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COMPTES RENDUS MATHEMATIQUE

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# Beilinson resolutions on weighted projective spaces

# Résolutions de Beilinson sur espaces projectifs à poids

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**Abstract**

Beilinson's theorem [Funct. Anal. Appl. 12 (1978) 214–216], which describes the bounded derived category of coherent sheaves on $\mathbb{P}^n$, is extended to weighted projective spaces. This result is obtained by considering, instead of the usual category of coherent sheaves, a suitable category of graded coherent sheaves (which is equivalent in the case of $\mathbb{P}^n$). ***To cite this article: A. Canonaco, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***


**Résumé**

On étend aux espaces projectifs à poids le théorème de Beilinson [Funct. Anal. Appl. 12 (1978) 214–216], qui décrit la catégorie derivée bornée des faisceaux cohérents sur $\mathbb{P}^n$. Pour obtenir ce résultat on considère, au lieu de la catégorie habituelle des faisceaux cohérents, une certaine catégorie de faisceaux cohérents gradués (qui lui est équivalente dans le cas de $\mathbb{P}^n$). ***Pour citer cet article : A. Canonaco, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***


## Version française abrégée

Soit $\mathbb{K}$ un corps. Pour $n \in \mathbb{N}$ et $\mathrm{w} = (\mathrm{w}_0, \dots, \mathrm{w}_n) \in \mathbb{N}_+^{n+1}$, on notera $\mathrm{P}(\mathrm{w})$ (ou simplement P) l'anneau de polinômes $\mathbb{K}[x_0, \dots, x_n]$ gradué par $\deg(x_i) = \mathrm{w}_i$, et $\mathbb{P}(\mathrm{w}) := \operatorname{Proj} \mathrm{P}(\mathrm{w})$ (ou simplement $\mathbb{P}$) l'espace projectif à poids w. On définit aussi $|\mathrm{w}| := \sum_{i=0}^n \mathrm{w}_i$. Si $X = \bigoplus_{d \in \mathbb{Z}} X_d$ est un objet gradué et si $i \in \mathbb{Z}$, on désigne par $X(i)$ l'objet gradué défini par $X(i)_d := X_{i+d}\ \forall d \in \mathbb{Z}$.

Le théorème de Beilinson (voir [2] ou [3]) donne des équivalences entre $D^b(\mathbf{Coh}(\mathbb{P}^n))$ (la catégorie derivée bornée des faisceaux cohérents sur $\mathbb{P}^n$) et deux catégories homotopes de modules. De façon plus précise, soit $\mathrm{P} = \mathrm{P}(1, \dots, 1)$ (c'est l'algèbre symétrique de $\mathrm{P}_1$) et soit $\Lambda$ l'algèbre extérieur du dual de $\mathrm{P}_1$. Si $A$ est un anneau gradué, on désigne par $\mathbf{M}_{[a,b]}(A)$ la sous-catégorie pleine de la catégorie des $A$-modules gradués $\overline{\mathbf{Mod}}(A)$ définie

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par les objets qui sont sommes directes finies de modules de la forme $A(j)$ pour $a \leqslant j \leqslant b$. On considère aussi les deux ensembles suivants de fibrés vectoriels sur $\mathbb{P}^n$ :

$$O := \{\mathcal{O}_{\mathbb{P}^n}(j) \mid -n \leqslant j \leqslant 0\}, \qquad D := \{\Omega^j_{\mathbb{P}^n}(j) \mid 0 \leqslant j \leqslant n\}.$$

**Théorème 0.1** (Beilinson). *Les foncteurs additifs naturels* $\mathbf{M}_{[-n,0]}(\mathrm{P}) \to \mathbf{Coh}(\mathbb{P}^n)$ *et* $\mathbf{M}_{[-n,0]}(\Lambda) \to \mathbf{Coh}(\mathbb{P}^n)$ (*définis sur les objets, respectivement, par* $\mathrm{P}(j) \mapsto \mathcal{O}_{\mathbb{P}^n}(j)$ *et* $\Lambda(j) \mapsto \Omega^{-j}_{\mathbb{P}^n}(-j)$ *pour* $-n \leqslant j \leqslant 0$) *s'étendent à des équivalences exactes entre catégories triangulées*

$$F_{\mathrm{P}} : K^b(\mathbf{M}_{[-n,0]}(\mathrm{P})) \to D^b(\mathbf{Coh}(\mathbb{P}^n)), \qquad F_{\Lambda} : K^b(\mathbf{M}_{[-n,0]}(\Lambda)) \to D^b(\mathbf{Coh}(\mathbb{P}^n)).$$

*De plus, pour tout* $\mathcal{G}^\bullet \in D^b(\mathbf{Coh}(\mathbb{P}^n))$, *il existe un unique complexe borné* (*à isomorphisme de complexes près*) $\mathcal{X}^\bullet = \mathcal{X}^\bullet(\mathcal{G}^\bullet)$ (*respectivement* $\mathcal{Y}^\bullet = \mathcal{Y}^\bullet(\mathcal{G}^\bullet)$) *isomorphe à* $\mathcal{G}^\bullet$ *dans* $D^b(\mathbf{Coh}(\mathbb{P}^n))$, *tel que chaque* $\mathcal{X}^i$ (*respectivement* $\mathcal{Y}^i$) *soit une somme directe finie de faisceaux de* $O$ (*respectivement* $D$), *et qui soit minimal. Explicitement on a* :

$$\mathcal{X}^i = \bigoplus_{0 \leqslant j \leqslant n} \mathcal{O}_{\mathbb{P}^n}(-j) \otimes_{\mathbb{K}} H^{i+j}(\mathbb{P}^n, \mathcal{G}^\bullet \otimes \Omega^j_{\mathbb{P}^n}(j)), \qquad \mathcal{Y}^i = \bigoplus_{0 \leqslant j \leqslant n} \Omega^j_{\mathbb{P}^n}(j) \otimes_{\mathbb{K}} H^{i+j}(\mathbb{P}^n, \mathcal{G}^\bullet(-j)).$$

