# RENEWAL THEOREM FOR A SYSTEM OF RENEWAL EQUATIONS 

# THÉORÈME DE RENOUVELLEMENT POUR UN SYSTÈME D'ÉQUATIONS DE RENOUVELLEMENT 

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AbStract. - We show that the classical renewal theorems of Feller hold in the case of a system of renewal equations, when the distributions involved are supported on the whole real line. We extend Feller's methods and also use Perron-Frobenius theory and potential theory. © 2003 Éditions scientifiques et médicales Elsevier SAS

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Résumé. - On généralise les théorèmes de renouvellement de Feller au cas d'un système d'équations de renouvellement faisant intervenir des mesures qui ont pour support toute la droite réelle. Pour cela on suit la même démarche que Feller en faisant intervenir de plus la théorie de Perron-Frobenius et la théorie du potentiel.
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## 1. Introduction

We study the asymptotic behavior, when $t$ tends to $+\infty$, of $Z(t)={ }^{t}\left(Z_{1}(t), \ldots, Z_{p}(t)\right)$ the solution of a system of renewal equations of the following type:

$$
\begin{equation*}
Z_{i}(t)=G_{i}(t)+\sum_{k=1}^{p} \int_{-\infty}^{\infty} Z_{k}(t-u) F_{i k}(d u), \quad \forall t \in \mathbb{R}, \forall 1 \leqslant i \leqslant p \tag{1}
\end{equation*}
$$

where $G(t)={ }^{t}\left(G_{1}(t), \ldots, G_{p}(t)\right)$ is a vector of real-valued Borel-measurable functions that are bounded on compact sets, and for each $1 \leqslant i, j \leqslant p, F_{i j}$ is a distribution: nonnegative, non-decreasing, right-continuous and tending to 0 in $-\infty$.

Such systems, with $F_{i j}: \mathbb{R} \rightarrow \mathbb{R}_{+}$, arise in the study of the tail of the stationary solution of the stochastic equation $Y_{n+1}=a_{n} Y_{n}+b_{n}$ where $\left(a_{n}\right)$ is a Markov chain
on a finite state space $\left\{e_{1}, \ldots, e_{p}\right\}$ with transition matrix $P=\left(p_{i j}\right)$. In this case, $F_{i j}(t)=\left|e_{i}\right|^{\lambda} p_{j i} \mathbf{1}_{t \geqslant \log \left|e_{i}\right|}$. This is what motivated this study.

The standard renewal equation corresponds to the case when $p=1$ and $F_{11}(\infty)=1$. Then Feller's renewal theorems (see [5], XI) are available for any directly Riemann integrable $G_{1}$. The multidimensional case for measures supported on the positive real line has also already been studied by Crump in [2] and Athreya et al. in [1]. They extended Feller's ideas and methods to derive a similar theorem.

For more recent works on systems of renewal equations, see [4] and [7]. In both papers, the authors study such systems in the special case when $F_{i j}$ are supported on the positive half-line and have a density. In [4], Engibaryan proves that the renewal theorems hold for a wider class of function $G$, namely integrable, essentially bounded functions tending to 0 in $+\infty$. His approach is based on similar results in dimension 1 , and the Gauss triangular factorization. In [7], Tsalyuk uses complex analysis. His functions are complex-valued and he uses the Laplace transform $\widehat{F}(z)$ of $F$. Under suitable assumptions, mainly that $I-\widehat{F}(z)$ is not invertible at a finite number of points in the closed half-plane $\operatorname{Re}(z) \geqslant 0$, he gives the structure of the resolvent $R$ of the renewal equation ( $R=U-F^{(0)}$, see our notations in the following part). However both proofs cannot be extended to wider classes of $F_{i j}$.

In this paper, we further extend Feller's methods to the case of measures supported on the whole real line. Here we only study the case when the matrix of $F_{i j}$ is non-lattice (see Definition 2).

In the following section, we state some definitions and the main results. In Sections 3 and 4 , we state and prove two preliminary results that we will need in the last part to prove our renewal theorems.

## 2. Hypotheses and main results

We start with a list of notations we are going to use throughout this paper.

### 2.1. Notations

Let $F=\left(F_{i j}\right)_{1 \leqslant i, j \leqslant p}$ be a matrix of distributions as above.
DEFINITION 1. - For any $p \times r$ matrix $H$ of Borel-measurable real-valued functions that are bounded on compact intervals, we define the convolution product $F * H$ by:

$$
(F * H)_{i j}(t)=\sum_{k=1}^{p} \int_{-\infty}^{\infty} H_{k j}(t-u) F_{i k}(d u),
$$

when the integrals exist.
We can then rewrite Eq. (1) as $Z=G+F * Z$.
For any real $t$ we define:

- the expectation of $F$ (when it exists): $B=\left(b_{i j}\right)_{1 \leqslant i, j \leqslant p}$ with $b_{i j}=\int u F_{i j}(d u)$,
- $F^{(0)}(t)=\left(\delta_{i j}(t)\right)_{1 \leqslant i, j \leqslant p}$ with $\delta_{i j}(t)=\mathbf{1}_{t \geqslant 0}$ if $i=j$ and 0 otherwise, so that $F^{(0)} * H=H$ for any $H$ as in the definition above,
- the n-fold convolution of $F: F^{(n)}(t)=F * F^{(n-1)}(t)$,
- the renewal function associated with $F: U(t)=\sum_{n=0}^{\infty} F^{(n)}(t)$.

We also recall the definition of a lattice matrix of distributions as given in [1].
DEFINITION 2. - $F$ is lattice if the following assertions are true:

- For each $i \neq j, F_{i j}$ is concentrated on a set of the form $b_{i j}+\lambda_{i j} \mathbb{Z}$.
- For each $i, F_{i i}$ is concentrated on a set of the form $\lambda_{i i} \mathbb{Z}$.
- The $\lambda_{i i}$ are integral multiples of some same number.

We take $\lambda$ to be the largest such number.

- If $a_{i j}, a_{j k}, a_{i k}$ are points of increase of $F_{i j}, F_{j k}$ and $F_{i k}$ respectively, then $a_{i j}+a_{j k}-a_{i k}$ is an integral multiple of $\lambda$.


### 2.2. Hypotheses

To get a renewal theorem similar to that of Feller in dimension 1, we need make some assumptions on the matrix $F$, as in [1], essentially to be able to use Perron-Frobenius theory (see [6]):

- an assumption of finiteness of measures,

$$
\begin{equation*}
\forall 1 \leqslant i, j \leqslant p, \quad F_{i j}(\infty)=\lim _{t \rightarrow \infty} F_{i j}(t)<\infty \tag{2}
\end{equation*}
$$

- an assumption of irreducibility.

