C. R. Acad. Sci. Paris, Ser. I 334 (2002) 661–666

Équations aux dérivées partielles/*Partial Differential Equations*

Viscosity solutions to the degenerate oblique derivative problem for fully nonlinear elliptic equations

Petar Popivanov, Nickolai Kutev

Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, 1113 Sofia, Bulgaria

Received 3 December 2001; accepted after revision 4 February 2002

Note presented by Pierre-Louis Lions.

Abstract In this paper we prove a comparison principle between the semicontinuous viscosity sub- and supersolutions of the tangential oblique derivative problem and the mixed Dirichlet–Neumann problem for fully nonlinear elliptic equations. By means of the comparison principle, the existence of a unique viscosity solution is obtained. ***To cite this article: P. Popivanov, N. Kutev, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 661–666.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Solutions visqueuses du problème avec une dérivée oblique dégénérée pour une classe d'équations complètement non linéaires

Résumé On démontre dans cette Note un principe de comparaison entre les sous et supersolutions visqueuses semi-continues du problème avec une dérivée oblique tangentielle et aussi le problème mixte du type de Dirichlet–Neumann pour une classe d'équations elliptiques complètement non-linéaires. En appliquant ce principe de comparaison on démontre l'existence d'une solution visqueuse unique. ***Pour citer cet article : P. Popivanov, N. Kutev, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 661–666.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Version française abrégée

Dans cette Note on considère le problème avec une dérivée oblique tangentielle

$$F\bigl(x,u,Du,D^2u\bigr)=0 \quad \text{dans } \Omega, \qquad B(x,u,Du)=0 \quad \text{sur } \partial\Omega\setminus E,$$

où $B(x,u,Du)=\partial u/\partial l+b(x,u)$, $\partial/\partial l=\sum_{i=1}^n a^i(x)\partial/\partial x_i$, F est un opérateur elliptique non linéaire, Ω est un domaine borné dans $\mathbf{R}^n$ et le champ vectoriel nondégénéré $l(x)$ est tangentiel à $\partial\Omega$ en les points d'une variété fermée et lisse E de $\partial\Omega$ de dimension $n-2$, mais $l(x)$ n'est pas tangentiel à E.

E-mail address: popivano@math.bas.bg (P. Popivanov).

S1631-073X(02)02321-X/FLA

Dans le cas linéaire ce problème a été considéré pour la première fois par H. Poincaré en étudiant les marées basses et les marées montantes dans les océans. D'un point de vue mathématique le problème est très intéressant car la condition de Schapiro–Lopatinskii est violée sur E.

Notons par $\nu(x)$ la normale unitaire extérieure à $\partial\Omega$ et par $\langle l(x), \nu(x)\rangle$ le produit scalaire entre $l(x)$ et $\nu(x)$. Dans la théorie linéaire on considère les trois cas suivants de la structure géométrique de $l(x)$:

(i) $\langle l(x), \nu(x)\rangle$ conserve son signe sur $\partial\Omega$,

(ii) $\langle l(x), \nu(x)\rangle$ change de signe sur $\partial\Omega$ de plus à moins en traversant E dans la direction du champ vectoriel $l|_E$,

(iii) $\langle l(x), \nu(x)\rangle$ change de signe sur $\partial\Omega$ de moins à plus en traversant E dans la direction de $l|_E$.

Les propriétés de notre problème au bord dans les trois cas précédents sont très différentes. Par exemple, dans le cas (ii) une condition supplémentaire $u|_E = \varphi(x)$ est toujours imposée, tandis que dans le cas (iii) la solution est discontinue sur E en général.

On démontre dans cette Note un principe de comparaison entre les sous et supersolutions visqueuses semi-continues de notre problème au bord non linéaire dans les cas (i) et (ii), $u|_E = \varphi$. En utilisant ce principe de comparaison et la méthode de Perron on obtient l'existence d'une solution visqueuse unique du problème non linéaire dans les cas (i) et (ii), $u|_E$.

