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Probabilités/*Probability Theory*

Measure transport on Wiener space and the Girsanov theorem

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Abstract Let (W, H, μ) be an abstract Wiener space, and assume that ν_i, $i = 1, 2$, are two probability measures on $(W, \mathcal{B}(W))$ which are absolutely continuous with respect to μ. Assume that the Wasserstein distance between ν_1 and ν_2 is finite. Then there exists a map $T = I_W + \xi$ of W into itself such that $\xi : W \to H$ is measurable and 1-cyclically monotone such that the image of ν_1 under T is ν_2. Moreover T is invertible on the support of ν_2. We give also some applications of this result such as the existence of the solutions of the Monge–Ampère equation in infinite dimensions. ***To cite this article: D. Feyel, A.S. Üstünel, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 1025–1028.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Transport de mesure sur l'espace de Wiener et théorème de Girsanov

Résumé Soit (W, H, μ) un espace de Wiener abstrait ; on suppose que ν_i, $i = 1, 2$, sont deux probabilités sur $(W, \mathcal{B}(W))$ qui sont absolument continues par rapport à μ et que la distance de Wasserstein entre ν_1 et ν_2 est finie. Alors il existe une application $T = I_W + \xi$ de W dans lui-même telle que $\xi : W \to H$ soit mesurable, 1-cycliquement monotone et l'image de ν_1 sous T soit égale à ν_2. De plus T est inversible sur le support de ν_2. Nous donnons aussi quelques applications de ce résultat comme l'existence de solutions de l'équation de Monge–Ampère. ***Pour citer cet article : D. Feyel, A.S. Üstünel, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 1025–1028.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

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Soit (W, H, μ) un espace de Wiener abstrait : W est un Fréchet localement convexe, μ est la mesure gaussienne standard et H est l'espace de Cameron-Martin dont le produit scalaire et la norme sont notés respectivement par $(\cdot,\cdot)_H$ et $|\cdot|_H$. Soit ν une autre probabilité et notons par $\Sigma(\mu, \nu)$ l'ensemble de probabilités sur $W \times W$ ayant les marginales μ et ν. On note J la fonctionnelle définie sur $\Sigma(\mu, \nu)$ par $J(\beta) = \int_{W\times W} |x - y|_H^2 \, \mathrm{d}\beta(x, y)$. Dans le cas où W est de dimension finie, le problème (dual) de Monge–Kantorovitch consiste de trouver une mesure $\gamma \in \Sigma(\mu, \nu)$ telle que la distance de Wasserstein :

$$d_W^2(\mu, \nu) = \inf\bigl\{J(\beta) : \beta \in \Sigma(\mu, \nu)\bigr\}$$

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soit atteinte en γ. Ce problème a été résolu dans [1] en dimension finie (*cf.* aussi [4]). Nous allons le résoudre dans le cas décrit ci-dessous. Rappelons d'abord quelques notions : soit $A \subset \mathbb{R}^n \times \mathbb{R}^n$, on dit que A est cycliquement monotone [5] si pour toute suite finie $(x_1, y_1), \ldots, (x_k, y_k)$ de A, on a

$$(y_1, x_2 - x_1)_{\mathbb{R}^n} + (y_2, x_3 - x_2)_{\mathbb{R}^n} + \cdots + (y_k, x_1 - x_k)_{\mathbb{R}^n} \leqslant 0.$$

Soit $(\pi_n,\ n \geqslant 1)$ une suite de projection de H de rangs finis, croissant vers l'identité de H, continûment prolongeable à W, telle que $\pi_n x \to x$ μ-p.s. dans W. On notera $x_n = \pi_n x$ et $x_n^\perp = x - x_n$ et si $z = (x, y) \in W \times W$, on écrira $z_n = q_n z = (\pi_n x, \pi_n y)$, $z_n^\perp = z - z_n$. Si $S \subset W \times W$, on notera par $S(z_n^\perp)$ la section de S en $z_n^\perp$, i.e. $S(z_n^\perp) = \{z_n \in W_n \times W_n : z_n + z_n^\perp \in S\}$. Nous dirons que l'ensemble S est cycliquement monotone si $S(z_n^\perp)$ est cycliquement monotone dans $W_n \times W_n \simeq \mathbb{R}^n \times \mathbb{R}^n$ pour tout $z_n^\perp$. Nous dirons qu'une variable aléatoire $\eta : W \to W$ est 1-cycliquement monotone si le graph de $x \to x + \eta(x)$ est cycliquement monotone dans $W \times W$. Une variable aléatoire $f : W \to \mathbb{R} \cup \{\infty\}$ est dite 1-convexe si $h \to \frac{1}{2}|h|_H^2 + f(w + h)$ est convexe sur H à valeurs dans $\mathbb{L}^0(\mu)$ [3]. Il est facile de voir que, si f a une dérivée de Sobolev ∇f [2,6], alors ∇f est 1-cycliquement monotone.