On notera $\overline{\mathbb{P}} = \overline{\mathbb{P}}(\mathrm{w})$ le « schéma gradué » suivant : $\overline{\mathbb{P}}$ est un espace annelé avec le même espace topologique que $\mathbb{P}$, mais avec un faisceau structural gradué $\mathcal{O}_{\overline{\mathbb{P}}} := \bigoplus_{d \in \mathbb{Z}} \mathcal{O}_{\mathbb{P}}(d)$. On peut considérer alors la catégorie $\overline{\mathbf{Mod}}(\overline{\mathbb{P}})$ des faisceaux de $\mathcal{O}_{\overline{\mathbb{P}}}$-modules gradués. Si $M \in \overline{\mathbf{Mod}}(\mathrm{P})$, soit $M^{\approx} := \bigoplus_{d \in \mathbb{Z}} M(d)^{\sim} \in \overline{\mathbf{Mod}}(\overline{\mathbb{P}})$ le faisceau gradué associé. On dit que $\mathcal{F} \in \overline{\mathbf{Mod}}(\overline{\mathbb{P}})$ est *localement libre* s'il est localement isomorphe à un faisceau de la forme $\bigoplus_{i \in I} \mathcal{O}_{\overline{\mathbb{P}}}(m_i)$ et qu'il est un *fibré vectoriel* s'il est localement libre de rang fini. On dit que $\mathcal{F}$ est *cohérent* s'il est localement isomorphe au conoyau d'un morphisme de fibrés vectoriels. On désigne par $\overline{\mathbf{Coh}}(\overline{\mathbb{P}})$ la sous-catégorie pleine de $\overline{\mathbf{Mod}}(\overline{\mathbb{P}})$ définie par les faisceaux cohérents.

**Proposition 0.2.** *Le foncteur naturel* $-_0 : \overline{\mathbf{Coh}}(\overline{\mathbb{P}}) \to \mathbf{Coh}(\mathbb{P})$ (*défini sur les objets par* $\mathcal{F} \mapsto \mathcal{F}_0$) *est exact et essentiellement surjectif. Il est une équivalence de catégories si et seulement si* $\mathrm{w} = (1, \ldots, 1)$.

Les modules des syzygies $Syz^j \in \overline{\mathbf{Mod}}(\mathrm{P})$ du complexe de Koszul $K^\bullet$ de la suite régulière $(x_0, \ldots, x_n)$ permettent de définir $\Omega^j_{\overline{\mathbb{P}}} := (Syz^j)^{\approx}$ pour $0 \leqslant j \leqslant n$. Au lieu de $O$ and $D$, on considère les deux ensembles suivants de fibrés vectoriels sur $\overline{\mathbb{P}}$ :

$$\overline{O} := \{\mathcal{O}_{\overline{\mathbb{P}}}(j) \mid -|\mathrm{w}| < j \leqslant 0\}, \qquad \overline{D} := \{\mathcal{O}_{\overline{\mathbb{P}}}(j) \mid n - |\mathrm{w}| < j < 0\} \cup \{\Omega^j_{\overline{\mathbb{P}}}(j) \mid 0 \leqslant j \leqslant n\}.$$

À partir de $K^\bullet$ on définit aussi des complexes bornés de fibrés vectoriels $\mathcal{M}^\bullet_{(j)}$ pour $-|\mathrm{w}| < j \leqslant 0$ et $\mathcal{N}^\bullet_{(j)}$ pour $n - |\mathrm{w}| < j < 0$. Enfin, on désigne par $\widehat{\mathbf{M}}_{]a,b[}(\mathrm{P})$ la sous-catégorie pleine de $\overline{\mathbf{Mod}}(\mathrm{P})$ définie par les objets qui sont sommes directes finies de modules de la forme $\mathrm{P}(j)$ pour $a < j < b$ ou $Syz^j(j)$ pour $0 \leqslant j \leqslant n$. Le théorème de Beilinson peut être alors généralisé comme suit.

**Théorème 0.3.** *La restriction de* $^{\approx} : \overline{\mathbf{Mod}}(\mathrm{P}) \to \overline{\mathbf{Mod}}(\overline{\mathbb{P}})$ *induit des foncteurs additifs* $\mathbf{M}_{[1-|\mathrm{w}|,0]}(\mathrm{P}) \to \overline{\mathbf{Coh}}(\overline{\mathbb{P}})$ *et* $\widehat{\mathbf{M}}_{]n-|\mathrm{w}|,0[}(\mathrm{P}) \to \overline{\mathbf{Coh}}(\overline{\mathbb{P}})$. *Ils s'étendent à des équivalences exactes entre catégories triangulées*

$$\overline{F}_{\mathrm{P}} : K^b(\mathbf{M}_{[1-|\mathrm{w}|,0]}(\mathrm{P})) \to D^b(\overline{\mathbf{Coh}}(\overline{\mathbb{P}})), \qquad \overline{G}_{\mathrm{P}} : K^b(\widehat{\mathbf{M}}_{]n-|\mathrm{w}|,0[}(\mathrm{P})) \to D^b(\overline{\mathbf{Coh}}(\overline{\mathbb{P}})).$$

