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Équations aux dérivées partielles/*Partial Differential Equations*

Global Carleman estimates for weak solutions of elliptic nonhomogeneous Dirichlet problems

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Abstract We consider a general second order elliptic equation with right-hand side $f+\sum_{j=0}^{N}\frac{\partial f_j}{\partial x_j}\in \mathrm{H}^{-1}(\Omega)$ where $f, f_j\in \mathrm{L}^2(\Omega)$ and Dirichlet boundary condition $g\in \mathrm{H}^{1/2}(\Gamma)$. We prove a global Carleman estimate for the solution y of this equation in terms of the weighted L^2 norms of f and f_j and the $\mathrm{H}^{1/2}$ norm of g. This estimate depends on two real parameters s and λ which are supposed to be large enough and is sharp with respect to the exponents of these parameters. This allows us to obtain, for example, sharper estimates on the pressure term in the linearized Navier–Stokes equations and it turns out to be very useful in the context of controllability problems. ***To cite this article: O.Y. Imanuvilov, J.-P. Puel, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 33–38.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Estimations de Carleman globales pour des solutions faibles de problèmes elliptiques avec condition de Dirichlet non homogène

Résumé On considère une équation elliptique du second ordre générale avec second membre $f+\sum_{j=0}^{N}\frac{\partial f_j}{\partial x_j}\in \mathrm{H}^{-1}(\Omega)$, $f, f_j\in \mathrm{L}^2(\Omega)$ et condition de Dirichlet $g\in \mathrm{H}^{1/2}(\Gamma)$. On montre une estimation de Carleman globale pour la solution y de cette équation en termes de normes L^2 à poids de f et f_j et de la norme $\mathrm{H}^{1/2}$ de g. Cette estimation dépend de deux paramètres réels s et λ qui sont supposés assez grands et est optimale en ce qui concerne les exposants de ces paramètres. Ceci nous permet d'obtenir, par exemple, des estimations fines sur la pression dans les équations de Navier–Stokes linéarisées et se révèle fort utile dans l'étude des problèmes de contrôlabilité. ***Pour citer cet article : O.Y. Imanuvilov, J.-P. Puel, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 33–38.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Version française abrégée

Soit Ω un ouvert borné de classe C^2 de $\mathbb{R}^{N+1}$ de frontière Γ et soit y solution de l'équation elliptique

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$$Py=\sum_{i,j=0}^{N}a_{ij}(x)\frac{\partial^2 y}{\partial x_i\partial x_j}+\sum_{j=0}^{N}b_j(x)\frac{\partial y}{\partial x_j}+\sum_{i=0}^{N}\frac{\partial}{\partial x_i}\big(c_i(x)y\big)+d(x)y=f+\sum_{j=0}^{N}\frac{\partial f_j}{\partial x_j}\quad \text{dans }\Omega, \tag{1}$$

$$y=g\quad \text{sur }\Gamma, \tag{2}$$

où $a_{ij}\in C^2(\overline{\Omega})$, $b_j,c_i,d\in L^\infty(\Omega)$ pour $i,j=0,\dots,N$ et où les a_{ij} vérifient

$$\exists\beta>0,\ \forall\xi\in\mathbb{R}^{N+1},\ \forall x\in\Omega,\quad \sum_{i,j=0}^{N}a_{ij}(x)\xi_i\xi_j\geqslant\beta|\xi|^2. \tag{3}$$

On suppose que $f\in L^2(\Omega)$, $f_j\in L^2(\Omega)$, $j=0,\dots,N$, $g\in H^{1/2}(\Gamma)$. Si $\psi\in C^2(\overline{\Omega})$ est tel que $\psi=0$ on Γ, $\psi(x)>0$ pour tout $x\in\Omega$, $|\nabla\psi(x)|>0$ pour tout $x\in\overline{\Omega-\omega}$ (de telles fonctions existent) nous posons $\varphi=e^{\lambda\psi(x)}$, $\lambda\in\mathbb{R}$, et nous montrons le résultat suivant.

