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# EQUILIBRIUM FLUCTUATIONS FOR LATTICE GASES

# FLUCTUATIONS À L'ÉQUILIBRE POUR DES GAZ RÉTICULÉS

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ABSTRACT. – The authors in a previous paper proved the hydrodynamic incompressible limit in  $d \ge 3$  for a thermal lattice gas, namely a law of large numbers for the density, velocity fields and energy. In this paper the equilibrium fluctuations for this model are studied and a central limit theorem is proved for a suitable modification of the vector fluctuation field  $\zeta^{\varepsilon}(t)$ , whose components are the density, velocity and energy fluctuations fields. We consider a modified fluctuation field  $\xi^{\varepsilon}(t) = \exp\{-\varepsilon^{-1}tE\}\zeta^{\varepsilon}$ , where *E* is the linearized Euler operator around the equilibrium, and prove that  $\xi^{\varepsilon}(t)$  converges to a vector generalized Ornstein–Uhlenbeck process  $\xi(t)$ , which is formally solution of the stochastic differential equation  $d\xi(t) = N\xi(t) dt + B dW_t$ , with  $BB^* = -2NC$ , where *C* is the compressibility matrix, *N* is a matrix whose entries are second order differential operators and  $W_t$  is a mean zero Gaussian field. The relation  $-2NC = BB^*$  is the fluctuation-dissipation relation.

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RÉSUMÉ. – Les auteurs, dans un précédent travail, ont prouvé la limite hydrodynamique incompressible en dimension  $d \ge 3$  pour un gaz réticulé avec phénomènes thermiques, c'est-àdire une loi des grands nombres pour les champs de densité, de vitesse ainsi que pour l'énergie. Dans cet article les fluctuations à l'équilibre sont étudiées et un théorème de la limite centrée est démontré pour une modification convenable du champ vectoriel  $\zeta^{\varepsilon}(t)$  des fluctuations, dont les composantes sont la densité, la vitesse et l'énergie. Nous considérons un champ de fluctuations modifié  $\xi^{\varepsilon}(t) = \exp\{-\varepsilon^{-1}tE\}\zeta^{\varepsilon}$ , où *E* est l'opérateur d'Euler linéarisé autour de l'équilibre, et nous prouvons que  $\xi^{\varepsilon}(t)$  converge vers un processus d'Ornstein–Uhlenbeck

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généralisé vectoriel  $\xi(t)$ , qui est formellement solution de l'équation différentielle stochastique  $d\xi(t) = N\xi(t) dt + B dW_t$ , avec  $BB^* = -2NC$ , où *C* est la matrice de compressibilité, *N* une matrice dont les coefficients sont des opérateurs différentiels du second ordre et  $W_t$  un champ gaussien de moyenne nulle. La relation  $-2NC = BB^*$  est la relation de fluctuation-dissipation. © 2003 Éditions scientifiques et médicales Elsevier SAS

Mots Clés : Fluctuations ; Gaz réticulés ; Équation de Navier-Stokes

# 1. Introduction

The behavior of fluids is usually described by phenomenological partial differential equations as Euler or Navier–Stokes equations. One of the great successes of the Non Equilibrium Statistical Mechanics has been to show that, at least formally, these equations can be obtained as hydrodynamic limit of the microscopic description based on the Liouville equation. This becomes a rigorous result in the form of a law of large numbers when the deterministic microscopic dynamics is replaced by suitable stochastic dynamics. More precisely, after introducing the scale parameter  $\varepsilon$ , defined as the ratio between the microscopic characteristic length and the macroscopic one, the law of large numbers arises in the study of the asymptotic behavior of the system as  $\varepsilon$  goes to zero. In this limit the local microscopic conserved quantities converge to hydrodynamic fields solution to the hydrodynamic equations. We focus in this paper on the hydrodynamic limit giving the incompressible Navier–Stokes equations (INS) and its fluctuations. The INS equation for the divergenceless velocity field u(x, t) is the PDE in 3 dimensions

$$\bar{\rho}\,\partial_t u + \bar{\rho}(u\cdot\underline{\partial}\,)u + \underline{\partial}P = v\Delta u \tag{1.1}$$

where  $\underline{\partial} = (\partial_{x_1}, \partial_{x_2}, \partial_{x_3})$ ,  $\Delta = \sum_{i=1}^3 \partial_{x_i}^2$ ,  $\nu$  is a constant called viscosity,  $\bar{\rho}$  the constant density and P(x, t) is the pressure which plays here the role of a Lagrangian multiplier for the incompressibility constraint div u = 0. The temperature  $\overline{T}$  of this fluid is constant and small perturbations  $\rho(x, t)$  and T(x, t) of the constant profiles of density and temperature are related through the Boussinesq condition

$$\bar{\rho}\,\underline{\partial}T + \overline{T}\,\underline{\partial}\rho = 0. \tag{1.2}$$

