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A sharp inequality for Sobolev functions

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Abstract Let $N \geqslant 5$, $a > 0$, Ω be a smooth bounded domain in $\mathbb{R}^N$, $2^* = \frac{2N}{N-2}$, $2^\# = \frac{2(N-1)}{N-2}$ and $\|u\|^2 = |\nabla u|_2^2 + a|u|_2^2$. We prove there exists an $\alpha_0 > 0$ such that, for all $u \in \mathrm{H}^1(\Omega) \setminus \{0\}$,

$$\frac{S}{2^{2/N}} \leqslant \frac{\|u\|^2}{|u|_{2^*}^2}\left(1 + \alpha_0 \frac{|u|_{2^\#}^{2^\#}}{\|u\| \cdot |u|_{2^*}^{2^*/2}}\right).$$

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Une inégalité dans un espace de Sobolev

Résumé Nous considérons $N \geqslant 5$, $a > 0$, Ω un ouvert borné régulier de $\mathbb{R}^N$, $2^* = \frac{2N}{N-2}$, $2^\# = \frac{2(N-1)}{N-2}$ et $||u||^2 = |\nabla u|_2^2 + a|u|_2^2$. Nous prouvons qu'il existe $\alpha_0 > 0$ tel que, pour toute fonction $u \in \mathrm{H}^1(\Omega) \setminus \{0\}$,

$$\frac{S}{2^{2/N}} \leqslant \frac{\|u\|^2}{|u|_{2^*}^2}\left(1 + \alpha_0 \frac{|u|_{2^\#}^{2^\#}}{\|u\| \cdot |u|_{2^*}^{2^*/2}}\right).$$

Cette inégalité implique l'inégalité de Cherrier. Pour citer cet article : P.M. Girão, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 105–108. © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Let $N \geqslant 5$, $a > 0$, $\alpha \geqslant 0$, Ω be a smooth bounded domain in $\mathbb{R}^N$, $2^* = \frac{2N}{N-2}$ and $2^\# = \frac{2(N-1)}{N-2}$. We regard a as fixed and α as a parameter. Denote the L^p and H^1 norms of u in Ω by

$$|u|_p := \left(\int |u|^p\right)^{1/p} \quad \text{and} \quad ||u|| := \left(|\nabla u|_2^2 + a|u|_2^2\right)^{1/2},$$

respectively. All our integrals are over Ω. We define the functional $\delta : \mathrm{H}^1(\Omega) \setminus \{0\} \to \mathbb{R}$, homogeneous of degree zero, by

$$\delta(u) := \frac{|u|_{2^\#}^{2^\#}}{\|u\| \cdot |u|_{2^*}^{2^*/2}}$$

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and consider the system

$$\begin{cases} \left(1+\frac{1}{2}\alpha\delta(u)\right)(-\Delta u+au)+\frac{2^{\#}}{2}\alpha u^{2^{\#}-1}=\left(1+\frac{2^{\#}+1}{2}\alpha\delta(u)\right)u^{2^*-1} & \text{in } \Omega,\\ u>0 & \text{in } \Omega,\\ \frac{\partial u}{\partial \nu}=0 & \text{on } \partial\Omega. \end{cases} \tag{1}_\alpha$$

We claim that the solutions of $(1)_\alpha$ correspond to critical points of the functional $\Phi_\alpha : \mathrm{H}^1(\Omega)\setminus\{0\}\to\mathbb{R}$, defined by

$$\Phi_\alpha(u):=\left(\frac{1}{2}\|u\|^2-\frac{1}{2^*}|u|_{2^*}^{2^*}\right)\left(1+\alpha\frac{|u|_{2^{\#}}^{2^{\#}}}{\|u\|\cdot|u|_{2^*}^{2^*/2}}\right)^{N/2}=\Phi_0(u)\big(1+\alpha\delta(u)\big)^{N/2}. \tag{2}$$