Recall that a $n \times n$ matrix $A=\left(a_{i j}\right)$ is irreducible if for any non-trivial partition $(I, J)$ of $\{1, \ldots, n\}$, we can find $i$ in $I$ and $j$ in $J$ so that $a_{i j} \neq 0$ (see [6]).

$$
\begin{equation*}
F(\infty) \text { is an irreducible matrix. } \tag{3}
\end{equation*}
$$

As $F(\infty)$ is a non-negative (component-wise) irreducible matrix, we can apply Perron-Frobenius theorem: its spectral radius $\rho(F(\infty))$ is an eigenvalue of algebraic multiplicity 1 , with a right-hand and a left-hand positive (component-wise) eigenvector. In the following, we will also assume that

$$
\begin{equation*}
\rho(F(\infty))=1 \tag{4}
\end{equation*}
$$

This very assumption enables us to deal with the matrix $F$ as with a "probability". Then we denote by $m$ and $u$ the Perron-Frobenius eigenvectors for the eigenvalue 1 :

$$
\left.\begin{array}{rlrl}
F(\infty) m & =m, & & { }^{t} u F(\infty)
\end{array}\right){ }^{t} u, ~ \sum_{i=1}^{p} m_{i}=1, \quad ~ \quad ~ \sum_{i=1}^{p} u_{i} m_{i}=1 .
$$

- Finally we make a transience-type assumption:

$$
\begin{equation*}
\forall t \in \mathbb{R}, \quad U(t)<\infty \tag{6}
\end{equation*}
$$

This last assumption does not appear in [1]. Indeed, it is automatically true for measures distributed on the positive half-line. However, this is no longer so in the general case, even in dimension 1 (for example it is false if $F$ has means zero).

### 2.3. Main results

We can now state the main theorems we are going to prove in the following parts.
THEOREM 1. - If assumptions (2), (3), (4), and (6) are true, if, in addition, $F$ is nonlattice and $Z$ is a bounded continuous (component-wise) solution of $Z=F * Z$, then $Z$ is a constant vector.

THEOREM 2. - If assumptions (2), (3), (4), and (6) are true, then for any $i, j$, for any bounded interval $I=] a ; b], U_{i j}(I+t)=U_{i j}(t+b)-U_{i j}(t+a)$ is uniformly (in $\left.t\right)$ bounded.

These first two theorems will help us to prove the following renewal theorems:
THEOREM 3 (Renewal theorem, first form). - If assumptions (2), (3), (4), and (6) are true, if, in addition, $F$ is non-lattice and $B$ exists, then ${ }^{t} u B m \neq 0$ and, for any $i, j$ and for any $h>0$, we have

$$
U_{i j}(t+h)-U_{i j}(t) \underset{t \rightarrow \infty}{\longrightarrow} c m_{i} u_{j} h
$$

where $m$ and $u$ are the eigenvectors defined in (5), and $c=\left({ }^{t} u B m\right)^{-1}$.
THEOREM 4 (Renewal theorem, second form). - Under the assumptions of Theorem 3, if $G$ is directly Riemann integrable (component-wise), and $Z=U * G$ exists, then

$$
\lim _{t \rightarrow \infty} Z_{i}(t)=c m_{i} \sum_{j=1}^{p} u_{j} \int_{-\infty}^{\infty} G_{j}(u) d u
$$

This last form is the useful one when we want to derive the asymptotic behavior of a function from a renewal equation it satisfies. Note that if $Z=U * G$ exists, then $Z$ is solution of the renewal equation (1). However in the case of measures supported on the whole line, we cannot prove the uniqueness of this solution. To know the limit of a function $Z$ satisfying a renewal equation of type (1), we have to prove first that $Z=U * G$, then we can apply the renewal theorem. A general method to prove this is iterating the renewal equation and prove that $F^{(n)} * Z \underset{n \rightarrow \infty}{\longrightarrow} 0$.

## 3. Equation $Z=F * Z$

As in dimension 1, the special form of the solutions of this equation will play an important part in the proof of the renewal theorems. This whole section is almost the same as in the case of measures supported on the positive half-line.

We start with a study of the points of increase of $U$.

### 3.1. Points of increase of $\boldsymbol{U}$

Lemma 1. - Let $\Sigma_{i j}$ be the set of all points of increase of the $F_{i j}^{(k)}$ for all $k \in \mathbb{N}$, i.e.

$$
\Sigma_{i j}=\left\{a \mid \exists k \in \mathbb{N}, F_{i j}^{(k)}(a+\varepsilon)-F_{i j}^{(k)}(a-\varepsilon)>0, \forall \varepsilon>0\right\}
$$

Then for any $i, j, k, \Sigma_{i k}+\Sigma_{k j} \subset \Sigma_{i j}$.

The proof is exactly the same as in the case of measures distributed on the positive half-line.

Proof. - Set $x$ in $\Sigma_{i k}$ and $y$ in $\Sigma_{k j}$. Then we can find integers $n$ and $m$ so that $x$ be a point of increase of $F_{i k}^{(n)}$ and $y$ a point of increase of $F_{k j}^{(m)}$. According to Lemma V.4.1 in [5], $x+y$ is then a point of increase of $F_{i k}^{(n)} * F_{k j}^{(m)}$, hence one of $\sum_{k=1}^{p} F_{i k}^{(n)} * F_{k j}^{(m)}=F_{i j}^{(n+m)}$. Thus $\Sigma_{i k}+\Sigma_{k j} \subset \Sigma_{i j}$.

The definition of a lattice matrix was chosen to have the following lemma work quite similarly to Lemma V.4.2 in [5] in dimension 1.

Lemma 2. - If assumption (3) is true, and if $F$ is non-lattice and the $F_{i j}$ are not all concentrated on $\mathbb{R}_{-}$, then for any $i, j, \Sigma_{i j}$ is asymptotically dense at infinity in the following sense:

$$
\left.\forall \varepsilon>0, \quad \exists \Delta_{\varepsilon}>0 \text { so that for any } x \geqslant \Delta_{\varepsilon}, \quad\right] x ; x+\varepsilon\left[\cap \Sigma_{i j} \neq \emptyset\right.
$$

The proof follows the same steps as that of Lemma 2 in [1].
Proof. - According to Lemma 1, if $\Sigma_{i_{0} j_{0}}$ is asymptotically dense at infinity, then so is $\Sigma_{i_{0} j}$ for any $j$ and $\Sigma_{i j_{0}}$ for any $i$, thus either all $\Sigma_{i j}$ are asymptotically dense at infinity, or none is.

Suppose none of the $\Sigma_{i j}$ is asymptotically dense at infinity, especially $\Sigma_{i i}$ is not asymptotically dense at infinity. It is a closed subset of $\mathbb{R}$ for addition according to Lemma 1, and it is not empty according to Lemma 1 and because $F(\infty)$ is not a zeromatrix thanks to assumption (3). Thus there is a $\delta_{i i}$ so that $\Sigma_{i i} \subset \delta_{i i} \mathbb{Z}$ and it contains $n \delta_{i i}$ for all large enough $n$ (see Lemma V.4.2 in [5]).