THÉORÈME 0.1. – *Nous supposons vérifiées les hypothèses ci-dessus. Soit $y\in H^1(\Omega)$ une solution de l'équation elliptique* (1), (2)*. Alors il existe une constante $C>0$ indépendante de s et de λ et des paramètres $\hat\lambda>1$ et $\hat s>1$ tels que pour tout $\lambda\geqslant\hat\lambda$, pour tout $s\geqslant\hat s$,*

$$\begin{aligned}&\int_\Omega|\nabla y|^2e^{2s\varphi}\,dx+s^2\lambda^2\int_\Omega\varphi^2|y|^2e^{2s\varphi}\,dx\\&\quad\leqslant C\bigg(s^{1/2}e^{2s}\|g\|^2_{H^{1/2}(\Gamma)}+\frac{1}{s\lambda^2}\int_\Omega\frac{|f|^2}{\varphi}e^{2s\varphi}\,dx+\sum_{j=0}^{N}s\int_\Omega|f_j|^2\varphi e^{2s\varphi}\,dx\\&\qquad+\int_\omega\big(|\nabla y|^2+s^2\lambda^2\varphi^2|y|^2\big)e^{2s\varphi}\,dx\bigg).\end{aligned}\tag{4}$$

Les principales difficultés de la démonstration apparaissent dans l'estimation au voisinage du bord, après localisation. On factorise alors l'opérateur en deux opérateurs du premier ordre, après un changement convenable de coordonnées. L'inégalité cherchée résulte alors d'une succession d'estimations qui utilisent d'une part des estimations d'énergie et des méthodes typiques aux inégalités de Carleman, d'autre part des arguments de dualité et l'utilisation de problèmes adéquats de minimisation de fonctionnelles.

Local Carleman inequalities for second order elliptic operators (*cf.* [3]) give weighted L^2 estimates of H^2 functions with compact support in terms of the corresponding norm of the elliptic operator evaluated on this function. A stronger version (for weaker norms) but still local for functions with compact support has been obtained by [1] giving estimates of the L^2 norm in terms of the H^{-1} norm of the operator evaluated on the function. On the other hand, global Carleman inequalities have been given by [2], then by [4] and [5], in the context of controllability problems, for solutions of elliptic (and parabolic) equations when the right-hand side is element of L^2. In order to find stability inequalities for inverse problems, a global Carleman inequality is proved in [7] for solutions of parabolic equations with right-hand sides in H^{-1} but with homogeneous Dirichlet boundary value. In the present work we give a global Carleman inequality for weak solutions of elliptic equations with right-hand sides in H^{-1} and nonhomogeneous Dirichlet boundary value in $H^{1/2}$. The inequalities we obtain turn out to be very useful in order to obtain sharper estimates on the pressure in linearized Navier–Stokes equations and they allow us to consider lower regularity on the velocity. The detailed proofs will be given in [6].

1. Precise statement of the result

Let Ω be a bounded open set of $\mathbb{R}^{N+1}$ with boundary Γ of class C^2. We consider a solution $y\in H^1(\Omega)$ of the general second order elliptic equation (1), (2) where $a_{ij}\in C^2(\overline{\Omega})$, $b_j,c_i,d\in L^\infty(\Omega)$ for $i,j=0,\dots,N$

and the coefficients a_{ij} satisfy the standard ellipticity condition (3). We also assume that $f \in \mathrm{L}^2(\Omega)$, $f_j \in \mathrm{L}^2(\Omega)$, for $j = 0, \dots, N$, $g \in \mathrm{H}^{1/2}(\Gamma)$. Our goal is to obtain a sharp global Carleman inequality for solutions of (1), (2). In order to formulate our main result, we first have to introduce a suitable weight function.

LEMMA 1.1. – *Let ω be an arbitrary nonempty open set such that $\omega \subset \Omega$. Then there exists a function $\psi \in \mathrm{C}^2(\overline{\Omega})$ such that*

$$\psi = 0 \quad \text{on } \Gamma, \quad \psi(x) > 0, \quad \forall x \in \Omega, \quad \left|\nabla\psi(x)\right| > 0, \quad \forall x \in \overline{\Omega - \omega}. \tag{5}$$

For the proof, see [2], Lemma 1.1.