In fact, since

$$\Phi_\alpha'=(1+\alpha\delta)^{N/2-1}\left[\Phi_0'(1+\alpha\delta)+\tfrac{N}{2}\Phi_0\alpha\delta'\right]$$

and, for $\varphi\in\mathrm{H}^1(\Omega)$,

$$\delta'(u)(\varphi)=-\frac{\delta(u)}{\|u\|^2}\int(\nabla u\cdot\nabla\varphi+au\varphi)+2^{\#}\frac{\delta(u)}{|u|_{2^{\#}}^{2^{\#}}}\int\big(|u|^{2^{\#}-2}u\varphi\big)-\frac{2^*}{2}\frac{\delta(u)}{|u|_{2^*}^{2^*}}\int\big(|u|^{2^*-2}u\varphi\big),$$

the critical points of Φ_α satisfy

$$\begin{aligned}&(-\Delta u+au)\left(1+\frac{4-N}{4}\alpha\delta(u)+\frac{N-2}{4}\frac{|u|_{2^*}^{2^*}}{\|u\|^2}\alpha\delta(u)\right)+\frac{2^{\#}N}{2}\alpha|u|^{2^{\#}-2}u\left(\frac{1}{2}\frac{||u||}{|u|_{2^*}^{2^*/2}}-\frac{1}{2^*}\frac{|u|_{2^*}^{2^*/2}}{||u||}\right)\\&-|u|^{2^*-2}u\left(1+\frac{4-N}{4}\alpha\delta(u)+\frac{2^*N}{8}\frac{\|u\|^2}{|u|_{2^*}^{2^*}}\alpha\delta(u)\right)=0\end{aligned}$$

in Ω, ($\partial u/\partial\nu=0$ on $\partial\Omega$). However, multiplying this equation by u and integrating over Ω (i.e., differentiating (2) along the radial direction) we get $\|u\|^2=|u|_{2^*}^{2^*}$. Conversely, the solutions of $(1)_\alpha$ are solutions of the previous equation: multiplying $(1)_\alpha$ by u and integrating over Ω we get a quadratic equation in $\|u\|/|u|_{2^*}^{2^*/2}$, whose solution is $\|u\|/|u|_{2^*}^{2^*/2}=1$. This proves our claim.

The functional Φ_α restricted to the Nehari manifold,

$$\mathcal{N}:=\left\{u\in\mathrm{H}^1(\Omega)\setminus\{0\}:\Phi_\alpha'(u)u=0\right\}=\left\{u\in\mathrm{H}^1(\Omega)\setminus\{0\}:\|u\|^2=|u|_{2^*}^{2^*}\right\},$$

is $\frac{1}{N}[\beta(1+\alpha\delta)]^{N/2}$, where $\beta:\mathrm{H}^1(\Omega)\setminus\{0\}\to\mathbb{R}$ is defined by

$$\beta(u):=\frac{\|u\|^2}{|u|_{2^*}^2}.$$

So, we consider the functional $\Psi_\alpha:\mathrm{H}^1(\Omega)\setminus\{0\}\to\mathbb{R}$, defined by

$$\Psi_\alpha:=\beta(1+\alpha\delta).$$

A *least energy* solution of $(1)_\alpha$ is a function $u \in \mathrm{H}^1(\Omega) \setminus \{0\}$, such that

$$\Phi_\alpha(u) = \inf_{\mathcal{N}} \Phi_\alpha = \inf_{\mathrm{H}^1(\Omega)\setminus\{0\}} \frac{1}{N}(\Psi_\alpha)^{N/2}.$$

We are interested in proving existence and nonexistence of least energy solutions of $(1)_\alpha$. We note that every critical point of Φ_α is a critical point of Ψ_α. It is easy to check that the Nehari manifold is a natural constraint for Φ_α. So conversely, if u is a critical point of Ψ_α, then there exists a unique $t(u) > 0$, such that $t(u)u$ is a critical point of Φ_α $(t(u) = (\|u\|^2/|u|_{2^*}^{2^*})^{(N-2)/4})$.