Set $c$ in $\Sigma_{i j}$, and $d$ in $\Sigma_{j i}$. Set a large enough $n$ so that $n \delta_{i i} \in \Sigma_{i i}$ and $(n+1) \delta_{i i} \in \Sigma_{i i}$, then according to Lemma $1, d-n \delta_{i i}+c$ and $d-n \delta_{i i}+c+\delta_{i i}$ are in $\Sigma_{j j}$, thus $\delta_{i i} \geqslant \delta_{j j}$, and by symmetry they are equal. Thus all $\delta_{j j}$ are equal. We set $\delta=\delta_{j j}$ for all $j$.

By a similar argument, we show that if $i \neq j$, then $\Sigma_{i j} \subset b_{i j}+\delta \mathbb{Z}$ (indeed $\Sigma_{i j}+\Sigma_{j j}$ is closed under addition), and according to Lemma $1, b_{i j}+b_{j k}=b_{i k}+n \delta$. Thus $F$ is lattice, which is impossible.

### 3.2. Proof of Theorem 1

We start with studying a more regular special case.
LEMMA 3. - Let $K$ be a vector of bounded uniformly continuous functions on $\mathbb{R}$ such that $K=F * K$. Under assumptions (2), (3), (4), and (6), if in addition $F$ is non-lattice and there is $i_{0}$ so that $a_{i_{0}}=\sup _{t \in \mathbb{R}} K_{i_{0}}(t)>0$, then there exists $\delta_{i_{0}}>0$ such that for any $h>0$, there exists an interval of length $h$ on which $K_{i_{0}}>\delta_{i_{0}}$.

Proof. - For any $1 \leqslant j \leqslant p$ we set $a_{j}=\sup _{t \in \mathbb{R}} K_{j}(t)$. Set $i_{0}$ such that $a_{i_{0}}>0$ and $j_{0}$ such that $\frac{a_{j_{0}}}{m_{j_{0}}}=\max _{1 \leqslant j \leqslant p} \frac{a_{j}}{m_{j}}>0$, where $m$ is the eigenvector of $F(\infty)$ defined in (5). As $F(\infty) m=m$, for any $i, n$ we get $\sum_{j=1}^{p} F_{i j}^{n}(\infty) m_{j}=m_{i}$, where $F_{i j}^{n}(\infty)$ are
the coordinates of the matrix $F(\infty)^{n}$. Then

$$
\sum_{j=1}^{p} F_{j_{0} j}^{n}(\infty) a_{j}=\sum_{j=1}^{p} F_{j_{0} j}^{n}(\infty) m_{j} \frac{a_{j}}{m_{j}} \leqslant\left(\sum_{j=1}^{p} F_{j_{0} j}^{n}(\infty) m_{j}\right) \frac{a_{j_{0}}}{m_{j_{0}}}=m_{j_{0}} \frac{a_{j_{0}}}{m_{j_{0}}}=a_{j_{0}}
$$

Thus we get

$$
\begin{equation*}
a_{j_{0}} \geqslant \sum_{j=1}^{p} F_{j_{0} j}^{n}(\infty) a_{j} \tag{7}
\end{equation*}
$$

We divide the rest of the proof in two cases depending on $a_{j_{0}}$ being reached or not. First case: $\exists t_{0} \in \mathbb{R}$ such that $K_{j_{0}}\left(t_{0}\right)=a_{j_{0}}$. Iterating $K=F * K$, we get

$$
\begin{aligned}
a_{j_{0}}=K_{j_{0}}\left(t_{0}\right) & =\sum_{r=1}^{p} \int K_{r}\left(t_{0}-u\right) F_{j_{0} r}^{(n)}(d u) \leqslant \sum_{r=1}^{p} a_{r} \int F_{j_{0} r}^{(n)}(d u) \\
& =\sum_{r=1}^{p} a_{r} F_{j_{0} r}^{(n)}(\infty) \\
& \leqslant \sum_{r=1}^{p} a_{r} F_{j_{0} r}^{n}(\infty) \text { as } F_{i j}^{(n)}(\infty) \leqslant F_{i j}^{n}(\infty) \\
& \leqslant a_{j_{0}} \quad \text { according to }(7) .
\end{aligned}
$$

All these inequalities are thus in fact equalities. Hence $\sum_{r=1}^{p} \int\left(a_{r}-K_{r}\left(t_{0}-\right.\right.$ $u)) F_{j_{0} r}^{(n)}(d u)=0$. As the integrated function is non-negative and continuous, we conclude that for any $u$, point of increase of a $F_{j_{0} r}^{(n)}$, i.e. for any $u \in \Sigma_{j_{0} r}$, we have $a_{r}=K_{r}\left(t_{0}-u\right)$. But according to Lemma 2, $\Sigma_{j_{0} r}$ is asymptotically dense at infinity. The uniform continuity of the functions $K_{r}$ now implies that

$$
\lim _{t \rightarrow-\infty} K_{r}(t)=a_{r}
$$

From the bounded convergence theorem applied to $K_{i}(t)=\sum_{r=1}^{p} \int K_{r}(t-u) F_{i r}^{(n)}(d u)$ when $t \rightarrow \infty$, we derive that $a_{i}=\sum_{r=1}^{p} a_{r} F_{i r}^{(n)}(\infty)$. Thus for any $t, r$ we get

$$
\begin{aligned}
K_{r}(t)-a_{r} & =\sum_{l=1}^{p} \int\left(K_{l}(t-u)-a_{l}\right) F_{r l}^{(n)}(d u), \\
\left|K_{r}(t)-a_{r}\right| & \leqslant \sum_{l=1}^{p} \int\left|K_{l}(t-u)-a_{l}\right| F_{r l}^{(n)}(d u) \\
& =\sum_{l=1}^{p} \int_{-\infty}^{T}\left|K_{l}(t-u)-a_{l}\right| F_{r l}^{(n)}(d u)+\sum_{l=1}^{p} \int_{T}^{\infty}\left|K_{l}(t-u)-a_{l}\right| F_{r l}^{(n)}(d u) .
\end{aligned}
$$

As $F(\infty)$ has spectral radius 1, and thus that $\lim _{n \rightarrow \infty}\left\|F(\infty)^{n}\right\|=1$, we get $\sup _{n, i, j} F_{i j}^{(n)}(\infty) \leqslant \sup _{n, i, j} F_{i j}^{n}(\infty)<\infty$. Set $\varepsilon>0$, we can choose $T$ so that for any $n$,
we have

$$
\sum_{l=1}^{p} \int_{T}^{\infty}\left|K_{l}(t-u)-a_{l}\right| F_{r l}^{(n)}(d u)<\varepsilon
$$

As $K$ is bounded and $\lim _{n \rightarrow \infty} F^{(n)}(T)=0$ because $U(T)<\infty$, we get

$$
\lim _{n \rightarrow \infty} \int_{-\infty}^{T}\left|K_{l}(t-u)-a_{l}\right| F_{r l}^{(n)}(d u) \leqslant M \lim _{n \rightarrow \infty} F_{r l}^{(n)}(T)=0
$$

Thus for any $1 \leqslant r \leqslant p, K_{r}$ is the constant function $a_{r}$. Especially, $K_{i_{0}}(t)=a_{i_{0}}>0$, from which we derive the expected result for $\delta_{i_{0}}=a_{i_{0}} / 2$.

