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On the Gibbs properties of the Erdös measure

Sur les propriétés de Gibbs de la mesure d'Erdös

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Abstract

We consider the infinite convolved Bernoulli measures (Bernoulli convolutions) related to β-numeration. A Markovian matrix decomposition of these measures is obtained when β is a Pisot number whose associated β-shift is of finite type. We study the special case of the Erdös measure (i.e., when β is the golden ratio) that we prove to be weak Gibbs, insuring the multifractal formalism to hold. ***To cite this article: E. Olivier, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Résumé

Nous considérons les mesures obtenues comme une convolution d'une infinité de mesures de Bernoulli (convolutions de Bernoulli) liées à la β-numération. Une décomposition matricielle markovienne de ces mesures est établie, quand β est un nombre de Pisot dont le β-shift associé est de type fini. Nous concluons en démontrant que la mesure d'Erdös (i.e., quand β est le nombre d'or) est faiblement de Gibbs, assurant ainsi que le formalisme multifractal est valide. ***Pour citer cet article : E. Olivier, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Version française abrégée

L'étude des convolutions de Bernoulli a été initiée à la fin des années 30 avec les travaux de Jesen et Wintner [4] ainsi que ceux de Erdös [2] et s'est développée récemment en géométrie fractale [1,6,5] et en théorie ergodique [9] (voir [8] pour plus de détails). Cette Note est consacrée à la présentation d'un cas particulier représentatif extrait de l'analyse plus complète faite dans [7] (voir aussi [3]). Il s'agit de démontrer que la mesure d'Erdös est faiblement de Gibbs au sens de Yuri [10], sans être toutefois de Gibbs ; l'intérêt du résultat tient au fait que le formalisme multifractal est valide pour cette classe de mesures (voir [3]).

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Plus précisément, pour tout nombre réel $1 < \beta < 2$, la convolution de Bernoulli μ est définie comme la distribution de la variable aléatoire réelle X définie sur l'espace produit $\{0,1\}^{\mathbf{N}}$, muni de la probabilité de Bernoulli de paramètre $\mathsf{p} = (\mathsf{p}_0, \mathsf{p}_1)$ (avec $\mathsf{p}_0\mathsf{p}_1 > 0$) et telle que, pour tout $\xi = (\xi_k)_{k=0}^{\infty}$,

$$X(\xi) = \frac{1}{\alpha}\sum_{k=0}^{\infty}\frac{\xi_k}{\beta^{k+1}} \quad \text{avec } \alpha = \frac{1}{\beta - 1}.$$

La probabilité μ est pleinement supportée par l'intervalle unité $I := [0,1]$; pour tout $i \in \mathbf{R}$ nous notons $\mathbb{T}_i$ la contraction affine telle que $\mathbb{T}_i(x) := (x+i)/\alpha$, de sorte que $\mathbb{T}_i(I) \cap I$ a un intérieur non vide uniquement pour $-1 < i < \alpha$. Lorsque β est un nombre de Pisot, nous montrons qu'il existe un ensemble fini $\mathcal{I}_\beta := \{\mathtt{i}_0, \ldots, \mathtt{i}_\mathtt{r}\}$ contenu dans $]-1, \alpha[$, ainsi que deux matrices M_0 et M_1 d'ordre $\mathtt{r}$ à coefficients positifs ou nuls tels que, pour tout borélien B de $\mathbf{R}$,

$$\begin{pmatrix} \mu \circ \mathbb{T}_{\mathtt{i}_0}(B) \\ \vdots \\ \mu \circ \mathbb{T}_{\mathtt{i}_\mathtt{r}}(B) \end{pmatrix} = M_\varepsilon \begin{pmatrix} \mu \circ \mathbb{T}_{\mathtt{i}_0}(\mathbb{R}_\varepsilon^{-1}(B)) \\ \vdots \\ \mu \circ \mathbb{T}_{\mathtt{i}_\mathtt{r}}(\mathbb{R}_\varepsilon^{-1}(B)) \end{pmatrix}, \tag{1}$$

