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Analyse numérique/*Numerical Analysis*

Global a priori convergence theory for reduced-basis approximations of single-parameter symmetric coercive elliptic partial differential equations

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Abstract We consider "Lagrangian" reduced-basis methods for single-parameter symmetric coercive elliptic partial differential equations. We show that, for a logarithmic-(quasi-)uniform distribution of sample points, the reduced-basis approximation converges *exponentially* to the exact solution *uniformly* in parameter space. Furthermore, the convergence rate depends only weakly on the continuity–coercivity ratio of the operator: thus very low-dimensional approximations yield accurate solutions even for very wide parametric ranges. Numerical tests (reported elsewhere) corroborate the theoretical predictions. ***To cite this article: Y. Maday et al., C. R. Acad. Sci. Paris, Ser. I 335 (2002) 289–294.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Résultats globaux a priori pour l'approximation d'équations aux dérivées partielles coercives symétriques elliptiques dépendant d'un paramètre

Résumé On considère des méthodes de bases réduites de type Lagrange pour des équations aux dérivées partielles coercives symétriques elliptiques et dépendant d'un paramètre. On montre que, pour une répartition logarithmiquement quasi uniforme des points d'échantillonage, l'approximation en base réduite converge de façon exponentielle vers la solution exacte uniformément par rapport au paramètre. De plus la convergence ne dépend que faiblement du rapport entre les coefficients de coercivité et de continuité de l'opérateur : ainsi une approximation de très basse dimension procure une solution très précise même dans le cas d'un large eventail de paramètres. Des test numériques (présentés ailleurs) corroborent ces prédictions numériques. ***Pour citer cet article : Y. Maday et al., C. R. Acad. Sci. Paris, Ser. I 335 (2002) 289–294.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

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Dans un espace de Hilbert H, muni du produit scalaire $(\cdot,\cdot)_Y$ et de la norme $\|\cdot\|_Y$ on se pose le problème de trouver $u \in Y$ vérifiant (1) où la forme bilinéaire $a\colon Y \times Y \times \mathcal{D} \to \mathbb{R}$ dépend d'un paramètre $\mu \in \mathcal{D} \equiv [0, \mu_{\max}]$. Sous des conditions classiques de continuité et de coercivité de a ce problème possède une solution unique. La méthode de base réduite consiste alors à choisir un entier N et un jeux de paramètres $S_N = \{\alpha_1, \ldots, \alpha_N\}$ pour lesquels, de façon préalable, on calcule — le plus exactement possible — les solutions associées $u(\alpha_k),\ k = 1, \ldots, N$. On résout alors le système (2) où $W_N = \mathrm{Vect}\{u(\alpha_k),\ k = 1, \ldots, N\}$. On analyse dans cette Note le cas d'un problème dépendant d'un seul paramètre du type (3) où $a_0 : Y \times Y \to \mathbb{R}$ et $a_1 : Y \times Y \to \mathbb{R}$ sont continues, symétriques, semi positives et de plus où a_0 est coercive induisant une norme $|||\cdot|||^2 = a_0(\cdot,\cdot)$ équivalente à celle de Y. Des exemples de problèmes entrant dans ce cadre sont présentés, analysés et simulés sur base réduite dans [11]. Plus particulièrement nous montrons ici que la convergence de cette méthode en base réduite est exponentielle en le nombre de paramètres N utilisé dans la base de W_N, et ce uniformément par rapport au paramètre. En particulier on a la borne suivante entre la solution exacte $u(\mu)$ et son approximation $u_N(\mu)$: il existe un entier N_{crit} tel que pour tout $N \geqslant N_{\text{crit}}$, on a (19) avec une constante c ne dépendant que des conditions d'ellipticité de a_0 et de $\mu_{\max}$.

La démonstration de ce résultat repose d'une part sur le lemme classique de Cea rappelé en (10) et une estimation a priori de la meilleure approximation donnée dans le Lemme 2.