Second case: For any $t, K_{j_{0}}(t) \neq a_{j_{0}}$.
Then we can find $\left(t_{n}\right)$, a sequence tending to $\pm \infty$ such that $K_{j_{0}}\left(t_{n}\right) \rightarrow a_{j_{0}}$. Let $\zeta_{n, i}(x)=K_{i}\left(t_{n}+x\right)$. As $K$ is bounded and uniformly continuous, $\left(\zeta_{n, i}\right)_{n, i}$ is a uniformly bounded and uniformly equi-continuous family. Ascoli theorem then gives us a subsequence $\left(t_{n_{j}}\right)$ of $\left(t_{n}\right)$ such that for any $n, i$, the sequence $\left(\zeta_{n_{j}, i}\right)_{j}$ converges uniformly on any compact set to $\eta_{i}$, a bounded uniformly continuous function. Now we get

$$
\begin{aligned}
\zeta_{n_{j}, i}(x) & =K_{i}\left(t_{n_{j}}+x\right)=\sum_{r=1}^{p} \int K_{r}\left(t_{n_{j}}+x-y\right) F_{i r}(d y) \\
& =\sum_{r=1}^{p} \int \zeta_{n_{j}, r}(x-y) F_{i r}(d y) .
\end{aligned}
$$

When $j$ tends to $\infty$, the bounded convergence theorems says

$$
\begin{equation*}
\eta_{i}(x)=\sum_{r=1}^{p} \int \eta_{r}(x-y) F_{i r}(d y) \tag{8}
\end{equation*}
$$

In addition, for any $x, i$, we get $\eta_{i}(x)=\lim _{j \rightarrow \infty} K_{i}\left(t_{n_{j}}+x\right) \leqslant a_{i}$, and by choice of $t_{n}, \eta_{j_{0}}(0)=\lim _{j \rightarrow \infty} K_{j_{0}}\left(t_{n_{j}}\right)=a_{j_{0}}$. Thus $\sup \eta_{j_{0}}=a_{j_{0}}>0$, hence $\eta_{j_{0}}$ satisfies the assumptions of this lemma in the first case. Each $\eta_{i}$ is thus a constant function, say $c_{i}$, with $c_{j_{0}}=a_{j_{0}}$.

From (8), we derive that $c={ }^{t}\left(c_{1}, \ldots, c_{p}\right)$ is a right eigenvector of $F(\infty)$ for the eigenvalue 1. As the corresponding eigenvectors sub-space is one-dimensional according to Perron-Frobenius theorem, we conclude that $c=\alpha m$. As $c_{j_{0}}=a_{j_{0}}>0$, we get $\alpha=c_{j_{0}} / m_{j_{0}}>0$ and thus $c$ has positive coordinates.

Set $h>0$. As $K_{i_{0}}\left(t_{n_{j}}+x\right) \rightarrow c_{i_{0}}$ uniformly on [0; $h$ ], for any large enough $j$ we have $K_{i_{0}}(x)>c_{i_{0}} / 2$ for any $x$ in $] t_{n_{j}} ; t_{n_{j}}+h[$.

Proof of Theorem 1. - Set $\phi_{\varepsilon}(t)=\frac{1}{\varepsilon \sqrt{2 \pi}} \exp \left(-\frac{t^{2}}{2 \varepsilon^{2}}\right)$. For any $i$, we set

$$
f_{\varepsilon, i}(t)=\phi_{\varepsilon} * Z_{i}(t)=\int_{-\infty}^{\infty} \phi_{\varepsilon}(t-y) Z_{i}(y) d y=\int_{-\infty}^{\infty} \phi_{\varepsilon}(y) Z_{i}(t-y) d y
$$

For any $\varepsilon>0$, and any $1 \leqslant i \leqslant p$, we have

$$
\begin{aligned}
f_{\varepsilon, i}(t) & =\sum_{r=1}^{p} \int_{-\infty}^{\infty} \phi_{\varepsilon}(y) \int_{-\infty}^{\infty} Z_{r}(t-y-u) F_{i r}(d u) d y \\
& =\sum_{r=1}^{p} \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} \phi_{\varepsilon}(y) Z_{r}(t-y-u) d y\right) F_{i r}(d u) \\
& =\sum_{r=1}^{p} \int_{-\infty}^{\infty} f_{\varepsilon, r}(t-u) F_{i r}(d u)
\end{aligned}
$$

In addition $f_{\varepsilon, i}$ is smooth, and its derivative is bounded, because $Z$ is bounded and uniformly continuous. Thus, $f_{\varepsilon, i}^{\prime}(t)=\sum_{r=1}^{p} \int f_{\varepsilon, r}^{\prime}(t-u) F_{i r}(d u)$, and we can use Lemma 3.

Set $a_{i}=\sup f_{\varepsilon, i}^{\prime}$. If there is a $i$ such that $a_{i}>0$, then we can find $\delta$ such that for any $h>0$ there is an interval $] t ; t+h$ [ on which $f_{\varepsilon, i}^{\prime}>\delta$. Integration on $] t ; t+h[$ yields $\delta h<f_{\varepsilon, i}(t+h)-f_{\varepsilon, i}(t)$. As $f_{\varepsilon, i}$ is bounded, we get $\delta h<M$ for any $h>0$, which is impossible. Thus for any $i, a_{i} \leqslant 0$.

Replacing $Z_{i}$ by $-Z_{i}$, we prove similarly that for any $i, a_{i} \geqslant 0$. Thus for any $i, t, \varepsilon$, we have $f_{\varepsilon, i}^{\prime}(t)=0$. For any $i$, $\varepsilon$, the convolution $f_{\varepsilon, i}$ is a constant function. Letting $\varepsilon$ tend to 0 , we obtain that $Z_{i}$ is a constant function for any $i$.

## 4. Potential theory

The aim of this section is to prove Theorem 2, i.e. that $U$ has uniformly bounded increments. It is easily proved for measures supported on the positive half-line, or in the one-dimensional case, thanks to special renewal equations. However these methods cannot be extended to the present case. This is the only technical difficulty we have met to extend the renewal theorems from the case of measures supported on the positive half-line to measures supported on the whole real line. We give here an original proof of Theorem 2 that involves the one-dimensional potential theory (see [3]), by extending it to the $d$-dimensional case.

### 4.1. Definitions and notations

DEFINITION 3. - $A$ kernel $N$ on $\mathbb{R}$ is a mapping of $\mathbb{R} \times \mathcal{B}(\mathbb{R})$ onto $[0,+\infty]$ such that

- $t \mapsto N(t, A)$ is measurable for any $A \in \mathcal{B}(\mathbb{R})$,
- $A \mapsto N(t, A)$ is a measure for any $t \in \mathbb{R}$.