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1. Main results

THEOREM 1. – *Assume that ζ and ν are two probability measures on W such that* $\lim_n \pi_n x = x$ *almost surely with respect to ζ and ν in the Fréchet topology of W. Assume moreover that*

$$d_W(\zeta, \nu) < \infty,$$

and that the finite dimensional projections of these measures are absolutely continuous with respect to the Lebesgue measure. Let $\zeta_n = \pi_n \zeta$ and $\nu_n = \pi_n \nu$, and let γ_n be the measure defined on $W \times W$ as the obvious extension of the solution of the Monge–Kantorovitch problem for (ζ_n, ν_n) on $W_n \times W_n$. Let γ be in a weak cluster point of $(\gamma_n,\ n \geqslant 1)$. Then there exists a unique, 1*-cyclically monotone measurable map $\xi : W \to W$ such that $(I_W \times T)\zeta = \gamma$, where $T = I_W + \xi$. Moreover T is γ-invertible in the sense that there exists S such that $S \circ T = I_W$ ζ-a.e., and that $T \circ S = I_W$ ν-a.e.*

Proof. – We have the obvious disintegration of γ:

$$\int f \, \mathrm{d}\gamma = \iint_{S_\gamma(z_n^\perp)} f\big(z_n + z_n^\perp\big)\gamma\big(\mathrm{d}z_n | z_n^\perp\big)\gamma_n^\perp\big(\mathrm{d}z_n^\perp\big),$$

where $\gamma_n^\perp$ is the image of γ under $I_{W \times W} - q_n$ and $\gamma(\mathrm{d}z_n | z_n^\perp)$ is the regular conditional probability $\gamma(\mathrm{d}z_n | q_n^\perp = z_n^\perp)$. Using an extension of the section theorem, we see that the measure $\gamma(\mathrm{d}z_n | z_n^\perp)$ is the solution of the Monge–Kantorovitch problem corresponding to its marginals. Hence its support $S_\gamma(z_n^\perp)$ is cyclically monotone. The cyclic monotonicity of $S_\gamma(z_n^\perp)$ implies the existence of a monotone map $T_n(x_n; z_n^\perp) = x_n + \xi_n(x_n; z_n^\perp)$ which is the derivative of a convex function on W_n [4] such that the the regular conditional probability $\gamma(\cdot\, | z_n^\perp)$ is supported by the set $\{(x_n, y_n) \in W_n \times W_n : y_n = T_n(x_n; z_n^\perp)\}$. Hence γ is carried by the set

$$\big\{(x, y) \in W \times W : y_n = T_n\big(x_n; z_n^\perp\big)\big\}.$$

Moreover $\pi_n x \to x$ and $\pi_n y \to y$ in W γ-almost surely, hence the sequence $(\xi_n(\pi_n x; z_n^\perp),\ n \geqslant 1)$ converges also γ-almost surely in W. Note that each ξ_n is measurable with respect to the sigma algebra $\mathcal{B}(W) \otimes \sigma(q_n^\perp)$. Hence the limit is $\bigcap_n \mathcal{B}(W) \otimes \sigma(q_n^\perp) = \mathcal{B}(W) \otimes \bigcap_n \sigma(q_n^\perp)$-measurable. Since $\bigcap_n \sigma(q_n^\perp)$ is γ-trivial, $\xi = \lim \xi_n$ is $\mathcal{B}(W)$-measurable. The fact that T is invertible is obvious from the symmetry of the construction of T. The uniqueness of T follows from the fact that the finite dimensional sections of the support of γ are cyclically monotone, hence the approximating sequence is unique by the finite dimensional results of Mc Cann [4]. □

THEOREM 2. – *Assume that ν is given with a density $L \in \mathbb{L}^1(\mu)$ such that $E[L\log^+ L] < \infty$. Then there exists a 1-convex function ϕ whose Sobolev derivative $\nabla\phi$ is in $\mathbb{L}^2(\mu, H)$ such that $T = I_W + \nabla\phi$ and the solution of the Monge–Kantorovitch problem is given by $\gamma = (I_W \times T)\mu$. If $L > 0$ almost surely, then $T^{-1} = I_W + \nabla\psi$ such that ψ is also 1-convex, $\nabla\phi(x) + \nabla\psi(y) = 0$ γ-a.s. and finally*

$$E\left[|\nabla\phi|_H^2\right] \leqslant 2E\left[L\log^+ L\right]. \tag{1.1}$$

Proof. – The existence of T and its invertibility is given by Theorem 1. To show the regularity of $\xi = \nabla\phi$, we use the inequality (1.1) whose validity is known in the finite dimensional case and this inequality is preserved when we pass to the limit. To show that T^{-1} is of the form $I_W + \nabla\psi$ is suffices to remark that the Radon–Nikodym derivative of $T^{-1}\mu$ with respect to μ is $(L \circ T)^{-1}$, hence it satisfies also the $\mathbb{L}\log\mathbb{L}$-condition. □

The following proposition gives some more information about the question studied in Theorem 2.7.1 of [7].