*De plus, pour tout* $\mathcal{F}^\bullet \in D^b(\overline{\mathbf{Coh}}(\overline{\mathbb{P}}))$, *il existe un unique complexe borné* (*à isomorphisme de complexes près*) $\overline{\mathcal{X}}^\bullet = \overline{\mathcal{X}}^\bullet(\mathcal{F}^\bullet)$ (*respectivement* $\overline{\mathcal{Y}}^\bullet = \overline{\mathcal{Y}}^\bullet(\mathcal{F}^\bullet)$) *isomorphe à* $\mathcal{F}^\bullet$ *dans* $D^b(\overline{\mathbf{Coh}}(\overline{\mathbb{P}}))$, *tel que chaque* $\overline{\mathcal{X}}^i$ (*respectivement* $\overline{\mathcal{Y}}^i$) *soit une somme directe finie de faisceaux de* $\overline{O}$ (*respectivement* $\overline{D}$), *et qui soit minimal. Explicitement on a* :

$$\overline{\mathcal{X}}^i = \bigoplus_{-|\mathrm{w}|<j\leqslant 0} \mathcal{O}_{\overline{\mathbb{P}}}(j) \otimes_{\mathbb{K}} H^i\big(\overline{\mathbb{P}}, \mathcal{F}^\bullet \otimes \mathcal{M}^\bullet_{(j)}\big),$$

$$\overline{\mathcal{Y}}^i = \bigoplus_{n-|\mathrm{w}|<j<0} \mathcal{O}_{\overline{\mathbb{P}}}(j) \otimes_{\mathbb{K}} H^i\big(\overline{\mathbb{P}}, \mathcal{F}^\bullet \otimes \mathcal{N}^\bullet_{(j)}\big) \bigoplus_{0\leqslant j\leqslant n} \Omega^j_{\overline{\mathbb{P}}}(j) \otimes_{\mathbb{K}} H^{i+j}\big(\overline{\mathbb{P}}, \mathcal{F}^\bullet(-j)\big).$$

## 1. Introduction

We will work over a fixed field $\mathbb{K}$. Given $n \in \mathbb{N}$ and $\mathrm{w} = (\mathrm{w}_0, \dots, \mathrm{w}_n) \in \mathbb{N}_+^{n+1}$, we will denote by $\mathrm{P}(\mathrm{w})$ (or simply by $\mathrm{P}$) the polynomial ring $\mathbb{K}[x_0, \dots, x_n]$ graded by $\deg(x_i) = \mathrm{w}_i$, and by $\mathbb{P}(\mathrm{w}) := \operatorname{Proj} \mathrm{P}(\mathrm{w})$ (or simply by $\mathbb{P}$) the weighted projective space of weights w (of course, $\mathbb{P}(\mathrm{w}) = \mathbb{P}^n$ if $\mathrm{w} = (1, \dots, 1)$). If $I \subseteq \{0, \dots, n\}$, we define $|\mathrm{w}_I| := \sum_{i\in I} \mathrm{w}_i$; we will also write $|\mathrm{w}|$ instead of $|\mathrm{w}_{\{0,\dots,n\}}|$.

If $X = \bigoplus_{d\in\mathbb{Z}} X_d$ is a graded object (module or sheaf, in our case) and $i \in \mathbb{Z}$, the twisted object $X(i)$ is defined by $X(i)_d := X_{i+d}$ $\forall d \in \mathbb{Z}$. If $Y^\bullet = (Y^j, \mathrm{d}^j_{Y^\bullet} : Y^j \to Y^{j+1})_{j\in\mathbb{Z}}$ is a complex (in some additive category $\mathbf{A}$), $Y^\bullet[i]$ will denote instead the shifted complex defined by $Y^\bullet[i]^j := Y^{i+j}$ and $\mathrm{d}^j_{Y^\bullet[i]} := (-1)^i \mathrm{d}^{i+j}_{Y^\bullet}$. An object of $\mathbf{A}$ will be regarded as a complex concentrated in position 0.

Beilinson's theorem (see [2] or [3]) gives equivalences between $D^b(\mathbf{Coh}(\mathbb{P}^n))$ (the bounded derived category of coherent sheaves on $\mathbb{P}^n$) and two homotopy categories of modules (see [7] or [8] for definitions and main properties of homotopy, derived and triangulated categories), showing in particular that $D^b(\mathbf{Coh}(\mathbb{P}^n))$ is generated (as a triangulated category) by each of the following two sets of vector bundles:

$$O := \big\{\mathcal{O}_{\mathbb{P}^n}(j) \mid -n \leqslant j \leqslant 0\big\}, \qquad D := \big\{\Omega^j_{\mathbb{P}^n}(j) \mid 0 \leqslant j \leqslant n\big\}.$$

More precisely, let $\mathrm{P} = \mathrm{P}(1, \dots, 1)$ (it is the symmetric algebra of $\mathrm{P}_1$) and let $\Lambda$ be the exterior algebra of the dual of $\mathrm{P}_1$. Moreover, if $A$ is a graded ring, $\mathbf{M}_{[a,b]}(A)$ will be the full subcategory of the category of graded $A$-modules $\overline{\mathbf{Mod}}(A)$ whose objects are finite direct sums of objects of the form $A(j)$ for $a \leqslant j \leqslant b$.