Il convient de noter que l'analyse de la meilleure approximation fait ici intervenir une approximation polynomiale de la solution, mais cette approximation polynomiale est proposée après un changement de variable approprié ($\mu = \mathrm{e}^{\tilde{\mu}} - \gamma^{-1}$). Le point qui doit être noté est que la méthode de Galerkin détermine naturellement une approximation dans W_N qui est (à une constante multiplicative près) aussi bonne que cette approximation polynomiale en une variable à définir. Ceci donne une supériorité et un caractère général à l'approche variationelle par rapport à une « simple » interpolation puisque aucune connaissance a priori de la forme de la solution en son paramètre n'est à connaître.

L'analyse faite ici suggère une répartition logarithmique du jeux de paramètres qui donne en effet de meilleurs résultats dans les applications comme celà est reporté dans [14]. On renvoit aussi à [11] pour plus de détails sur la mise en oeuvre et les applications.

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1. Introduction

Let Y be an Hilbert space with inner product and norm $(\cdot,\cdot)_Y$ and $\|\cdot\|_Y = (\cdot,\cdot)_Y^{1/2}$, respectively. Consider a parametrized "bilinear" form $a : Y \times Y \times \mathcal{D} \to \mathbb{R}$, where $\mathcal{D} \equiv [0, \mu_{\max}]$, and a bounded linear form $f : Y \to \mathbb{R}$. We introduce the problem to be solved: Given $\mu \in \mathcal{D}$, find $u \in Y$ such that

$$a\big(u(\mu), v; \mu\big) = f(v), \quad \forall v \in Y. \tag{1}$$

Under natural conditions on the bilinear form a (e.g. continuity and coercivity) it is readily shown that this problem admits a unique solution.

We introduce an approximation index N, the parameter sample $S_N = \{\alpha_1, \ldots, \alpha_N\}$, and the solutions $u(\alpha_k),\ k = 1, \ldots, N$, of problem (1) for this set of parameters. We next define the *reduced-basis approximation space* $W_N = \mathrm{span}\{u(\alpha_k),\ k = 1, \ldots, N\}$. Our reduced-basis approximation is then: given $\mu \in \mathcal{D}$, find $u_N(\mu) \in W_N$ such that:

$$a\big(u_N(\mu), v; \mu\big) = f(v), \quad \forall v \in W_N. \tag{2}$$

This discrete problem is well posed under the same former continuity and coercivity conditions.

The reduced-basis approach, as earlier developped, is typically local in parameter space in both practice and theory [1,2,4,9,10,12]. To wit, the α_k are chosen in the vicinity of a particular parameter point μ^* and the associated *a priori* convergence theory relies on asymptotic arguments in sufficiently small neighborhoods of μ^* [4]. In this Note we present, for single-parameter symmetric coercive elliptic partial differential equations, a first theoretical *a priori* convergence result that demonstrates exponential convergence of reduced-basis approximations *uniformly* over an extended parameter domain. The proof requires, and thus suggests, a point distribution in parameter space which does, indeed, exhibit superior convergence properties in a variety of numerical tests [14]. We refer also to [5–7,11] for further discussions of these results and related work and applications.

2. Problem formulation

Let us define the parametrized "bilinear" form $a : Y \times Y \times \mathcal{D} \to \mathbb{R}$ as

$$a(w, v; \mu) \equiv a_0(w, v) + \mu a_1(w, v), \tag{3}$$

where the bilinear forms $a_0 : Y \times Y \to \mathbb{R}$ and $a_1 : Y \times Y \to \mathbb{R}$ are continuous, symmetric and positive semi-definite; suppose moreover that a_0 is coercive, inducing a (Y-equivalent) norm $|||\cdot|||^2 = a_0(\cdot,\cdot)$. It follows from our assumptions that there exists a real positive constant γ_1 such that

$$0 \leqslant \frac{a_1(v, v)}{a_0(v, v)} \leqslant \gamma_1, \quad \forall v \in Y. \tag{4}$$

For these hypotheses, it is readily demonstrated that the problem (1) has a unique solution.

