C. R. Acad. Sci. Paris, Ser. I 334 (2002) 733–738

Algèbre homologique/*Homological Algebra*
(Algèbres de Lie/*Lie Algebras*)

The Lie algebra structure of the first Hochschild cohomology group for monomial algebras

Claudia Strametz

Laboratoire GTA UMR 5030, Département de Mathématiques, Université de Montpellier 2, 34095 Montpellier cedex 5, France

Received 21 January 2002; accepted 4 March 2002

Note presented by Alain Connes.

Abstract We study the Lie algebra structure of the first Hochschild cohomology group of a finite dimensional monomial algebra Λ, in terms of the combinatorics of its quiver, in any characteristic. This allows us also to examine the identity component of the algebraic group of outer automorphisms of Λ in characteristic zero. Criteria for the solvability, the (semi-)simplicity, the commutativity and the nilpotency are given. ***To cite this article: C. Strametz, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 733–738.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

La structure d'algèbre de Lie du premier groupe de la cohomologie de Hochschild pour des algèbres monomiales

Résumé Nous étudions la structure d'algèbre de Lie du premier groupe de la cohomologie de Hochschild d'une algèbre monomiale de dimension finie Λ, en termes combinatoires de son carquois, en quelconque caractéristique. Cela nous permet aussi d'examiner la composante de l'identité du groupe algébrique des automorphismes extérieurs de Λ en caractéristique zéro. Nous donnons des critères pour la résolubilité, la (semi-)simplicité, la commutativité et la nilpotence. ***Pour citer cet article : C. Strametz, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 733–738.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Version française abrégée

Soient k un corps et Λ une k-algèbre monomiale de dimension finie. Alors Λ est isomorphe à un quotient d'une algèbre des chemins $kQ/\langle Z\rangle$ où Q désigne un carquois fini et $\langle Z\rangle$ un idéal bilatère engendré par un ensemble Z des chemins de Q de longueur $\geqslant 2$ tel qu'aucun sous-chemin propre d'un chemin de Z n'est contenu dans Z. Soit B l'ensemble des chemins de Q qui ne contiennent aucun chemin de Z comme sous-chemin. Pour tout $n \in \mathbb{N}$ on note par B_n l'ensemble des chemins de B de longueur n. Deux chemins γ et ε de Q sont dits parallèles s'ils ont la même source et le même terminus. Si X et Y sont deux ensembles de chemins de Q, l'ensemble $X//Y$ est formé par les paires $(\gamma, \varepsilon) \in X \times Y$ telles que les chemins γ et ε sont parallèles.

NOTATIONS 0.1. – Soient ε un chemin de Q et $(a, \gamma) \in Q_1//B$. On note par $\varepsilon^{(a,\gamma)}$ la somme de tous les chemins différents de 0 (i.e. de B) obtenus en remplaçant une occurrence de la flèche a dans ε par le chemin γ. Si le chemin ε ne contient pas la flèche a ou si tout remplacement de a dans ε par γ n'est pas

E-mail address: strametz@math.univ-montp2.fr (C. Strametz).

S1631-073X(02)02346-4/FLA

un chemin dans B, nous posons $\varepsilon^{(a,\gamma)} = 0$. Supposons que $\varepsilon^{(a,\gamma)} = \sum_{i=1}^{n} \varepsilon_i$, où $\varepsilon_i \in B$ et soit η un chemin de Z parallèle à ε. Par abus de langage nous notons par $(\eta, \varepsilon^{(a,\gamma)})$ la somme $\sum_{i=1}^{n} (\eta, \varepsilon_i)$ dans $k(Z//B)$ (avec la convention que $(\eta, \varepsilon^{(a,\gamma)}) = 0$ si $\varepsilon^{(a,\gamma)} = 0$).

Le premier groupe de la cohomologie de Hochschild $\mathrm{H}^1(\Lambda, \Lambda) = \mathrm{Ext}^1_{\Lambda^e}(\Lambda, \Lambda) = \mathrm{Der}_k(\Lambda)/\mathrm{Ad}_k(\Lambda)$ a une structure d'algèbre de Lie qui est donnée par le commutateur des dérivations. Si on utilise la résolution projective minimale de Λ en tant que Λ-bimodule donnée par Bardzell dans [1] on voit que les k-espaces vectoriels $\mathrm{H}^1(\Lambda, \Lambda)$ et $\mathrm{Ker}\,\psi_1/\mathrm{Im}\,\psi_0$ sont isomorphes où

$$\psi_0 : k(Q_0//B) \to k(Q_1//B), \qquad (e, \gamma) \mapsto \sum_{\substack{a \in Q_1 e \\ a\gamma \in B}} (a, a\gamma) - \sum_{\substack{a \in eQ_1 \\ \gamma a \in B}} (a, \gamma a) \quad \text{et}$$

$$\psi_1 : k(Q_1//B) \to k(Z//B), \qquad (a, \gamma) \mapsto \sum_{p \in Z} \big(p, p^{(a,\gamma)}\big).$$