où $\mathbb{R}_\varepsilon$ est la contraction affine telle que $\mathbb{R}_\varepsilon(x) := (x+\varepsilon)/\beta$. Nous considérons le cas particulier de la mesure d'Erdös, c'est-à-dire la mesure μ associée à $\beta = (1+\sqrt{5})/2$ et $\mathsf{p} = (1/2, 1/2)$. Soit $\mu_i(\cdot) := \mu \circ \mathbb{T}_i(\cdot \cap I)$: c'est une mesure positive de support inclus dans I, dès que $-1 < i < \alpha$. Pour $\mathbb{R}'_0 := \mathbb{R}_0 \circ \mathbb{R}_0$, $\mathbb{R}'_1 := \mathbb{R}_0 \circ \mathbb{R}_1 \circ \mathbb{R}_0$ et $\mathbb{R}'_2 := \mathbb{R}_1 \circ \mathbb{R}_0$, on vérifie que $\{\mathbb{R}'_0[0,1[, \mathbb{R}'_1[0,1[, \mathbb{R}'_2[0,1[\}$ forme une partition de $[0,1[$: étant donné $m = \omega_0 \cdots \omega_{n-1} \in \{0,1,2\}^n$ ($n \geqslant 1$), le cylindre de base m est par définition l'intervalle $[\![m]\!] := \mathbb{R}_{\omega_0} \circ \cdots \circ \mathbb{R}_{\omega_{n-1}}[0,1[$ et nous déduisons de (1) que

$$\begin{pmatrix} \mu_0[\![m]\!] \\ \mu_{1/\beta}[\![m]\!] \end{pmatrix} = P_{\omega_0} \cdots P_{\omega_{n-1}} \begin{pmatrix} \mu_0(I) \\ \mu_{1/\beta}(I) \end{pmatrix}, \tag{2}$$

où le calcul des produits matriciels M_0M_0, $M_0M_1M_0$ et M_1M_0 donne :

$$P_0 = \begin{pmatrix} 1/4 & 0 \\ 1/4 & 1/4 \end{pmatrix}, \quad P_1 = \begin{pmatrix} 1/8 & 1/8 \\ 1/8 & 1/8 \end{pmatrix} \quad \text{and} \quad P_2 = \begin{pmatrix} 1/4 & 1/4 \\ 0 & 1/4 \end{pmatrix}.$$

Nous montrons tout d'abord que μ n'est pas une mesure de Gibbs. Cependant, en utilisant la représentation matricielle (2), il est possible de prouver, grâce au Théorème 2.1, que la suite (ϕ_n) des potentiels markoviens d'ordre n (n-step potentials) de la probabilité $\mu_* := \frac{3}{4}(\mu_0 + \mu_{1/\beta})$ est uniformément convergente sur l'espace produit $\{0,1,2\}^{\mathbf{N}}$: on en déduit alors successivement que μ_* et μ sont faiblement gibssiennes (Théorème 4.1).

1. Introduction

The Bernoulli convolutions have been studied since the early 30s [4,2] and more recently with applications in Fractal Geometry [1,6,5] and Ergodic Theory [9]: we refer to [8] for further details and references. In this Note we present a special case taken from the more detailed analysis in [7] (see also [3]). More precisely, we prove that the Erdös measure, that is the symmetric Bernoulli convolution associated with the golden ratio, is not a Gibbs measure, but displays the weak Gibbs property in the sense of Yuri [10]. Our interest for such a property comes from the fact that a weak Gibbs measure satisfies the so-called multifractal formalism (see [3]).