Now, using this function ψ, we can consider a weight function $\varphi(x) = \mathrm{e}^{\lambda\psi(x)}$ where $\lambda \in \mathbb{R}$, $\lambda \geqslant 1$ will be chosen later on large enough. Our main result is the following.

THEOREM 1.2. – *Let us assume the above hypotheses and let $y \in \mathrm{H}^1(\Omega)$ be a solution of* (1), (2). *Then there exists a constant $C > 0$ independent of s and λ and parameters $\hat\lambda > 1$ and $\hat s > 1$ such that $\forall\lambda \geqslant \hat\lambda$ $\forall s \geqslant \hat s$,*

$$\begin{aligned}&\int_\Omega |\nabla y|^2 \mathrm{e}^{2s\varphi}\,\mathrm{d}x + s^2\lambda^2\int_\Omega \varphi^2|y|^2\mathrm{e}^{2s\varphi}\,\mathrm{d}x\\ &\quad\leqslant C\Bigg(s^{1/2}\mathrm{e}^{2s}\|g\|^2_{\mathrm{H}^{1/2}(\Gamma)} + \frac{1}{s\lambda^2}\int_\Omega \frac{|f|^2}{\varphi}\mathrm{e}^{2s\varphi}\,\mathrm{d}x + \sum_{j=0}^N s\int_\Omega |f_j|^2\varphi\,\mathrm{e}^{2s\varphi}\,\mathrm{d}x\\ &\qquad + \int_\omega \big(|\nabla y|^2 + s^2\lambda^2\varphi^2|y|^2\big)\mathrm{e}^{2s\varphi}\,\mathrm{d}x\Bigg).\end{aligned} \tag{6}$$

2. Ideas of proof of Theorem 1.2

By a standard localization procedure we are lead to prove the estimate in two different cases:

(i) Supp $y \subset \Omega_0$,

(ii) Supp $y \subset B_\delta(\hat x)$, $\hat x \in \Gamma$, with δ small.

Case (i) does not involve boundary terms and corresponds to functions with compact support in Ω. It is already treated in [1] or in [7]. Only the constants and their dependence on the parameters have to be checked carefully.

Concerning case (ii), for δ small enough we have $B_\delta(\hat x) \cap \omega = \emptyset$ and therefore, for some index i_0,

$$\frac{\partial\psi}{\partial x_{i_0}}(x) \neq 0, \quad \forall x \in B_\delta(\hat x).$$

After renumbering we can take a new system of coordinates

$$\hat x_0 = \psi(x), \quad \hat x_i = x_i, \quad i = 1, \dots, N. \tag{7}$$

If we omit from now on the notation ˆ and normalize $\hat\delta$ to 1 for simplicity, we then obtain

$$Py = \frac{\partial^2 y}{\partial x_0^2} + \sum_{j=1}^N a_{0j}\frac{\partial^2 y}{\partial x_0\partial x_j} + Ay + By = f + \sum_{j=0}^N \frac{\partial f_j}{\partial x_j} \quad \text{in }]0,1[\times\mathbb{R}^N, \tag{8}$$

$$y(0,x') = g(x') \quad \text{on } \{0\}\times\mathbb{R}^N \tag{9}$$

and

$$y \text{ vanishes in the neigborhood of the set } [0,1]\times\partial B'_\delta \cup \big(\{1\}\times B'_\delta\big), \tag{10}$$

where $Ay = \sum_{i,j=1}^N a_{ij}(x)\frac{\partial^2 y}{\partial x_i\partial x_j}$ and where B is a first order operator which can be neglected. Writing $a(\xi^1,\xi^2) = a(x,\xi^1,\xi^2) = \sum_{i,j=1}^N a_{ij}(x)\xi_i^1\xi_j^2$ from the ellipticity condition, there exists $\beta > 0$, such that for all $\tau \in \mathbb{R}$, for all $\xi \in \mathbb{R}^N$,

$$\tau^2 + \sum_{j=1}^{N} a_{0j}\tau\xi_j + a(\xi,\xi) \geqslant \beta\left(\tau^2 + |\xi|^2\right), \quad \forall (x_0, x') \in [0,1] \times B'_\delta. \tag{11}$$

