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Équations aux dérivées partielles/*Partial Differential Equations*

Stabilization for viscous compressible heat-conducting media equations with nonmonotone state functions

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Abstract We consider the system of quasilinear equations for 1d-motion of viscous compressible heat-conducting media. The state function has the form $p(\eta,\theta) = p_0(\eta) + p_1(\eta)\theta$, with general nonmonotone p_0 and p_1, which allows us to treat both nuclear fluids and thermoviscoelastic solids (for fluids, p, η, and θ are the pressure, specific volume, and temperature). For an initial boundary value problem with large data, we establish stabilization as $t \to \infty$: pointwise and in L^q for η, in L^q for v (the velocity), for any $q \in [2,\infty)$, and in L^2 for θ. To cite this article: B. Ducomet, A. Zlotnik, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 119–124. © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Stabilisation pour un milieu continu compressible avec pression non monotone

Résumé Nous étudions l'évolution 1d d'un milieu continu compressible conducteur de la chaleur. La pression est donnée par $p(\eta,\theta) = p_0(\eta) + p_1(\eta)\theta$, où p_0 et p_1 sont des fonctions non monotones assez générales pour permettre de traiter à la fois des modèles de fluides nucléaires et des solides thermo-visco-élastiques. Pour un problème aux limites d'évolution associé, avec grandes données, nous prouvons la stabilisation pour $t \to \infty$ au sens suivant : convergence ponctuelle et dans L^q pour le volume spécifique η, dans L^q pour la vitesse v, pour tout $q \in [2,\infty)$, et dans L^2 pour la temperature θ. Pour citer cet article : B. Ducomet, A. Zlotnik, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 119–124. © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

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Les travaux concernant le comportement aux grands temps des solutions des équations gouvernant le mouvement des milieux continus compressibles et conducteurs de la chaleur supposent généralement que la loi de pression utilisée est monotone en densité. C'est le cas des gaz polytropiques. En revanche, dans certains modèles récents de fluides nucléaires de type Van der Waals [3] ou de solides thermoviscoélastiques [5], l'hypothèse de monotonie n'est pas satisfaite, ce qui, du point de vue physique, ne fait que traduire l'apparition de nouveaux aspects dynamiques dans les modèles : transition liquide–gaz, matériaux à mémoire de forme, etc.

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Précisons que l'abandon de la monotonie complique de manière essentielle (*voir* [4]) la preuve de l'existence globale d'une solution (grandes données) ainsi que l'étude de son comportement asymptotique (multiplicité variable des états stationnaires accessibles).

Dans cette Note, nous considérons une loi de pression $p(\eta,\theta) = p_0(\eta) + p_1(\eta)\theta$ assez générale pour permettre de traiter de manière assez complète à la fois le cas fluide [3] et le cas solide [5]. Nous supposons de plus que le milieu est soumis à une (grande) force de volume, de type autogravitation. Nous montrons, pour un problème aux limites d'évolution associé, et dans les deux situations physiques précitées, que la solution faible régulière converge, pour $t \to \infty$ au sens suivant : ponctuellement et dans L^q pour le volume specifique η, dans L^q pour la vitesse v, pour tout $q \in [2,\infty)$, et dans L^2 pour la température θ.

1. Statement of the problem and main results

Many papers are devoted to the problem of large-time behaviour of global solutions to the equations of 1d-motion of viscous compressible heat-conducting media, see [2,8,7,10] etc. The case of state function $p = p(\eta,\theta)$, particular or general but decreasing in η, was investigated; for fluids, p, η, and θ are the pressure, specific volume, and absolute temperature, respectively. The case of p nonmonotone in η, being important for a lot of applications, is essentially more difficult. For simplified equations of thermoviscoelasticity and $p(\eta,\theta) = p_0(\eta) + p_1(\eta)\theta$, its study was begun in [11,6,12]. In this paper we consider p of the same form with general nonmonotone p_0 and p_1 and treat both nuclear fluids [3] and thermoviscoelastic solids [5] (without the above mentioned simplifications). The mass force of "self-gravitation" type is also taken into account.