La comparaison de la résolution minimale avec la résolution standard de Λ en tant que Λ-bimodule nous permet de transférer le crochet de Lie de $\mathrm{H}^1(\Lambda, \Lambda)$ à $\mathrm{Ker}\,\psi_1/\mathrm{Im}\,\psi_0$:

THÉORÈME 0.2. – *Le crochet*

$$\big[(a, \gamma), (b, \varepsilon)\big] = \big(b, \varepsilon^{(a,\gamma)}\big) - \big(a, \gamma^{(b,\varepsilon)}\big)$$

pour tout $(a, \gamma), (b, \varepsilon) \in Q_1//B$ *induit une structure d'algèbre de Lie sur* $\mathrm{Ker}\,\psi_1/\mathrm{Im}\,\psi_0$ *telle que* $\mathrm{H}^1(\Lambda, \Lambda)$ *et* $\mathrm{Ker}\,\psi_1/\mathrm{Im}\,\psi_0$ *sont des algèbres de Lie isomorphes.*

Si $(a, \gamma) \in Q_1//B_n$ et $(b, \varepsilon) \in Q_1//B_m$, alors $[(a, \gamma), (b, \varepsilon)] \in k(Q_1//B_{n+m-1})$. En définissant, pour tout $i \geqslant -1$, $L_i = k(Q_1//B_{i+1}) \cap \mathrm{Ker}\,\psi_1/k(Q_1//B_{i+1}) \cap \mathrm{Im}\,\psi_0$ nous obtenons une graduation sur $\mathrm{H}^1(\Lambda, \Lambda) = \bigoplus_{i \geqslant -1} L_i$ telle que $[L_i, L_j] \subset L_{i+j}$ pour tout $i, j \geqslant -1$ où $L_{-2} = 0$. Si $L_{-1} = 0$ (ce qui est par exemple le cas si Q n'a pas de boucle ou si la caractéristique du corps k est zéro), alors $\mathrm{Rad}\,\mathrm{H}^1(\Lambda, \Lambda) = \mathrm{Rad}\,L_0 \oplus \bigoplus_{i \geqslant 1} L_i$.

DÉFINITION 0.3. – Deux flèches parallèles a et b sont dites équivalentes si $p^{(a,b)} = 0 = p^{(b,a)}$ pour tout $p \in Z$. L'idéal $\langle Z \rangle$ est dit complètement saturé si toutes les flèches parallèles sont équivalentes. L'ensemble des classes de flèches parallèles de Q est noté $Q_1//$ et nous notons $\overline{\alpha}/\approx$ l'ensemble des classes d'équivalence de $\overline{\alpha} \in Q_1//$.

THÉORÈME 0.4. – *Le radical de l'algèbre de Lie* L_0 *est engendré en tant que* k*-espace vectoriel par les éléments suivants de* L_0 : *pour toute classe de flèches parallèles* $\overline{\alpha} = \{\alpha_1, \ldots, \alpha_n\}$

- $\sum_{\alpha_i \in S} (\alpha_i, \alpha_i)$ *pour toute classe d'équivalence* $S \in \overline{\alpha}/\approx$;
- (α_i, α_j), $i \neq j$, *si* $(\alpha_i, \alpha_j) \in L_0$ *et* $(\alpha_j, \alpha_i) \notin L_0$;
- *si* $\mathrm{car}\,k = 2$, $(\alpha_{i_1}, \alpha_{i_2})$, $(\alpha_{i_2}, \alpha_{i_1})$ *et* $(\alpha_{i_1}, \alpha_{i_1})$ *pour toute* $S = \{\alpha_{i_1}, \alpha_{i_2}\} \in \overline{\alpha}/\approx$.

La démonstration de ce théorème montre que l'algèbre de Lie semi-simple $\mathrm{H}^1(\Lambda, \Lambda)/\mathrm{Rad}\,\mathrm{H}^1(\Lambda, \Lambda) = L_0/\mathrm{Rad}\,L_0$ est le produit d'algèbres de Lie ayant un facteur $\mathfrak{pgl}(|S|, k) := \mathfrak{gl}(|S|, k)/k1$ pour chaque classe d'équivalence $S \in \overline{\alpha}/\approx$ d'une classe de flèches parallèles $\overline{\alpha}$ telle que $|S| \geqslant 2$ si $\mathrm{car}\,k \neq 2$ et $|S| > 2$ si $\mathrm{car}\,k = 2$.