For a given non-negative measurable function $f$ on $\mathbb{R}$, we define the mapping $N f$ by

$$
N f(t)=\int f(y) N(t, d y)
$$

We also define the composition of kernels: given two kernels $M$ and $N$ on $\mathbb{R}$, their product $M N$ is defined by

$$
M N(t, A)=\int N(y, A) M(t, d y)
$$

DEFINITION 4. $-N=\left(N_{i, j}\right)_{1 \leqslant i, j \leqslant p}$ is a kernel on $\mathbb{R}^{p}$ if each of its components $N_{i j}$ is a kernel on $\mathbb{R}$ in the sense of Definition 3.

For any measurable non-negative (component-wise) vector of functions $f={ }^{t}\left(f_{1}, \ldots\right.$, $f_{p}$ ), the mapping $N f$ is defined by $N f={ }^{t}\left((N f)_{1}, \ldots,(N f)_{p}\right)$, with

$$
(N f)_{i}(t)=\sum_{j=1}^{p} N_{i j} f_{j}(t)
$$

If $M$ and $N$ are two kernels on $\mathbb{R}^{p}$, their product is $M N=\left((M N)_{i j}\right)$, where

$$
(M N)_{i j}=\sum_{k=1}^{p} M_{i k} N_{k j}
$$

We also define a special kernel $I$ by

$$
\begin{aligned}
& I_{i j}(t, A)=0 \quad \text { if } i \neq j, \\
& I_{i i}(t, A)=\mathbf{1}_{A}(t),
\end{aligned}
$$

where

$$
\mathbf{1}_{A}(t)= \begin{cases}1 & \text { if } t \in A \\ 0 & \text { otherwise }\end{cases}
$$

Thus for any function $f: \mathbb{R} \rightarrow \mathbb{R}^{p}$, we have $I f=f$.
In the following, $N$ will always denote a kernel on $\mathbb{R}^{p}$. Let $N^{k}$ be its powers for the composition product defined above, with $N^{0}=I$.

DEFINITION 5. - The potential kernel associated with the kernel $N$ is the following kernel

$$
G=\sum_{k=0}^{\infty} N^{k}
$$

On the set of measurable function from $\mathbb{R}$ onto $\mathbb{R}^{p}$ we define the following partial order relationship:

$$
u \preceq v \quad \text { if, } \quad \forall 1 \leqslant i \leqslant p, \quad u_{i} \leqslant v_{i}
$$

This order has the following good property: if $u \preceq v$ then for any kernel $M$, we have $M u \preceq M v$.

DEFINITION 6. - Let $u: \mathbb{R} \rightarrow \mathbb{R}_{+}^{p}$ be a non-negative (component-wise) function. It is excessive for kernel $N$ if $N u \preceq u$.

### 4.2. Maximum principle

Let $A \subset \mathbb{R}$ and $A^{c}$ be its complementary set. We denote $J_{A}$ the kernel on $\mathbb{R}^{p}$ that satisfies $\left(J_{A} f\right)_{i}(t)=f_{i}(t) \mathbf{1}_{A}(t)$, i.e.

$$
\begin{aligned}
& \left(J_{A}\right)_{i j}(t, B)=0 \quad \text { if } i \neq j, \\
& \left(J_{A}\right)_{i i}(t, B)=\mathbf{1}_{A \cap B}(t)
\end{aligned}
$$

Notice that $J_{A} f$ depends only on the values of $f$ on $A$.
Let $G_{A}$ be the potential kernel associated with $N J_{A}$ and $G^{A}$ that associated with $J_{A} N$. We have $N G^{A}=G_{A} N$ and $J_{A} G_{A}=G^{A} J_{A}$. We also define the similar potential kernels for $A^{c}$.

DEFINITION 7. - We set $H_{A}=J_{A}+J_{A^{c}} G_{A^{c}} N J_{A}=G^{A^{c}} J_{A}$.
We now give a series of propositions as a preliminary to the maximum principle.
Proposition 1. - The measures $\left(H_{A}\right)_{i j}$ are supported on $A$, and for any $t$ in $A$,

$$
\begin{aligned}
& \left(H_{A}\right)_{i j}(t, B)=0 \quad \text { if } i \neq j, \\
& \left(H_{A}\right)_{i i}(t, B)=\mathbf{1}_{B}(t)
\end{aligned}
$$

Proof. - It is an easy consequence of the definition of $H_{A}$ and $J_{A}$.
Proposition 2. - If $u$ is an excessive function, then $H_{A} u \preceq u$.
Proof. - We prove by induction on $k$ that

$$
\begin{equation*}
J_{A} u+\sum_{m=0}^{k} J_{A^{c}}\left(N J_{A^{c}}\right)^{m} N J_{A} u \preceq u \tag{9}
\end{equation*}
$$

If $k=0$, as $u$ is excessive and $J_{A} u \preceq u$, we have $N J_{A} u \preceq N u \preceq u$. Then $J_{A^{c}} N J_{A} u \preceq$ $J_{A^{c}} u$ and $J_{A} u+J_{A^{c}} N J_{A} u \preceq J_{A} u+J_{A^{c}} u=u$.

Suppose it is true at rank $k: J_{A} u+\sum_{m=0}^{k} J_{A^{c}}\left(N J_{A^{c}}\right)^{m} N J_{A} u \preceq u$.
At rank $k+1$, we apply $N$ then $J_{A^{c}}$ to the two members of the inequality in the induction hypotheses. We get

$$
J_{A^{c}} u \succeq J_{A^{c}} N u \succeq J_{A^{c}} N J_{A} u+\sum_{m=0}^{k} J_{A^{c}} N J_{A^{c}}\left(N J_{A^{c}}\right)^{m} N J_{A} u=\sum_{m=0}^{k+1} J_{A^{c}}\left(N J_{A^{c}}\right)^{m} N J_{A} u .
$$

Adding $J_{A} u$ to both sides of the equation, we get:

$$
J_{A^{c}} u+J_{A} u=u \succeq J_{A} u+\sum_{M=0}^{k+1} J_{A^{c}}\left(N J_{A^{c}}\right)^{m} N J_{A} u
$$

which ends the induction.
Letting $k$ tend to $+\infty$ in (9), we get the expected equation $H_{A} u \preceq u$.
Proposition 3. $-H_{A}=J_{A}+J_{A^{c}} N H_{A}$, thus $N H_{A}=H_{A}$ on $A^{c}, N H_{A}=G_{A^{c}} N J_{A}$.

Proof. - We have $G_{A^{c}}=I+N J_{A^{c}} G_{A^{c}}$, thus

$$
N H_{A}=N J_{A}+N J_{A^{c}} G_{A^{c}} N J_{A}=G_{A^{c}} N J_{A} .
$$

It yields that

$$
J_{A^{c}} N H_{A}=J_{A^{c}} G_{A^{c}} N J_{A}=H_{A}-J_{A}
$$

Thus $H_{A}=J_{A}+J_{A^{c}} N H_{A}$.
PROPOSITION 4. - Let $u$ be an excessive function. Then $H_{A} u$ is the smallest (for $\preceq$ ) excessive function greater than or equal to $u$ on $A$.