Many situations may be modeled by our rather simple problem statement (1), (3). For example, if we take $Y = \mathrm{H}_0^1(\Omega)$ where Ω is a smooth bounded subdomain of $\mathbb{R}^{d=2}$, and set $a_0(w, v) = \int_\Omega \nabla w \cdot \nabla v$, $a_1 = \int_\Omega wv$, we model conduction in thin plates; here μ represents the convective heat transfer coefficient. Other choices of a_0 and a_1 can model variable rectilinear geometry, variable orthotropic properties, and variable Robin boundary conditions.

The space Y is typically of infinite dimension so $u(\mu)$ is, in general, not exactly calculable. In order to construct our reduced-basis space W_N, we must therefore replace $u(\mu) \in Y$ by a "truth approximation" $u^{\mathcal{N}}(\mu) \in Y^{\mathcal{N}} \subset Y$, solution of the Galerkin approximation $a(u^{\mathcal{N}}(\mu), v; \mu) = f(v),\ \forall v \in Y^{\mathcal{N}}$. Here $Y^{\mathcal{N}}$, of finite (but typically very high) dimension $\mathcal{N}$, is a sufficiently rich approximation subspace such that $|||u(\mu) - u^{\mathcal{N}}(\mu)|||$ is sufficiently small for all μ in $\mathcal{D}$; for example, for $Y = \mathrm{H}_0^1(\Omega)$ we know that, for any desired $\varepsilon > 0$, we can indeed construct a finite-element approximation space, $Y^{\mathcal{N}(\varepsilon)}$, such that $|||u(\mu) - u^{\mathcal{N}(\varepsilon)}(\mu)||| \leqslant \varepsilon$.

It shall prove convenient in what follows to introduce a generalized eigenvalue problem: find $(\varphi_i^{\mathcal{N}} \in Y^{\mathcal{N}},\ \lambda_i^{\mathcal{N}} \in \mathbb{R})$, $i = 1, \ldots, \mathcal{N}$, satisfying $a_1(\varphi_i^{\mathcal{N}}, v) = \lambda_i^{\mathcal{N}} a_0(\varphi_i^{\mathcal{N}}, v)$, $\forall v \in Y^{\mathcal{N}}$. We shall order the (perforce real, non-negative) eigenvalues as $0 \leqslant \lambda_{\mathcal{N}}^{\mathcal{N}} \leqslant \lambda_{\mathcal{N}-1}^{\mathcal{N}} \leqslant \cdots \leqslant \lambda_1^{\mathcal{N}} \leqslant \gamma_1$, where the last inequality follows directly from (4). We may choose our eigenfunctions such that

$$a_0\big(\varphi_i^{\mathcal{N}}, \varphi_j^{\mathcal{N}}\big) = \delta_{ij}, \tag{5}$$

and hence $a_1(\varphi_i^{\mathcal{N}}, \varphi_j^{\mathcal{N}}) = \lambda_i^{\mathcal{N}} \delta_{ij}$, where δ_{ij} is the Kronecker-delta symbol; and such that $Y^{\mathcal{N}}$ can be expressed as $\mathrm{span}\{\varphi_i,\ i = 1, \ldots, \mathcal{N}\}$. Note that, thanks to the *finite* dimension of our approximation space $Y^{\mathcal{N}}$, we preclude (the complications associated with) a continuous spectrum — and, as we shall see, at no loss in rigor.