COROLLAIRE 0.5. – *Si* $L_{-1} = k(Q_0//Q_1) \cap \mathrm{Ker}\,\psi_1 = 0$, *alors les conditions suivantes sont équivalentes* :

(1) *L'algèbre de Lie* $\mathrm{H}^1(\Lambda, \Lambda)$ *est résoluble.*
(2) *Toute classe d'équivalence* S *d'une classe de flèches parallèles* $\overline{\alpha} \in Q_1//$ *de* Q *contient une et une seule flèche si* $\mathrm{car}\,k \neq 2$. *Si* $\mathrm{car}\,k = 2$ *nous avons* $|S| \leqslant 2$ *pour toute* $S \in \overline{\alpha}/\approx$, $\overline{\alpha} \in Q_1//$.

Soit $\overline{Q}$ le sous-carquois de Q obtenu en prenant un représentant pour chaque classe de flèches parallèles.

COROLLAIRE 0.6. – *Si* $L_{-1} = k(Q_0//Q_1) \cap \mathrm{Ker}\,\psi_1 = 0$, *alors les conditions suivantes sont équivalentes* :

(1) *L'algèbre de Lie* $\mathrm{H}^1(\Lambda, \Lambda)$ *est semi-simple.*
(2) *Le carquois* $\overline{Q}$ *est un arbre,* Q *a au moins une classe de flèches parallèles non triviale et l'ideal* $\langle Z\rangle$ *est complètement saturé. Si* $\operatorname{car} k = 2$*, alors* Q *n'a pas de classe de flèches parallèles contenant exactement deux flèches.*
(3) *L'algèbre de Lie* $\mathrm{H}^1(\Lambda, \Lambda)$ *est isomorphe au produit non trivial d'algèbres de Lie* $\prod_{\overline{\alpha}\in Q_1//} \mathfrak{pgl}(|\overline{\alpha}|, k)$ *où* $|\overline{\alpha}| \neq 2$ *si* $\operatorname{car} k = 2$.

Comme $\mathrm{H}^1(\Lambda, \Lambda)$ peut être considerée en caractéristique zéro comme l'algèbre de Lie du groupe algébrique des automorphismes extérieurs $\mathrm{Out}(\Lambda) = \mathrm{Aut}(\Lambda)/\mathrm{Inn}(\Lambda)$ nous pouvons déduire en caractéristique zéro des critères de résolubilité et de (semi-)simplicité pour la composante de l'identité de $\mathrm{Out}(\Lambda)$.

We give some notation and terminology which we keep throughout the paper. Let Q denote a finite quiver (that is a finite oriented graph) and k a field. The path algebra kQ is the k-linear span of the set of paths of Q where the multiplication of $\beta \in Q_i$ and $\alpha \in Q_j$ is provided by the concatenation $\beta\alpha \in Q_{i+j}$ if possible and 0 otherwise. We denote by Λ a finite dimensional monomial k-algebra, that is a finite dimensional k-algebra which is isomorphic to a quotient of a path algebra kQ/I where the two-sided ideal I of kQ is generated by a set Z of paths of length $\geqslant 2$. We shall assume that Z is minimal, i.e. no proper subpath of a path in Z is again in Z. Let B be the set of paths of Q which do not contain any element of Z as a subpath. The (classes modulo $I = \langle Z\rangle$ of the) elements of B form a basis of Λ. We shall denote by B_n the set of paths of B of length n. Two paths ε, γ of Q are called parallel if they have the same source and terminus vertex. If X and Y are sets of paths of Q, the set $X//Y$ of parallel paths is formed by the couples $(\varepsilon, \gamma) \in X \times Y$ such that ε and γ are parallel paths. For instance, $Q_0//Q_n$ is the set of oriented cycles of Q of length n. Let $E \simeq kQ_0$ be the separable subalgebra of Λ generated by the (classes modulo I of the) vertices of Q.

