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Équations aux dérivées partielles/*Partial Differential Equations*

On the nondegeneracy of the critical points of the Robin function in symmetric domains

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Abstract Let Ω be a smooth bounded domain of $\mathbb{R}^N$, $N \geqslant 2$, which is symmetric with respect to the origin. In this Note we prove that, under some geometrical condition on Ω (for example convexity in the directions $x_1, \ldots, x_N$), the Hessian matrix of the Robin function computed at zero is diagonal and strictly negative definite. ***To cite this article: M. Grossi, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 157–160.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Sur le non-dégénérescence des points critiques de la fonction de Robin dans les domaines symétriques

Résumé Soit Ω un domain borné et régulier de $\mathbb{R}^N$, $N \geqslant 2$, qui est symétrique par rapport à l'origine. Dans cette Note, nous montrons que, sous certaines hypothèses sur Ω (par exemple convexité dans les directions $x_1, \ldots, x_N$), la matrice hessienne calculée à zero est diagonale et strictement négative. ***Pour citer cet article : M. Grossi, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 157–160.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

1. Introduction

Let Ω be a smooth bounded domain of $\mathbb{R}^N$, $N \geqslant 2$. Let $G(x,y)$ be the Green function of the operator $-\Delta$ in $\mathrm{H}_0^1(\Omega)$. It is known that $G(x,y)$ can be splitted as follows

$$G(x,y) = \begin{cases} \frac{1}{N(2-N)\omega_N |x-y|^{N-2}} - H(x,y) & \text{if } N \geqslant 3, \\ \frac{1}{2\pi} \log|x-y| - H(x,y) & \text{if } N = 2, \end{cases} \tag{1.1}$$

where ω_N is the area of the unit ball in $\mathbb{R}^N$. The function $H(x,y)$ is the regular part of the Green function and it is not difficult to show that $H(x,y) \in \mathrm{C}^\infty(\Omega \times \Omega)$. The *Robin function* $R(x) : \Omega \mapsto \mathbb{R}$ is defined as follows:

$$R(x) = H(x,x). \tag{1.2}$$

This function plays an important role in various fields of the mathematics, e.g., geometric function theory, capacity theory, concentration problems (*see* [2] and the references therin). In particular, concerning

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problems involving critical Sobolev exponent (*see*, for example, [6–8,1,5]) it is important to extablish when the Hessian matrix of the Robin function computed at a critical point is nondegenerate. In this Note we study this problem when Ω is a symmetric domain and we obtain the following result:

THEOREM 1.1. – *Let Ω be a smooth bounded domain of $\mathbb{R}^N$, symmetric with respect to x_1 and satisfying the following geometric condition*:

$$\text{assume that } x_1\nu_1(x) \leqslant 0 \text{ for any } x \in \partial\Omega. \tag{1.3}$$

Then, for $\bar{y} \in \Omega \cap \{x_1 = 0\}$ we have

$$\frac{\partial R(\bar{y})}{\partial y_1} = 0 \tag{1.4}$$

and

$$\frac{\partial^2 R(\bar{y})}{\partial y_1 \partial y_i} = \begin{cases} 0 & \text{if } i \neq 1, \\ a < 0 & \text{if } i = 1. \end{cases} \tag{1.5}$$

From the previous theorem we immediately get

THEOREM 1.2. – *Let Ω be a smooth bounded domain of $\mathbb{R}^N$, symmetric with respect to $x_1, \dots, x_N$ and satisfying the condition $x_i\nu_i(x) \leqslant 0$ for any $x \in \partial\Omega$, $i = 1, \dots, N$. Then*

$$\nabla R(0) = 0 \tag{1.6}$$

and

$$\frac{\partial^2 R(0)}{\partial y_j \partial y_i} = \begin{cases} 0 & \text{if } i \neq j, \\ a_i < 0 & \text{if } i = j. \end{cases} \tag{1.7}$$

COROLLARY 1.3. – *Let Ω be a smooth bounded domain of $\mathbb{R}^N$, symmetric with respect to $x_1, \dots, x_N$ and convex with respect to x_i, for any $i = 1, \dots, N$. Then* (1.6) *and* (1.7) *hold.*

2. Proof of Theorem 1.1

Let us assume that Ω is a symmetric domain with respect to the plane $x_1 = 0$ and set $\Omega_0 = \Omega \cap \{x_1 = 0\}$.

LEMMA 2.1. – *For $\bar{y} \in \Omega_0$, $x = (x_1, x')$, $x_1 \in \mathbb{R}$, $x' \in \mathbb{R}^{N-1}$, we get*

$$G(x_1, x', \bar{y}) = G(-x_1, x', \bar{y}). \tag{2.1}$$

Proof. – By the definition of the Green function we have

$$\int_\Omega \nabla G(x, \bar{y}) \nabla\phi(x)\, \mathrm{d}x = \phi(\bar{y}). \tag{2.2}$$

Let us fix $\phi \in C_0^\infty(\Omega)$. Thus, since Ω is symmetric with respect to the plane $x_1 = 0$, we get by (2.2)

$$\int_\Omega \nabla G(-x_1, x', \bar{y}) \nabla\phi(x)\, \mathrm{d}x = \int_\Omega \nabla G(x, \bar{y}) \nabla\phi(-x_1, x')\, \mathrm{d}x = \phi(\bar{y}) \tag{2.3}$$

since $\bar{y}$ belongs to the plane $x_1 = 0$. Hence

$$\int_\Omega \big(\nabla G(-x_1, x', \bar{y}) - \nabla G(x, \bar{y})\big) \nabla\phi(x)\, \mathrm{d}x = 0 \quad \text{for any } \phi \in C_0^\infty(\Omega) \tag{2.4}$$

and this gives the claim. □

Fix $\bar{y} \in \Omega_0$ and let u_1 be the solution of the following problem

$$\begin{cases} \Delta u_1 = 0 & \text{in } \Omega, \\ u_1(x) = \frac{\partial G}{\partial x_1}(x, \bar{y}) & \text{on } \partial\Omega. \end{cases} \tag{2.5}$$

Denote by $\nu(x) = (\nu_1(x), \dots, \nu_N(x))$ the unit outward normal at a point $x \in \partial\Omega$.