**Theorem 1.1** (Beilinson). *There are natural additive functors* $\mathbf{M}_{[-n,0]}(\mathrm{P}) \to \mathbf{Coh}(\mathbb{P}^n)$, $\mathbf{M}_{[-n,0]}(\Lambda) \to \mathbf{Coh}(\mathbb{P}^n)$ (*defined on objects, respectively, by* $\mathrm{P}(j) \mapsto \mathcal{O}_{\mathbb{P}^n}(j)$ *and* $\Lambda(j) \mapsto \Omega^{-j}_{\mathbb{P}^n}(-j)$ *for* $-n \leqslant j \leqslant 0$). *They extend to exact equivalences between triangulated categories*

$$F_{\mathrm{P}} : K^b\big(\mathbf{M}_{[-n,0]}(\mathrm{P})\big) \to D^b\big(\mathbf{Coh}\big(\mathbb{P}^n\big)\big), \qquad F_\Lambda : K^b\big(\mathbf{M}_{[-n,0]}(\Lambda)\big) \to D^b\big(\mathbf{Coh}\big(\mathbb{P}^n\big)\big).$$

*Moreover, given* $\mathcal{G}^\bullet \in D^b(\mathbf{Coh}(\mathbb{P}^n))$, *there exists a unique* (*up to isomorphism of complexes*) *bounded complex* $\mathcal{X}^\bullet = \mathcal{X}^\bullet(\mathcal{G}^\bullet)$ (*respectively* $\mathcal{Y}^\bullet = \mathcal{Y}^\bullet(\mathcal{G}^\bullet)$) *isomorphic to* $\mathcal{G}^\bullet$ *in* $D^b(\mathbf{Coh}(\mathbb{P}^n))$, *such that each* $\mathcal{X}^i$ (*respectively* $\mathcal{Y}^i$) *is a finite direct sum of sheaves of* $O$ (*respectively* $D$), *and which is minimal. Explicitly,*

$$\mathcal{X}^i = \bigoplus_{0\leqslant j\leqslant n} \mathcal{O}_{\mathbb{P}^n}(-j) \otimes_{\mathbb{K}} H^{i+j}\big(\mathbb{P}^n, \mathcal{G}^\bullet \otimes \Omega^j_{\mathbb{P}^n}(j)\big), \qquad \mathcal{Y}^i = \bigoplus_{0\leqslant j\leqslant n} \Omega^j_{\mathbb{P}^n}(j) \otimes_{\mathbb{K}} H^{i+j}\big(\mathbb{P}^n, \mathcal{G}^\bullet(-j)\big).$$

The difficulties in trying to extend this result to a weighted projective space $\mathbb{P}(\mathrm{w})$ mainly come from the fact that the sheaves of the form $\mathcal{O}_{\mathbb{P}(\mathrm{w})}(j)$ are not always invertible and there is in general no way to define twist functors so that they satisfy all the usual properties. In order to overcome such problems it is useful to endow $\mathbb{P}(\mathrm{w})$ with a (natural) $\mathbb{Z}$-graded structure sheaf and to replace the categories of ordinary sheaves with categories of graded sheaves. This allows us to obtain a good generalization of Beilinson's theorem (much better than our previous version [4]). A similar approach (with the main difference that the graduation is over an Abelian group depending on w, and not over $\mathbb{Z}$) is used in [1], where however only a part of Beilinson's theorem is generalized (namely, the fact that $F_{\mathrm{P}}$ is an equivalence).

Proofs will be omitted or only sketched (full proofs are given in [5] and will appear elsewhere).

## 2. The graded weighted projective space $\overline{\mathbb{P}}(\mathrm{w})$

Having fixed $\mathrm{w} \in \mathbb{N}_+^{n+1}$, we will denote by $\overline{\mathbb{P}} = \overline{\mathbb{P}}(\mathrm{w})$ the following "graded scheme": $\overline{\mathbb{P}}$ is a ringed space having the same topological space as $\mathbb{P} = \mathbb{P}(\mathrm{w})$, but with a graded structure sheaf $\mathcal{O}_{\overline{\mathbb{P}}} := \bigoplus_{d\in\mathbb{Z}} \mathcal{O}_{\mathbb{P}}(d)$. It is then natural to consider the (Abelian) category $\overline{\mathbf{Mod}}(\overline{\mathbb{P}})$ of sheaves of graded $\mathcal{O}_{\overline{\mathbb{P}}}$-modules, which is much better-behaved than $\mathbf{Mod}(\mathbb{P})$ (the usual category of sheaves of $\mathcal{O}_{\mathbb{P}}$-modules), essentially because the natural twist functors in $\overline{\mathbf{Mod}}(\overline{\mathbb{P}})$ satisfy all the expected properties: for instance, they are exact and $(i) \circ (j) = (i + j)$ $\forall i, j \in \mathbb{Z}$. Moreover, if $M$ is a graded $\mathrm{P} = \mathrm{P}(\mathrm{w})$-module, one can define the associated sheaf of graded $\mathcal{O}_{\overline{\mathbb{P}}}$-modules as $M^{\overline{\sim}} := \bigoplus_{d\in\mathbb{Z}} M(d)^{\sim}$, and then $M(i)^{\overline{\sim}} \cong M^{\overline{\sim}}(i)$ $\forall i \in \mathbb{Z}$.