2. Gibbs properties

Let $\mathcal{A}^*$ denote the set of words on the alphabet $\mathcal{A} := \{0, \ldots, s-1\}$ (for a given integer $s \geqslant 2$), that is, $\mathcal{A}^* := \{\emptyset\} \cup \bigcup_{n=1}^{\infty} \mathcal{A}^n$, where $\emptyset$ is the empty word. The concatenation of the two words w, m is denoted by wm

so that $\mathcal{A}^*$ endowed with the concatenation is a monoid of unit element $\emptyset$. Whenever $x_0, \ldots, x_{s-1}$ are s elements of a monoid $(X, \star)$ with identity element e, we note $x_\emptyset := \mathsf{e}$ and $x_w := x_{\xi_0} \star \cdots \star x_{\xi_{n-1}}$, for any non-empty word $w = \xi_0 \cdots \xi_{n-1} \in \mathcal{A}^*$.

We now consider that $\mathcal{S} := \{\mathbb{S}_i\}_{i=0}^{s-1}$ is a system of (orientation preserving) affine contractions of the real line; we assume that $\mathcal{S}$ is *adapted* to the interval $[0, 1[$, meaning that $\{\mathbb{S}_i[0, 1[\}_{i=0}^{s-1}$ form a partition of $[0, 1[$. For any word $w \in \mathcal{A}^*$, we define the cylinder $[\![w]\!] := \mathbb{S}_w[0, 1[$ and the $\mathcal{S}$-net is $\mathfrak{F} := \{\mathfrak{F}_n\}_{n=0}^{\infty}$, where $\mathfrak{F}_n := \{[\![w]\!];\ w \in \mathcal{A}^n\}$ (by convention $\mathcal{A}^0 := \{\emptyset\}$). We denote by σ the shift transformation defined on the product space $\mathcal{A}^{\mathbf{N}}$; the product topology on $\mathcal{A}$ is given by the metric such that the distance between ξ and ζ is 2^{-k}, where k is the length of the largest common prefix of ξ and ζ.

Suppose that η is a finite positive Borel measure supported by the unit interval $I := [0, 1]$. For any $\xi \in \mathcal{A}^{\mathbf{N}}$ we set $\phi_1(\xi) := \log \eta[\![\xi_0]\!]$ and $\phi_n(\xi) := \log \eta[\![\xi_0 \cdots \xi_{n-1}]\!]/\eta[\![\xi_1 \cdots \xi_{n-1}]\!]$, for $n \geqslant 2$. The so-called n-step potential ϕ_n is clearly continuous on $\mathcal{A}^{\mathbf{N}}$. Under the assumption that ϕ_n converges uniformly to a potential Φ, it is easily checked that for $n \geqslant 1$,

$$\frac{1}{K_n} \leqslant \frac{\eta[\![\xi_0 \cdots \xi_{n-1}]\!]}{\exp(\sum_{k=0}^{n-1} \Phi(\sigma^k \xi))} \leqslant K_n, \tag{3}$$

where $K_n := \exp(\sum_{k=1}^{n} \|\Phi - \phi_n\|_\infty)$. Since $\|\Phi - \phi_n\|_\infty$ tends to 0, when n goes to infinity, by a classical lemma on the Cesàro sums, the sequence (K_n) is sub-exponential in the sense that $\lim_n (\log K_n)/n = 0$; in such a case, (3) means that η is a weak Gibbs measure; when (K_n) is constant, η is said to be a Gibbs measure: this clearly happens when $\|\Phi - \phi_n\|_\infty$ is the term of a convergent series.

We now assume that $s = 3$. For any $i \in \mathcal{A}$, we let $M_i := \gamma_i P_i$, where $\gamma_i > 0$ and

$$P_0 := \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad P_1 := \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad P_2 := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix};$$

we suppose in addition that there exists a column vector R with positive entries such that $(M_0 + M_1 + M_2)R = R$. Let L be a vector with non-negative entries such that ${}^tLR = 1$; then, by Kolmogorov's consistency theorem, by setting $\nu[\![w]\!] = {}^tLM_wR$, for any word $w \in \mathcal{A}^*$, one defines a Borel probability measure on $\mathbf{R}$ which is supported by I.