We can extend the coefficients of operator P up to periodic functions of the x' variable on the cube $K' = \prod_{j=1}^{N}[-\delta', \delta']$, $\delta' > \delta$, keeping the ellipticity property (11). We also extend f, f_j and g on K' and therefore we consider Eq. (8) as set on $]0,1[\times M$ where M is the compact smooth torus $\Pi_{\delta'}^N$. We now factorize operator P using regularized roots of its symbol

$$r_\pm(x,\xi) = \frac{-\mathrm{i}\sum_{j=1}^{N} a_{0j}(x)\xi_j \pm \mu(|\xi|)\sqrt{4a(x,\xi,\xi) - (\sum_{j=1}^{N} a_{0j}(x)\xi_j)^2}}{2}, \tag{12}$$

where $\mu \in \mathrm{C}^\infty(\mathbb{R}^+)$ is such that

$$\mu(t) = 0 \quad \text{for } t \in \left[0, \frac{1}{2}\right], \quad \mu(t) = 1 \quad \text{for } t > 1, \quad 0 \leqslant \mu(t) \leqslant 1, \quad \forall t \in \mathbb{R}^+,$$

and we obtain

$$Py = \left(\frac{\partial}{\partial x_0} - R_+(x,D)\right)\left(\frac{\partial}{\partial x_0} - R_-(x,D)\right)y + Ky, \tag{13}$$

where, $R_\pm$ are first order operators with principal symbol $r_\pm$ and, from [9], Proposition 4.2.A, $K \in \mathrm{C}([0,1]; \mathcal{L}(\mathrm{H}^1(M), \mathrm{L}^2(M)))$. Since in the new variables the weight is $\varphi = \mathrm{e}^{\lambda x_0}$, we can again neglect the first order operator K. Considering the new function $w = \mathrm{e}^{s\varphi} y$ we obtain, changing f in $f - s\lambda\varphi f_0$

$$P_s w = \mathrm{e}^{s\varphi} P\left(\mathrm{e}^{-s\varphi} w\right) = \mathrm{e}^{s\varphi} f + \sum_{j=0}^{N} \frac{\partial}{\partial x_j}\left(\mathrm{e}^{s\varphi} f_j\right) \quad \text{in } G =]0,1[\times M, \tag{14}$$

$$w(0,x') = \mathrm{e}^s g(x'), \quad x' \in M. \tag{15}$$

Let us define z by

$$\left(\frac{\partial}{\partial x_0} - s\lambda\varphi - R_-(x,D)\right)w = z. \tag{16}$$

We obtain, as w vanishes in a neighborhood of $\{1\} \times M$

$$\left(\frac{\partial}{\partial x_0} - s\lambda\varphi - R_+(x,D)\right)z = \mathrm{e}^{s\varphi} f + \sum_{j=0}^{N} \frac{\partial}{\partial x_j}\left(\mathrm{e}^{s\varphi} f_j\right) \quad \text{in } G =]0,1[\times M, \tag{17}$$

$$z(1,x') = 0, \quad x' \in M, \tag{18}$$

and we have

PROPOSITION 2.1. – *There exists a constant C independent of s and λ and there exists $s_0 \geqslant 0$ such that for all $\lambda \geqslant 1$, for all $s \geqslant s_0$*

$$\int_G \varphi|z|^2\,\mathrm{d}x \leqslant C\left(\frac{1}{s^2\lambda^2}\int_G \frac{|f|^2}{\varphi}\,\mathrm{e}^{2s\varphi}\,\mathrm{d}x + \sum_{j=0}^{N}\int_G |f_j|^2\varphi\,\mathrm{e}^{2s\varphi}\,\mathrm{d}x\right). \tag{19}$$