**Definition 2.1.** A sheaf of graded $\mathcal{O}_{\overline{\mathbb{P}}}$-modules $\mathcal{F}$ is *locally free* if it is locally isomorphic to a sheaf of the form $\bigoplus_{i\in I} \mathcal{O}_{\overline{\mathbb{P}}}(m_i)$ (where $m_i \in \mathbb{Z}$); if $|I|$ is finite and constant, $\mathcal{F}$ is said to be of *rank* $|I|$. A locally free sheaf of finite rank will be also called a *vector bundle*. A sheaf of graded $\mathcal{O}_{\overline{\mathbb{P}}}$-modules $\mathcal{F}$ is *coherent* if it is locally isomorphic to the cokernel of a morphism of vector bundles.

We will denote by $\overline{\mathbf{Coh}}(\overline{\mathbb{P}})$ the full subcategory of $\overline{\mathbf{Mod}}(\overline{\mathbb{P}})$ consisting of coherent graded sheaves.

**Proposition 2.2.** *The natural exact functor* $-_0 : \overline{\mathbf{Mod}}(\overline{\mathbb{P}}) \to \mathbf{Mod}(\mathbb{P})$ (*defined on objects by* $\mathcal{F} \mapsto \mathcal{F}_0$) *admits left and right adjoints*

$$\mathcal{O}_{\overline{\mathbb{P}}} \otimes_{\mathcal{O}_{\mathbb{P}}} - : \mathbf{Mod}(\mathbb{P}) \to \overline{\mathbf{Mod}}(\overline{\mathbb{P}}), \qquad \overline{\mathcal{H}om}_{\mathcal{O}_{\mathbb{P}}}(\mathcal{O}_{\overline{\mathbb{P}}}, -) : \mathbf{Mod}(\mathbb{P}) \to \overline{\mathbf{Mod}}(\overline{\mathbb{P}})$$

(*defined on objects by* $(\mathcal{O}_{\overline{\mathbb{P}}} \otimes_{\mathcal{O}_{\mathbb{P}}} \mathcal{G})_d := \mathcal{O}_{\mathbb{P}}(d) \otimes_{\mathcal{O}_{\mathbb{P}}} \mathcal{G}$, $\overline{\mathcal{H}om}_{\mathcal{O}_{\mathbb{P}}}(\mathcal{O}_{\overline{\mathbb{P}}}, \mathcal{G})_d := \mathcal{H}om_{\mathcal{O}_{\mathbb{P}}}(\mathcal{O}_{\mathbb{P}}(-d), \mathcal{G})$ $\forall d \in \mathbb{Z}$), *which satisfy* $-_0 \circ (\mathcal{O}_{\overline{\mathbb{P}}} \otimes_{\mathcal{O}_{\mathbb{P}}} -) \cong -_0 \circ \overline{\mathcal{H}om}_{\mathcal{O}_{\mathbb{P}}}(\mathcal{O}_{\overline{\mathbb{P}}}, -) \cong \mathrm{id}_{\mathbf{Mod}(\mathbb{P})}$. *Moreover,* $-_0$ *sends* $\overline{\mathbf{Coh}}(\overline{\mathbb{P}})$ *into* $\mathbf{Coh}(\mathbb{P})$ *and* $\mathcal{O}_{\overline{\mathbb{P}}} \otimes_{\mathcal{O}_{\mathbb{P}}} -$ *and* $\overline{\mathcal{H}om}_{\mathcal{O}_{\mathbb{P}}}(\mathcal{O}_{\overline{\mathbb{P}}}, -)$ *send* $\mathbf{Coh}(\mathbb{P})$ *into* $\overline{\mathbf{Coh}}(\overline{\mathbb{P}})$.

**Proposition 2.3.** *The functor* $-_0 : \overline{\mathbf{Mod}}(\overline{\mathbb{P}}) \to \mathbf{Mod}(\mathbb{P})$ *is an equivalence of categories* (*in which case* $\mathcal{O}_{\overline{\mathbb{P}}} \otimes_{\mathcal{O}_{\mathbb{P}}} -$ *and* $\overline{\mathcal{H}om}_{\mathcal{O}_{\mathbb{P}}}(\mathcal{O}_{\overline{\mathbb{P}}}, -)$ *are isomorphic, exact and quasi-inverse of* $-_0$) *if and only if* $\mathrm{w} = (1, \ldots, 1)$. *The same is true for the restriction* $-_0 : \overline{\mathbf{Coh}}(\overline{\mathbb{P}}) \to \mathbf{Coh}(\mathbb{P})$.

**Proof.** To prove the "if" part it is enough to show that, given $\mathcal{F} \in \overline{\mathbf{Mod}}(\overline{\mathbb{P}}^n)$, the natural map $\mathcal{O}_{\overline{\mathbb{P}}^n} \otimes_{\mathcal{O}_{\mathbb{P}^n}} \mathcal{F}_0 \to \mathcal{F}$ is an isomorphism, which is an easy consequence of the fact that every stalk of $\mathcal{O}_{\overline{\mathbb{P}}^n}$ contains an invertible element of degree 1. Conversely, we can assume $\mathrm{w}_0 > 1$, and then $M := \mathrm{P}/(x_1, \ldots, x_n)\mathrm{P} \in \overline{\mathbf{Mod}}(\mathrm{P})$ is such that $0 \neq M(1)^{\overline{\sim}} \in \overline{\mathbf{Coh}}(\overline{\mathbb{P}})$, but $(M(1)^{\overline{\sim}})_0 = M(1)^{\sim} = 0$. □

Since $\overline{\mathbf{Mod}}(\overline{\mathbb{P}})$ has enough injectives, $\forall \mathcal{F} \in \overline{\mathbf{Mod}}(\overline{\mathbb{P}})$ we can consider the right derived functors $\mathrm{Ext}^i_{\overline{\mathbb{P}}}(\mathcal{F}, -)$ of $\mathrm{Hom}_{\overline{\mathbf{Mod}}(\overline{\mathbb{P}})}(\mathcal{F}, -)$. In the case $\mathcal{F} = \mathcal{O}_{\overline{\mathbb{P}}}$, $\mathrm{Ext}^i_{\overline{\mathbb{P}}}(\mathcal{O}_{\overline{\mathbb{P}}}, -)$ will be denoted by $H^i(\overline{\mathbb{P}}, -)$.