Theorem 2.1. *ν is $\mathfrak{F}$-weak Gibbs when L has positive entries.*

Proof. For $\omega \in \Omega$ and $n \geqslant 1$ we note $\widetilde{P}_n^\omega := P_{\omega_0 \cdots \omega_{n-1}}$; for any vector $V \neq 0$ with non-negative entries, we define $\theta_n(\omega|V) := \arctan(\tau(\widetilde{P}_n^\omega V))$ where $\tau(\widetilde{P}_n^\omega V)$ is the ratio of the y-coordinate of $\widetilde{P}_n^\omega V$ divided by its x-coordinate. Let ${}^tL = (u \quad v)$ and ${}^tLP_i := (u' \quad v')$ for a fixed $i \in \mathcal{A}$; then, for any $\xi = (\xi_k)_{k=0}^\infty \in \mathcal{A}^{\mathbf{N}}$, $\xi_0 = i$ and $n \geqslant 2$, one has with $\sigma\xi = (\xi_{k+1})_{k=0}^\infty$:

$$\phi_n(\xi) = \log \gamma_i + \Lambda_i\big(\theta_n(\sigma\xi|R)\big), \quad \text{where } \Lambda_i(x) = \log\left(\frac{u' \cos x + v' \sin x}{u \cos x + v \sin x}\right).$$

If L is an eigenvector of ${}^tP_i$, the map Λ_i is constant and the sequence of the n-step potentials ϕ_n is constant on $\{\xi_0 = i\}$. Otherwise, Λ_i is a homeomorphism from $[0, \pi/2]$ onto the closed interval $\Lambda_i[0, \pi/2]$; by the Mean Value Theorem, there exists a constant $c > 0$, independent of ξ such that

$$\big|\phi_p(\xi) - \phi_q(\xi)\big| \leqslant c\big|\theta_p(\sigma\xi|R) - \theta_q(\sigma\xi|R)\big|. \tag{4}$$

Since the matrix $\widetilde{P}_n^\omega$ has no null column it is possible (with the convention $x/0 = \infty$ for $x > 0$) to define the homography $\mathsf{h}_n^\omega : [0, \infty] \mapsto [0, \infty]$ by setting

$$\mathsf{h}_n^\omega(x) := \frac{\gamma_n^\omega + \delta_n^\omega x}{\alpha_n^\omega + \beta_n^\omega x}, \quad \text{where } \widetilde{P}_n^\omega = \begin{pmatrix} \alpha_n^\omega & \beta_n^\omega \\ \gamma_n^\omega & \delta_n^\omega \end{pmatrix} \tag{5}$$

(and $\mathsf{h}_n^\omega(\infty) = \delta_n^\omega/\beta_n^\omega$). Since $\det(\widetilde{P}_n^\omega) \geqslant 0$, the homography h_n^ω is non-decreasing on $[0,\infty]$ and for any V with positive entries, the quantity $\tau(\widetilde{P}_p^\omega V)$ belongs to the interval $\mathtt{I}_n^\omega := [\mathsf{h}_n^\omega(0), \mathsf{h}_n^\omega(\infty)]$.

Lemma 2.2. *The sequence $\theta_n(\cdot\,|R)$ converges uniformly on Ω whenever, for any $\omega \in \Omega$, the decreasing intersection $\mathtt{I}^\omega := \bigcap_{n=1}^\infty \mathtt{I}_n^\omega$ is reduced to one point in $[0,\infty]$.*