Moreover T behaves like a passive scalar

$$\partial_t T + u \cdot \underline{\partial} T = \kappa \,\Delta T \tag{1.3}$$

where  $\kappa$  is a constant called conductivity. We observe that these equations are invariant under the scaling

$$x' = \varepsilon^{-1}x, \quad t' = \varepsilon^{-2}t, \quad u' = \varepsilon u, \quad \rho' = \varepsilon \rho, \quad T' = \varepsilon T$$

and that it is this invariance property of the equations that makes possible to obtain them as hydrodynamic limit.

In [1] we propose a stochastic lattice gas model whose hydrodynamic limit is described by suitable hydrodynamic equations (see (2.15), (2.16) in Section 2) similar

to (1.1), (1.2) and (1.3). It is a model of particles with discrete velocities, jumping on the lattice: a particle with a given velocity moves on the 3-d cubic lattice as the asymmetric simple exclusion process with the jump intensity chosen so to have a drift equal to its velocity. Particles collide independently in each site at Poisson times, exchanging velocities in such a way to conserve the number of particles, the momentum in each direction and the energy. This model generalizes the one in [9] to include the case of particles with different kinetic energies.

In [1] we prove the law of large numbers for this model in the following form. The set  $\mathcal{V}$  of velocities is a finite cardinality subset of  $\mathbb{R}^3$ .  $\eta(x) = \{\eta(x, v) \mid v \in \mathcal{V}\}$  denotes the configuration in the site  $x \in \Lambda$ ,  $\Lambda$  cube of the lattice  $\mathbb{Z}^3$  of size  $\varepsilon^{-1}$ , while  $\eta(x, v) = 0, 1$  is the occupation number of the particles with velocity v in x. The quantities

$$I_0(\eta(x)) = \sum_{v \in \mathcal{V}} \eta(x, v), \qquad I_\alpha = \sum_{v \in \mathcal{V}} \eta(x, v) v_\alpha, \quad \alpha = 1, \dots, 3,$$
$$I_4 = \sum_{v \in \mathcal{V}} \eta(x, v) \frac{|v|^2}{2}$$

are the mass, momentum and energy at x. We choose as initial state a Bernoulli measure with average density and energy small perturbation (of order  $\varepsilon$ ) of constant profiles and average momentum of order  $\varepsilon$ . Then the empirical fields

$$\nu_{\beta}^{\varepsilon}(u,t) = \varepsilon^{2} \sum_{x \in \Lambda} \left( I_{\beta} \left( \eta_{\varepsilon^{-2}t}(x) \right) - \bar{I}_{\beta} \right) \delta(u - \varepsilon x), \quad \beta = 0, \dots, 4$$

of density, momentum and energy, with  $\bar{I}_0$  and  $\bar{I}_4$  suitable constants and  $\bar{I}_{\alpha} = 0$  for  $\alpha = 1, ..., 3$ , converge weakly in probability as  $\varepsilon$  goes to 0 to the solution of the hydrodynamic equations for this model, which are the incompressible Navier–Stokes equation for the velocity field and a diffusive equation (including the transport along the velocity field) for the energy. The dissipative terms in these equations are given in terms of a diffusion tensor  $D_{\alpha,\gamma}^{\beta,\nu}$ ,  $\beta, \nu = 0, ..., 4, \alpha, \gamma = 1, ..., 3$ , which is expressed by the Green–Kubo formulas.

