}$ means that there exist a vector $\boldsymbol{a} \in \boldsymbol{E}^3$ and a matrix $\boldsymbol{Q} \in \mathbb{O}^3$ such that $\boldsymbol{\Theta}(x) = \boldsymbol{a} + \boldsymbol{Q}\widetilde{\boldsymbol{\Theta}}(x)$ for all $x \in \Omega$.

The set $\dot{\mathcal{C}}^3(\Omega; \boldsymbol{E}^3)$ becomes a metric space when it is equipped with the distance $\dot{d}_3$ defined by

$$\dot{d}_3(\dot{\boldsymbol{\Theta}}, \dot{\boldsymbol{\psi}}) := \inf_{\left\{\substack{\boldsymbol{\kappa} \in \dot{\boldsymbol{\Theta}} \\ \boldsymbol{\chi} \in \dot{\boldsymbol{\psi}}}\right.} d_3(\boldsymbol{\kappa}, \boldsymbol{\chi}),$$

where $\dot{\boldsymbol{\Theta}}$ denotes the equivalence class of $\boldsymbol{\Theta}$ *modulo* $\mathcal{R}$.

The announced continuity of a deformation as a function of its metric tensor is then an easy corollary to Theorem 3. If d is a metric defined on a set X, the associated metric space is denoted $\{X; d\}$.

THEOREM 4. – *Let Ω be a connected and simply connected open subset of $\mathbb{R}^3$. Let*

$$\mathcal{C}_0^2\big(\Omega; \mathbb{S}_>^3\big) := \big\{(g_{ij}) \in \mathcal{C}^2\big(\Omega; \mathbb{S}_>^3\big);\ R_{qijk} = 0 \text{ in } \Omega\big\},$$

where the functions R_{qijk} are defined in terms of the functions g_{ij} as in Theorem 1. *Given any matrix field $\boldsymbol{C} = (g_{ij}) \in \mathcal{C}_0^2(\Omega; \mathbb{S}_>^3)$, let $\mathcal{F}(\boldsymbol{C}) \in \dot{\mathcal{C}}^3(\Omega; \boldsymbol{E}^3)$ denote the equivalence class modulo $\mathcal{R}$ of any $\boldsymbol{\Theta} \in \mathcal{C}^3(\Omega; \boldsymbol{E}^3)$ that satisfies $\boldsymbol{\nabla}\boldsymbol{\Theta}^{\mathrm{T}}\boldsymbol{\nabla}\boldsymbol{\Theta} = \boldsymbol{C}$ in Ω. Then the mapping*

$$\mathcal{F} : \big\{\mathcal{C}_0^2\big(\Omega; \mathbb{S}_>^3\big); d_2\big\} \longrightarrow \big\{\dot{\mathcal{C}}^3(\Omega; \boldsymbol{E}^3); \dot{d}_3\big\}$$

defined in this fashion is continuous.

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