Proof. Suppose that $\mathtt{I}^\omega$ is reduced to $\{t_\omega\}$, for any $\omega \in \Omega$; given $\varepsilon > 0$, we define the integer $n(\omega,\varepsilon)$, such that $|\mathtt{I}_{n(\omega,\varepsilon)}^\omega| \leqslant \varepsilon$ if $t_\omega < \infty$ and $\mathtt{I}_{n(\omega,\varepsilon)}^\omega \subseteq [1/\varepsilon, \infty]$ if $t_\omega = \infty$. Let $C(\omega,\varepsilon)$ denote the open set of the $\xi \in \Omega$ satisfying $\xi_0 \cdots \xi_{n(\omega,\varepsilon)-1} = \omega_0 \cdots \omega_{n(\omega,\varepsilon)-1}$; since Ω is compact, there exists a finite $X \subset \Omega$ such that $\Omega = \bigcup_{\xi\in X} C(\xi,\varepsilon)$. For p, q bounded from below by $n_\varepsilon := \max_{\xi\in X}\{n(\xi,\varepsilon)\}$, both $\tau(\widetilde{P}_p^\omega R)$ and $\tau(\widetilde{P}_q^\xi R)$ belong to $\mathtt{I}_{n(\xi,\varepsilon)}^\omega = \mathtt{I}_{n(\xi,\varepsilon)}^\xi$, with $\xi \in X$ and $\omega \in C(\xi,\varepsilon)$. By the definition of n_ε, one deduces that $|\theta_p(\omega|R) - \theta_q(\omega|R)| \leqslant \varepsilon$: the proof of Lemma 2.2 is complete. □

According to (4) and Lemma 2.2, Theorem 2.1 will be established if we prove that $\mathtt{I}^\omega$ is always reduced to one point. First, if $\omega_{n_0} = 1$ then, for $n \geqslant n_0$ one has $\det(\widetilde{P}_n^\omega) = 0$: the homography h_n^ω is constant and $\mathtt{I}^\omega$ is reduced to one point. It remains to consider the case when $\omega \in \{0,2\}^{\mathbf{N}}$. If $\sigma^{n_0}\omega = \bar{0}$ (or similarly $\bar{2}$) then, for any $m \geqslant 0$ one has $\mathsf{h}_{n_0+m}^\omega(x) = \mathsf{h}_{n_0}^\omega(x+m)$, so that $\mathtt{I}_{n_0+m}^\omega = [\mathsf{h}_{n_0}^\omega(m), \mathsf{h}_{n_0}^\omega(\infty)]$ and thus $\mathtt{I}^\omega = \{\mathsf{h}_{n_0}^\omega(\infty)\}$. Otherwise, when $\sigma^k\omega$ always differs from $\bar{0}$ or $\bar{2}$, there exists an infinite sequence of integers $a_0, a_1, \ldots$ with $a_0 \geqslant 0$ and $a_1 a_2 \cdots > 0$ such that, $\omega = 2^{a_0}0^{a_1}2^{a_2}\cdots$ (by convention $2^0 = \emptyset$); for p_k and q_k the relatively prime integers such that

$$\frac{p_k}{q_k} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_k}}}$$

and $n_k = a_0 + \cdots + a_{2k-1}$, a direct computation, yields:

$$\widetilde{P}_{n_k}^\omega = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_{2k-1} & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} q_{2k-2} & q_{2k-1} \\ p_{2k-2} & p_{2k-1} \end{pmatrix};$$

therefore $[\mathsf{h}_{n_k}^\omega(0), \mathsf{h}_{n_k}^\omega(\infty)] = [p_{2k-2}/q_{2k-2}, p_{2k-1}/q_{2k-1}]$ and $\mathtt{I}^\omega$ is reduced to the common limit of p_{2k-2}/q_{2k-2} and p_{2k-1}/q_{2k-1} when k goes to infinity: Theorem 2.1 is proved. □