Proof. - Set $v=H_{A} u$. As $u$ is excessive, we have $v=H_{A} u \preceq u$ according to Proposition 2, and thus $N v \preceq N u \preceq u$. As $u=v$ on $A$ by Proposition 1, especially we have $N v \preceq v$ on $A$. Proposition 3 yields $N H_{A}=H_{A}$ on $A^{c}$, therefore on this set $N v=v$. Thus $N v \preceq v$ everywhere and $v$ is excessive.

If $w$ is excessive and greater than or equal to $u$ on $A$, Proposition 1 yields $H_{A} u \preceq$ $H_{A} w$, and $H_{A} w \preceq w$ by Proposition 2. Hence $H_{A} u \preceq H_{A} w \preceq w$ everywhere.

PROPOSITION 5. $-G=H_{A} G+J_{A^{c}} G_{A^{c}}=H_{A} G+G^{A^{c}} J_{A^{c}}$.
Proof. - Multiplying equality $I-J_{A^{c}} N=I-N+J_{A} N$ on the left by $G^{A^{c}}$ and on the right by $G$ yields:

$$
G^{A^{c}}\left(I-J_{A^{c}} N\right) G=G^{A^{c}}(I-N) G+G^{A^{c}} J_{A} N G
$$

But by definition we have $G^{A^{c}}\left(I-J_{A^{c}} N\right)=I=(I-N) G$. Thus

$$
\begin{aligned}
G & =G^{A^{c}}+G^{A^{c}} J_{A} N G=G^{A^{c}} J_{A^{c}}+G^{A^{c}} J_{A}(I+N G) \\
& =G^{A^{c}} J_{A^{c}}+H_{A} G=J_{A^{c}} G_{A^{c}}+H_{A} G .
\end{aligned}
$$

PROPOSITION 6. - If $f$ is any non-negative (component-wise) excessive function, $v$ an excessive function, and $A=\bigcup_{i=1}^{p}\left\{f_{i}>0\right\}$, then

$$
G f \preceq v \quad \text { on } A \quad \Rightarrow \quad G f \preceq v \quad \text { on } \mathbb{R} .
$$

Proof. - As $G f \preceq v$ on $A$, Proposition 1 yields $H_{A} G f \preceq H_{A} v$. But $v$ is excessive, thus Proposition 2 yields $H_{A} v \preceq v$. Finally Proposition 5 yields $G f=H_{A} G f+G^{A^{c}} J_{A^{c}} f=$ $H_{A} G f$ as by definition of $A$, we have $J_{A^{c}} f=0$. Thus $G f=H_{A} G f \preceq H_{A} v \preceq v$.

DEFINITION 8. - Let $f$ be a non-negative (component-wise) function, and $A \subset \mathbb{R}$. We define $\sup _{t \in A} f(t) b y$ :

$$
\sup _{t \in A} f(t)=\max _{1 \leqslant i \leqslant p}\left(\sup _{t \in A}\left(f_{i}(t)\right)\right) .
$$

With this definition, on the set $A$ we have $f \preceq \sup _{t \in A} f(t) \mathbf{1}$, where $\mathbf{1}=(\mathbf{1}, \ldots, \mathbf{1})$, the function with all coordinates equal to the constant function 1 .

Corollary 1 (Maximum Principle). - If $\mathbf{1}$ is excessive, then for any non-negative (component-wise) function $f$, if $A=\bigcup_{i=1}^{p}\left\{f_{i}>0\right\}$, we have

$$
\sup _{t \in \mathbb{R}} G f(t)=\sup _{t \in A} G f(t)
$$

Proof. - Set $\alpha=\sup _{t \in A} G f(t)$. If $\alpha$ is infinite, it is obviously true. Otherwise, we have $G f \preceq \alpha \mathbf{1}$ on $A$. As $\mathbf{1}$ is excessive, Proposition 6 yields $G f \preceq \alpha \mathbf{1}$ on $\mathbb{R}$. Thus $\sup _{t \in \mathbb{R}} G f(t) \leqslant \alpha$, and then $\sup _{t \in \mathbb{R}} G f(t)=\sup _{t \in A} G f(t)$.

### 4.3. Increments of $\boldsymbol{U}$

Now we can give the proof of Theorem 2. Let $m$ be the eigenvector defined in (5), and $N=\left(N_{i j}\right)$ the following kernel:

$$
N_{i j}(t, A)=\frac{m_{j}}{m_{i}} \int \mathbf{1}_{A}(t-x) F_{i j}(d x)
$$

Then $(N f)_{i}(t)=\sum_{j=1}^{p} \frac{m_{j}}{m_{i}} \int f_{j}(t-x) F_{i j}(d x)=\left(\left(\frac{m_{j}}{m_{i}} F_{i j}\right) * f\right)_{i}(t)$.
Function 1 is excessive for $N$. Indeed,

$$
\begin{aligned}
(N \mathbf{1})_{i}(t) & =\sum_{j=1}^{p} \frac{m_{j}}{m_{i}} \int F_{i j}(d x)=\frac{1}{m_{i}} \sum_{j=1}^{p} m_{j} F_{i j}(\infty) \\
& =\frac{m_{i}}{m_{i}} \quad \text { by definition of } m \\
& =1
\end{aligned}
$$

The potential kernel $G$ associated with $N$ satisfies $G_{i j}=\frac{m_{j}}{m_{i}} U_{i j}$.
Set $h>0, A=[-h ; h]$, and $f_{i}=\mathbf{1}_{A}$ for $1 \leqslant i \leqslant p$. Then we have

$$
\begin{aligned}
(G f)_{i}(t) & =\sum_{j=1}^{p} \frac{m_{j}}{m_{i}} \int \mathbf{1}_{[-h ; h]}(t-x) U_{i j}(d x) \\
& =\sum_{j=1}^{p} \frac{m_{j}}{m_{i}}\left(U_{i j}(t+h)-U_{i j}(t-h)\right)
\end{aligned}
$$

In the sense of Definition $8, G f$ has finite upper bound, say $\alpha$, on the bounded interval $A=[-h ; h]$, because $U$ is finite according to assumption (6). The maximum principle yields then $\sup _{t \in \mathbb{R}} G f(t)=\sup _{t \in[-h ; h]} G f(t)$. Denote $i_{0}$ the number of a coordinate of $G f$ that reaches this upper bound. Set $\left(t_{n}\right)$ a series of points in $A$ such that $(G f)_{i_{0}}\left(t_{n}\right)$ tends to this upper bound. Then majoring $\mathbf{1}_{[-h ; h]}\left(t_{n}-x\right)$ by $\mathbf{1}_{[-2 h ; 2 h]}(x)$, we get, for any $t \in \mathbb{R}$ and any $1 \leqslant i \leqslant p$,

$$
\begin{aligned}
\sum_{j=1}^{p} \frac{m_{j}}{m_{i}}\left(U_{i j}(t+h)-U_{i j}(t-h)\right) & =(G f)_{i}(t) \leqslant \sup _{t \in[-h ; h]}(G f)_{i_{0}}(t) \\
& \leqslant \sum_{j=1}^{p} \frac{m_{j}}{m_{i_{0}}}\left(U_{i_{0} j}(2 h)-U_{i_{0} j}(-2 h)\right)<\infty
\end{aligned}
$$

All these terms are non-negative and $m_{i}>0$ for any $i$, thus each $U_{i j}(t+h)-U_{i j}(t-h)$ is uniformly (in $t$ ) bounded. To get the expected result on any finite interval $I$, just include $I$ in a larger symmetric interval.