This result is proved by duality, considering the adjoint Cauchy problem

$$\left(-\frac{\partial}{\partial x_0} - s\lambda\varphi - R_+^*(x,D)\right)u = z \quad \text{in } G, \tag{20}$$

$$u(0,x') = 0, \quad x' \in M, \tag{21}$$

for which we show the following

LEMMA 2.2. – *For every $z \in \mathrm{L}^2(G)$, there exists a unique solution $u \in \mathrm{H}^1(G)$ of problem* (20), (21) *and u satisfies the following estimate*: *there exists C independent of s and λ and there exists $s_0 \geqslant 0$ such that for every $\lambda \geqslant 1$, for every $s \geqslant s_0$*

$$\int_G \big(|\nabla u|^2 + s^2\lambda^2\varphi^2|u|^2\big)\,\mathrm{d}x \leqslant C\int_G |z|^2\,\mathrm{d}x. \tag{22}$$

Then we know that w satisfies

$$\left(\frac{\partial}{\partial x_0} - s\lambda\varphi - R_-(x,D)\right)w = z \quad \text{in } G, \tag{23}$$

$$w(0,x') = \mathrm{e}^s g(x'), \quad w(1,x') = 0, \quad x' \in M, \tag{24}$$

and we decompose

$$Lw = \left(\frac{\partial}{\partial x_0} - s\lambda\varphi - R_-(x,D)\right)w = L_1 w + L_2 w, \tag{25}$$

where

$$L_1 = -\left(s\lambda\varphi + \frac{R_-(x,D) + R_-^*(x,D)}{2}\right), \tag{26}$$

$$L_2 = \frac{\partial}{\partial x_0} - \left(\frac{R_-(x,D) - R_-^*(x,D)}{2}\right). \tag{27}$$

We have two cases. The first one can be treated easily.

PROPOSITION 2.3. – *If $(L_1(0,\cdot)g, g)_{\mathrm{L}^2(M)} \leqslant 0$, there exists $\lambda_1 > 0$ and there exists $s_1 > 0$ such that for every $\lambda \geqslant \lambda_1$, for every $s \geqslant s_1$,*

$$\frac{1}{2}\big|L_1(\sqrt{\varphi}w)\big|^2_{\mathrm{L}^2(G)} + \big|L_2(\sqrt{\varphi}w)\big|^2_{\mathrm{L}^2(G)} + \frac{1}{2}s\lambda^2\int_G \varphi^2|w|^2\,\mathrm{d}x \leqslant 2|\sqrt{\varphi}z|^2_{\mathrm{L}^2(G)}. \tag{28}$$

In the second case we have $(L_1(0,\cdot)g, g)_{\mathrm{L}^2(M)} \geqslant 0$ so that $s\lambda\int_M |g|^2\,\mathrm{d}x' \leqslant C\|g\|^2_{\mathrm{H}^{1/2}(M)}$. We consider the following (adjoint) problem

$$L^* p = \left(-\frac{\partial}{\partial x_0} - s\lambda\varphi - R_-^*(x,D)\right)p = \varphi w \quad \text{in } G, \tag{29}$$

and we prove the following

LEMMA 2.4. – *There exists a constant $C > 0$, and there exists $\lambda_2 > 0$, such that for every $\lambda \geqslant \lambda_2$ and for every $s \geqslant 1$, there exists p satisfying* (29) *such that*

$$s\lambda^2\int_G |p|^2\,\mathrm{d}x + \sqrt{s}\lambda\int_M \big|p(0,x')\big|^2\,\mathrm{d}x' \leqslant C\int_G \varphi|w|^2\,\mathrm{d}x. \tag{30}$$

This lemma is shown by considering the minimization problem $J_\varepsilon(p_\varepsilon) = \min_{p\in U} J_\varepsilon(p)$, where $U = \{p \in \mathrm{L}^2(G),\ \partial p/\partial x_0 + s\lambda\varphi p + R_-^*(x,D)p + \varphi w \in \mathrm{L}^2(G)\}$, and

$$J_\varepsilon(p) = \frac{1}{2}|p|^2_{\mathrm{L}^2(G)} + \frac{1}{2\varepsilon}\left|\frac{\partial p}{\partial x_0} + s\lambda\varphi p + R_-^*(x,D)p + \varphi w\right|^2_{\mathrm{L}^2(G)}. \tag{31}$$