The next natural step is to prove the space-time central limit theorem, namely that the fluctuation fields starting from the equilibrium state converge to a stationary multidimensional Gaussian process with a given space-time covariance. In this paper we study the equilibrium fluctuations for the model introduced in [1]. Since the macroscopic behavior of this model is very close to the real hydrodynamics we face the main difficulty of the hydrodynamic fluctuations: the Euler terms and the Navier–Stokes terms live on different time scales. The same feature is responsible for the impossibility of obtaining the compressible dissipative hydrodynamic equations as scaling limit. In fact, the previous result on the law of large numbers is true for an initial condition which is a small perturbation of the global equilibrium. This perturbation remains small at later times of order  $\varepsilon^{-2}$  and evolves macroscopically according to the incompressible hydrodynamics. The case of the fluctuations is different because a small perturbation of the equilibrium may become very large and be of order  $\varepsilon^{-1}$  on times of order  $\varepsilon^{-2}$ . We go now in some details to explain better this point. The fluctuation fields under diffusive scaling are defined by

$$\zeta_{\beta}^{\varepsilon}(t,G) = \varepsilon^{3/2} \sum_{x} G(\varepsilon x) \big[ I_{\beta} \big( \eta_{\varepsilon^{-2}t}(x) \big) - \mathbb{E}[I_{\beta}] \big],$$

 $\beta = 0, ..., 4$ , where G is a suitable test function,  $\eta_t(x)$  is the configuration in x at time t and  $\mathbb{E}$  is the equilibrium expectation.

At time zero the limiting fluctuation fields

$$\lim_{\varepsilon \to 0} \zeta_{\beta}^{\varepsilon}(0, G) = \zeta_{\beta}(0, G)$$

are jointly Gaussian with covariance

$$\mathbb{E}[\zeta_{\beta}(0,G)\zeta_{\nu}(0,H)] = C_{\beta,\nu}\int d^3x \,G(x)H(x).$$

The matrix  $C = (C_{\beta,\nu})$  is called the compressibility matrix. The limit is in the sense of weak convergence of path measures.

It can be shown (it is indeed a by-product of the results and estimates in this paper) that the equilibrium fluctuations under Euler time scale are trivial in the sense that they satisfy in the limit a deterministic equation. This is a general feature first showed in [11, 10]. More precisely, the limiting field  $\zeta^E = (\zeta^E_\beta), \ \beta = 0, \dots, 4$ 

$$\zeta^{E}(\tau) = \lim_{\varepsilon \to 0} \zeta^{E}_{\varepsilon}(\tau, G) = \lim_{\varepsilon \to 0} \zeta^{\varepsilon}(\varepsilon\tau, G)$$

is solution of the deterministic equation

$$d\zeta^{E}(\tau) = E\zeta^{E}(\tau) \, d\tau, \qquad (1.4)$$

where E is the linearized Euler operator around the global equilibrium given explicitly by

$$E = \begin{pmatrix} 0 & -a_0\underline{\partial} & 0\\ -b_0\underline{\partial} & 0 & -b_4\underline{\partial}\\ 0 & -a_4\underline{\partial} & 0 \end{pmatrix}$$

where the real constants  $a_i$  and  $b_i$  are defined in (2.6) below. Eq. (1.4) are a system of linear hyperbolic equations. The stochastic noise should appear as a correction of order  $\varepsilon$ :

$$d\zeta_{\varepsilon}^{E}(\tau) = (E + \varepsilon \mathcal{D})\zeta_{\varepsilon}^{E}(\tau) d\tau + \sqrt{\varepsilon} B dW_{\tau} + O(\varepsilon^{2}),$$

where D is the linearized Navier–Stokes operator around the global equilibrium defined, for any vector function  $\zeta$ , as

$$\mathcal{D}\zeta = \sum_{\alpha,\gamma=1}^{3} D_{\alpha,\gamma} \partial_{\alpha} \partial_{\gamma} \zeta,$$

 $D_{\alpha,\beta}$  being, for any  $\alpha, \beta = 1, ..., 3, 5 \times 5$  matrices, whose elements  $D_{\alpha,\beta}^{\gamma,\nu}$  are defined in (2.10) and (2.11) below and the positive operator *B* is defined by the relation

$$BB^* = -2\mathcal{D}C$$

which represents the fluctuation-dissipation relation. Hence, to see a finite noise one has to look at longer times  $\tau = \varepsilon^{-1}t$ . Formally, since  $\zeta^{\varepsilon}(t) = \zeta_{\varepsilon}^{E}(\varepsilon^{-1}t)$ , we get

$$d\zeta^{\varepsilon}(t) = (\varepsilon^{-1}E + \mathcal{D})\zeta^{\varepsilon}(t) dt + B dW_t + O(\varepsilon).$$