We conclude this section by noting that, if we set $f_i^{\mathcal{N}} = f(\varphi_i^{\mathcal{N}})$, then $u^{\mathcal{N}}(\mu)$ can be expressed as

$$u^{\mathcal{N}}(\mu) = \sum_{i=1}^{\mathcal{N}} \frac{f_i^{\mathcal{N}} \varphi_i^{\mathcal{N}}}{1 + \mu \lambda_i^{\mathcal{N}}}. \tag{6}$$

3. A priori convergence theory

Let us set $\delta_N = \ln(\gamma \mu_{\max} + 1)/N$ with γ any finite upper bound for γ_1.[1] We propose here to choose the sample points α_k, $k = 1, \ldots, N$, log-equidistributed in $\mathcal{D}$, $\alpha_k = \exp\{-\ln\gamma + \sum_{\ell=1}^{k} \tilde{\delta}_{\ell N}\} - \gamma^{-1}$, where $\sum_{\ell=1}^{N} \tilde{\delta}_{\ell N} = \ln(\gamma \mu_{\max} + 1)$ and $\tilde{\delta}_{kN}/\delta_N \leqslant c^*,\ k = 1, \ldots, N$, where c^* is a real positive constant.

Denote the reduced-basis approximation space as $W_N^{\mathcal{N}} = \text{span}\{u^{\mathcal{N}}(\alpha_k),\ k = 1, \ldots, N\}$. Although in general $\dim(W_N^{\mathcal{N}}) \leqslant N$, we can suppose that $\dim(W_N^{\mathcal{N}}) = N$ (otherwise we eliminate elements from $W_N^{\mathcal{N}}$ until it contains only linearly independent vectors). Then, the (reduced basis) problem is: given $\mu \in \mathcal{D}$, find $u_N^{\mathcal{N}}(\mu) \in W_N^{\mathcal{N}}$ such that

$$a\big(u_N^{\mathcal{N}}(\mu), v; \mu\big) = f(v), \quad \forall v \in W_N^{\mathcal{N}}. \tag{7}$$

This problem admits a unique solution.

Our goal is to (sharply) bound $|||u^{\mathcal{N}}(\mu) - u_N^{\mathcal{N}}(\mu)|||$, for all $\mu \in \mathcal{D}$, as a function of N (and ultimately $\mathcal{N}$ as well). This error bound in the energy norm can be readily translated into error bounds on continuous-linear-functional outputs [11]; we do not consider this extension further here.

We shall need two standard results from the theory of Galerkin approximation of symmetric coercive problems [13]:

$$a\big(u^{\mathcal{N}} - u_N^{\mathcal{N}}, u^{\mathcal{N}} - u_N^{\mathcal{N}}; \mu\big) = \inf_{w_N^{\mathcal{N}} \in W_N^{\mathcal{N}}} a\big(u^{\mathcal{N}} - w_N^{\mathcal{N}}, u^{\mathcal{N}} - w_N^{\mathcal{N}}; \mu\big); \tag{8}$$

$$a\big(u^{\mathcal{N}}, u^{\mathcal{N}}; \mu\big) \leqslant a(u, u; \mu). \tag{9}$$

From the positive semidefiniteness of a_1, (3), (4) and (8) we can write

$$\begin{aligned} \big|\big|\big|u^{\mathcal{N}}(\mu) - u_N^{\mathcal{N}}(\mu)\big|\big|\big|^2 &\leqslant a\big(u^{\mathcal{N}}(\mu) - u_N^{\mathcal{N}}(\mu), u^{\mathcal{N}}(\mu) - u_N^{\mathcal{N}}(\mu), \mu\big) \\ &= \inf_{w_N^{\mathcal{N}} \in W_N^{\mathcal{N}}} a\big(u^{\mathcal{N}}(\mu) - w_N^{\mathcal{N}}, u^{\mathcal{N}}(\mu) - w_N^{\mathcal{N}}, \mu\big) \\ &\leqslant (1 + \mu_{\max}\gamma_1) \inf_{w_N^{\mathcal{N}} \in W_N^{\mathcal{N}}} \big|\big|\big|u^{\mathcal{N}}(\mu) - w_N^{\mathcal{N}}\big|\big|\big|^2, \quad \forall \mu \in \mathcal{D}. \end{aligned} \tag{10}$$