This problem has a solution (cf. [8]) and writing $q_\varepsilon = \frac{1}{\varepsilon}(\partial p_\varepsilon/\partial x_0 + s\lambda\varphi p_\varepsilon + R_-^*(x,D)p_\varepsilon + \varphi w)$, we have

$$Lq_\varepsilon = (L_1 + L_2)q_\varepsilon = p_\varepsilon \quad \text{in } G, \tag{32}$$

$$q_\varepsilon(0,x') = 0, \quad q_\varepsilon(1,x') = 0, \quad x' \in M, \tag{33}$$

and we can obtain estimates on q_ε and on $\frac{\partial q_\varepsilon}{\partial x_0}(0,x') = p_\varepsilon(0,x')$ to obtain Lemma 2.4.

We then have

$$\int_G \varphi |w|^2\,\mathrm{d}x = (L^* p, w)_{\mathrm{L}^2(G)} = (p, Lw)_{\mathrm{L}^2(G)} + \int_M p(0,x')w(0,x')\,\mathrm{d}x'$$
$$= (p,z)_{\mathrm{L}^2(G)} + \int_M p(0,x')\,\mathrm{e}^s g(x')\,\mathrm{d}x',$$

and therefore

PROPOSITION 2.5. – *There exists $C>0$, there exists $\overline{\lambda}>0$ and there exists $\overline{s}>0$ such that for every $\lambda \geqslant \overline{\lambda}$, for every $s \geqslant \overline{s}$, the solution w of* (23) *satisfies*

$$s^2\lambda^2 \int_G \varphi^2 |w|^2\,\mathrm{d}x \leqslant C\,\mathrm{e}^{2s}\sqrt{s}\|g\|^2_{\mathrm{H}^{1/2}(M)} + C\left(\frac{1}{s\lambda^2}\int_G \frac{|f|^2}{\varphi}\,\mathrm{e}^{2s\varphi}\,\mathrm{d}x + \sum_{j=0}^N s\int_G |f_j|^2\varphi\,\mathrm{e}^{2s\varphi}\,\mathrm{d}x\right). \tag{34}$$

Now in order to complete the proof of Theorem 1.2, it only remains to give an estimate on ∇w. We rewrite (23) as

$$\frac{\partial w}{\partial x_0} - R_-(x,D)w = z + s\lambda\varphi w \quad \text{in } G, \tag{35}$$

$$w(0,x') = \mathrm{e}^s g(x'), \quad x' \in M. \tag{36}$$

Using the same argument as in Lemma 2.2, taking care of the nonzero initial data for the Cauchy problem (35) we first obtain

$$\int_0^1 \|w\|^2_{\mathrm{H}^1(M)}\,\mathrm{d}x_0 \leqslant C\int_G |z|^2\,\mathrm{d}x + Cs^2\lambda^2\int_G \varphi^2|w|^2\,\mathrm{d}x + C\,\mathrm{e}^{2s}\|g\|^2_{\mathrm{H}^{1/2}(M)}$$

and then, using again Eq. (35) we obtain

$$\int_G |\nabla w|^2\,\mathrm{d}x \leqslant C\int_G |z|^2\,\mathrm{d}x + Cs^2\lambda^2\int_G \varphi^2|w|^2\,\mathrm{d}x + C\,\mathrm{e}^{2s}\|g\|^2_{\mathrm{H}^{1/2}(M)}.$$

This estimate, together with (34) gives

$$\int_G \left(|\nabla w|^2 + s^2\lambda^2\varphi^2|w|^2\right)\mathrm{d}x \leqslant C\left(s^{1/2}\,\mathrm{e}^{2s}\|g\|^2_{\mathrm{H}^{1/2}(M)} + \frac{1}{s\lambda^2}\int_G \frac{|f|^2}{\varphi}\,\mathrm{e}^{2s\varphi}\,\mathrm{d}x + \sum_{j=0}^N s\int_G |f_j|^2\varphi\,\mathrm{e}^{2s\varphi}\,\mathrm{d}x\right)$$

and this finishes the proof of Theorem 1.2.

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