We study a system of quasilinear equations

$$\eta_t = v_x, \quad v_t = \sigma_x + g, \quad e[\eta,\theta]_t = \pi_x + \sigma v_x, \tag{1}$$

where $(x,t) \in Q \equiv \Omega \times \mathbf{R}^+ = (0,M) \times (0,\infty)$ are the Lagrangian mass coordinates. The unknown quantities are $\eta > 0$, v (the velocity), and $\theta > 0$. We also denote by $\rho = \eta^{-1}$ the density (for fluids), $\sigma = \nu\rho v_x - p[\eta,\theta]$ the stress, $e[\eta,\theta]$ the internal energy, $-\pi = -\kappa[\eta,\theta]\rho\theta_x$ the heat flux, and $g = g(x)$ the mass force. Hereafter the notation $\mu[\eta,\theta](x,t) = \mu(\eta(x,t),\theta(x,t))$ is adopted, for $\mu = e, p, \kappa$, etc.

Let us use the following Helmholtz free energy $\Psi(\eta,\theta) = -c_V\theta\log\theta - P_0(\eta) - P_1(\eta)\theta$, from which one gets $p(\eta,\theta) = p_0(\eta) + p_1(\eta)\theta$ and $e(\eta,\theta) = -P_0(\eta) + c_V\theta$, with $p_0 = P_0'$ and $p_1 = P_1'$.

First we consider the case of nuclear fluids. Suppose that $p_0, p_1 \in \mathrm{C}^1(\mathbf{R}^+)$ and

$$\lim_{\eta\to 0^+} p_0(\eta) = +\infty, \quad \lim_{\eta\to\infty} p_0(\eta) = 0; \qquad p_1(\eta) \geqslant 0, \quad \eta p_1(\eta) = \mathrm{O}(1) \quad \text{as } \eta\to\infty. \tag{2}$$

Suppose also that $\nu = \text{const} > 0$ and $\kappa \in \mathrm{C}^2(\mathbf{R}^+\times\mathbf{R}^+)$, with $0 < \underline{\kappa} \leqslant \kappa(\eta,\theta) \leqslant \overline{\kappa}$.

Let us supplement equations (1) with the following boundary and initial conditions

$$v|_{x=0} = 0, \quad \sigma|_{x=M} = -p_\Gamma, \quad \theta|_{x=0} = \theta_\Gamma, \quad \pi|_{x=M} = 0, \quad (\eta,v,\theta)|_{t=0} = \big(\eta^0(x), v^0(x), \theta^0(x)\big), \tag{3}$$

with an outer pressure $p_\Gamma = \text{const}$ and a given temperature $\theta_\Gamma = \text{const} > 0$.

We define the stationary pressure $p_S := p_\Gamma - I^*g$ with $(I^*g)(x) := \int_x^M g(\xi)\,\mathrm{d}\xi$, for $x \in \overline{\Omega}$, and set $\underline{p}_S := \min_{\overline{\Omega}} p_S$ and $\overline{p}_S := \max_{\overline{\Omega}} p_S$.

Let $\eta^0, v^0, \theta^0 \in \mathrm{H}^1(\Omega)$ with $\min_{\overline{\Omega}} \eta^0 > 0$, $\min_{\overline{\Omega}} \theta^0 > 0$, and $v^0(0) = 0$, $\theta^0(0) = \theta_\Gamma$ as well as $g \in \mathrm{L}^2(\Omega)$, $\underline{p}_S > 0$. Then we can assert that there exists a unique regular weak solution to problem (1), (3) such that, for any $T > 0$, $\eta \in \mathrm{C}(\overline{Q}_T)$, $\eta_x, \eta_t \in \mathrm{C}([0,T];\mathrm{L}^2(\Omega))$, $v, \theta \in \mathrm{H}^{2,1}(Q_T)$ with $\min_{\overline{Q}_T} \eta > 0$,

$\min_{\overline{Q}_T} \theta > 0$. Here $\mathrm{H}^{2,1}(Q_T) = \{w \in \mathrm{H}^1(Q_T) \mid w_{xx} \in \mathrm{L}^2(Q_T)\}$ and $Q_T = \Omega \times (0, T)$. Notice that really it is possible to prove our results for discontinuous data and weak solutions such as in [1].

Let $N > 1$ be an arbitrarily large parameter, $K_i = K_i(N)$ and $K^{(i)} = K^{(i)}(N)$, $i \geqslant 0$, be positive nondecreasing functions of N, which can also depend on M, ν, $\underline{\kappa}$, $\overline{\kappa}$ etc. but not on η^0, v^0, θ^0, and g.