Also from the definition of the $|||\cdot|||$ norm and the positive-semidefiniteness of a_1, (3), (4) and (9), we obtain

$$\big|\big|\big|u^{\mathcal{N}}(\mu)\big|\big|\big| \leqslant (1 + \mu_{\max}\gamma_1)^{1/2}\big|\big|\big|u(\mu)\big|\big|\big|, \quad \forall \mu \in \mathcal{D}. \tag{11}$$

We first state a preparatory result (*see* [8] for the proof)

LEMMA 1. – *Let* $g(z, \lambda) = 1/(1 - \lambda/\gamma + \lambda\, \mathrm{e}^z)$ *for* $z \in Z \equiv [\ln(\gamma^{-1}), \infty]$ *and* $\lambda \in \Lambda \equiv [0, \gamma]$ *(recall* γ *is our strictly positive upper bound for* γ_1*). Then, for any* $q \geqslant 0$, $|D_1^q g(z, \lambda)| \leqslant 2^q q!$, $\forall z \in Z$, $\forall \lambda \in \Lambda$, *where* $D_1^q g$ *denotes the* q*-th derivative of* g *with respect to the first argument.*

We now prove a bound for the best approximation result in

LEMMA 2. – *For* $N \geqslant N_{\text{crit}} \equiv c^* \mathrm{e} \ln(\gamma\, \mu_{\max} + 1)$

$$\inf_{w_N^{\mathcal{N}} \in W_N^{\mathcal{N}}} \big|\big|\big|u^{\mathcal{N}}(\mu) - w_N^{\mathcal{N}}\big|\big|\big| \leqslant \big|\big|\big|u^{\mathcal{N}}(0)\big|\big|\big| \exp\left\{-\frac{N}{2N_{\text{crit}}}\right\}, \quad \forall \mu \in \mathcal{D}. \tag{12}$$

Proof. – To facilitate the proof, we shall effect a change of coordinates in parameter space. To wit, we let $\widetilde{\mathcal{D}} \equiv [\ln\gamma^{-1}, \ln(\mu_{\max} + \gamma^{-1})]$, and introduce $\tau : \widetilde{\mathcal{D}} \to \mathcal{D}$ as $\tau(\tilde{\mu}) = \mathrm{e}^{\tilde{\mu}} - \gamma^{-1}$ so that $\tau^{-1}(\mu) = \ln(\mu + \gamma^{-1})$. We then set $\tilde{u}(\tilde{\mu}) = u(\tau(\tilde{\mu}))$, $\tilde{u}^{\mathcal{N}}(\tilde{\mu}) = u^{\mathcal{N}}(\tau(\tilde{\mu}))$, and $\tilde{u}_N^{\mathcal{N}}(\tilde{\mu}) = u_N^{\mathcal{N}}(\tau(\tilde{\mu}))$. We note that

$$\tilde{u}^{\mathcal{N}}(\tilde{\mu}) = \sum_{i=1}^{\mathcal{N}} \frac{f_i^{\mathcal{N}} \varphi_i^{\mathcal{N}}}{1 - \lambda_i^{\mathcal{N}}/\gamma + \lambda_i^{\mathcal{N}}\, \mathrm{e}^{\tilde{\mu}}} = \sum_{i=1}^{\mathcal{N}} f_i^{\mathcal{N}} \varphi_i^{\mathcal{N}} g\big(\tilde{\mu}, \lambda_i^{\mathcal{N}}\big), \tag{13}$$

from (6), our change of variable, and the definition of g.