3. Bernoulli convolutions

We denote by $\sigma : \Sigma \to \Sigma$ the shift on the product space $\Sigma := \{0,1\}^{\mathbf{N}}$. Assume that Σ is weighted by the Bernoulli measure of parameter $\mathsf{p} = (\mathsf{p}_0, \mathsf{p}_1)$, with $\mathsf{p}_0\mathsf{p}_1 > 0$; for a given real number $1 < \beta < 2$, the Bernoulli convolution μ is by definition the distribution of the random variable $X : \Sigma \to \mathbf{R}$, such that $X(\xi) := (\sum_{i=0}^\infty \xi_i/\beta^{i+1})/\alpha$, with $\alpha := 1/(\beta-1)$. For any $i \in \mathbf{R}$ we let $\mathbb{T}_i$ be the affine contraction such that $\mathbb{T}_i(x) := (x+i)/\alpha$; for $\varepsilon = 0$ or 1 and any Borel set $B \subset \mathbf{R}$, one has $X(\omega) \in \mathbb{T}_i(B)$ if and only if $X(\sigma\omega) \in \mathbb{T}_{i_{\omega_0}^\varepsilon}(\mathbb{R}_\varepsilon^{-1}(B))$ where $i_{\omega_0}^\varepsilon := \beta i + (\varepsilon - \omega_0)$ and $\mathbb{R}_\varepsilon(x) = (x+\varepsilon)/\beta$. Therefore, using the properties of Bernoulli measures, one gets:

$$\mu \circ \mathbb{T}_i(B) = \mathsf{p}_0 \cdot \mu \circ \mathbb{T}_{i_0^\varepsilon}\big(\mathbb{R}_\varepsilon^{-1}(B)\big) + \mathsf{p}_1 \cdot \mu \circ \mathbb{T}_{i_1^\varepsilon}\big(\mathbb{R}_\varepsilon^{-1}(B)\big). \tag{6}$$

Notice that $\mu \circ \mathbb{T}_i(B) = 0$ for each Borel set $B \subset I$ whenever $i \notin\,]-1,\alpha[$; we define $\mathcal{I}_\beta$ as the set of the real numbers $i \in\,]-1,\alpha[$ for which there exists a sequence $i_1, \ldots, i_n$ in $]-1,\alpha[$ such that $0 \rhd i_1 \rhd \cdots \rhd i_n \rhd i$, where

$x \rhd y$ means that exists $\varepsilon \in \{-1, 0, 1\}$ such that $y = \beta x + \varepsilon$. The set $\mathcal{I}_\beta$ is clearly countable and we shall write $0 = \mathtt{i}_0, \mathtt{i}_1, \ldots$ the sequence of its elements: it follows from (6) that for any $B \subset I$ such that $\mathbb{R}_\varepsilon^{-1}(B) \subset I$

$$\mu \circ \mathbb{T}_{\mathtt{i}_i}(B) = \sum_j M_\varepsilon(i, j) \cdot \mu \circ \mathbb{T}_{\mathtt{i}_j}\big(\mathbb{R}_\varepsilon^{-1}(B)\big), \tag{7}$$

where i and j are respectively the row and the column indexes of the matrix M_ε such that

$$M_\varepsilon(i, j) = \begin{cases} \mathsf{p}_k & \text{if } k = \varepsilon + \beta \mathtt{i}_i - \mathtt{i}_j \in \{0, 1\}, \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

For any real number i we define the measure $\mu_i(\cdot) := \mu \circ \mathbb{T}_i(\cdot \cap I)$; since μ is supported by I, the measure μ_i is positive (and of support contained in I) if and only if $-1 < i < \alpha$. When $\mathcal{I}_\beta := \{0 = \mathtt{i}_0, \ldots, \mathtt{i}_\mathsf{r}\}$, one deduces from (7) that for $\varepsilon \in \{0, 1\}$ and $B \subset I$ such that $\mathbb{R}_\varepsilon^{-1}(B) \subset I$,

$$\begin{pmatrix} \mu_{\mathtt{i}_0}(B) \\ \vdots \\ \mu_{\mathtt{i}_\mathsf{r}}(B) \end{pmatrix} = M_\varepsilon \cdot \begin{pmatrix} \mu_{\mathtt{i}_0}(\mathbb{R}_\varepsilon^{-1}(B)) \\ \vdots \\ \mu_{\mathtt{i}_\mathsf{r}}(\mathbb{R}_\varepsilon^{-1}(B)) \end{pmatrix}. \tag{9}$$