A l'aide de la méthode des solutions visqueuses on étudie d'une manière similaire le problème mixte du type de Dirichlet–Neumann pour la même équation avec les données au bord suivantes :

$B_1(x,u,Du) = 0$ sur $\partial\Omega$, où $B_1(x,u,Du) = \partial u/\partial l + b(x,u)$ sur Γ_1, $B_1(x,u,Du) = u - \varphi(x)$ sur Γ_2 et $\Gamma_1 = \{x \in \partial\Omega;\ \langle l(x), \nu(x)\rangle > 0\}$, $\Gamma_1 \cup \Gamma_2 = \partial\Omega$, $\Gamma_1 \cap \Gamma_2 = \emptyset$, $\overline{\Gamma}_1 \cap \overline{\Gamma}_2 = E$.

Let Ω be a bounded domain in $\mathbf{R}^n$ with $C^{2,1}$ smooth boundary. We consider the tangential oblique derivative problem

$$F\big(x,u,Du,D^2u\big) = 0 \quad \text{in } \Omega, \qquad B(x,u,Du) = 0 \quad \text{on } \partial\Omega \setminus E, \tag{1}$$

$$B(x,u,Du) = \frac{\partial u}{\partial l} + b(x,u), \qquad \frac{\partial}{\partial l} = \sum_{i=1}^{n} a^i(x)\frac{\partial}{\partial x_i}, \tag{2}$$

where $a^i(x) \in C^{1,1}(\overline{\Omega})$, $b(x,u) \in C^{0,1}(\overline{\Omega} \times \mathbf{R})$, $F(x,r,p,X) \in C(\overline{\Omega} \times \mathbf{R} \times \mathbf{R}^n \times S^n)$. Here S^n is the space of all $n \times n$ symmetric matrices and the nonzero vector field l is tangential to $\partial\Omega$ at some closed $n-2$-dimensional $C^{2,1}$ smooth submanifold E of $\partial\Omega$, but is not tangential to E, i.e., $\langle l(x), \nu(x)\rangle \neq 0$ on $\partial\Omega \setminus E$, $\langle l(x), \nu(x)\rangle = 0$ on E, but $\langle l(x), \tau(x)\rangle \neq 0$ on E for every nonzero vector $\tau(x)$ tangential to E, where $\nu(x)$ is the unit outer normal to $\partial\Omega$.

In the theory of the tangential oblique derivative problem three cases are possible:

$$\big\langle l(x), \nu(x)\big\rangle \text{ preserves its sign on } \partial\Omega; \tag*{(3)$_{\rm i}$}$$

$$\big\langle l(x), \nu(x)\big\rangle \text{ changes its sign on } \partial\Omega \text{ through } E \text{ from minus to plus in the direction of the vector field } l(x)|_E; \tag*{(3)$_{\rm ii}$}$$

$$\big\langle l(x), \nu(x)\big\rangle \text{ changes its sign on } \partial\Omega \text{ through } E \text{ from plus to minus in the direction } l(x)|_E. \tag*{(3)$_{\rm iii}$}$$

The qualitative properties of the solutions in the above cases are quite different, for example, in $(3)_{\rm ii}$ in order to have uniqueness of the classical solutions for linear elliptic equations some extra condition is necessary on E, i.e.,

$$u(x) = \varphi(x) \quad \text{on } E \tag{4}$$

while in $(3)_{\rm iii}$ the solutions are in general discontinuous functions on E (*see* [11]).

To explain the big difference between $(3)_{\mathrm{ii}}$ and $(3)_{\mathrm{iii}}$ we consider the simplest case of linear elliptic operator A with smooth coefficients instead of F and linear term $b(x,u)=hu$:

$$Au=f(x),\quad x\in\Omega,\qquad \frac{\partial u}{\partial l}+hu=g(x),\quad x\in\partial\Omega. \tag{$*$}$$

