C. R. Acad. Sci. Paris, Ser. I 335 (2002) 17–22

Analyse mathématique/*Mathematical Analysis*

Log-Lipschitz regularity and uniqueness of the flow for a field in $(\mathrm{W}^{n/p+1,p}_{\mathrm{loc}}(\mathbb{R}^n))^n$

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Received 19 March 2002; accepted 28 April 2002

Note presented by Pierre-Louis Lions.

Abstract We consider the initial value problem $\dot{x} = b(x),\ t > 0;\ x(0) = x_0$, with $x = x(t) \in \mathbb{R}^n$. We prove that local existence and uniqueness of solutions holds when the field b belongs to $(\mathrm{W}^{n/p+1,p}_{\mathrm{loc}}(\mathbb{R}^n))^n$. This case corresponds to the limit regularity one in Sobolev terms since uniqueness may fail when $b \in (\mathrm{W}^{s,p}_{\mathrm{loc}}(\mathbb{R}^n))$ with $s < n/p + 1$ but holds immediately when $s > n/p + 1$ because of the Sobolev imbedding from $(\mathrm{W}^{s,p}_{\mathrm{loc}}(\mathbb{R}^n))^n$ into the space of locally Lipschitz fields. The proof of uniqueness relies on a Log-Lipschitz continuity property we prove for vector fields in this Sobolev class. When $p = 2$ the proof is carried out by means of Fourier series, decomposing the field into the low and high frequencies. When $p \neq 2$ the proof uses Trudinger's inequality and the strategy of proof of Morrey's theorem. ***To cite this article: E. Zuazua, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 17–22.***

Régularité Log-Lipschitz et unicité du flot pour les champs de vecteurs $(\mathrm{W}^{n/p+1,p}_{\mathrm{loc}}(\mathbb{R}^n))^n$

Résumé On considère le problème de Cauchy pour un système d'équations différentielles ordinaires $\dot{x} = b(x),\ t > 0;\ x(0) = x_0$ où l'état $x = x(t) \in \mathbb{R}^n$ et où b est un champ de vecteurs dans $(\mathrm{W}^{n/p+1,p}_{\mathrm{loc}}(\mathbb{R}^n))^n$. On démontre que, pour tout $x_0 \in \mathbb{R}^n$, il existe une unique solution locale (en temps). Ceci correspond à un cas limite du point de vue de l'appartenance à des espaces de Sobolev. En effet, si $s < n/p + 1$ il existe des champs de vecteurs $b \in (\mathrm{W}^{s,p}_{\mathrm{loc}}(\mathbb{R}^n))^n$ pour lesquels l'unicité n'est pas satisfaite. Par contre, lorsque $s > n/p+1$ l'unicité est trivialement vraie car b est localement Lipschitz grâce aux inclusions de Sobolev. La preuve consiste à démontrer que le champ de vitesses vérifie une condition de continuité de type Log-Lipschitz permettant de vérifier que la condition classique d'unicité d'Osgood est satisfaite. Lorsque $p = 2$ la preuve se fait à l'aide des séries de Fourier. Lorsque $p \neq 2$ on utilise l'inégalité de Trudinger et la stratégie de la preuve du théorème de Morrey. ***Pour citer cet article : E. Zuazua, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 17–22.***

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S1631-073X(02)02426-3/FLA

Version française abrégée

On considère le problème de Cauchy pour un système d'équations différentielles ordinaires

$$\dot{x}(t) = b\big(x(t)\big), \quad t > 0\,; \quad x(0) = x_0, \tag{1}$$

où l'état $x = x(t) \in \mathbb{R}^n$ et où b est un champ de vecteurs $b = (b_1, \ldots, b_n)$.

On démontre le résultat suivant :

THÉORÈME 1. – *Si*

$$b \in \big(\mathrm{W}^{n/p+1,p}_{\mathrm{loc}}\big(\mathbb{R}^n\big)\big)^n, \tag{2}$$

avec $p \geqslant 1$*, alors, pour tout* $x_0 \in \mathbb{R}^n$ *il existe une unique solution locale de* (1).