**Proposition 2.4.** *There are natural isomorphisms* $H^i(\overline{\mathbb{P}}, \mathcal{F}^\bullet) \cong H^i(\mathbb{P}, \mathcal{F}^\bullet_0)$ $\forall \mathcal{F}^\bullet \in D^+(\overline{\mathbf{Mod}}(\overline{\mathbb{P}}))$.

## 3. Beilinson's theorem on $\overline{\mathbb{P}}(\mathrm{w})$

We define $K^\bullet$ to be the Koszul complex (of graded P-modules) of the sequence $(x_0, \ldots, x_n)$: denoting by $\mathrm{d}x_I$ (for $I \subseteq \{0, \ldots, n\}$) elements of degree $|\mathrm{w}_I|$, $K^\bullet$ is given by $K^j = \bigoplus_{|I|=-j} \mathrm{P}\,\mathrm{d}x_I \cong \bigoplus_{|I|=-j} \mathrm{P}(-|\mathrm{w}_I|)$ with differential

$$\begin{aligned} \mathrm{d}^j_{K^\bullet} : K^j &\to K^{j+1} \\ \mathrm{d}x_I &\mapsto \textstyle\sum_{i\in I} (-1)^{|\{i'\in I \,|\, i'<i\}|} x_i \,\mathrm{d}x_{I\setminus\{i\}}. \end{aligned}$$

As $(x_0, \dots, x_n)$ is a regular sequence, $K^\bullet$ is exact everywhere, except that $H^0(K^\bullet) = \mathrm{P}/(x_0, \dots, x_n)\mathrm{P} \cong \mathbb{K}$ (in degree zero). We will consider also the sheafification $\mathcal{K}^\bullet := (K^\bullet)^{\widetilde{\ }}$: as $\widetilde{\ }$ is an exact functor and $\mathbb{K}^{\sim} = 0$, $\mathcal{K}^\bullet$ is an exact complex. Let moreover $Syz^j := \ker \mathrm{d}_{K^\bullet}^{-j}$ and $\Omega_{\overline{\mathbb{P}}}^j := (Syz^j)^{\sim} \cong \ker \mathrm{d}_{\mathcal{K}^\bullet}^{-j} = \operatorname{im} \mathrm{d}_{\mathcal{K}^\bullet}^{-j-1}$. Therefore we have short exact sequences

$$0 \to \Omega_{\overline{\mathbb{P}}}^j \to \mathcal{K}^{-j} \cong \bigoplus_{|I|=j} \mathcal{O}_{\overline{\mathbb{P}}}\big(-|\mathrm{w}_I|\big) \to \Omega_{\overline{\mathbb{P}}}^{j-1} \to 0, \tag{1}$$

from which it is easy to see that $\Omega_{\overline{\mathbb{P}}}^j$ is locally free of rank $\binom{n}{j}$ (notice that $\Omega_{\overline{\mathbb{P}}}^0 \cong \mathcal{O}_{\overline{\mathbb{P}}}$ and $\Omega_{\overline{\mathbb{P}}}^n \cong \mathcal{O}_{\overline{\mathbb{P}}}(-|\mathrm{w}|)$).

**Remark 1.** The sheaves $\Omega_{\overline{\mathbb{P}}}^j(i)_0$ on $\mathbb{P}$ (which are not locally free in general) are denoted by $\overline{\Omega}_{\mathbb{P}}^j(i)$ in [6], and they are the usual $\Omega_{\mathbb{P}^n}^j(i)$ if $\mathrm{w} = (1, \dots, 1)$.

The sets $O$ and $D$ of vector bundles on $\mathbb{P}^n$ will be replaced by the following sets of vector bundles on $\overline{\mathbb{P}}$:

$$\overline{O} := \big\{\mathcal{O}_{\overline{\mathbb{P}}}(j) \mid -|\mathrm{w}| < j \leqslant 0\big\}, \qquad \overline{D} := \big\{\mathcal{O}_{\overline{\mathbb{P}}}(j) \mid n - |\mathrm{w}| < j < 0\big\} \cup \big\{\Omega_{\overline{\mathbb{P}}}^j(j) \mid 0 \leqslant j \leqslant n\big\}.$$

**Lemma 3.1.** *If* $\mathcal{L}, \mathcal{M} \in \overline{O}$ *or* $\mathcal{L}, \mathcal{M} \in \overline{D}$, *then* $\mathrm{Ext}_{\overline{\mathbb{P}}}^i(\mathcal{L}, \mathcal{M}) = 0$ *for* $i > 0$.