In a standard way $(*)$ can be reduced locally to the solvability of the following first order pseudodifferential equation on the boundary $\partial\Omega$ near E: $P_\pm(y,D)w=\psi(y)$, $w=u|_{\partial\Omega}$, $y\in\mathbf{R}^{n-1}$ and $P_\pm$ has the principal symbol $\eta_1+\mathrm{i}y_1\theta(y,\eta)$, $\theta<0$ $(\theta>0)$ in the case $(3)_{\mathrm{ii}}$ ($(3)_{\mathrm{iii}}$). According to [6] the operator P_- is locally solvable and possesses an infinite dimensional kernel, while P_+ is hypoelliptic but locally nonsolvable in Schwartz distributions space. In order to kill the infinite dimensional $\ker P_-$ usually Cauchy type extra condition is prescribed: $w|_{y_1=0}=\varphi$. The situation in case $(3)_{\mathrm{iii}}$ is much more complicated because in general a solution of $(*)$ does not exist. As it is shown in [11], a classical solution of $(*)$ with $(3)_{\mathrm{iii}}$ exists if and only if the right-hand side (f,g) satisfies infinitely many compatibility conditions.

The previous comment motivates our considerations in the nonlinear case (1), (2) for vector fields satisfying the conditions $(3)_{\mathrm{i}}$, $(3)_{\mathrm{ii}}$. The case $(3)_{\mathrm{iii}}$ will not be considered here.

As for Eq. (1) we suppose that $F(x,r,p,X)$ is degenerate elliptic and uniformly monotone with respect to r, i.e.,

$$F(x,r,p,X)\leqslant F(x,r,p,Y), \tag{5}$$

$$\gamma(r-s)\leqslant F(x,r,p,X)-F(x,s,p,X) \tag{6}$$

whenever $Y\leqslant X$, $r\geqslant s$, $x\in\overline{\Omega}$, $r,s\in\mathbf{R}$, $p\in\mathbf{R}^n$, $X,Y\in S^n$, $\gamma=\text{const.}>0$.

Moreover, for the most general first order boundary operator (2) we suppose that $B(x,r,p)$ is monotone with respect to r, i.e.,

$$B(x,s,p)\leqslant B(x,r,p) \tag{7}$$

whenever $s\leqslant r$, $x\in\partial\Omega$, $r,s\in\mathbf{R}$, $p\in\mathbf{R}^n$, and B is oriented in the direction of the outer normal $\nu(x)$ to $\partial\Omega$, i.e.,

$$B\big(x,r,p-t\nu(x)\big)\leqslant B\big(x,r,p-s\nu(x)\big) \tag{8}$$

whenever $t\geqslant s$, $t,s\in\mathbf{R}$, $(x,r,p)\in\partial\Omega\times\mathbf{R}\times\mathbf{R}^n$.

The tangential oblique derivative problem (linear case) was considered for the first time by Poincaré in connection with the study of the high and low tides on the surface of the earth. However, the first important results on the subject were obtained about 50 years later. Another application of the same problem is in the probability theory (*see* the references in [11]). The problem (1), (2) is interesting from mathematical point of view even in the linear case because the well known Schapiro–Lopatinskii condition is violated on E and we have a nonclassical elliptic boundary value problem.

For the time being, the tangential oblique derivative problem was investigated with energy methods in the Sobolev spaces (*see* [3,5,11]) or with the Schauder technique in the Hölder spaces (*see* [4,12–14]) mainly for linear or weakly nonlinear elliptic and parabolic equations. Unfortunately, the Sobolev technique is applicable only for linear equations, while the Schauder method is useful for nonlinear problems, but under very restrictive conditions.

These problems were the motivation for us to investigate the tangential oblique derivative problem (1) with the method of viscosity solutions introduced by Crandall and Lions (*see* [1]). The advantages of the method of viscosity solutions are the minimal smoothness of the coefficients of the equations, the boundary operator and the solutions which are continuous. Nevertheless, the weak solution is unique. Moreover, the existence result is based on the Perron method [7,8,10] so that the subelliptic estimates of Hörmander and Egorov type are not necessary. In this way, general fully nonlinear, even degenerate elliptic equations can be investigated. Note that in the cases $(3)_{\mathrm{i}}$ and $(3)_{\mathrm{ii}}$ the method of viscosity solutions is naturally applicable while in the last case $(3)_{\mathrm{iii}}$ this method does not work.