Ce résultat est en fait une conséquence du théorème suivant :

THÉORÈME 2. – *Si* b *vérifie* (2) *avec* $p > 1$*, alors, pour tout compact* K *de* $\mathbb{R}^n$*, il existe une constante* $C = C(K) > 0$ *telle que*

$$\big|b(x) - b(y)\big| \leqslant C(K)|x-y|\big|\log|x-y|\big|^{(p-1)/p}, \quad \forall x, y \in K. \tag{3}$$

Lorsque $p = 1$ *et* $p = \infty$ *le champ est localement Lipschitz.*

Remarques. –

(a) Le Théorème 1 est optimal. En effet, par exemple, si $p = 2$, quelque soit $s < 1 + n/2$, on peut trouver un champ de vecteurs $b \in (\mathrm{H}^s_{\mathrm{loc}}(\mathbb{R})^n)^n$ tel que l'unicité ne soit pas satisfaite.

(b) Lorsque $s > n/p + 1$, le résultat d'unicité du Théorème 1 est trivialement vrai car, grâce aux inclusions de Sobolev, tout champ de vecteurs de $(\mathrm{W}^{s,p}_{\mathrm{loc}}(\mathbb{R}^n))^n$ est localement Lipschitz.

(c) L'inégalité (3) garantit que le champ de vecteurs b satisfait la condition d'Osgood (*voir* [2] pour le cas des équations différentielles dans un space de Banach). On en déduit donc immédiatement le résultat d'unicité du Théorème 1.

(d) Lorsque $p = 2$ la preuve du Théorème 2 est inspirée du travail de Sedenko [10] sur l'unicité des solutions des plaques de von Kármán. Il consiste à développer b en séries de Fourier et le décomposer en basses et hautes fréquences.
Lorsque $p \neq 2$ la preuve utilise l'inégalité de Trudinger et l'argument de Morrey pour démontrer la régularité C^α des fonctions dans $\mathrm{W}^{1,q}$ avec $q > n$.

(e) Lorsque $p = 1$ le champ est Lipschitz à cause de l'inclusion de Sobolev de $\mathrm{W}^{n,1}$ dans l'espace de fonctions continues et bornées (Th. 5.4 de [1]). Lorsque $p = \infty$ c'est conséquence du bien connu caractère Lipschitz des fonctiones de $\mathrm{W}^{1,\infty}$.

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1. Introduction and main result

Let us consider the following initial value problem for a system of ordinary differential equations:

$$\dot{x}(t) = b\big(x(t)\big), \quad t > 0; \quad x(0) = x_0. \tag{1}$$

Here the state $x = x(t) \in \mathbb{R}^n$, $n \geqslant 1$, is a vector with components $x_j = x_j(t)$, and $b = (b_1, \ldots, b_n)$ is a vector field.

The two most classical and widely used results on the well-posedness of (1) are as follows:

- *Cauchy–Lipschitz*: If b is locally Lipschitz continuous, (1) has a unique local (in time) solution for any initial value $x_0 \in \mathbb{R}^n$.
- *Peano*: If b is continuous, for every $x_0 \in \mathbb{R}^n$ there is at least a local solution of (1).

It is also well known that the Lipschitz continuity condition on b may be weakened and still the uniqueness property to hold. This fact is expressed for instance by the classical Osgood condition for uniqueness that requests that the integral of $1/\omega$ diverges at the origin, ω being the modulus of continuity of b. In this way uniqueness is guaranteed, for instance, for Log-Lipschitz fields b, not only in finite dimensions but in general Banach spaces [2]. There are also clear limits when relaxing the Lipschitz condition since, for instance, it is well known that uniqueness may fail for Hölder continuous vector fields. Indeed, for instance, uniqueness fails when $n = 1$, $x_0 = 0$ and $b(x) = x^\alpha$ for any $\alpha < 1$.