1. Projective resolutions and the Lie bracket

The Hochschild cohomology $\mathrm{H}^*(A, A) = \mathrm{Ext}^*_{A^e}(A, A)$ of a k-algebra A can be computed using different projective resolutions of A over its enveloping algebra $A^e = A \otimes_k A^{\mathrm{op}}$. The standard resolution $\mathcal{P}_{\mathrm{Hoch}}$ is

$$\cdots \to A^{\otimes_k^n} \xrightarrow{\delta} A^{\otimes_k^{n-1}} \to \cdots \to A \otimes_k A \xrightarrow{\varepsilon} A \to 0,$$

where $\varepsilon(x_1 \otimes_k x_2) = x_1x_2$ and $\delta(x_1 \otimes_k \cdots \otimes_k x_n) = \sum_{i=1}^{n-1}(-1)^{i+1}x_1 \otimes_k \cdots \otimes_k x_ix_{i+1} \otimes_k \cdots \otimes_k x_n$ for $x_1, \ldots, x_n \in A$. Applying the functor $\mathrm{Hom}_{A^e}(\_, A)$ and identifying $\mathrm{Hom}_{A^e}(A \otimes_k A^{\otimes_k^n} \otimes_k A, A)$ and $\mathrm{Hom}_k(A^{\otimes_k^n}, A)$ for all $n \in \mathbb{N}$, yields the cochain complex $\mathcal{C}_{\mathrm{Hoch}}$ defined by Hochschild:

$$0 \to \Lambda \xrightarrow{d_0} \mathrm{Hom}_k(\Lambda, \Lambda) \to \cdots \to \mathrm{Hom}_k\big(A^{\otimes_k^n}, A\big) \xrightarrow{d_n} \mathrm{Hom}_k\big(A^{\otimes_k^{n+1}}, A\big) \to \cdots,$$

where $(d_0a)(x) = ax - xa$ for $a, x \in A$ and $(d_nf)(x_1 \otimes_k \cdots \otimes_k x_{n+1}) = x_1f(x_2 \otimes_k \cdots \otimes_k x_{n+1}) + \sum_{i=1}^n(-1)^i f(x_1 \otimes_k \cdots \otimes_k x_ix_{i+1} \otimes_k \cdots \otimes_k x_{n+1}) + (-1)^{n+1}f(x_1 \otimes_k \cdots \otimes_k x_n)x_{n+1}$ for all $f \in \mathrm{Hom}_k(A^{\otimes_k^n}, A)$, $n \in \mathbb{N}$, and $x_1, \ldots, x_{n+1} \in A$. In 1962 the structure of a Gerstenhaber algebra was introduced on the Hochschild cohomology $\mathrm{H}^*(A, A)$ by Gerstenhaber in [3]. In particular the first cohomology group $\mathrm{H}^1(A, A)$ which is the quotient of the derivations $\mathrm{Der}_k(A) = \operatorname{Ker} d_1 = \{f \in \mathrm{Hom}_k(A, A) \mid f(ab) = af(b) + f(a)b,\ \forall a, b \in \Lambda\}$ modulo the inner derivations $\mathrm{Ad}_k(A) = \operatorname{Im} d_0 = \{f \in \mathrm{Hom}_k(A, A) \mid \exists a \in A \text{ such that } f(x) = ax - xa,\ \forall x \in A\}$ of A is a Lie algebra whose bracket is induced by the commutator on $\mathrm{Hom}_k(A, A)$. Keller [7] showed that Hochschild cohomology is preserved under derived equivalence as a graded (super) Lie algebra. From [6], 13.2 we deduce:

PROPOSITION 1.1. – *Let* k *be a field of characteristic* 0 *and* A *a finite dimensional* k*-algebra. The derivations* $\mathrm{Der}_k(A)$ *form the Lie algebra of the algebraic group of* k*-algebra automorphisms* $\mathrm{Aut}(A)$ *and its Lie ideal* $\mathrm{Ad}_k(A)$ *forms the Lie algebra of the algebraic group of inner automorphisms* $\mathrm{Inn}(A)$*. The Lie algebra* $\mathrm{H}^1(A, A)$ *can be regarded as the Lie algebra of the algebraic group of outer automorphisms* $\mathrm{Out}(A)$ *or as the Lie algebra of its identity component* $\mathrm{Out}(A)^\circ$.

In order to compute the Lie algebra structure on the first Hochschild cohomology group $\mathrm{H}^1(\Lambda,\Lambda)$ of the finite dimensional monomial algebra Λ, we shall use the minimal projective resolution of the Λ-bimodule Λ given by Bardzell in [1]. The part of this resolution $\mathcal{P}_{\min}$ in which we are interested is the following:

$$\cdots \to \Lambda\otimes_E kZ\otimes_E\Lambda \xrightarrow{\delta_1} \Lambda\otimes_E kQ_1\otimes_E\Lambda \xrightarrow{\delta_0} \Lambda\otimes_E\Lambda \xrightarrow{\pi} \Lambda\to 0,$$