## 5. The renewal theorems

Now we can prove the renewal Theorems 3 and 4. Thanks to the result of the preceding section, the proof is now again the same as in the case of measures supported on the positive half-line, at least for the first two steps. The renewal equation used in the third step is slightly different as it involves $F(\infty) \mathbf{1}_{t \geqslant 0}$ instead of $F(\infty)$, and $Z(t)=m \mathbf{1}_{t \geqslant 0}$ instead of $Z(t)=m$. However the method is essentially the same.

Proof of Theorem 3. - For any interval $I=] a ; b]$, any $1 \leqslant i, j \leqslant p$, and $t \in \mathbb{R}$, we set $U_{i j}^{(t)}(I)=U_{i j}(t+b)-U_{i j}(t+a)$. Theorem 2 yields that the family $\left(U_{i j}^{(t)}(I)\right)_{t}$ is bounded. Theorem VIII.6.2 in [5] gives us a sequence $\left(t_{n}\right)$ tending to $+\infty$ and measures $V_{i j}$ such that for any $1 \leqslant i, j \leqslant p$ and any interval $I, U_{i j}^{\left(t_{n}\right)}(I) \xrightarrow[n \rightarrow \infty]{\longrightarrow} V_{i j}(I)$.

First step: Show that $V_{i j}$ are multiples of Lebesgue measure.
Set $k_{0} \in\{1, \ldots, p\}$ and $a>0$. Let $G(t)$ be the vector defined by $G_{k}(t)=0$ for any $k \neq k_{0}$ and $G_{k_{0}}$ is a continuous non-zero function that vanishes outside $[0 ; a]$. Then $Z=U * G$ is well defined, and $Z$ is solution of the renewal equation

$$
\begin{equation*}
\forall 1 \leqslant i \leqslant p, \quad Z_{i}(t)=G_{i}(t)+\sum_{k=1}^{p} \int Z_{k}(t-u) F_{i k}(d u) \tag{10}
\end{equation*}
$$

For any $i$, we have:

$$
\begin{aligned}
Z_{i}\left(t_{n}+x\right) & =\int G_{k_{0}}\left(t_{n}+x-y\right) U_{i k_{0}}(d y) \\
& =\int G_{k_{0}}(x-y) U_{i k_{0}}^{\left(t_{n}\right)}(d y) \\
& \xrightarrow[n \rightarrow \infty]{\longrightarrow} G_{k_{0}}(x-y) V_{i k_{0}}(d y)
\end{aligned}
$$

Set $\zeta_{i}(x)=\int G_{k_{0}}(x-y) V_{i k_{0}}(d y)$. Then $\zeta_{i}$ is a bounded continuous function, and $Z_{i}\left(t_{n}+x\right) \rightarrow \zeta_{i}(x)$. The bounded convergence theorem applied to Eq. (10) yields

$$
\forall 1 \leqslant i \leqslant p, \quad \zeta_{i}(t)=\sum_{k=1}^{p} \int \zeta_{k}(t-u) F_{i k}(d u)
$$

Now Theorem 1 yields that $\zeta_{i}$ is a constant function for any $i$. Thus $\int G_{k_{0}}(x-$ $y) V_{i k_{0}}(d y)$ does not depend on $x$, and this is true for any continuous function $G_{k_{0}}$ that vanishes outside a compact set. Thus $V_{i k_{0}}$ is finite on compact sets, and unchanged by translation, therefore it is a multiple of Lebesgue measure. Denote Lebesgue measure by $l$. Hence there are $a_{i j} \in \mathbb{R}$ such that:

$$
\forall i, j, \quad V_{i j}=a_{i j} l
$$

Second step: Show that $a_{i j}=c m_{i} u_{j}$.
Again we set $k_{0}$ and we define $G$ by $G_{k}(t)=0$ for any $k \neq k_{0}$ and $G_{k_{0}}(t)=\mathbf{1}_{[0 ; 1]}(t)$. Then $Z=U * G$ is well defined and $Z$ is solution of the renewal equation $Z=G+F * Z$. For any $x$, we have

$$
\begin{aligned}
Z_{i}\left(t_{n}-x\right) & =\int G_{k_{0}}\left(t_{n}-x-y\right) U_{i k_{0}}(d y) \\
& =U_{i k_{0}}\left(t_{n}-x\right)-U_{i k_{0}}\left(t_{n}-x-1\right) \\
& \xrightarrow[n \rightarrow \infty]{\longrightarrow} a_{i k_{0}}
\end{aligned}
$$

The bounded convergence theorem applied to equation $Z\left(t_{n}\right)=G\left(t_{n}\right)+F * Z\left(t_{n}\right)$ yields $a_{i k_{0}}=\sum_{k=1}^{p} a_{k k_{0}} F_{i k}(\infty)$. Thus $\left(a_{1 k_{0}}, \ldots, a_{p k_{0}}\right)$ is an eigenvector of $F(\infty)$ for eigenvalue 1. As the corresponding eigenvectors subspace is one-dimensional, there is a $r_{k_{0}}$ such that for any $i, a_{i k_{0}}=r_{k_{0}} m_{i}$. Replacing $F$ by ${ }^{t} F$, we prove similarly that there is a $s_{k_{0}}$ such that for any $j, a_{k_{0} j}=s_{k_{0}} u_{j}$. Thus for any $i, k_{0}$, we have $a_{i k_{0}}=r_{k_{0}} m_{i}=s_{i} u_{k_{0}}$. Hence the quotient $\frac{s_{i}}{m_{i}}=\frac{r_{k_{0}}}{u_{k_{0}}}=c$ does not depend on $i$, and $a_{i j}=r_{j} m_{i}=c m_{i} u_{j}$.