Finally, let us note that the viscosity method allows us in a similar way to investigate the mixed Dirichlet–Neumann problem (9)

$$F\left(x, u, Du, D^2u\right) = 0 \quad \text{in } \Omega, \qquad B_1(x, u, Du) = 0 \quad \text{on } \partial\Omega, \tag{9}$$

where

$$B_1(x, u, Du) = \frac{\partial u}{\partial l} + b(x, u) \quad \text{on } \Gamma_1, \qquad B_1(x, u, Du) = u - \varphi(x) \quad \text{on } \Gamma_2$$

and

$$\Gamma_1 = \left\{x \in \partial\Omega;\ \left\langle l(x), \nu(x)\right\rangle > 0\right\}, \quad \Gamma_1 \cup \Gamma_2 = \partial\Omega, \quad \Gamma_1 \cap \Gamma_2 = \emptyset, \quad \overline{\Gamma}_1 \cap \overline{\Gamma}_2 = E, \quad \overline{\Gamma}_2 = \Gamma_2.$$

Further on, we will use the notations and definitions in [1] in order to formulate the main results.

Let us recall the main assumptions guaranteeing a comparison principle for semicontinuous viscosity sub- and supersolutions of (1) with nontangential Neumann's conditions (*see* [2, Theorem 7.5] or [9, Theorem 6.1]).

Suppose that

$$\left|F(x, r, p, X) - F(x, r, q, Y)\right| \leqslant \omega\left(|p - q| + \|X - Y\|\right) \tag{10}$$

for $x \in \overline{V}$, $p, q \in \mathbf{R}^n$, $X, Y \in S^n$, for some modulus of continuity $\omega(s)$, $\omega \in \mathrm{C}([0, \infty))$, $\omega(s) > 0$ for $s > 0$, $\omega(0) = 0$, where V is some oneside neighbourhood of $\partial\Omega$;

$$F(y, r, p, Y) - F(x, r, p, X) \leqslant \omega\left(N|x - y|^2 + |x - y|\left(|p| + 1\right)\right) \tag{11}$$

whenever $x, y \in \overline{\Omega}$, $r \in \mathbf{R}$, $p \in \mathbf{R}^n$, $X, Y \in S^n$ and the inequalities

$$-3N \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leqslant \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leqslant 3N \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} \tag{12}$$

hold for every constant $N \geqslant 1$.

Under conditions (5)–(8), (10)–(12) it was proved in [9, Theorem 6.2] (*see* also [2, Theorem 7.5]) that the comparison principle for semicontinuous viscosity sub- and supersolutions for the nontangential oblique derivative problem (1) is valid.

The following theorem shows that under the same conditions the comparison principle holds for the tangential oblique derivative problem in the case $(3)_{\mathrm{ii}}$ but with additional Dirichlet condition (4) on E.

THEOREM 1. – *Suppose* $(3)_{\mathrm{ii}}$, (5)–(8), (10)–(12) *hold. If* $u \in \mathrm{USC}(\overline{\Omega})$, $v \in \mathrm{LSC}(\overline{\Omega})$ *are, respectively, bounded viscosity sub- and supersolutions of* (1), (2) *with*

$$B(x, r, p) = \left\langle l(x), p\right\rangle + b(x, r) \quad \text{on } \partial\Omega_+ = \left\{x \in \partial\Omega;\ \left\langle l(x), \nu(x)\right\rangle > 0\right\},$$
$$B(x, r, p) = -\left\langle l(x), p\right\rangle - b(x, r) \quad \text{on } \partial\Omega_- = \left\{x \in \partial\Omega;\ \left\langle l(x), \nu(x)\right\rangle < 0\right\},$$

and $B(x, r, p) = r - \varphi(x)$ *on* E, *then* $u \leqslant v$ *in* $\overline{\Omega}$.