We now observe that in our mapped coordinate, the sample points $\tilde{\alpha}_k \equiv \tau^{-1}(\alpha_k)$, $k = 1, \ldots, N$, are equi-distributed with separation $\tilde{\alpha}_{k+1} - \tilde{\alpha}_k \simeq \ln(\gamma\mu_{\max} + 1)/N$. It thus follows that, given any $\tilde{\mu} \in \widetilde{\mathcal{D}}$

we can construct a closed interval $\widetilde{I}^{\tilde{\mu}}_{\widetilde{\Delta}}$ of length $\widetilde{\Delta}$, that includes $\tilde{\mu}$ and $M^{\tilde{\mu}}(\widetilde{\Delta},\delta_N)$ distinct points $\tilde{\alpha}_{P^{\tilde{\mu}}_n}$, $n=1,\ldots,M^{\tilde{\mu}}(\widetilde{\Delta},\delta_N)$. Here $M^{\tilde{\mu}}(\widetilde{\Delta},\delta_N)$ is of the order of $\widetilde{\Delta}/\delta_N$; more precisely,

$$M^{\tilde{\mu}}(\widetilde{\Delta},\delta_N) \geqslant \frac{\widetilde{\Delta}}{c^*\delta_N}. \tag{14}$$

In what follows, we shall often abbreviate $M^{\tilde{\mu}}(\widetilde{\Delta},\delta_N)$ as M.

Now, for any $\tilde{\mu}\in\widetilde{\mathcal{D}}$, we introduce $\hat{u}^{\tilde{\mu}}\in W_N^{\mathcal{N}}$ given by

$$\hat{u}^{\tilde{\mu}} \equiv \sum_{n=1}^{M} \widetilde{Q}^{\tilde{\mu}}_n(\tilde{\mu})u^{\mathcal{N}}\big(\tau(\tilde{\alpha}_{P^{\tilde{\mu}}_n})\big) = \sum_{n=1}^{M} \widetilde{Q}^{\tilde{\mu}}_n(\tilde{\mu})\tilde{u}^{\mathcal{N}}(\tilde{\alpha}_{P^{\tilde{\mu}}_n}) = \sum_{n=1}^{M} \widetilde{Q}^{\tilde{\mu}}_n(\tilde{\mu})\sum_{i=1}^{\mathcal{N}} f_i^{\mathcal{N}}\varphi_i^{\mathcal{N}} g\big(\tilde{\alpha}_{P^{\tilde{\mu}}_n},\lambda_i^{\mathcal{N}}\big),$$

where the characteristic functions $\widetilde{Q}^{\tilde{\mu}}_n$ are uniquely determined by $\widetilde{Q}^{\tilde{\mu}}_n\in\mathbb{P}_{M-1}(\widetilde{I}^{\tilde{\mu}}_{\widetilde{\Delta}})$, $n=1,\ldots,M$, and $\widetilde{Q}^{\tilde{\mu}}_n(\tilde{\alpha}_{P^{\tilde{\mu}}_{n'}})=\delta_{nn'}$, $1\leqslant n,n'\leqslant M$; here $\mathbb{P}_{M-1}(\widetilde{I}^{\tilde{\mu}}_{\widetilde{\Delta}})$ refers to the space of polynomials of degree $\leqslant M-1$ over $\widetilde{I}^{\tilde{\mu}}_{\widetilde{\Delta}}$. We thus obtain

$$\hat{u}^{\tilde{\mu}} = \sum_{i=1}^{\mathcal{N}} f_i^{\mathcal{N}}\varphi_i^{\mathcal{N}}\big[\widetilde{\mathcal{I}}^{\tilde{\mu}}_{M-1} g\big(\cdot,\lambda_i^{\mathcal{N}}\big)\big](\tilde{\mu}), \tag{15}$$

where, for given λ, $\widetilde{\mathcal{I}}^{\tilde{\mu}}_{M-1}g(\cdot,\lambda)$ is the $(M-1)$-th order polynomial interpolant of $g(\cdot,\lambda)$ through the $\tilde{\alpha}_{P^{\tilde{\mu}}_n}$, $n=1,\ldots,M$; more precisely, $\widetilde{\mathcal{I}}^{\tilde{\mu}}_{M-1}g(\cdot,\lambda)\in\mathbb{P}_{M-1}(\widetilde{I}^{\tilde{\mu}}_{\widetilde{\Delta}})$, and $(\widetilde{\mathcal{I}}^{\tilde{\mu}}_{M-1}g(\cdot,\lambda))(\tilde{\alpha}_{P^{\tilde{\mu}}_n})=g(\tilde{\alpha}_{P^{\tilde{\mu}}_n},\lambda)$, $n=1,\ldots,M$. Note that $[\widetilde{\mathcal{I}}^{\tilde{\mu}}_{M-1}g(\cdot,\lambda)](\tau^{-1}(\mu))$ is *not* a polynomial in μ.