PROPOSITION 1. – *Let μ be the Wiener measure and denote by ν_i, $i = 1, 2$, two probability measures on $(W, \mathcal{B}(W))$, which are absolutely continuous with respect to μ with the respective Radon–Nikodym derivatives L_i, $i = 1, 2$. Assume that*

$$E_\mu[L_i \log L_i] < \infty, \quad i = 1, 2.$$

Then there exists an invertible map $T_{1,2} = I_W + \xi_{1,2}$ such that $\xi_{1,2}$ is with values in H and that $T_{1,2}\nu_1 = \nu_2$.

Proof. – The existence of $T_{1,2}$ follows from Theorem 1. The only claim to prove is the regularity of $\xi_{1,2}$. This follows from the triangle inequality

$$\begin{aligned} d_W(\nu_1, \nu_2) &= \left\{\int_{W\times W} |x-y|_H^2 \gamma_{1,2}(\mathrm{d}x, \mathrm{d}y)\right\}^{1/2} \\ &\leqslant d_W(\nu_1, \mu) + d_W(\nu_2, \mu) < \infty, \end{aligned}$$

where d_W denotes the Wasserstein metric and the finiteness of $d_W(\nu_i, \mu)$, $i = 1, 2$, follow from the $\mathbb{L}\log\mathbb{L}$-integrability of the Radon–Nikodym densities. □

2. Some applications

2.1. Monge–Ampère equation

Assume that in Theorem 3, ν_1 is equivalent to μ and $L_i \in \mathbb{L}\log\mathbb{L}$ then $T = I_W + \nabla\phi$ is μ-almost surely invertible, hence $\sigma(T) = \mathcal{B}(W)$. Assume also that $\nabla\phi$ is regular enough in such a way that $\Lambda_T\,\mathrm{d}\mu$ is a Girsanov measure for T and μ (*cf.* [7], p. 14), where

$$\Lambda_T = \det{}_2(I_H + \nabla\xi)\exp\left\{-\delta\xi - \frac{1}{2}|\xi|_H^2\right\}.$$

Then T is a monotone solution of the following infinite dimensional of Monge–Ampère equation:

$$\Lambda_T\, L_2 \circ T = L_1$$

μ-almost surely.

2.2. An inequality for $d_W(\mu, \nu)$

Assume that W is equipped with a filtration of sigma algebras in such a way that it becomes a classical Wiener space as $W = \mathrm{C}_0([0,1], \mathbb{R}^d)$. Let L be a strictly positive random variable with $E[L] = 1$ and that $E[L \log^+ L] < \infty$. We can represent it as

$$L = \exp\left[-\delta u - \frac{1}{2}|u|_H^2\right],$$

where u is an H-valued, adapted random variable with $E[|u|_H^2] < \infty$. Let $U : W \to W$ be the map $U(x) = x + u(x)$. Note that, due to the adaptedness of u, its divergence δu can be interpreted as an Ito integral:

$$\delta u = \int_0^1 \left(\dot{u}_s, \mathrm{d}W_s\right).$$

Let also $T = I_W + \nabla\phi$ be the transport map corresponding to L, i.e., the map which satisfies

$$E[f \circ T] = E[f L],$$

for any $f \in \mathrm{C}_b(W)$. It follows from the Girsanov theorem and Theorem 2, that

$$E[f \circ U \circ T] = E[f \circ U L] = E[f],$$

hence the map $s = U \circ T$ is a rotation, i.e., it preserves the Wiener measure. The Girsanov theorem for U implies immediately that

$$E[L \log L] = \frac{1}{2} E\left[L|u|_H^2\right].$$

Again by the Girsanov theorem we have $U\nu = \mu$, let β be the measure on $W \times W$ defined as $\beta = (U \times I_W)\nu$. Then the first marginal of β is μ and the second one is ν. Consequently, denoting by γ the solution of $\Sigma(\mu, \nu)$, we have

$$\begin{aligned}\int_W |\nabla\phi|_H^2 \,\mathrm{d}\mu &= \int_{W\times W} |x-y|_H^2 \,\mathrm{d}\gamma(x,y) \leqslant \int_{W\times W} |x-y|_H^2 \,\mathrm{d}\beta(x,y) = \int_W |u|_H^2 L \,\mathrm{d}\mu \\ &= 2\int_W L \log L \,\mathrm{d}\mu.\end{aligned}$$

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