C. R. Acad. Sci. Paris, Ser. I 334 (2002) 957–960

Analyse complexe/*Complex Analysis*

An inner function which is not weak outer

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Received 23 January 2002; accepted 22 March 2002

Note presented by Jean-Pierre Demailly.

Abstract We construct a nonvanishing inner function I in the unit ball $B \subset \mathbb{C}^n$ such that the subspace $I\mathrm{H}^p(B)$ is not weakly dense in the Hardy space $\mathrm{H}^p(B)$, with $0 < p < 1$. ***To cite this article: E. Doubtsov, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 957–960.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Une fonction intérieure qui n'est pas faiblement extérieure

Résumé On construit une fonction intérieure I dans la boule unité $B \subset \mathbb{C}^n$ telle que $I(z) \neq 0$ pour $z \in B$ et le sous-espace $I\mathrm{H}^p(B)$ n'est pas faiblement dense dans la classe de Hardy $\mathrm{H}^p(B)$, pour $0 < p < 1$. ***Pour citer cet article : E. Doubtsov, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 957–960.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Version française abrégée

1. Fonctions faiblement extérieures

Soit $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Supposons $0 < p < 1$, alors l'espace de Hardy $\mathrm{H}^p(\mathbb{D})$ n'est pas localement convexe, et le théorème de Hahn–Banach ne reste pas vrai pour $\mathrm{H}^p(\mathbb{D})$. En fait, soit θ une fonction intérieure (c'est-à-dire une fonction holomorphe bornée dans $\mathbb{D}$ dont les valeurs au bord sont de module presque partout égal à 1). On dit que θ est faiblement extérieure si le sous-espace fermé $\theta\mathrm{H}^p(\mathbb{D}) \subset \mathrm{H}^p(\mathbb{D})$ est faiblement dense (*voir* [3]). Pour une telle θ on a $\theta(z) \neq 0$, $z \in \mathbb{D}$. Supposons que $\theta(0) > 0$. Alors on a :

$$\theta(z) = \theta_\mu(z) = \exp\left(-\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z}\, \mathrm{d}\mu(\zeta)\right), \quad z \in \mathbb{D},$$

où μ est une mesure positive singulière sur $\mathbb{T} = \partial\mathbb{D}$. Duren, Romberg et Shields [3] démontrent que θ_μ est faiblement extérieure si la mesure μ est suffisamment régulière.

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S1631-073X(02)02374-9/FLA

2. Cas de plusieurs variables

Soit I une fonction intérieure dans la boule complexe $B = B_n = \{z \in \mathbb{C}^n : |z| < 1\}$, $n \geqslant 2$. Supposons que $I(0) > 0$ et $I(z) \neq 0$ pour $z \in B$. Alors on a :

$$I(z) = \exp\left(-\int_S \left[\frac{2}{(1 - \langle z, \zeta \rangle)^n} - 1\right] \mathrm{d}\nu(\zeta)\right), \quad z \in B,$$

où ν est une mesure positive singulière *pluriharmonique* sur la sphère unité $S = \partial B$. Les mesures pluriharmoniques sont suffisamment régulières. Donc est naturel de poser la question si toute fonction intérieure I dans B_n, $I(z) \neq 0$ pour $z \in B_n$, est faiblement extérieure pour $n \geqslant 2$ (Shapiro, *voir* [5], Problème 24.2).

Dans cet Note nous démontrons le résultat suivant :

THÉORÈME 1. – *Il existe une fonction intérieure $I \in \mathrm{H}^\infty(B)$ telle que $I(z) \neq 0$ pour $z \in B_n$ et*

$$I(\lambda, 0) = \exp\left(\frac{\lambda + 1}{\lambda - 1}\right), \quad \lambda \in \mathbb{D},\ 0 \in \mathbb{C}^{n-1}.$$

La fonction intérieure I n'est pas faiblement extérieure.