It now follows from (5), (6), (13) and (15) that

$$\begin{aligned} \big|\big|\big|\tilde{u}^{\mathcal{N}}(\tilde{\mu})-\hat{u}^{\tilde{\mu}}\big|\big|\big| &\leqslant \left|\left|\left|\sum_{i=1}^{\mathcal{N}} f_i^{\mathcal{N}}\varphi_i^{\mathcal{N}}\big(g\big(\tilde{\mu},\lambda_i^{\mathcal{N}}\big)-\big[\widetilde{\mathcal{I}}^{\tilde{\mu}}_{M-1}g\big(\cdot,\lambda_i^{\mathcal{N}}\big)\big](\tilde{\mu})\big)\right|\right|\right| \\ &\leqslant \sup_{\lambda\in\Lambda}\big|g(\tilde{\mu},\lambda)-\big[\widetilde{\mathcal{I}}^{\tilde{\mu}}_{M-1}g(\cdot,\lambda)\big](\tilde{\mu})\big|\,\big|\big|\big|u^{\mathcal{N}}(0)\big|\big|\big|. \end{aligned} \tag{16}$$

We next invoke the standard polynomial interpolation remainder formula [3] and Lemma 1 to obtain

$$\sup_{\lambda\in\Lambda}\big|g(\tilde{\mu},\lambda)-\big[\widetilde{\mathcal{I}}^{\tilde{\mu}}_{M-1}g(\cdot,\lambda)\big](\tilde{\mu})\big| \leqslant \sup_{\lambda\in\Lambda}\sup_{z\in Z}\frac{1}{M!}\big|D_1^M g(z,\lambda)\big|\widetilde{\Delta}^M \leqslant (2\widetilde{\Delta})^{M^{\tilde{\mu}}(\widetilde{\Delta},\delta_N)}. \tag{17}$$

We now assume that $\frac{c^*\delta_N}{2}\leqslant\widetilde{\Delta}$ and $\widetilde{\Delta}\leqslant\frac{1}{2}$; under these conditions (recall (14)) we obtain $(2\widetilde{\Delta})^{M^{\tilde{\mu}}(\widetilde{\Delta},\delta_N)}\leqslant(2\widetilde{\Delta})^{\widetilde{\Delta}/c^*\delta_N}$, and hence, from (16) and (17), we can write

$$\big|\big|\big|\tilde{u}^{\mathcal{N}}(\tilde{\mu})-\hat{u}^{\tilde{\mu}}\big|\big|\big| \leqslant \big|\big|\big|u^{\mathcal{N}}(0)\big|\big|\big|\big(2\widetilde{\Delta}\big)^{\widetilde{\Delta}/c^*\delta_N}. \tag{18}$$

It remains to select a best $\widetilde{\Delta}$ satisfying $c^*\delta_N/2\leqslant\widetilde{\Delta}\leqslant 1/2$.

To provide the sharpest possible bound, we choose $\widetilde{\Delta}=\widetilde{\Delta}^*\equiv\frac{1}{2\mathrm{e}}$, the minimizer (over all positive $\widetilde{\Delta}$) of $(2\widetilde{\Delta})^{\widetilde{\Delta}/c^*\delta_N}$. Our conditions on $\widetilde{\Delta}$ are readily verified: $c^*\delta_N/2\leqslant\widetilde{\Delta}^*$ follows directly from the hypothesis of our lemma, $N\geqslant N_{\mathrm{crit}}$; and $\widetilde{\Delta}^*\leqslant\frac{1}{2}$ follows from inspection. We now insert $\widetilde{\Delta}=\widetilde{\Delta}^*$ into (18) to obtain $|||\tilde{u}^{\mathcal{N}}(\tilde{\mu})-\hat{u}^{\tilde{\mu}}|||\leqslant|||u^{\mathcal{N}}(0)|||\,\mathrm{e}^{-N/2N_{\mathrm{crit}}}$, $\forall\tilde{\mu}\in\widetilde{\mathcal{D}}$.