where the Λ-bimodule morphisms are given by

$$\begin{aligned}\pi(\lambda\otimes_E\mu) &= \lambda\mu,\\ \delta_0(\lambda\otimes_E a\otimes_E\mu) &= \lambda a\otimes_E\mu-\lambda\otimes_E a\mu \quad\text{and}\\ \delta_1(\lambda\otimes_E p\otimes_E\mu) &= \sum_{d=1}^{n}\lambda a_n\ldots a_{d+1}\otimes_E a_d\otimes_E a_{d-1}\ldots a_1\mu\end{aligned}$$

for all $\lambda,\mu\in\Lambda$, $a,a_n,\ldots,a_1\in Q_1$ and $p=a_n\ldots a_1\in Z$ (with $a_{n+1}=t(a_n)$ and $a_0=s(a_1)$).

NOTATIONS 1.2. – Let ε be a path in Q and $(a,\gamma)\in Q_1//B$. We denote by $\varepsilon^{(a,\gamma)}$ the sum of all nonzero paths (i.e. paths in B) obtained by replacing one occurrence of the arrow a in ε by the path γ. If the path ε does not contain the arrow a or if every replacement of a in ε by γ is not a path in B, we set $\varepsilon^{(a,\gamma)}=0$. Suppose that $\varepsilon^{(a,\gamma)}=\sum_{i=1}^{n}\varepsilon_i$, where $\varepsilon_i\in B$ and let η be a path of Z parallel to ε. By abuse of language we denote by $(\eta,\varepsilon^{(a,\gamma)})$ the sum $\sum_{i=1}^{n}(\eta,\varepsilon_i)$ in $k(Z//B)$ (with the convention that $(\eta,\varepsilon^{(a,\gamma)})=0$ if $\varepsilon^{(a,\gamma)}=0$).

Applying the functor $\mathrm{Hom}_{\Lambda^e}(\_,\Lambda)$ to $\mathcal{P}_{\min}$ and using the fact, that $\mathrm{Hom}_{\Lambda^e}(\Lambda\otimes_E kX\otimes_E\Lambda,\Lambda)$ and $k(X//B)$ are isomorpic vector spaces for every set X of paths of Q, yields that the beginning of the cochain complex $\mathcal{C}_{\min}$ can be characterized in the following way

$$0\to k(Q_0//B)\xrightarrow{\psi_0} k(Q_1//B)\xrightarrow{\psi_1} k(Z//B)\xrightarrow{\psi_2}\cdots,$$

where the maps are given by

$$\begin{aligned}&\psi_0: k(Q_0//B)\to k(Q_1//B), && (e,\gamma)\mapsto \textstyle\sum_{\substack{a\in Q_1e\\ a\gamma\in B}}(a,a\gamma)-\sum_{\substack{a\in eQ_1\\ \gamma a\in B}}(a,\gamma a)\quad\text{and}\\ &\psi_1: k(Q_1//B)\to k(Z//B), && (a,\gamma)\mapsto \textstyle\sum_{p\in Z}\big(p,p^{(a,\gamma)}\big).\end{aligned}$$

In particular, we have $\mathrm{H}^1(\Lambda,\Lambda)\simeq\operatorname{Ker}\psi_1/\operatorname{Im}\psi_0$. Note that the dimension of the vector space $\mathrm{H}^1(\Lambda,\Lambda)$ has been obtained by Cibils and Saorín in [2], Theorem 1, using different methods.

THEOREM 1.3. – *The bracket*

$$\big[(a,\gamma),(b,\varepsilon)\big]=\big(b,\varepsilon^{(a,\gamma)}\big)-\big(a,\gamma^{(b,\varepsilon)}\big)$$

for all $(a,\gamma),(b,\varepsilon)\in Q_1//B$ *induces a Lie algebra structure on* $\operatorname{Ker}\psi_1/\operatorname{Im}\psi_0$ *such that* $\mathrm{H}^1(\Lambda,\Lambda)$ *and* $\operatorname{Ker}\psi_1/\operatorname{Im}\psi_0$ *are isomorphic Lie algebras.*