We consider the minimization problem corresponding to

$$S_\alpha := \inf\bigl\{\Psi_\alpha(u) \mid u \in \mathrm{H}^1(\Omega) \setminus \{0\}\bigr\}.$$

We recall that $S := \inf\{|\nabla u|_2^2/|u|_{2^*}^2 \mid u \in \mathrm{L}^{2^*}(\mathbb{R}^N),\ \nabla u \in \mathrm{L}^2(\mathbb{R}^N), u \neq 0\}$ is achieved by the instanton $U(x) := (N(N-2)/(N(N-2)+|x|^2))^{(N-2)/2}$. Our main result is

THEOREM 1. – *There exists a positive real number $\alpha_0 = \alpha_0(a, \Omega) = \min\{\alpha \mid \Psi_\alpha = S/2^{2/N}\}$ such that*
- (i) *if $\alpha < \alpha_0$, then $(1)_\alpha$ has a least energy solution u_α;*
- (ii) *if $\alpha > \alpha_0$, then $(1)_\alpha$ does not have a least energy solution and $S_\alpha \geqslant S/2^{2/N}$. The constant $S/2^{2/N}$ is sharp.*

Remark 1. – Obviously, α_0 is a nonincreasing function of a. By testing Ψ_α with constant functions and instantons we can prove that $\alpha_0 \geqslant \max\{[S/(2|\Omega|)^{2/N} - a]/\sqrt{a}, C(N)\max_{\partial\Omega} H\}$ where $|\Omega|$ is the Lebesgue measure of Ω, H is the mean curvature of $\partial\Omega$ and $C(N)$ is a constant that only depends on N. The least energy solutions might be constant $(a^{(N-2)/4})$ for $a \leqslant S/(2|\Omega|)^{2/N}$ if $\alpha \leqslant [S/(2|\Omega|)^{2/N} - a]/\sqrt{a}$.

COROLLARY 2. – *For all $u \in \mathrm{H}^1(\Omega) \setminus \{0\}$,*

$$\frac{S}{2^{2/N}} \leqslant \frac{\|u\|^2}{|u|_{2^*}^2}\left(1 + \alpha_0 \frac{|u|_{2^\#}^{2^\#}}{\|u\| \cdot |u|_{2^*}^{2^*/2}}\right).$$

COROLLARY 3. – *For all $u \in \mathrm{H}^1(\Omega)$,*

$$\frac{S}{2^{2/N}}|u|_{2^*}^2 \leqslant \|u\|^2 + \alpha_0\|u\| \cdot |u|_2 \quad \textit{and} \quad \frac{S}{2^{2/N}}|u|_{2^*}^2 \leqslant \bigl(|\nabla u|_2 + c_{a,\alpha_0}|u|_2\bigr)^2,$$

with $c_{a,\alpha_0} = \max\{\alpha_0/2, \sqrt{a + \alpha_0\sqrt{a}}\}$.

Proof. – From Hölder's inequality $|u|_{2^\#}^{2^\#} \leqslant |u|_2|u|_{2^*}^{2^*/2}$, so $\delta(u) \leqslant |u|_2/\|u\|$. □

COROLLARY 4 (Cherrier's inequality). – *Let $\varepsilon > 0$. For all $u \in \mathrm{H}^1(\Omega)$,*

$$\frac{S}{2^{2/N}}|u|_{2^*}^2 \leqslant (1+\varepsilon)\|u\|^2 + \frac{\alpha_0^2}{4\varepsilon}|u|_2^2 = (1+\varepsilon)|\nabla u|_2^2 + \left(\frac{\alpha_0^2}{4\varepsilon} + a\varepsilon\right)|u|_2^2.$$

Sketch of the proof of Theorem 1. – By testing Ψ_α with instantons, $S_\alpha \leqslant S/2^{2/N}$, for all $\alpha \geqslant 0$. We claim that if $S_\alpha < S/2^{2/N}$, then S_α is achieved. This is a consequence of the concentration-compactness principle. A minimizing sequence u_k with $|u_k|_{2^*} = 1$ is bounded and we can assume $u_k \rightharpoonup u$ in $\mathrm{H}^1(\Omega)$, $\lim_{k\to\infty}|\nabla u_k|_2^2 = |\nabla u|_2^2 + \|\mu\|$ and $\lim_{k\to\infty}|u_k|_{2^*}^{2^*} = |u|_{2^*}^{2^*} + \|\nu\| = 1$, where $(S/2^{2/N})\|\nu\|^{(2/2^*)} \leqslant \|\mu\|$ (we remark that this inequality follows from Cherrier's inequality). We can write $\beta(u)\delta(u) = \gamma(u)\sqrt{\beta(u)}$