**Proof.** Using long exact sequences of Ext induced by (twists of) short exact sequences of the form (1), one can reduce to prove the vanishing of some $\mathrm{Ext}_{\overline{\mathbb{P}}}^i(\mathcal{O}_{\overline{\mathbb{P}}}(l), \Omega_{\overline{\mathbb{P}}}^j(j'))$. Then the key point is that, $\mathcal{O}_{\overline{\mathbb{P}}}(l)$ being invertible, $\mathrm{Ext}_{\overline{\mathbb{P}}}^i(\mathcal{O}_{\overline{\mathbb{P}}}(l), \Omega_{\overline{\mathbb{P}}}^j(j')) \cong H^i(\overline{\mathbb{P}}, \Omega_{\overline{\mathbb{P}}}^j(j' - l)) \cong H^i(\mathbb{P}, \Omega_{\overline{\mathbb{P}}}^j(j' - l)_0)$ by 2.4 (these cohomology groups were computed in [6]). □

**Definition 3.2.** For $-|\mathrm{w}| < l \leqslant 0$ let $\mathcal{M}_{(l)}^\bullet$ be the subcomplex of $\mathcal{K}^\bullet(-l)$ defined $\forall j \in \mathbb{Z}$ by

$$\mathcal{M}_{(l)}^j := \bigoplus_{|I|=-j,\ |\mathrm{w}_I| \leqslant -l} \mathcal{O}_{\overline{\mathbb{P}}}\big(-l - |\mathrm{w}_I|\big) \subseteq \bigoplus_{|I|=-j} \mathcal{O}_{\overline{\mathbb{P}}}\big(-l - |\mathrm{w}_I|\big) = \mathcal{K}^j(-l)$$

and for $n - |\mathrm{w}| < l < 0$ let $\mathcal{N}_{(l)}^\bullet$ be the subcomplex of $\mathcal{K}^\bullet(-l)$ defined $\forall j \in \mathbb{Z}$ by

$$\mathcal{N}_{(l)}^j := \bigoplus_{|I|=-j,\ |\mathrm{w}_I| < -l-j} \mathcal{O}_{\overline{\mathbb{P}}}\big(-l - |\mathrm{w}_I|\big) \subseteq \bigoplus_{|I|=-j} \mathcal{O}_{\overline{\mathbb{P}}}\big(-l - |\mathrm{w}_I|\big) = \mathcal{K}^j(-l)$$

($\mathcal{M}_{(l)}^\bullet$ and $\mathcal{N}_{(l)}^\bullet$ are subcomplexes of $\mathcal{K}^\bullet(-l)$ because $\mathrm{d}_{\mathcal{K}^\bullet(-l)}^j(\mathcal{M}_{(l)}^j) \subseteq \mathcal{M}_{(l)}^{j+1}$ and $\mathrm{d}_{\mathcal{K}^\bullet(-l)}^j(\mathcal{N}_{(l)}^j) \subseteq \mathcal{N}_{(l)}^{j+1}$). We set also $\mathcal{N}_{(l)}^\bullet := \mathcal{O}_{\overline{\mathbb{P}}}(-l)[l]$ for $0 \leqslant l \leqslant n$.

**Lemma 3.3.** *Let* $\mathcal{E}_j := \mathcal{O}_{\overline{\mathbb{P}}}(j)$ *if* $n - |\mathrm{w}| < j < 0$ *and* $\mathcal{E}_j := \Omega^j(j)$ *if* $0 \leqslant j \leqslant n$. *Then*

(1) $h^i(\overline{\mathbb{P}}, \mathcal{M}_{(l)}^\bullet(j)) = \delta_{j,l}\delta_{i,0}$ *for* $-|\mathrm{w}| < j, l \leqslant 0$ *and* $\forall i \in \mathbb{Z}$;
(2) $h^i(\overline{\mathbb{P}}, \mathcal{N}_{(l)}^\bullet \otimes \mathcal{E}_j) = \delta_{j,l}\delta_{i,0}$ *for* $n - |\mathrm{w}| < j, l \leqslant n$ *and* $\forall i \in \mathbb{Z}$.

We will denote by $\widehat{\mathbf{M}}_{]a,b[}(\mathrm{P})$ the full subcategory of $\overline{\mathbf{Mod}}(\mathrm{P})$ whose objects are finite direct sums of graded P-modules of the form $\mathrm{P}(j)$ for $a < j < b$ or $Syz^j(j)$ for $0 \leqslant j \leqslant n$.

**Theorem 3.4.** *Restricting* $\widetilde{\ } : \overline{\mathbf{Mod}}(\mathrm{P}) \to \overline{\mathbf{Mod}}(\overline{\mathbb{P}})$ *induces additive functors* $\mathbf{M}_{[1-|\mathrm{w}|,0]}(\mathrm{P}) \to \overline{\mathbf{Coh}}(\overline{\mathbb{P}})$, $\widehat{\mathbf{M}}_{]n-|\mathrm{w}|,0[}(\mathrm{P}) \to \overline{\mathbf{Coh}}(\overline{\mathbb{P}})$. *They extend to exact equivalences between triangulated categories*

$$\overline{F}_{\mathrm{P}} : K^b\big(\mathbf{M}_{[1-|\mathrm{w}|,0]}(\mathrm{P})\big) \to D^b\big(\overline{\mathbf{Coh}}(\overline{\mathbb{P}})\big), \qquad \overline{G}_{\mathrm{P}} : K^b\big(\widehat{\mathbf{M}}_{]n-|\mathrm{w}|,0[}(\mathrm{P})\big) \to D^b\big(\overline{\mathbf{Coh}}(\overline{\mathbb{P}})\big).$$