Moreover, if $\underline{u}$, $\overline{u}$, $\underline{u} \leqslant \overline{u}$, *are continuous viscosity sub- and supersolutions of* (1), (2) *satisfying the same Dirichlet data on* E, *i.e.*, $\underline{u} = \overline{u} = \varphi$ *on* E, *then there exists a unique viscosity solution* $u \in \mathrm{C}(\overline{\Omega})$ *such that* $\underline{u} \leqslant u \leqslant \overline{u}$ *in* $\overline{\Omega}$, $u = \varphi$ *on* E.

In order to prove the same result in the case $(3)_{\mathrm{i}}$ we need some precise anisotropic conditions of the type (11), (12) which take into account the direction $l(x)$ on E where the boundary operator is tangential to $\partial\Omega$. For this purpose, in a small compact neighbourhood V_0 of every boundary point $x_0 \in E$ we change the variables

$$x = \Phi(t), \quad \frac{D(\Phi^1, \ldots, \Phi^n)}{D(x_1, \ldots, x_n)} \neq 0 \tag{13}$$

so that the vector field $l(x)$ transforms into the constant direction $(0, \ldots, 1)$ in V_0 (*see* [3]). Suppose that in the new variables Eq. (1) transforms into the equation $\widetilde{F}(t, v(t), D_t v(t), D_t^2 v(t)) = 0$ in $\widetilde{V}_0$ where

$$\widetilde{F} = F\big(\Phi(t), u\big(\Phi(t)\big), D_t v \cdot \big(D_x\Phi^{-1}\big)\big(\Phi(t)\big), D_t^2 v \cdot \big(D_x\Phi^{-1}\big)\big(\Phi(t)\big) + D_t v \cdot \big(D_x^2\Phi^{-1}\big)\big(\Phi(t)\big)\big)$$

and V_0 transforms into $\widetilde{V}_0$ with the equation $\widetilde{V}_0 = \{t;\ t_n < \rho(t_1, \ldots, t_{n-1})\}$ and the modulus of continuity $\omega_0(\tau)$ of the function $\rho(t')$ near t_0', $t_0 = \Phi(x_0)$.

Using (13) we reformulate (11), (12) in a small neighbourhood of E in the following way

$$\widetilde{F}(t, r, p, Y) - \widetilde{F}(s, r, p, X) \leqslant \omega\big(N\big\langle I_N(t-s), t-s\big\rangle + |t'-s'||p'| + |t_n - s_n||p_n| + |t-s|\big) \tag{14}$$

whenever $t, s \in \widetilde{V}_0$, $r \in \mathbf{R}$, $p \in \mathbf{R}^n$, $X, Y \in S^n$ and the inequalities

$$-3N\begin{pmatrix} I_N & 0 \\ 0 & I_N \end{pmatrix} \leqslant \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leqslant 3N\begin{pmatrix} I_N & -I_N \\ -I_N & I_N \end{pmatrix} \tag{15}$$

hold for each constant $N \geqslant 1$, where $t = (t', t_n)$, $s = (s', s_n)$, $I_N = \{\mathrm{diag}(1, \ldots, 1, C_N)\}$, $\mathrm{C}_N = (N\omega_0(N^{-1/2}))^{-1}$.

THEOREM 2. – *Suppose* $(3)_{\mathrm{i}}$, (5)–(8) *and* (10)–(12), (14), (15) *hold. If* $u \in \mathrm{USC}(\overline{\Omega})$, $v \in \mathrm{LSC}(\overline{\Omega})$ *are, respectively, bounded viscosity sub- and supersolutions of* (1), (2) *with* $B(x, r, p) = \langle l(x), p\rangle + b(x, u)$ *on* $\partial\Omega$, *then* $u \leqslant v$ *in* $\overline{\Omega}$.

Moreover, if $\underline{u}$, $\overline{u}$ *are continuous viscosity sub- and supersolutions of* (1), (2), $\underline{u} \leqslant \overline{u}$, *then there exists a unique viscosity solution* $u \in \mathrm{C}(\overline{\Omega})$ *such that* $\underline{u} \leqslant u \leqslant \overline{u}$.