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1. Weak outer functions

Consider the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. When $0 < p < 1$ the Hardy space $\mathrm{H}^p(\mathbb{D})$ is not locally convex, and the Hahn–Banach theorem fails. In fact, there exist weakly dense closed subspaces of H^p. Indeed, let θ be an inner function; that is, $\theta \in \mathrm{H}^\infty(\mathbb{D})$ and $|\theta^*| = 1$ a.e., where θ^* denotes the boundary values. Then the closed subspace $\theta\mathrm{H}^p(\mathbb{D}) \subset \mathrm{H}^p(\mathbb{D})$ can be weakly dense (here and in what follows, we assume that θ is not a constant); (*see* [7]). Such inner functions are said to be weak outer. The term *outer* is motivated by the following theorem of Beurling: the polynomial multiples of a function $f \in \mathrm{H}^p$ form a dense subset of H^p if and only if f is outer.

Assume θ is a nontrivial weak outer inner function. For $z \in \mathbb{D}$, the evaluation functional $\lambda_z(f) = f(z)$ is continuous on H^p, therefore, $\theta(z) \neq 0$ for all $z \in \mathbb{D}$. In other words, θ is a singular inner function. So, up to a unimodular multiplicative constant, we have:

$$\theta(z) = \theta_\mu(z) = \exp\left(-\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \,\mathrm{d}\mu(\zeta)\right), \quad z \in \mathbb{D},$$

where μ is the associated positive singular measure on the unit circle $\mathbb{T} = \partial\mathbb{D}$. In 1969 Duren, Romberg and Shields [3] discovered weak outer inner functions and proved that θ is weak outer if μ vanishes on the Carleson sets. Later Roberts [4] and Korenblum proved that the Carleson smoothness condition characterizes the weak outer functions.

2. Case of several variables

Consider a nonvanishing inner function I in the complex ball $B = B_n = \{z \in \mathbb{C}^n : |z| < 1\}$, $n \geqslant 2$. Assume $I(0) > 0$, then:

$$I(z) = \exp\left(-\int_S \left[\frac{2}{(1 - \langle z, \zeta \rangle)^n} - 1\right] \mathrm{d}\nu(\zeta)\right), \quad z \in B,$$

where ν is a positive singular *pluriharmonic* measure on the unit sphere $S = \partial B$. Recall that a measure is called pluriharmonic if its Poisson integral is a pluriharmonic function in the ball. Such measures are known to be sufficiently smooth, so Shapiro ([5], Problem 24.2) raised the following question.

Is every nonvanishing inner function in B_n weakly outer when $n \geqslant 2$?

In the present Note we give a negative answer to the above question. The idea is to reduce the problem to an argument in dimension one. Namely, assume that a slice-function of I, say $I(z,0)$, is a singular inner function $\theta_\mu(z)$, $z \in \mathbb{D}$. If the associated measure μ is sufficiently rough, then the argument below works. To avoid technicalities, we consider the simplest case of an atomic measure.

THEOREM 1. – *There exists a nonvanishing inner function $I \in \mathrm{H}^\infty(B)$ such that*

$$I(\lambda, 0) = \exp\left(\frac{\lambda+1}{\lambda-1}\right), \quad \lambda \in \mathbb{D},\ 0 \in \mathbb{C}^{n-1}.$$

Proof. – For $\xi \in \mathbb{D}$, put $\theta(\xi) = \xi$. Since $\|\theta\|_\infty \leqslant 1$, there exists an inner function $F \in \mathrm{H}^\infty(B)$ such that $F(\xi, 0) = \theta(\xi) = \xi$ for all $\xi \in \mathbb{D}$ ([1], Corrollary 1). Define:

$$I(z) = \exp\left(\frac{F(z)+1}{F(z)-1}\right), \quad z \in B.$$

Since $(\zeta+1)/(\zeta-1)$ maps conformally the unit disc $\mathbb{D}$ onto the left half-plane, the function I is inner. On the other hand, the slice-function $I(\lambda, 0)$ has the form required. □

PROPOSITION 2. – *The inner function I provided by Theorem* 1 *is not weak outer.*

Proof. – We have to show that the subspace $I\mathrm{H}^p(B)$ is not weakly dense in $\mathrm{H}^p(B)$, with $0 < p < 1$. So we construct a nontrivial continuous linear functional $\Lambda : \mathrm{H}^p(B) \to \mathbb{C}$ such that $\Lambda(IF) = 0$ for all $F \in \mathrm{H}^p(B)$.