We have the following lemma:

LEMMA 2.2. – *Let u_1 be a solution of* (2.5) *and let us assume that Ω verifies the condition* (1.3)*. Then* $\frac{\partial u_1}{\partial x_1}(\bar y) < 0$ *and* $\frac{\partial u_1}{\partial x_i}(\bar y) = 0$ *for any $i = 2, \dots, N$.*

Proof. – By Lemma 2.1 we get that $\frac{\partial G}{\partial x_1}(x, \bar y)$ is odd in x_1 and then also u_1 is odd in x_1. Hence $u_1 = 0$ on the plane $x_1 = 0$. Thus $\frac{\partial u_1}{\partial x_i}(\bar y) = 0$ for $i = 2, \dots, N$. To compute $\frac{\partial u_1}{\partial x_1}(\bar y)$ we remark that from $G(x, \bar y) = 0$ for $x \in \partial\Omega$ and $G(x, y) > 0$ for $x, y \in \Omega$, the assumption (1.3) implies that $\frac{\partial G}{\partial x_1}(x, \bar y) \geqslant 0$ for $\{x_1 < 0\} \cap \partial\Omega$ and $\frac{\partial G}{\partial x_1}(x, \bar y) \leqslant 0$ for $\{x_1 > 0\} \cap \partial\Omega$. Then, the maximum principle provides that $u_1(x) > 0$ for $x_1 < 0$; applying the Hopf lemma to u_1 in the domain $\Omega^- = \Omega \cap \{x_1 < 0\}$ we have that $\frac{\partial u_1}{\partial x_1}(\bar y) < 0$.

Now we recall a useful result on the Robin function:

LEMMA 2.3. – *We have that, for any $y \in \Omega$,*

$$\frac{\partial R(y)}{\partial y_i} = \int_{\partial\Omega} \nu_i(x) \left(\frac{\partial G(x, y)}{\partial \nu_x}\right)^2 \mathrm{d}S_x \tag{2.6}$$

and

$$\frac{\partial^2 R(y)}{\partial y_i \partial y_j} = 2 \int_{\partial\Omega} \frac{\partial G(y, x)}{\partial y_i} \frac{\partial}{\partial y_j} \left(\frac{\partial G(x, y)}{\partial \nu_x}\right) \mathrm{d}S_x. \tag{2.7}$$

Proof. – See [3,6] or [2] for the proof of (2.6). Differentiating (2.6) with respect to y_j we get

$$\frac{\partial^2 R(y)}{\partial y_i \partial y_j} = 2 \int_{\partial\Omega} \nu_i(x) \frac{\partial G(x, y)}{\partial \nu_x} \frac{\partial}{\partial y_j} \left(\frac{\partial G(x, y)}{\partial \nu_x}\right) \mathrm{d}S_x. \tag{2.8}$$

Since the Green function $G(x, y)$ is zero on the boundary $\partial\Omega$ and $G(x, y) = G(y, x)$ we have $\nu_i(x) \frac{\partial G(x,y)}{\partial \nu_x} = \frac{\partial G(x,y)}{\partial x_i} = \frac{\partial G(y,x)}{\partial y_i}$ and it proves the lemma. □

Now we can prove our main result

Proof of Theorem 1.1. – By the rapresentation formula for harmonic functions [4] we get

$$u_1(y) = \int_{\partial\Omega} \frac{\partial G}{\partial x_1}(x, \bar y) \frac{\partial G(x, y)}{\partial \nu_x} \mathrm{d}S_x. \tag{2.9}$$

Since u_1 is odd with respect to x_1 (see proof of Lemma 2.2), we have

$$0 = u_1(\bar y) = \int_{\partial\Omega} \nu_1(x) \left(\frac{\partial G(x, \bar y)}{\partial \nu_x}\right)^2 \mathrm{d}S_x \tag{2.10}$$

and then (1.4) follows by (2.6).

Differentiating (2.9) with respect to y_i and using that $G(x, y) = G(y, x)$ we deduce by (2.7)

$$\begin{aligned} \frac{\partial u_1}{\partial y_i}(\bar y) &= \int_{\partial\Omega} \frac{\partial G}{\partial x_1}(x, \bar y) \frac{\partial}{\partial y_i} \left(\frac{\partial G(x, \bar y)}{\partial \nu_x}\right) \mathrm{d}s_x \\ &= \int_{\partial\Omega} \frac{\partial G}{\partial y_1}(\bar y, x) \frac{\partial}{\partial y_i} \left(\frac{\partial G(x, \bar y)}{\partial \nu_x}\right) \mathrm{d}s_x = \frac{1}{2} \frac{\partial^2 R(\bar y)}{\partial y_1 \partial y_i}. \end{aligned} \tag{2.11}$$

From Lemma 2.2 and (2.7) we deduce (1.5). □

Remark 2.4. – By the proof of Theorem 1.1 and Lemma 2.2 we get that if we only assume that Ω is symmetric with respect to $x_1, \dots, x_N$ and $0 \in \Omega$ then the hessian matrix of the Robin function computed at zero is diagonal.

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