It immediately follows that, for any $\mu\in\mathcal{D}$,

$$\begin{aligned} \inf_{w_N^{\mathcal{N}}\in W_N^{\mathcal{N}}}\big|\big|\big|u^{\mathcal{N}}(\mu)-w_N^{\mathcal{N}}\big|\big|\big| &= \inf_{w_N^{\mathcal{N}}\in W_N^{\mathcal{N}}}\big|\big|\big|\tilde{u}^{\mathcal{N}}\big(\tau^{-1}(\mu)\big)-w_N^{\mathcal{N}}\big|\big|\big| \leqslant \big|\big|\big|\tilde{u}^{\mathcal{N}}\big(\tau^{-1}(\mu)\big)-\hat{u}^{\tau^{-1}(\mu)}\big|\big|\big| \\ &\leqslant \big|\big|\big|u^{\mathcal{N}}(0)\big|\big|\big|\,\mathrm{e}^{-N/2N_{\mathrm{crit}}} \end{aligned}$$

since $\hat{u}^{\cdot}\in W_N^{\mathcal{N}}$ and, for $\mu\in\mathcal{D}$, $\tau^{-1}(\mu)\in\widetilde{\mathcal{D}}$. This concludes the proof. □

Then, from (10), (11) and Lemma 2, we obtain

THEOREM 3. – *For* $N \geqslant N_{\text{crit}} \equiv c^* \,\mathrm{e} \ln(\gamma \mu_{\max} + 1)$,

$$\left|\left|\left|u^{\mathcal{N}}(\mu) - u_N^{\mathcal{N}}(\mu)\right|\right|\right| \leqslant (1 + \mu_{\max}\gamma_1)^{1/2} \left|\left|\left|u^{\mathcal{N}}(0)\right|\right|\right| \mathrm{e}^{-N/2N_{\text{crit}}}, \quad \forall \mu \in \mathcal{D};$$

furthermore for $\mathcal{N}(\varepsilon)$ *such that* $|||u(\mu) - u^{\mathcal{N}(\varepsilon)}(\mu)||| \leqslant \varepsilon$,

$$\left|\left|\left|u(\mu) - u_N^{\mathcal{N}(\varepsilon)}(\mu)\right|\right|\right| \leqslant \varepsilon + (1 + \mu_{\max}\gamma_1) \left|\left|\left|u(0)\right|\right|\right| \mathrm{e}^{-N/2N_{\text{crit}}}, \quad \forall \mu \in \mathcal{D}.$$

Remark 4. – By letting ε go to zero, we also have

$$\left|\left|\left|u(\mu) - u_N(\mu)\right|\right|\right| \leqslant c \left|\left|\left|u(0)\right|\right|\right| \mathrm{e}^{-N/2N_{\text{crit}}}, \quad \forall \mu \in \mathcal{D}, \tag{19}$$

for any $N \geqslant N_{\text{crit}}$ with a constant c that depends only on γ_1 and $\mu_{\max}$.

Remark 5. – It must be pointed out that the analysis of the best fit in Lemma 2 involves a simple polynomial approximation of the solution, but this is a polynomial in the $\tilde{\mu}$ variable. The Galerkin approximation provides this best fit, up to a multiplicative constant, regardless of any *a priori* knowledge of the dependence of the solution on the parameter. This demonstrates the superiority of the reduced basis method with respect to a "simple" interpolation approximation $N \geqslant 2N_{\text{crit}}$.

[1] Note that γ_1, γ, and hence δ_N, are *independent* of $\mathcal{N}$.

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