Recently, mainly motivated by the work by DiPerna and Lions [5] and its potential applications in transport equations, this subject has attracted much attention. In [5], roughly speaking, it was proved that under the assumption that $b \in (\mathrm{W}^{1,1})^n$ and $\operatorname{div} b \in \mathrm{L}^\infty$, the flow is well defined. This point of view has been pursued by Lions in [9] in the case of incompressible fields ($\operatorname{div} b = 0$) and has been further analyzed by Colombini and Lerner [3] where the $\mathrm{W}^{1,1}$ condition in [5] has been weakened to the hypothesis that b is BV, together with $\operatorname{div} b \in \mathrm{L}^\infty$.

In this Note we go back to the classical problem of giving sufficient conditions on the field b guaranteeing that (1) has a unique local solution for all $x_0 \in \mathbb{R}^n$.

We consider fields b in Sobolev spaces. More precisely, we assume that

$$b \in \left(\mathrm{W}^{s,p}_{\mathrm{loc}}\left(\mathbb{R}^n\right)\right)^n. \tag{2}$$

Three cases have to be distinguished. Let us explain this in the particular case $p = 2$:

Case 1: $s > n/2 + 1$. In this case, thanks to Sobolev imbeddings, b is Lipschitz and (1) is well-posed.

Case 2: $s < n/2 + 1$. In this case, uniqueness may fail. Indeed, consider a field b with a single nontrivial component of the form $b(x) = (|x|^\delta, 0)$ with $\delta < 1$. Then, uniqueness fails but $b \in \mathrm{H}^{s(\delta)+n/2}_{\mathrm{loc}}(\mathbb{R}^2)$ with $s(\delta) \nearrow 1$ as $\delta \nearrow 1$.

Case 3: The limit case $s = n/2 + 1$. In this case, according to Sobolev imbeddings, b belongs to $\mathrm{W}^{1,p}_{\mathrm{loc}}$ for all $p < \infty$ but may fail to be locally Lipschitz. This can be seen easily on the field $b(x) = x|\log|x||^\alpha$, with $\alpha < 1/2$ that, in space dimension $n = 2$ is in $\mathrm{H}^2_{\mathrm{loc}}(\mathbb{R}^2)$ but it is not locally Lipschitz. However, it is easy to see that the Osgood condition is satisfied in this case and, therefore, that system (1) is well posed.

Our main result shows that this is systematically the case for fields in $(\mathrm{W}^{n/p+1,p}_{\mathrm{loc}}(\mathbb{R}^n))^n$:

THEOREM 1 (Uniqueness of the flow). – *Assume that*

$$b \in \left(\mathrm{W}^{n/p+1,p}_{\mathrm{loc}}\left(\mathbb{R}^n\right)\right)^n, \tag{3}$$

with $p \geqslant 1$. *Then,* (1) *is well-posed. In other words, for any* $x_0 \in \mathbb{R}^n$ *there exists* $T > 0$ *and a unique solution* $x \in \mathrm{C}^1([0,T);\mathbb{R}^n)$ *of* (1).

Remarks. –

(a) Note that, under assumption (3), b is continuous. Then by Peano's theorem the existence of a solution of (1) is guaranteed. Therefore, Theorem 1 says simply that this solution is unique.

(b) The same method applies and yields the same result in the non-autonomous case where $b = b(t,x)$ provided $b \in \mathrm{L}^1(0,T;(\mathrm{W}^{n/p+1,p}_{\mathrm{loc}}(\mathbb{R}^n))^n)$.

(c) Condition (3) may be relaxed. We will explain this once the proof of Theorem 1 is complete.

Theorem 1 is in fact a direct consequence of the following one, which provides an upper bound on the modulus of continuity of the fields in the class (3).