Our analysis of the Bernoulli convolution μ relies on finiteness of the set $\mathcal{I}_\beta^{\mathsf{d}}$. In this Note, we shall consider that β is a Pisot number of degree s, that is an algebraic integer greater than 1 and whose conjugates β_i $(1 \leqslant i < s)$ have modulus strictly less than 1. The key property of such a number is that, for any polynomial $A(X)$ with integral coefficients bounded by M in absolute value, one has

$$A(\beta) \neq 0 \;\Rightarrow\; \big|A(\beta)\big| \geqslant \prod_{i=1}^{s-1} \frac{1 - |\beta_i|}{M} > 0. \tag{10}$$

Notice that for any element i in $\mathcal{I}_\beta$, there exists a finite sequence $\varepsilon_0, \ldots, \varepsilon_m$ of integers $\{-1, 0, 1\}$ such that $i = \varepsilon_0 + \varepsilon_1 \beta + \cdots + \varepsilon_m \beta^m$; by (10), the distance between two different elements in $\mathcal{I}_\beta$ is bounded from below by $\prod_{i=1}^{s-1}(1 - |\beta_i|)/2$:

Proposition 3.1. *The set $\mathcal{I}_\beta$ has finite cardinality when β is a Pisot number.*

4. The Erdös measure

We now consider that $\mathsf{p}_0 = \mathsf{p}_1 = 1/2$ and $\beta = (1 + \sqrt{5})/2$, so that μ is the Erdös measure. It is readily checked from its definition that $\mathcal{I}_\beta$ has three elements say $\mathtt{i}_0 = 0$, $\mathtt{i}_1 = 1$ and $\mathtt{i}_2 = \beta - 1 = 1/\beta$. Notice that the system of affine contractions $\mathcal{R} := \{\mathbb{R}_0', \mathbb{R}_1', \mathbb{R}_2'\}$, where $\mathbb{R}_0' = \mathbb{R}_{00}$, $\mathbb{R}_1' = \mathbb{R}_{010}$ and $\mathbb{R}_2' = \mathbb{R}_{10}$, is adapted to $[0, 1[$: we shall denote by $\mathfrak{F}$ the $\mathcal{R}$-net. According to (9), one deduces that for any word $w \in \{0, 1, 2\}^*$,

$$\begin{pmatrix} \mu_0[\![w]\!] \\ \mu_{1/\beta}[\![w]\!] \end{pmatrix} = P_w \begin{pmatrix} \mu_0(I) \\ \mu_{1/\beta}(I) \end{pmatrix}, \tag{11}$$

where the computation of the matrices M_{00}, M_{010} and M_{10} yields:

$$P_0 = \begin{pmatrix} 1/4 & 0 \\ 1/4 & 1/4 \end{pmatrix}, \quad P_1 = \begin{pmatrix} 1/8 & 1/8 \\ 1/8 & 1/8 \end{pmatrix} \quad \text{and} \quad P_2 = \begin{pmatrix} 1/4 & 1/4 \\ 0 & 1/4 \end{pmatrix}.$$

Using (6) with $i = \varepsilon = 0$, one deduces the classical self-similar equation satisfied by μ:

$$\mu = \frac{1}{2} \cdot \mu \circ \mathbb{T}_0^{-1} + \frac{1}{2} \cdot \mu \circ \mathbb{T}_{1/\beta}^{-1}. \tag{12}$$