THEOREM 1. – 1. *Suppose that the initial data, g, and p_Γ are such that*

$$N^{-1} \leqslant \eta^0 \leqslant N, \quad \|v^0\|_{\mathrm{L}^4(\Omega)} + \|\log\theta^0\|_{\mathrm{L}^1(\Omega)} + \|\theta^0\|_{\mathrm{L}^2(\Omega)} + \|g\|_{\mathrm{L}^1(\Omega)} \leqslant N, \quad N^{-1} \leqslant \underline{p}_S. \tag{4}$$

Then the following estimates in Q together with $\mathrm{L}^2(\Omega)$-stabilization properties hold

$$\left(K^{(1)}\right)^{-1} \leqslant \eta(x,t) \leqslant K^{(2)} \quad \text{in } \overline{Q}, \tag{5}$$

$$\|v\|_{V_2(Q)} \leqslant K^{(3)}, \quad \|\log\theta\|_{\mathrm{L}^{1,\infty}(Q)} + \|(\log\theta)_x\|_{\mathrm{L}^2(Q)} \leqslant K^{(4)}, \tag{6}$$

$$\|v^2\|_{V_2(Q)} + \|\theta - \theta_\Gamma\|_{V_2(Q)} \leqslant K^{(5)}, \quad \|p[\eta,\theta] - p_S\|_{\mathrm{L}^2(Q)} \leqslant K^{(6)}, \tag{7}$$

$$\|v^2(\cdot,t)\|_{\mathrm{L}^2(\Omega)} + \|\theta(\cdot,t) - \theta_\Gamma\|_{\mathrm{L}^2(\Omega)} \to 0, \quad \|p[\eta,\theta](\cdot,t) - p_S(\cdot)\|_{\mathrm{L}^2(\Omega)} \to 0 \quad \text{as } t \to \infty. \tag{8}$$

2. *Suppose that additionally, for any $c \in [\underline{p}_S, \overline{p}_S]$, there exists no interval (η_1, η_2) such that $p(\eta, \theta_\Gamma) \equiv c$ on (η_1, η_2). Then there exists a function $\eta_S \in \mathrm{L}^\infty(\Omega)$ satisfying*

$$p\big(\eta_S(x), \theta_\Gamma\big) = p_S(x) \quad \text{and} \quad \left(K^{(1)}\right)^{-1} \leqslant \eta_S(x) \leqslant K^{(2)} \quad \text{on } \overline{\Omega},$$

such that $\eta(x,t) \to \eta_S(x)$ as $t \to \infty$, for all $x \in \overline{\Omega}$. Consequently $\|\eta(\cdot,t) - \eta_S(\cdot)\|_{\mathrm{L}^q(\Omega)} \to 0$ as $t \to \infty$, for any $q \in [1, \infty)$.

3. *Suppose that, additionally to the the hypotheses of claim* 1, $\|v^0\|_{\mathrm{L}^q(\Omega)} \leqslant N$, *for some $q \in (4, \infty)$. Then the following estimate in Q together with $\mathrm{L}^q(\Omega)$-stabilization property hold*

$$\|v\|_{\mathrm{L}^{q,\infty}(Q)} + \|v\|_{\mathrm{L}^{\infty,q}(Q)} \leqslant qK^{(7)}, \quad \|v(\cdot,t)\|_{\mathrm{L}^q(\Omega)} \to 0 \quad \text{as } t \to \infty,$$

where $K^{(7)}$ does not depend on q.

In this result, $\|w\|_{V_2(Q)} = \|w\|_{\mathrm{L}^{2,\infty}(Q)} + \|w_x\|_{\mathrm{L}^2(Q)}$ and $\|\cdot\|_{\mathrm{L}^{q,r}(Q)} = \big\| \|\cdot\|_{\mathrm{L}^q(\Omega)} \big\|_{\mathrm{L}^r(\mathbf{R}^+)}$.

Let us justify that the third condition (4) is essential in Theorem 1, supposing that there exists a regular weak solution. Set $m(\theta_\Gamma) := \inf_{\eta > 0} p(\eta, \theta_\Gamma)$.

PROPOSITION 1. – *Let the hypotheses of Theorem* 1, *claim* 1, *be valid, excluding the third condition* (4).
1. *If $\underline{p}_S < m(\theta_\Gamma)$, then $\limsup_{t\to\infty} V(t) = \infty$, for $V(t) := \|\eta(\cdot,t)\|_{\mathrm{L}^1(\Omega)}$.*
2. *If $p(\eta, \theta_\Gamma) > 0$ for $\eta > 0$, $\int_1^\infty p(\eta, \theta_\Gamma)\,\mathrm{d}\eta < \infty$, and $\underline{p}_S = p_S(0) = 0$, then $\eta(0,t) \to \infty$ as $t \to \infty$.*