Consider a sequence $a = \{a_k\}_{k=1}^\infty \subset \mathbb{C}$. Given $G \in \mathrm{H}^p(B)$, define:

$$\Lambda_a(G) = \sum_{k=0}^\infty a_k \hat{g}(k),$$

where $g(z) = G(z, 0)$ for $z \in \mathbb{D}$.

It is well known that the slice-function g is in a weighted Bergman space (*see*, e.g., [2], Chapter 2, Theorem 1.3.1). Namely, let m_2 denote the normalized area measure, then we have:

$$\|g\|^p_{A^p_{n-2}} := (n-1)\int_{\mathbb{D}} \left|g(z)\right|^p \left(1-|z|^2\right)^{n-2} \mathrm{d}m_2(z) \leqslant \|G\|^p_{\mathrm{H}^p} < \infty.$$

Now recall that the Fourier coefficients of an A^p_{n-2}-function are $\mathcal{O}(k^{\beta(p,n)})$ with $\beta(p,n) < \infty$. For example, A^p_0 is the classical Bergman space, so $\beta(p,2) = 2/p - 1$ (*see* [8] for details). In other words, the following estimate holds:

$$\left|\hat{g}(k)\right| \leqslant C(p,n)k^{\beta(p,n)}\|g\|_{A^p_{n-2}} \leqslant C(p,n)k^{\beta(p,n)}\|G\|_{\mathrm{H}^p}.$$

Thus, the linear functional Λ_a is continuous on the Hardy class $\mathrm{H}^p(B)$ provided $|a_k| = \mathcal{O}(k^{-2-\beta(p,n)})$. To finish the proof, we find such a nontrivial sequence a that $\Lambda_a(IF) = 0$ for all $F \in \mathrm{H}^p(B)$. This argument is due to Shapiro [6].

Put $a_k = \hat{h}(-k)$, where

$$h(\lambda) = \lambda(1-\lambda)^q \overline{I(\lambda, 0)}, \quad \lambda \in \mathbb{T},\ q \in \mathbb{N}.$$

1. Fix q so large that h has at least $2+\beta(p,n)$ continuous derivatives. Then Λ_a is continuous on $\mathrm{H}^p(B)$, since $|a_k| = \mathcal{O}(k^{-2-\beta(p,n)})$.

2. The functional Λ_a is not trivial. Indeed, assume $a_k = 0$ for all $k \geqslant 0$, then $h \in \mathrm{H}^2(\mathbb{D})$. In particular, we have $h(r) = \mathcal{O}(1/\sqrt{1-r})$ as $r \to 1-$. On the other hand, $I(\cdot, 0)$ is inner, so $z(1-z)^q = h(z)I(z,0)$ for all $z \in \mathbb{D}$. The last identity yields a contradiction, since $I(r,0) = \mathcal{O}(\exp[-1/(1-r)])$ as $r \to 1-$.

3. Let m denote the normalized Lebesgue measure on the circle $\mathbb{T}$. Given a holomorphic polynomial P on $\mathbb{C}^n$, we have:

$$\Lambda_a(IP) = \sum_{k\geqslant 0} a_k \big(I(\cdot,0)P(\cdot,0)\big)\widehat{\ }(k) = \int_{\mathbb{T}} h(\lambda) I(\lambda,0) P(\lambda,0)\,\mathrm{d}m(\lambda)$$

$$= \int_{\mathbb{T}} \lambda\big((1-\lambda)^q P(\lambda,0)\big)\,\mathrm{d}m(\lambda) = 0,$$

since $I(\lambda,0)$ is an inner function. Finally, the polynomials are dense in $\mathrm{H}^p(B)$, so Λ_a vanishes on $I\mathrm{H}^p(B)$. □

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