As for the mixed Dirichlet–Neumann problem (1), (9) we do not need the extra conditions (14), (15) and the comparison principle holds under the same assumptions as in Theorem 1.

THEOREM 3. – *Suppose* (5)–(8), (10)–(12) *hold. If* $u \in \mathrm{USC}(\overline{\Omega})$, $v \in \mathrm{LSC}(\overline{\Omega})$ *are, respectively, bounded viscosity sub- and supersolutions of* (1), (9), *then* $u \leqslant v$ *in* $\overline{\Omega}$.

Moreover, if $\underline{u}$, $\overline{u}$ $\underline{u} \leqslant \overline{u}$, *are continuous viscosity sub- and supersolutions of* (1), (9), *such that* $\underline{u} = \overline{u} = \varphi$ *on* Γ_2, *then there exists a unique viscosity solution* $u \in \mathrm{C}(\overline{\Omega})$, $\underline{u} \leqslant u \leqslant \overline{u}$, $u = \varphi$ *on* Γ_2.

From Theorems 1–3, it follows that the existence of a viscosity solution holds under the additional condition that the corresponding boundary value problem has a viscosity sub- and supersolution, respectively, $\underline{u}$, $\overline{u}$, such that $\underline{u} \leqslant \overline{u}$. The construction of upper and lower barriers, in general, is not a trivial problem because of the nonstandard boundary conditions. However under additional structural assumptions on the equations or on the domain we can construct such functions. For example, barriers exist if Eq. (1) is uniformly elliptic with subquadratic growth with respect to the gradient variables and the boundary vector field $l \neq 0$ is constant. The only case when no extra conditions are necessary is the case when the assumptions of Theorem 2 are fulfilled.

Acknowledgements. The second author was partially supported by EC project “Viscosity solutions and their applications.”

References

[1] M. Crandall, P.-L. Lions, Viscosity solutions of Hamilton–Jacobi equations, Trans. Amer. Math. Soc. 277 (1983) 1–42.
[2] M. Crandall, H. Ishii, P.-L. Lions, User's guide to viscosity solutions of second order partial differential equations, Bull. Amer. Math. Soc. 27 (1992) 1–67.
[3] Yu. Egorov, V. Kondratiev, On the oblique derivative problem, Math. Sb. 78 (1969) 148–176.
[4] P. Guan, E. Sawyer, Regularity estimates for the oblique derivative problem, Ann. Math. 137 (1993) 1–70.
[5] L. Hörmander, On the existence and regularity of solutions of linear pseudodifferential equations, Enseign. Math. 17 (1971) 99–163.
[6] L. Hörmander, The Analysis of Linear Differential Operators, Vol. IV, Springer-Verlag, Berlin, 1985.
[7] H. Ishii, Perron's method for Hamilton–Jacobi equations, Duke Math. J. 55 (1987) 369–384.
[8] H. Ishii, On uniqueness and existence of viscosity solutions of fully nonlinear second order elliptic PDE's, Comm. Pure Appl. Math. 42 (1989) 14–45.
[9] H. Ishii, P.-L. Lions, Viscosity solutions of fully nonlinear second-order elliptic partial differential equations, J. Differential Equations 83 (1990) 26–78.
[10] P.-L. Lions, Neumann type boundary conditions for Hamilton–Jacobi equations, Duke Math. J. 52 (1985) 793–820.
[11] B. Paneah, The Oblique Derivative Problem, Ser. Math. Topics, Vol. 17, Wiley–VCH, Berlin, 2000.
[12] P. Popivanov, N. Kutev, The tangential oblique derivative problem for nonlinear elliptic equations, Comm. Partial Differential Equations 16 (1989) 413–428.
[13] P. Popivanov, D. Palagachev, The Degenerate Oblique Derivative Problem for Elliptic and Parabolic Equations, Academie-Verlag–VCH, 1997.
[14] B. Winzell, A boundary value problem with an oblique derivative, Comm. Partial Differential Equations 6 (1981) 305–328.