Proof. – As $\mathcal{P}_{\mathrm{Hoch}}$ and $\mathcal{P}_{\min}$ are projective resolutions of the Λ-bimodule Λ, there exist chain maps $\omega:\mathcal{P}_{\mathrm{Hoch}}\to\mathcal{P}_{\min}$ and $\varsigma:\mathcal{P}_{\min}\to\mathcal{P}_{\mathrm{Hoch}}$. The maps $\mathrm{Hom}_{\Lambda^e}(\omega,\Lambda)$ and $\mathrm{Hom}_{\Lambda^e}(\varsigma,\Lambda)$ induce inverse linear isomorphisms at the cohomology level. Taking into account the effected identifications we obtain k-linear maps $\overline{\omega_1}: k(Q_1//B)\to\mathrm{Hom}_k(\Lambda,\Lambda)$ and $\overline{\varsigma_1}:\mathrm{Hom}_k(\Lambda,\Lambda)\to k(Q_1//B)$ which induce inverse linear isomorphisms between $\mathrm{H}^1(\Lambda,\Lambda)=\operatorname{Ker}d_1/\operatorname{Im}d_0$ and $\mathrm{H}^1(\Lambda,\Lambda)=\operatorname{Ker}\psi_1/\operatorname{Im}\psi_0$. This allows us to transfer the Lie algebra structure of $\operatorname{Ker}d_1/\operatorname{Im}d_0$ to $\operatorname{Ker}\psi_1/\operatorname{Im}\psi_0$ by defining on $k(Q_1//B)$ the Lie bracket $[(a,\gamma),(b,\varepsilon)]=\overline{\varsigma_1}([\overline{\omega_1}(a,\gamma),\overline{\omega_1}(b,\varepsilon)])=(b,\varepsilon^{(a,\gamma)})-(a,\gamma^{(b,\varepsilon)})$ for all $(a,\gamma),(b,\varepsilon)\in Q_1//B$. $\square$

2. The Lie algebra $\mathrm{H}^1(\Lambda, \Lambda)$ of a monomial algebra Λ

Since Hochschild cohomology is additive and since its Lie algebra structure follows this additive decomposition we will assume henceforth that the quiver Q is connected. If $(a, \gamma) \in Q_1//B_n$ and $(b, \varepsilon) \in Q_1//B_m$, we see that $[(a, \gamma), (b, \varepsilon)]$ is an element of $k(Q_1//B_{n+m-1})$. Thus, we have a graduation on the Lie algebra $k(Q_1//B) = \bigoplus_{i \in \mathbb{N}} k(Q_1//B_i)$ considering the elements of $k(Q_1//B_i)$ having degree $i-1$ for all $i \in \mathbb{N}$. It is clear that the Lie subalgebra $\operatorname{Ker}\psi_1$ of $k(Q_1//B)$ preserves this graduation and that $\operatorname{Im}\psi_0$ is a graded ideal. Therefore, the Lie algebra $\mathrm{H}^1(\Lambda, \Lambda) = \operatorname{Ker}\psi_1/\operatorname{Im}\psi_0$ has also a graduation. If we note

$$L_i = k(Q_1//B_{i+1}) \cap \operatorname{Ker}\psi_1 \Big/ \Big\langle \sum_{\substack{a \in Q_1 e \\ \gamma a \in B}} (a, \gamma a) - \sum_{\substack{a \in e Q_1 \\ a\gamma \in B}} (a, a\gamma) \mid (e, \gamma) \in Q_0//Q_i \Big\rangle$$

for all $i \geqslant -1$ we obtain $\mathrm{H}^1(\Lambda, \Lambda) = \bigoplus_{i \geqslant -1} L_i$ and $[L_i, L_j] \subset L_{i+j}$ for all $i, j \geqslant -1$ where $L_{-2} = 0$.

Remark 2.1. – The Lie subalgebra L_{-1} of $\mathrm{H}^1(\Lambda, \Lambda)$ equals zero if and only if there exists for every loop $(a, e) \in Q_1//Q_0$ a path p in Z such that $p^{(a,e)} \neq 0$. This is for example the case if Q does not have a loop or if the characteristic of the field k is zero.

This remark shows that the case where L_{-1} is different from zero is quite exceptional. We will assume henceforth that $L_{-1} = 0$. In that case $\bigoplus_{i \geqslant 1} L_i$ is a solvable Lie ideal of $\mathrm{H}^1(\Lambda, \Lambda)$ since $\mathrm{H}^1(\Lambda, \Lambda)$ is finite dimensional. It is obvious that L_0 is a Lie subalgebra of $\mathrm{H}^1(\Lambda, \Lambda)$ whose bracket is

$$\big[(a, c), (b, d)\big] = \delta_{a,d}(b, c) - \delta_{b,c}(a, d)$$

for all $(a, c), (b, d) \in L_0$. Hence $\operatorname{Rad}\mathrm{H}^1(\Lambda, \Lambda) = \operatorname{Rad} L_0 \oplus \bigoplus_{i \geqslant 1} L_i$ and $\mathrm{H}^1(\Lambda, \Lambda)/\operatorname{Rad}\mathrm{H}^1(\Lambda, \Lambda) = L_0/\operatorname{Rad} L_0$ where $\operatorname{Rad}\mathrm{H}^1(\Lambda, \Lambda)$ (resp. $\operatorname{Rad} L_0$) denotes the radical of $\mathrm{H}^1(\Lambda, \Lambda)$ (resp. L_0). As a consequence the study of the Lie algebra $\mathrm{H}^1(\Lambda, \Lambda)$ can be reduced often to the study of the Lie algebra L_0. A basis $\mathcal{B}$ of the Lie algebra L_0 is given by the union of the following sets:

(1) all the couples $(a, b) \in L_0$ such that the parallel arrows a and b are different,

(2) for every class $\overline{\alpha} = \{\alpha_1, \ldots, \alpha_n\} \in Q_1//$ all the elements $(\alpha_i, \alpha_i) \in L_0$ such that $i < n$,

(3) $|Q_1//| - |Q_0| + 1$ linearly independent elements $(c, c) \in L_0$ different from those in (2).

We recall a definition introduced by Guil-Asensio and Saorín (*see* 2.3 in [5] and 25 in [4]):

DEFINITION 2.2. – We shall call two parallel arrows equivalent if $p^{(a,b)} = 0 = p^{(b,a)}$ for all $p \in Z$. The ideal $\langle Z \rangle$ is called completely saturated if all parallel arrows are equivalent. The set of classes of parallel arrows is denoted by $Q_1//$ and we denote by $\overline{\alpha}/\approx$ the set of equivalence classes of $\overline{\alpha} \in Q_1//$.

THEOREM 2.3. – *The radical of the Lie algebra L_0 is generated as a k-vector space by the following elements of L_0: for every class of parallel arrows $\overline{\alpha} = \{\alpha_1, \ldots, \alpha_n\}$*

- $\sum_{\alpha_i \in S} (\alpha_i, \alpha_i)$ *for every equivalence class $S \in \overline{\alpha}/\approx$;*
- (α_i, α_j), $i \neq j$, *if $(\alpha_i, \alpha_j) \in L_0$ and $(\alpha_j, \alpha_i) \notin L_0$;*
- $(\alpha_{i_1}, \alpha_{i_2})$, $(\alpha_{i_2}, \alpha_{i_1})$ *and* $(\alpha_{i_1}, \alpha_{i_1})$ *for all $S = \{\alpha_{i_1}, \alpha_{i_2}\} \in \overline{\alpha}/\approx$ if* $\operatorname{char} k = 2$.

The proof of this theorem uses the explicit basis of L_0 and the fact that Λ is a finite dimensional algebra (for more details see the proof of the Theorem 4.10 in [8]). It shows explicitly that the semi-simple Lie algebra $\mathrm{H}^1(\Lambda, \Lambda)/\operatorname{Rad}\mathrm{H}^1(\Lambda, \Lambda) = L_0/\operatorname{Rad} L_0$ is the product of Lie algebras having a factor $\mathfrak{pgl}(|S|, k) := \mathfrak{gl}(|S|, k)/k1$ for every equivalence class $S \in \overline{\alpha}/\approx$ of a class of parallel arrows $\overline{\alpha}$ such that $|S| \geqslant 2$ if $\operatorname{char} k \neq 2$ and $|S| > 2$ if $\operatorname{char} k = 2$.

COROLLARY 2.4. – *If $L_{-1} = k(Q_0//Q_1) \cap \operatorname{Ker}\psi_1 = 0$, then the following conditions are equivalent*:

(1) *The Lie algebra $\mathrm{H}^1(\Lambda, \Lambda)$ is solvable.*

(2) *Every equivalence class S of a class of parallel arrows $\overline{\alpha} \in Q_1//$ of Q contains one and only one arrow if the characteristic of the field k is not* 2. *In the case* $\operatorname{char} k = 2$ *we have $|S| \leqslant 2$ for all $S \in \overline{\alpha}/\approx$, $\overline{\alpha} \in Q_1//$.*

Using Propostion 1.1 and Remark 2.1 we recover the following result due to Guil-Asensio and Saorín (*see* Corollary 2.22 in [5]):

COROLLARY 2.5. – *Let k be a field of characteristic* 0*. Then the identity component of the algebraic group of outer automorphisms* $\mathrm{Out}(\Lambda)^\circ$ *is solvable if and only if every two parallel arrows of Q are not equivalent.*

Let $\overline{Q}$ be the subquiver of Q obtained by taking a representative for every class of parallel arrows of Q.