Finally, we consider the thermoviscoelastic case. Let $\underline{p}_S \leqslant \overline{p}_S$ be fixed (the condition $\underline{p}_S > 0$ is not required). Let, for some $0 < \check{\eta} \leqslant \hat{\eta} < \infty$, the following standard type conditions hold (instead of (2)):

$$\overline{p}_S \leqslant p_0(\eta) \quad \text{and} \quad 0 \leqslant p_1(\eta) \quad \text{for } 0 < \eta \leqslant \check{\eta}; \qquad p_0(\eta) \leqslant \underline{p}_S \quad \text{and} \quad p_1(\eta) \leqslant 0 \quad \text{for } \hat{\eta} \leqslant \eta. \tag{9}$$

THEOREM 2. – *All the claims* 1–3 *of Theorem* 4 *remain valid under conditions* (9), *and without the third condition* (4).

2. Proof of the results

Let us describe the scheme of the proof for Theorem 1. We start from the energy relation

$$D_t \int_\Omega \left(\frac{1}{2}v^2 + c_V\theta_\Gamma\left(\tilde\theta - \log\tilde\theta\right) + p_S\eta - P[\eta,\theta_\Gamma]\right)\mathrm{d}x + \theta_\Gamma\int_\Omega\left(\nu\rho\theta^{-1}v_x^2 + \kappa[\eta,\theta]\rho\theta^{-2}\theta_x^2\right)\mathrm{d}x = 0,$$

with $D_t \equiv \mathrm{d}/\mathrm{d}t$, $\tilde\theta := \theta/\theta_\Gamma$, and $P(\eta,\theta) := P_0(\eta) + P_1(\eta)\theta$. By using (4) and (2), it implies the estimates

$$\|\eta\|_{\mathrm{L}^{1,\infty}(Q)} + \|v\|_{\mathrm{L}^{2,\infty}(Q)} + \|\theta\|_{\mathrm{L}^{1,\infty}(Q)} + \|\log\theta\|_{\mathrm{L}^{1,\infty}(Q)} \leqslant K^{(8)}, \tag{10}$$

$$\left\|\sqrt{\rho\theta^{-1}}v_x\right\|_Q + \left\|\sqrt{\rho}\theta^{-1}\theta_x\right\|_Q \leqslant K^{(9)}. \tag{11}$$

Hereafter $\|\cdot\|_G = \|\cdot\|_{\mathrm{L}^2(G)}$.

Next we turn to the important equation (which follows from (1))

$$(\nu\log\eta)_t = p[\eta,\theta] - p_S - I^*v_t. \tag{12}$$

By putting $y := \nu\log\eta$ and fixing $x \in \overline{\Omega}$, we get the ordinary differential inequality

$$D_t y \geqslant p_0\left(\exp\left(\nu^{-1}y\right)\right) - \overline{p}_S - D_t I^* v.$$

The application to it of a slightly generalized Lemma 1.3 in [14] together with the estimate $|I^*v| \leqslant K_0 := M^{1/2}K^{(8)}$ lead to the lower bound $-K_1 \leqslant y$, i.e., $0 < \underline{\eta} := (K^{(1)})^{-1} \leqslant \eta$.

Let $\delta > 0$ be a parameter. The first and second equations in (1) yield the formulas

$$\eta = \exp\left(\nu^{-1}I_0(\sigma+\delta)\right)\left\{\eta^0 + I_0\left[\exp\left(-\nu^{-1}I_0(\sigma+\delta)\right)\eta\left(p[\eta,\theta]-\delta\right)\right]\right\},$$
$$I_0\sigma = -p_S t - I^*\left(v - v^0\right)$$

with $(I_0\phi)(t) := \int_0^t \phi(\tau)\,\mathrm{d}\tau$. We choose $\delta := \frac{1}{2}\underline{p}_S$, exploit the estimates $\underline{\eta} \leqslant \eta$ and $\|\theta\|_{\mathrm{L}^\infty(\Omega)} \leqslant K_2(1 + a\|\eta\|_{\mathrm{L}^\infty(\Omega)})$ with $a := \|\sqrt{\rho}\theta^{-1}\theta_x\|_\Omega^2$ (*see* [2]), and deduce the integral inequality

$$\left\|\eta(\cdot,t)\right\|_{\mathrm{L}^\infty(\Omega)} \leqslant K_3\left(1 + \mathrm{e}^{-\alpha_0 t}\int_0^t \mathrm{e}^{\alpha_0\tau}a(\tau)\left\|\eta(\cdot,\tau)\right\|_{\mathrm{L}^\infty(\Omega)}\mathrm{d}\tau\right)$$

with $\alpha_0 := \frac{1}{2}\underline{p}_S$. By using the estimate (11), it implies the upper bound $\eta \leqslant \overline{\eta} := K^{(2)}$, i.e., bounds (5) are proved. As a consequence, $\|\theta^{-1/2}v_x\|_Q + \|(\log\theta)_x\|_Q \leqslant \overline{\eta}^{1/2}K^{(9)}$.