Third step: Identification of $c$.
Now we set $G(t)=\left(F(\infty) \mathbf{1}_{t \geqslant 0}-F(t)\right) m$. Let $Z(t)=m \mathbf{1}_{t \geqslant 0}$. Then

$$
\begin{aligned}
& G_{i}(t)+\sum_{k=1}^{p} \int Z_{k}(t-x) F_{i k}(d x) \\
& \quad= \begin{cases}m_{i}-\sum_{j=1}^{p} F_{i j}(t) m_{j}+\sum_{k=1}^{p} m_{k} F_{i k}(t), & \text { if } t \geqslant 0, \\
-\sum_{j=1}^{p} F_{i j}(t) m_{j}+\sum_{k=1}^{p} m_{k} F_{i k}(t), & \text { if } t<0,\end{cases}
\end{aligned}
$$

and thus

$$
G_{i}(t)+\sum_{k=1}^{p} \int Z_{k}(t-x) F_{i k}(d x)=m_{i} \mathbf{1}_{t \geqslant 0}=Z_{i}(t)
$$

Thus $G+F * Z=Z$. Iterating this equality yields

$$
Z=G+F * Z=G+F * G+F^{(2)} * Z=\cdots=\sum_{k=0}^{n-1} F^{(k)} * G+F^{(n)} * Z
$$

But we have

$$
\begin{aligned}
\left(F^{(n)} * Z\right)_{i}(t) & =\int_{-\infty}^{\infty} \sum_{k=1}^{p} Z_{k}(t-x) F_{i k}^{(n)}(d x)=\sum_{k=1}^{p} m_{k} \int_{-\infty}^{t} F_{i k}^{(n)}(d x) \\
& =\sum_{k=1}^{p} m_{k} F_{i k}^{(n)}(t) \underset{n \rightarrow \infty}{\longrightarrow} 0
\end{aligned}
$$

as $U(t)=\sum_{n=0}^{\infty} F^{(n)}(t)$ is finite for any $t$. Thus $Z=U * G$. As $G$ is non-increasing and integrable on $\mathbb{R}_{+}$and on $\mathbb{R}_{-}, G$ is directly Riemann integrable (see [5], XI). To conclude, we need the following lemma.

LEMMA 4. - Let $G$ be directly Riemann integrable, and $U$ a matrix of distributions such that for any real $x$, any $h>0$ and any $1 \leqslant i, j \leqslant p, U_{i j}\left(t_{n}+x+h\right)-U_{i j}\left(t_{n}+\right.$ h) $\underset{n \rightarrow \infty}{\longrightarrow} a_{i j} h$. If $Z=U * G$ exists, then

$$
Z_{i}\left(t_{n}\right) \underset{n \rightarrow \infty}{\longrightarrow} \sum_{k=1}^{p} a_{i k} \int_{-\infty}^{\infty} G_{k}(y) d y
$$

This lemma and the result of the first step yield

$$
m_{i}=Z_{i}\left(t_{n}\right) \underset{n \rightarrow \infty}{\longrightarrow} \sum_{k=1}^{p} a_{i k} \int_{-\infty}^{\infty} G_{k}(y) d y
$$

But

$$
\begin{aligned}
\int_{-\infty}^{\infty} G_{k}(y) d y & =\int_{-\infty}^{\infty} \sum_{j=1}^{p}\left(F_{k j}(\infty) \mathbf{1}_{y \geqslant 0}-F_{k j}(y)\right) m_{j} d y \\
& =\sum_{j=1}^{p} m_{j} \int_{-\infty}^{\infty} y F_{k j}(d y)=\sum_{j=1}^{p} m_{j} b_{k j} .
\end{aligned}
$$

As $a_{i j}=c m_{i} u_{j}$, we get $m_{i}=c \sum_{k=1}^{p} \sum_{j=1}^{p} m_{i} u_{j} b_{j k} m_{k}$. But $\sum_{k, j} u_{j} b_{j k} m_{k} \neq 0$ as $m_{i}>0$ thus $c=\left(\sum_{k, j} u_{j} b_{j k} m_{k}\right)^{-1}$. This value does not depend on the choice of the sequence $\left(t_{n}\right)$. As $\left(U_{i j}(I+t)\right)_{t}$ is bounded, from any sequence $(t)$, we can extract a convergent sub-sequence. Hence we have proved the weak convergence of $U_{i j}^{(t)}$ to $a_{i j} l$ as $t$ tend to $+\infty$.

Proof of Lemma 4. - Set $h>0$. For any $k \in \mathbb{Z}$, we set $g_{k}(x)=\mathbf{1}_{[(k-1) h ; k h]}, \quad G_{i}^{k}=g_{k}$ for any $i$, and $Z^{k}=U * G^{k}$. Then

$$
\begin{aligned}
Z_{i}^{k}\left(t_{n}\right) & =\sum_{j=1}^{p} \int G_{j}^{k}\left(t_{n}-y\right) U_{i j}(d y) \\
& =\sum_{j=1}^{p} U_{i j}\left(t_{n}-(k-1) h\right)-U_{i j}\left(t_{n}-k h\right) \\
& \xrightarrow[n \rightarrow \infty]{\longrightarrow} \sum_{j=1}^{p} a_{i j} h .
\end{aligned}
$$

This limit is independent of $n$ and $k$, thus for any $n, k, i, Z_{i}^{k}\left(t_{n}\right) \leqslant M_{h}$.
Let $\underline{m}_{k}^{i}$ and $\bar{m}_{k}^{i}$ be respectively the minimum and maximum of $G_{i}$ on $[(k-1) h ; k h]$. As $G$ is directly Riemann integrable, the series $\underline{\sigma}^{i}=h \sum \underline{m}_{k}^{i}$ and $\bar{\sigma}^{i}=h \sum \bar{m}_{k}^{i}$ are absolutely convergent, and their difference tends to 0 as $h$ tends to 0 . For any $i$, we have:

$$
\sum_{j=-k}^{k} \underline{m}_{j}^{i} g_{j}\left(t_{n}\right) \leqslant G_{i}\left(t_{n}\right) \leqslant \sum_{j=-k}^{k} \bar{m}_{j}^{i} g_{j}\left(t_{n}\right)+\sum_{|j|>k} \bar{m}_{j}^{i} g_{j}\left(t_{n}\right),
$$

$$
\begin{gathered}
\sum_{r=1}^{p} \sum_{j=-k}^{k} \underline{m}_{j}^{r} \int g_{j}\left(t_{n}-y\right) U_{i r}(d y) \leqslant Z_{i}\left(t_{n}\right) \\
\leqslant \sum_{r=1}^{p} \sum_{j=-k}^{k} \bar{m}_{j}^{r} \int g_{j}\left(t_{n}-y\right) U_{i r}(d y)+M_{h} \sum_{r=1}^{p} \sum_{|j|>k} \bar{m}_{j}^{r}, \\
n \rightarrow \infty, \quad \sum_{r=1}^{p} \sum_{j=-k}^{k} \underline{m}_{j}^{r} a_{i r} h \leqslant \limsup Z_{i}\left(t_{n}\right) \leqslant \sum_{r=1}^{p} \sum_{j=-k}^{k} \bar{m}_{j}^{r} a_{i r} h+M_{h} \sum_{r=1}^{p} \sum_{|j|>k} \bar{m}_{j}^{r}, \\
k \rightarrow \infty, \quad \sum_{r=1}^{p} \underline{\sigma}^{r} a_{i r} \leqslant \limsup Z_{i}\left(t_{n}\right) \leqslant \sum_{r=1}^{p} \bar{\sigma}^{r} a_{i r} .
\end{gathered}
$$

Letting $h$ tend to 0 we get $\lim \sup Z_{i}\left(t_{n}\right)=\sum_{r=1}^{p} a_{i r} \int G_{r}(u) d u$. We get the same value for the inferior limit. Thus $\lim Z_{i}\left(t_{n}\right)=\sum_{r=1}^{p} a_{i r} \int G_{r}(u) d u$.

Lemma 4 and Theorem 3 easily yield the second form of the renewal theorem.

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