The specific algebraic properties of the golden ratio give the following identities:

$$\mathbb{T}_0\circ\mathbb{T}_0=\mathbb{R}'_0,\quad \mathbb{T}_0\circ\mathbb{T}_{1/\beta}\circ\mathbb{T}_{1/\beta}=\mathbb{T}_{1/\beta}\circ\mathbb{T}_0\circ\mathbb{T}_0=\mathbb{R}'_1\quad\text{and}\quad \mathbb{T}_{1/\beta}\circ\mathbb{T}_{1/\beta}=\mathbb{R}'_2$$

and an iteration of (12) applied to $\mu\circ\mathbb{R}'_i$ for $i=0,1,2$, yields:

$$\mu\circ\mathbb{R}'_0=\frac{\mu_0}{4},\quad \mu\circ\mathbb{R}'_1=\frac{\mu_0+\mu_{1/\beta}}{4}\quad\text{and}\quad \mu\circ\mathbb{R}'_2=\frac{\mu_{1/\beta}}{4}.$$

Hence, for any $w\in\{0,1,2\}^*$ and $\varepsilon\in\{0,1,2\}$, one has:

$$\mu[\![\varepsilon w]\!]={}^tV_\varepsilon P_w\begin{pmatrix}1\\1\end{pmatrix},\quad \text{where } V_0=\begin{pmatrix}1/6\\0\end{pmatrix},\ V_1=\begin{pmatrix}1/3\\1/3\end{pmatrix} \text{ and } V_2=\begin{pmatrix}0\\1/6\end{pmatrix}$$

(the constants are obtained from the fact that μ is a probability). By a direct computation, one gets $\mu[\![10^{n-1}]\!]=(n+1)/(3\cdot 4^{n-1})$, for any $n\geqslant 0$. If one supposes that μ is a Gibbs measure of some potential ψ, then the sequence $\log\{4^n\cdot\exp(n\psi(\bar{0}))/n\}$ would be bounded: this is impossible and one concludes that the Erdös measure μ is not Gibbs. We conclude this Note by proving the following theorem:

Theorem 4.1. *The Erdös measure μ is $\mathfrak{F}$-weak Gibbs.*

Proof. Consider the probability $\mu_*:=\frac{3}{4}(\mu_0+\mu_{1/\beta})$: then, for any word $w\in\{0,1,2\}^*$,

$$\mu_*[\![w]\!]=\frac{1}{2}\,{}^tUP_wU,\quad \text{where } U:=\begin{pmatrix}1\\1\end{pmatrix},\tag{13}$$

and by Theorem 2.1, one deduces that μ_* is $\mathfrak{F}$-weak Gibbs. Let $w=\varepsilon^a\hat{\varepsilon}w'\in\{0,1,2\}^n$ with $\varepsilon\in\{0,2\}$, $a>0$ and $\hat{\varepsilon}=2$ if $\varepsilon=0$ and $\hat{\varepsilon}=0$ if $\varepsilon=2$; for $P_{w'}U:={}^t(p_0\quad p_2)$

$$\frac{\mu[\![w]\!]}{\mu_*[\![w]\!]}=\frac{{}^tV_\varepsilon P_\varepsilon^{a-1}P_{\hat{\varepsilon}}P_{w'}U}{{}^tUP_\varepsilon^aP_{\hat{\varepsilon}}P_{w'}U}\geqslant\frac{1}{3}\frac{p_0+p_2}{(a+1)(p_0+p_2)+p_{\hat{\varepsilon}}}\geqslant\frac{1}{3(a+2)}$$

and since $a\leqslant n$ one gets:

$$\frac{\mu[\![w]\!]}{\mu_*[\![w]\!]}\geqslant\frac{1}{3(n+2)}.\tag{14}$$

A similar argument shows that (14) is still valid when $w=\varepsilon^a$ or $w=1w'$. The upper bound $\mu[\![w]\!]/\mu_*[\![w]\!]\leqslant 8/3$ being always valid, for any $w\in\{0,1,2\}^n$ with $n\geqslant 1$, one has:

$$\frac{1}{3(n+2)}\leqslant\frac{\mu[\![w]\!]}{\mu_*[\![w]\!]}\leqslant\frac{8}{3}.\tag{15}$$

The measure μ_* being $\mathfrak{F}$-weak Gibbs, one concludes from (15) that μ is also $\mathfrak{F}$-weak Gibbs. □

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