THEOREM 2 (Log-Lipschitz regularity). – *Assume that* (3) *holds, with* $p > 1$. *Then, for every compact set* K *of* $\mathbb{R}^n$ *there exists a constant* $C = C(K) > 0$ *such that*

$$\left|b(x) - b(y)\right| \leqslant C(K)|x-y|\left|\log|x-y|\right|^{(p-1)/p}, \quad \forall x, y \in K. \tag{4}$$

When $p = 1, \infty$ the field b is locally Lipschitz.

Remarks. –

(a) The locally Lipschitz regularity of the field b may not be achieved since $W^{n/p+1,p}_{loc}(\mathbb{R}^n)$, is not imbedded in $W^{1,\infty}_{loc}(\mathbb{R}^n)$ unless $p = 1, \infty$ (*see* Theorem 5.4 of [1]). Inequality (4) expresses the fact that the field b is "almost" locally Lipschitz.

(b) When $p = 2$ the proof of Theorem 2 relies on an idea used by Sedenko in [10] to prove the uniqueness of weak solutions for the dynamical von Kármán system of elastic plates. It consists in developping b in Fourier series and decomposing it into the low and the high frequencies.

(c) In view of (4), the classical Osgood condition for uniqueness is satisfied by fields in the class (3) and Theorem 1 holds automatically.

(d) In view of Theorem 2, for instance, when $p = 2$, one could think that for any function $b \in H^{n/2}_{loc}(\mathbb{R}^n)$, it follows $|b(x) - b(y)| \leqslant C(K)|\log|x - y||^{1/2}$, for all $x, y \in K$. However, this is not true since one may build functions in $H^{n/2}_{loc}(\mathbb{R}^n)$ with logarithmic singularities in a dense set of points ([11], p. 160).

2. Proof of Theorem 2: The case $p = 2$

Without loss of generality we may assume that b is compactly supported in $[0, 2\pi)^n$. We develop b in Fourier series

$$b(x) = \sum_{k\in\mathbb{Z}^n} b^k e^{ik\cdot x}, \tag{5}$$

where $(b^k)_{k\in\mathbb{Z}^n}$ are the Fourier coefficients. Following [10] we set $b = b_N + \varepsilon_N$ with

$$b_N = \sum_{|k|\leqslant N} b^k e^{ik\cdot x}; \qquad \varepsilon_N = \sum_{|k|>N} b^k e^{ik\cdot x}. \tag{6}$$

It is then easy to see that

$$\begin{aligned}\|b_N\|_{(W^{1,\infty})^N} &\leqslant \sum_{|k|\leqslant N} |b^k|\big(1+|k|\big) \leqslant \Big[\sum_{|k|\leqslant N} |b^k|^2\big(1+|k|\big)^{n+2}\Big]^{1/2}\Big[\sum_{|k|\leqslant N}\big(1+|k|\big)^{-n}\Big]^{1/2}\\ &\leqslant C(\log N)^{1/2}\|b\|_{H^{n/2+1}} \quad \text{as } N\to\infty.\end{aligned} \tag{7}$$

Moreover

$$\begin{aligned}\|\varepsilon_N\|_{(L^\infty)^n} &\leqslant \sum_{|k|\geqslant N} |b^k| \leqslant \Big[\sum_{|k|\geqslant N} |b^k|^2\big(1+|k|\big)^{n+2}\Big]^{1/2}\Big[\sum_{|k|\geqslant N}\big(1+|k|\big)^{-n-2}\Big]^{1/2}\\ &\leqslant \frac{C}{N}\|b\|_{H^{n/2+1}} \quad \text{as } N\to\infty.\end{aligned} \tag{8}$$

Given x and y we have

$$\big|b(x) - b(y)\big| \leqslant \big|b_N(x) - b_N(y)\big| + \big|\varepsilon_N(x) - \varepsilon_N(y)\big|. \tag{9}$$