THEOREM 2.6. – *If $L_{-1} = k(Q_0//Q_1) \cap \operatorname{Ker} \psi_1 = 0$, then the following conditions are equivalent*:

(1) *The Lie algebra* $\mathrm{H}^1(\Lambda, \Lambda)$ *is semi-simple.*
(2) *The quiver $\overline{Q}$ is a tree, Q has at least one non trivial class of parallel arrows and the ideal $\langle Z \rangle$ is completely saturated. If the characteristic of the field k is* 2*, then Q does not have a class of parallel arrows containing exactly two arrows.*
(3) *The Lie algebra* $\mathrm{H}^1(\Lambda, \Lambda)$ *is isomorphic to the non trivial product of Lie algebras* $\prod_{\overline{\alpha} \in Q_1//} \mathfrak{pgl}(|\overline{\alpha}|, k)$ *where $|\overline{\alpha}| \neq 2$ if the characteristic of k is equal to* 2*.*

Remark 2.7. – The Lie algebra $\mathfrak{pgl}(n, k) := \mathfrak{gl}(n, k)/k1$, $n \geqslant 2$, is isomorphic to the classical simple Lie algebra $\mathfrak{sl}(n, k)$ of $n \times n$-matrices having trace zero if $\operatorname{char} k$ does not divide n. If $\operatorname{char} k$ divides n and $n \neq 2$, then $\mathfrak{pgl}(n, k)$ is a semi-simple algebra without being a direct product of simple Lie algebras.

Using Proposition 1.1 and the main Theorem 2.20 of the article [5] we get the following Corollary (*see* Corollary 4.9 in [5] for a partial result on hereditary algebras):

COROLLARY 2.8. – *Let k be a field of characteristic* 0*. The following conditions are equivalent*:

(1) *The algebraic group* $\mathrm{Out}(\Lambda)^\circ$ *is semi-simple.*
(2) *The quiver $\overline{Q}$ is a tree, the quiver Q has at least one non trivial class of parallel arrows and the ideal $\langle Z \rangle$ is completely saturated.*
(3) $\mathrm{Out}(\Lambda)^\circ$ *is isomorphic to the non trivial product of algebraic groups* $\prod_{\overline{\alpha} \in Q_1//} \mathbf{PGL}(|\overline{\alpha}|, k)$.

Finally we give a criterion for the commutativity and the nilpotency:

PROPOSITION 2.9. – *If the characteristic of the field k is* 0*, then the following conditions are equivalent*:

(1) *The Lie algebra* $\mathrm{H}^1(\Lambda, \Lambda)$ *is abelian.*
(2) *The Lie algebra* $\mathrm{H}^1(\Lambda, \Lambda)$ *is nilpotent.*
(3) *The dimension of* $\mathrm{H}^1(\Lambda, \Lambda)$ *equals the Euler characteristic* $|Q_1| - |Q_0| + 1$.
(4) *For every path $\gamma \in B$ parallel to an arrow $a \neq \gamma$ there exists a path p of Z such that* $p^{(a,\gamma)} \neq 0$.
(5) *The algebraic group* $\mathrm{Out}(\Lambda)^\circ$ *is nilpotent.*
(6) *The algebraic group* $\mathrm{Out}(\Lambda)^\circ$ *is abelian.*

Acknowledgements. The results will form part of my thesis. I am happy to thank my advisor Claude Cibils for his help and comments.

References

[1] M.J. Bardzell, The alternating syzygy behavior of monomial algebras, J. Algebra 188 (1) (1997) 69–89.
[2] C. Cibils, M. Saorín, The first cohomology group of an algebra with coefficients in a bimodule, J. Algebra 237 (1) (2001) 121–141.
[3] M. Gerstenhaber, The cohomology structure of an associative ring, Ann. Math. 78 (2) (1963) 267–288.
[4] F. Guil-Asensio, M. Saorín, The group of outer automorphisms and the Picard group of an algebra, Algebr. Represent. Theory 2 (4) (1999) 313–330.
[5] F. Guil-Asensio, M. Saorín, The automorphism group and the Picard group of a monomial algebra, Comm. Algebra 27 (2) (1999) 857–887.
[6] J.E. Humphreys, Linear Algebraic Groups, Graduate Texts in Math., Vol. 21, Springer, Berlin, 1975.
[7] B. Keller, Hochschild cohomology and derived Picard groups, Lecture at the international workshop “Representation Theory around the Channel”, Amiens, France, 31 August–1 September 2001.
[8] C. Strametz, The Lie algebra structure on the first Hochschild cohomology group of a monomial algebra, see http://arXiv.org/abs/math.RT/0111060.