Set $w := \frac{1}{2}v^2 + c_V(\theta - \theta_\Gamma)$. The total energy equation $w_t = (\sigma v + \pi)_x + p_0[\eta]v_x + gv$ together with the second equation (1) yield the relation

$$\frac{1}{2}D_t\int_\Omega\left(w^2 + \frac{\delta}{2}v^4\right)\mathrm{d}x + \int_\Omega\left[(1+3\delta)\nu\rho v^2 v_x^2 + c_V\kappa[\eta,\theta]\rho\theta_x^2\right]\mathrm{d}x = -\int_\Omega\left(\nu c_V + \kappa[\eta,\theta]\right)\rho v v_x\theta_x\,\mathrm{d}x$$
$$+ \int_\Omega\left[p_0[\eta]v_x w + p[\eta,\theta]\left((1+3\delta)v^2v_x + c_V v\theta_x\right)\right]\mathrm{d}x + \int_\Omega gv\left(w + \delta v^2\right)\mathrm{d}x - p_\Gamma\left[v\left(w+\delta v^2\right)\right]\Big|_{x=M}.$$

Introduce the quantities

$$Y := \int_\Omega\left(w^2 + \frac{\delta}{2}v^4\right)\mathrm{d}x, \qquad Z := \int_\Omega\left(v^2v_x^2 + \theta_x^2\right)\mathrm{d}x.$$

By applying the estimates $\underline{\eta} \leqslant \eta \leqslant \overline{\eta}$ and $\|\theta\|_{L^{1,\infty}} \leqslant K^{(8)}$ and choosing $\delta := K_4$ large enough, we get

$$D_t Y + K_5^{-1} Z \leqslant K_6(bY + h)$$

with $b := \|v\|^2_{L^\infty(\Omega)}$ and $h := \|\theta^{-1/2} v_x\|^2_\Omega$. Moreover

$$K_7^{-1}\left(\left\|v^2\right\|^2_\Omega + \|\theta - \theta_\Gamma\|^2_\Omega\right) \leqslant Y \leqslant K_7\left(\left\|v^2\right\|^2_\Omega + \|\theta - \theta_\Gamma\|^2_\Omega\right) \leqslant K_8 Z.$$

As $\|b\|_{L^1(\mathbf{R}^+)} \leqslant K^{(8)}\|h\|_{L^1(\mathbf{R}^+)} \leqslant K_9$, the inequalities for Y and Z imply the first estimate (7) and the first property (8).

The second equation (1) yields the relation

$$D_t \int_\Omega \left(\frac{1}{2}v^2 + p_S\eta - P[\eta, \theta_\Gamma]\right) \mathrm{d}x + \int_\Omega \nu\rho v_x^2 \,\mathrm{d}x = \int_\Omega p_1[\eta](\theta - \theta_\Gamma)\, v_x \,\mathrm{d}x.$$

By using the bounds $\underline{\eta} \leqslant \eta \leqslant \overline{\eta}$ and $\|\theta_x\|_Q \leqslant K^{(5)}$ we get $\|v\|_{V_2(Q)} \leqslant K^{(3)}$, i.e., the first estimate (6).

We rewrite equation (12) in the form $p[\eta, \theta] - p_S - I^* v_t = \nu\rho v_x$. It implies the following equality

$$\begin{aligned}&\left\|p[\eta, \theta] - p_S\right\|^2_{Q_T} + \left\|I^* v_t\right\|^2_{Q_T}\\ &\quad = \|\nu\rho v_x\|^2_{Q_T} + 2\left[\int_\Omega \big(p[\eta, \theta] - p_S\big) I^* v \,\mathrm{d}x\Big|_0^T - \int_{Q_T} \big(p_\eta[\eta, \theta]\eta_t I^* v - p_1[\eta](\theta - \theta_\Gamma) I^* v_t\big)\,\mathrm{d}x\,\mathrm{d}t\right],\end{aligned}$$

for any $T > 0$. By using the estimates $\underline{\eta} \leqslant \eta \leqslant \overline{\eta}$, $\|v\|_{V_2(Q)} \leqslant K^{(3)}$, and $\|\theta_x\|_Q \leqslant K^{(5)}$, we obtain the second estimate (7).