On the other hand, according to (7), (8),

$$\big|b_N(x) - b_N(y)\big| \leqslant C\|b\|_{H^{n/2+1}}(\log N)^{1/2}|x - y| \tag{10}$$

and

$$\big|\varepsilon_N(x) - \varepsilon_N(y)\big| \leqslant 2\|\varepsilon_N\|_{L^\infty} \leqslant \frac{C\|b\|_{H^{n/2+1}}}{N}. \tag{11}$$

Combining (9)–(11) we get

$$\big|b(x) - b(y)\big| \leqslant C\|b\|_{H^{n/2+1}}\Big[(\log N)^{1/2}|x - y| + \frac{1}{N}\Big]. \tag{12}$$

We now choose N so that both terms in the right-hand side of (12) are of the same order. Thus $\log^{1/2} N \sim 1/[N|x-y|]$ or, equivalently,

$$N \sim \frac{1}{|x-y||\log|x-y||^{1/2}}. \tag{13}$$

Combining (12), (13) we easily deduce (4).

Remark. – When the Fourier coefficients (b^k) of the field b (localized in space) satisfy the condition

$$\sum_{k\in\mathbb{Z}^n} \frac{|b^k|^2(1+|k|)^{n+2}}{\log(1+|k|)} < \infty, \tag{14}$$

which is a weaker condition than $b \in \mathrm{H}^{n/2+1}(\mathbb{R}^n) \Leftrightarrow \sum_{k\in\mathbb{Z}^n} |b^k|^2(1+|k|)^{n+2} < \infty$, the field b satisfies

$$\bigl|b(x)-b(y)\bigr| \leqslant C(K)|x-y|\log|x-y|, \quad \forall x,y\in K. \tag{15}$$

The proof of (15) is the same as that in Theorem 2. Inequality (15) suffices to check that Osgood's condition is satisfied. Thus, uniqueness of solutions of (1) holds under the weaker condition (14) on the field b.

3. Proof of Theorem 2: The general case

We now consider the case $p \neq 2$.

First of all we note that, when $p < 2$, $\mathrm{W}^{n/p+1,p}$ is continuously imbedded into $\mathrm{H}^{n/2+1}$. Then, (4) holds with $p = 2$. But this is a weaker result than the one we state in (3). Thus, it is convenient to address directly each case.

Our proof combines Trudinger's inequality, and the strategy of proof of Morrey's result on the Hölder continuous regularity of functions in $\mathrm{W}^{1,q}$ with $q > n$. Some of the ingredients of this proof have been already used in [4] when analyzing the well-posedness of transport equations.

To simplify the presentation we consider the case $p = n$, $n \geqslant 3$. Then, $b \in (\mathrm{W}^{2,n}_{\mathrm{loc}}(\mathbb{R}^n))^n$. Without loss of generality, we can assume that b is of compact support. We then have $\nabla b \in (\mathrm{W}^{1,n}(\mathbb{R}^n))^n$, and by Trudinger's inequality (*see* [8], p. 162), there exist constants $C_1, C_2 > 0$ such that

$$\int_\Omega \exp\bigl[\bigl|\nabla b(x)\bigr|/C_1||b||_{\mathrm{W}^{2,n}}\bigr]^{n/(n-1)}\,\mathrm{d}x \leqslant C_2|\Omega|, \tag{16}$$

for any domain Ω. We now argue as in the proof of Morrey's theorem guaranteeing the Hölder continuity of functions in $\mathrm{W}^{1,p}$ with $p > n$ (see, for instance, Evans [6], pp. 266–268). Given two points x, y we set $r = |x-y|$ and W the intersection of the balls of centers in x and y and radius r. Then,

$$\bigl|b(x)-b(y)\bigr| \leqslant \frac{1}{|W|}\left[\int_W \bigl|b(x)-b(z)\bigr|\,\mathrm{d}z + \int_W \bigl|b(y)-b(z)\bigr|\,\mathrm{d}z\right]. \tag{17}$$