*Moreover, given* $\mathcal{F}^\bullet \in D^b(\overline{\mathbf{Coh}}(\overline{\mathbb{P}}))$*, there exists a unique (up to isomorphism of complexes) bounded complex* $\overline{\mathcal{X}}^\bullet = \overline{\mathcal{X}}^\bullet(\mathcal{F}^\bullet)$ *(respectively* $\overline{\mathcal{Y}}^\bullet = \overline{\mathcal{Y}}^\bullet(\mathcal{F}^\bullet)$*) isomorphic to* $\mathcal{F}^\bullet$ *in* $D^b(\overline{\mathbf{Coh}}(\overline{\mathbb{P}}))$*, such that each* $\overline{\mathcal{X}}^i$ *(respectively* $\overline{\mathcal{Y}}^i$*) is a finite direct sum of sheaves of* $\overline{O}$ *(respectively* $\overline{D}$*), and which is minimal.*[1] *Explicitly,*

$$\overline{\mathcal{X}}^i = \bigoplus_{-|\mathrm{w}|<j\leqslant 0} \mathcal{O}_{\overline{\mathbb{P}}}(j) \otimes_{\mathbb{K}} H^i(\overline{\mathbb{P}}, \mathcal{F}^\bullet \otimes \mathcal{M}^\bullet_{(j)}),$$

$$\overline{\mathcal{Y}}^i = \bigoplus_{n-|\mathrm{w}|<j<0} \mathcal{O}_{\overline{\mathbb{P}}}(j) \otimes_{\mathbb{K}} H^i(\overline{\mathbb{P}}, \mathcal{F}^\bullet \otimes \mathcal{N}^\bullet_{(j)}) \bigoplus_{0\leqslant j\leqslant n} \Omega^j_{\overline{\mathbb{P}}}(j) \otimes_{\mathbb{K}} H^{i+j}\big(\overline{\mathbb{P}}, \mathcal{F}^\bullet(-j)\big).$$

**Proof.** Fully faithfulness of $\overline{F}_{\mathrm{P}}$ and $\overline{G}_{\mathrm{P}}$ is a formal consequence of 3.1 (see [2] or [3]), whereas essential surjectivity corresponds to the fact that $\overline{O}$ and $\overline{D}$ generate $D^b(\overline{\mathbf{Coh}}(\overline{\mathbb{P}}))$ as a triangulated category. Now, by Hilbert's syzygy theorem $D^b(\overline{\mathbf{Coh}}(\overline{\mathbb{P}}))$ is generated by the $\mathcal{O}_{\overline{\mathbb{P}}}(j)$ ($j \in \mathbb{Z}$), and the exactness of $\mathcal{K}^\bullet$ implies that $|\mathrm{w}|$ consecutive of them are actually enough, so that $\overline{O}$ generates $D^b(\overline{\mathbf{Coh}}(\overline{\mathbb{P}}))$. From this and from the definition of $\Omega^j_{\overline{\mathbb{P}}}(j)$ it is also easy to see that $\overline{D}$ generates $D^b(\overline{\mathbf{Coh}}(\overline{\mathbb{P}}))$, too. Then it is a standard fact that $\overline{\mathcal{X}}^\bullet$ and $\overline{\mathcal{Y}}^\bullet$ exist unique, and using 3.3 one can show that they must have the claimed form. □

**Remark 2.** If $\mathrm{w} = (1, \ldots, 1)$, this result coincides with 1.1 (taking into account the equivalence of 2.3, the isomorphism of 2.4 and the fact that $\mathcal{M}^\bullet_{(-j)}$ is isomorphic to $\Omega^j_{\overline{\mathbb{P}}^n}(j)[j]$ in $D^b(\mathbf{Coh}(\overline{\mathbb{P}}^n))$), except that, in order to identify $\overline{G}_{\mathrm{P}}$ with $F_\Lambda$, it is also necessary to consider the equivalence between $\mathbf{M}_{[-n,0]}(\Lambda)$ and $\widehat{\mathbf{M}}_{]-1,0[}(\mathrm{P})$ (given by $\Lambda(-j) \mapsto Syz^j(j)$).

**Remark 3.** Given $\mathcal{G}^\bullet \in D^b(\mathbf{Coh}(\mathbb{P}))$, by 2.2 there exists (in general not unique) $\mathcal{F}^\bullet \in D^b(\overline{\mathbf{Coh}}(\overline{\mathbb{P}}))$ such that $\mathcal{G}^\bullet = \mathcal{F}^\bullet_0$, whence $\overline{\mathcal{X}}^\bullet(\mathcal{F}^\bullet)_0$ and $\overline{\mathcal{Y}}^\bullet(\mathcal{F}^\bullet)_0$ are (minimal) resolutions of $\mathcal{G}^\bullet$ (but they are no more unique, unless $\mathrm{w} = (1, \ldots, 1)$).

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[1] Here minimal means that in the differential of the complex no morphism from $\mathcal{O}_{\overline{\mathbb{P}}}(j)$ to $\mathcal{O}_{\overline{\mathbb{P}}}(j)$ or from $\Omega^j_{\overline{\mathbb{P}}}(j)$ to $\Omega^j_{\overline{\mathbb{P}}}(j)$ is an isomorphism. This implies that each morphism from $\mathcal{O}_{\overline{\mathbb{P}}}(j)$ to $\mathcal{O}_{\overline{\mathbb{P}}}(j)$ is 0, whereas morphisms from $\Omega^j_{\overline{\mathbb{P}}}(j)$ to $\Omega^j_{\overline{\mathbb{P}}}(j)$ are nilpotent, but not necessarily 0 (they are 0 if $\mathrm{w} = (1, \ldots, 1)$).