The last obtained estimate implies that $\|p[\eta, \theta_\Gamma] - p_S\|_Q \leqslant K_{10}$. Therefore

$$\int_0^\infty \left|D_t\left(\left\|p[\eta, \theta_\Gamma] - p_S\right\|^2_\Omega\right)\right| \mathrm{d}x \leqslant 2\left\|p_\eta[\eta, \theta_\Gamma]\right\|_{L^\infty(Q)} \|v_x\|_Q K_{10} \leqslant K_{11}.$$

So $\|p[\eta, \theta_\Gamma](\cdot, t) - p_S(\cdot)\|^2_\Omega \to 0$ as $t \to \infty$ which together with the first property (8) give the second one.

We need the following modification of the well known Ball–Pego lemma [9].

LEMMA 1. – *Let $f \in C(\mathbf{R})$ be such that, for a given constant f_S, there exists no interval (z_1, z_2) such that $f(z) \equiv f_S$ on (z_1, z_2). Let also $\alpha, h \in C(\mathbf{R}^+)$ be two functions such that $\alpha(t) \to 0$ as $t \to \infty$ and $|h| \leqslant |a| + |\beta|$, with $a, \beta \in C(\mathbf{R}^+)$, $a \in L^1(\mathbf{R}^+)$, and $\beta(t) \to 0$ as $t \to \infty$.*

If a function y satisfies $\sup_{t>0} |y(t)| < \infty$, $y \in W^{1,1}(0, T)$ for all $T > 0$, and

$$D_t y = f(y + \alpha) - f_S + h \quad \text{on } \mathbf{R}^+,$$

then $y(t) \to y_S$ as $t \to \infty$, and $f(y_S) = f_S$.

For any fixed $x \in \overline{\Omega}$, we rewrite equation (12) in the following form

$$D_t y = f(y + \alpha) - p_S + p_1[\eta](\theta - \theta_\Gamma),$$

with $y := \nu \log \eta - \alpha$, $\alpha := -I^* v$, and $f(z) := p(\exp(\nu^{-1} z), \theta_\Gamma)$. The bounds $\underline{\eta} \leqslant \eta \leqslant \overline{\eta}$ and $|I^* v| \leqslant K_0$ imply the estimates $\sup_{t \geqslant 0} |y(t)| \leqslant K_{12}$ and

$$\left|p_1[\eta](\theta - \theta_\Gamma)\right| \leqslant K_{13}\|\theta_x\|^{1/2}_\Omega \|\theta - \theta_\Gamma\|^{1/2}_\Omega \leqslant \|\theta_x\|^2_\Omega + K_{13}^{4/3}\|\theta - \theta_\Gamma\|^{2/3}_\Omega =: a + \beta.$$

It follows from the first property (8) that $\alpha(t) \to 0$ and $\beta(t) \to 0$ as $t \to \infty$; moreover $\|a\|_{L^1(\mathbf{R}^+)} = \|\theta_x\|_Q^2 \leqslant (K^{(5)})^2$. So Lemma 1 yields claim 2 of Theorem 4.

Claim 3 is proved in the same way as for the barotropic case in [13,15]. Proposition 1 also has its counterparts in [15].

To prove Theorem 2 only minor changes are required in the above proofs of the energy estimates (10) (without the first summand), (11), and the bounds (5). To this end we need the following estimates:

$$p_S\eta - P(\eta,\theta_\Gamma) \geqslant C := \min\Big\{-P(\check{\eta},\theta_\Gamma) + \overline{p}_S\check{\eta}, -\max_{\check{\eta}\leqslant\eta\leqslant\hat{\eta}} P(\eta,\theta_\Gamma), -P(\hat{\eta},\theta_\Gamma) + \underline{p}_S\hat{\eta}\Big\} \quad \text{for } \eta > 0,$$

$$p(\eta,\theta) - p_S(x) \geqslant 0 \quad \text{for } 0 < \eta \leqslant \check{\eta}; \qquad p(\eta,\theta) - p_S(x) \leqslant 0 \quad \text{for } \hat{\eta} \leqslant \eta,$$

with any $\theta > 0$ and $x \in \overline{\Omega}$, see conditions (9). Full proofs are given in [4].

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