It is then sufficient to get upper bounds in these integrals. We estimate the first one, the second one being completely similar. Note that

$$\frac{1}{|W|}\int_W \bigl|b(x)-b(z)\bigr|\,\mathrm{d}z \leqslant C\frac{1}{|B(x,r)|}\int_{B(x,r)} \bigl|b(x)-b(z)\bigr|\,\mathrm{d}z. \tag{18}$$

On the other hand, according to (21) in p. 266 in [6], we have

$$\frac{1}{|B(x,r)|}\int_{B(x,r)} \bigl|b(x)-b(z)\bigr|\,\mathrm{d}z \leqslant C\int_{B(x,r)} \frac{|\nabla b(y)|}{|x-y|^{n-1}}\,\mathrm{d}y = CI. \tag{19}$$

We now apply Young's inequality with $f(t) = \|b\|_{\mathrm{W}^{2,n}} \exp[t/C_1\|b\|_{\mathrm{W}^{2,n}}]^{n/(n-1)}$ and f^* its convex conjugate. We have,

$$I \leqslant \|b\|_{\mathrm{W}^{2,n}} \int_{B(x,r)} \exp\bigl[\bigl|\nabla b(y)\bigr|/C_1\|b\|_{\mathrm{W}^{2,n}}\bigr]^{n/(n-1)}\,\mathrm{d}y + \int_{B(x,r)} f^*\bigl(|x-y|^{-(n-1)}\bigr)\,\mathrm{d}y. \tag{20}$$

According to (16), we also have

$$\|b\|_{W^{2,n}} \int_{B(x,r)} \exp\left[\left|\nabla b(y)\right| / C_1 \|b\|_{W^{2,n}}\right]^{n/(n-1)} \mathrm{d}y \leqslant C \|b\|_{W^{2,n}} r^n. \tag{21}$$

We now observe that $f^*(t) \sim C\|b\|_{W^{2,n}} t[\log[C((n-1)/n)t]]^{(n-1)/n}$ as $t \to \infty$. This allows to show that

$$\int_{B(x,r)} f^*\left(|x-y|^{-(n-1)}\right) \mathrm{d}y \sim C\|b\|_{W^{2,n}} r \left[\log\left[C(n-1)/n\, r^{n-1}\right]\right]^{(n-1)/n} \quad \text{as } r \to 0. \tag{22}$$

Combining (18)–(22) we easily deduce the statement in Theorem 2, when $p = n$.

When $p \neq n$, the proof is similar since Trudinger's inequality (see Theorem 8.25, [1]) guarantees that

$$\int_{\Omega} \exp\left[\left|\nabla b(x)\right| / C_1 \|b\|_{W^{n/p+1,p}}\right]^{p/(p-1)} \mathrm{d}x \leqslant C_2|\Omega|, \tag{23}$$

and allows to estimate the integral in (19).

Remark. – The hypotheses may be weakened. In fact, we only need that the integral involved in Trudinger's inequality to be finite and the same result holds when $b \in W^{1,1}_{\mathrm{loc}}$ is such that

$$\int_{\Omega} \exp\left[\left|\nabla b(x)\right| / C_1\right]^{n/(n-1)} \mathrm{d}x \leqslant C_2|\Omega|, \tag{24}$$

holds with finite constants $C_1,\ C_2$ for any open set Ω.

Similar results can also be proved under the weaker condition introduced in [7]. Namely,

$$\int_{\Omega} \left|D^2 u\right|^n \log^{-\sigma}\left(\mathrm{e} + \left|D^2 u\right|\right) \mathrm{d}x < \infty$$

for all bounded domain Ω. This is indeed weaker that the condition $W^{2,n}$ we impose in the case $p = n$. In that case, according to [7], an inequality like (23) holds but with the exponent $n/(n-1+\sigma)$. Thus the Log-Lipschitz property holds provided $\sigma \leqslant 1$.

Acknowledgements. The author thanks J.M. Arrieta for fruitful comments. The author has been partially supported by grant PB96-0663 of the DGES (MCYT, Spain) and the TMR Network of the EU "Homogenization and Multiple Scales (HMS2000)".

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