for $\gamma(u)=|u|_{2^\#}^{2^\#}/|u|_{2^*}^{2^\#}$. The key step in the proof of the claim is the following observation. Define f and $g:[0,1]\to\mathbb{R}$, by

$$\begin{aligned}f(x) &:= \beta x^{2/2^*}+\frac{S}{2^{2/N}}(1-x)^{2/2^*}+\alpha\gamma x^{2^\#/2^*}\sqrt{\beta x^{2/2^*}+\frac{S}{2^{2/N}}(1-x)^{2/2^*}}\\ &\geqslant \beta x+\frac{S}{2^{2/N}}(1-x)+\alpha\gamma x\sqrt{\beta x+\frac{S}{2^{2/N}}(1-x)}=:g(x).\end{aligned}$$

Suppose $\min f<S/2^{2/N}$. It follows that $\beta<S/2^{2/N}$, and this in turn implies that g is concave. Since f and g coincide at 0 and 1, the minimum of f occurs at 1. This proves the claim.

A similar argument shows $\alpha\mapsto S_\alpha$ is continuous. In particular, the supremum $\alpha_0:=\sup\{\alpha\,|\,S_\alpha<S/2^{2/N}\}$ is either $+\infty$ or a maximum. The map $\alpha\mapsto S_\alpha$ is strictly increasing on $[0,\alpha_0]$. If $\alpha\in[0,\alpha_0[$, then $(1)_\alpha$ has a least energy solution. If $\alpha\in]\alpha_0,+\infty[$, then $(1)_\alpha$ does not have a least energy solution. It remains to prove that α_0 is finite.

Suppose $\alpha_0=+\infty$. Choose $\alpha_k\to+\infty$ as $k\to+\infty$ and let u_k be minimizer for S_{α_k} satisfying $(1)_{\alpha_k}$. It is easy to prove that $\lim_{\alpha\to\infty}S_\alpha=S/2^{2/N}$, $M_k:=\max_{\overline{\Omega}}u_k=u_k(P_k)\to+\infty$, $\lim_{k\to\infty}|\nabla u_k|_2^2=\lim_{k\to\infty}|u_k|_{2^*}^{2^*}=S^{N/2}/2$ and $\alpha_k\delta(u_k)\to 0$. We can apply the Gidas–Spruck blow up technique to $(1)_{\alpha_k}$ because $\alpha_k\delta(u_k)\to 0$. Define $\varepsilon_k:=M_k^{-2/(N-2)}$ and $U_{\varepsilon,y}:=\varepsilon^{-(N-2)/2}U\left(\frac{x-y}{\varepsilon}\right)$. We can prove that $\lim_{k\to\infty}\alpha_k\varepsilon_k=0$, $\lim_{k\to\infty}|\nabla u_k-\nabla U_{\varepsilon_k,P_k}|_2=0$ and $P_k\in\partial\Omega$, for large k.

At this point, using the ideas of [2], we follow the argument in [7], which applies with no modification. We show $\Psi_{\alpha_k}(u_k)>S/2^{2/N}$, for large k. This is impossible. Therefore α_0 is finite. □

Remark 2. – The functional behavior behind this inequality is also present, for example, in the Dirichlet problem for $-\Delta u-au+\alpha u^{1/3}=u^{7/3}$ in $\Omega\subset\mathbb{R}^5$, with $0<a<\lambda_1(-\Delta,\mathrm{H}_0^1(\Omega))$. In this case $s:=[t(u)]^{2/3}$ is the solution of a cubic equation $(|\nabla u|_2^2-a|u|_2^2)s+\alpha|u|_{4/3}^{4/3}-|u|_{10/3}^{10/3}s^3=0$.

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