Then the limit  $\lim_{\varepsilon \to 0} \zeta^{\varepsilon}(t)$  does not exist because the Euler modes are too big on this time scale. A similar difficulty is present also in the case of ASEP but the analogous of *E* is simply an operator of the form  $v \cdot \underline{\partial}$  with  $v_i = (p_i - q_i)(1 - 2\alpha)$ ,  $p_i$ ,  $q_i$  the rates of jumping to the left and right respectively and  $\alpha = \mathbb{E}[\eta]$ . Therefore, a Galilean shift is sufficient to remove the divergence and in fact in [5] the central limit theorem is proved for a fluctuation field of the form

$$Y^{\varepsilon}(t,G) = \varepsilon^{d/2} \sum_{x} G\left(\varepsilon x - \varepsilon^{-1} v t\right) \left[\eta_{\varepsilon^{-2}t}(x) - \mathbb{E}[\eta]\right].$$

In our case a possible way to subtract the Euler modes is to consider a modified fluctuation field which moves together with the waves solutions of (1.4), traveling with velocity of order  $\varepsilon^{-1}$ . Denoting by  $E^*$  the adjoint operator of E with respect the usual scalar product in  $L^2(\mathbb{T}_3, \mathbb{R}^5)$ , we define the fluctuation field as

$$\xi^{\varepsilon}(t,G) = \zeta^{\varepsilon}(t,e^{-\frac{t}{\varepsilon}E^{*}}G).$$

We prove that the limit  $\varepsilon \to 0$  exists and satisfies a suitable stochastic differential equation. Before writing the equation, we consider the same problem in a very simple case: let *A* and *M* be  $K \times K$  matrix with complex entries such that  $A = -A^*$ , where the adjoint is relative to the scalar product in  $\mathbb{C}^K$ . Consider the linear ODE system

$$\dot{x}_{\varepsilon} = (\varepsilon^{-1}A + M)x_{\varepsilon}, \quad x_{\varepsilon}(0) = \bar{x}.$$

Then,  $y_{\varepsilon} = e^{-\frac{t}{\varepsilon}A}x_{\varepsilon}$  is solution of

$$\dot{y}_{\varepsilon} = e^{-\frac{t}{\varepsilon}A} M e^{\frac{t}{\varepsilon}A} y_{\varepsilon}, \quad y_{\varepsilon}(0) = \bar{x}.$$

Consider the limit

$$U := \lim_{\varepsilon \to 0} \int_{0}^{1} ds \, \mathrm{e}^{-\frac{s}{\varepsilon}A} M \mathrm{e}^{\frac{s}{\varepsilon}A} = \lim_{\varepsilon \to 0} \varepsilon \int_{0}^{1/\varepsilon} ds \, \mathrm{e}^{-sA} M \mathrm{e}^{sA}.$$

An asymptotic average theorem [7] states that for any  $\delta > 0$  and T > 0 there exists  $\varepsilon_0 > 0$  such that the solution *z* of

$$\dot{z} = Uz, \quad z(0) = \bar{x}$$

satisfies

$$\sup_{0 < t < T} |y_{\varepsilon} - z| < \delta, \quad 0 < \varepsilon < \varepsilon_0.$$

Therefore,  $y = \lim_{\varepsilon \to 0} y_{\varepsilon}$  is solution of

$$\dot{y} = Uy, \quad y(0) = \bar{x}.$$

The limit *U* can be characterized in the following way: Let  $\mathcal{N}$  be the space of the  $K \times K$  matrices with complex entries.  $\mathcal{N}$  is a Hilbert space under the inner product

$$(A, B) = \sum_{1 \leq i, j \leq K} A_{ij}^* B_{ij}.$$

For  $A \in \mathcal{N}$  define  $\Pi_A$  as the orthogonal projection onto the subspace of the matrices which commute with A

$$\{B \in \mathcal{N} \colon [B, A] = 0\}.$$

Since the spectrum of A is imaginary one can show that

$$U = \lim_{\varepsilon \to 0} \varepsilon \int_{0}^{1/\varepsilon} ds \, \mathrm{e}^{-sA} M \mathrm{e}^{sA}$$

is the projector  $\Pi_A M$ .

Applying this kind of considerations to our problem, by Fourier analysis, we prove that  $\xi(t, G) = \lim_{\varepsilon \to 0} \xi^{\varepsilon}(t, G)$  exists and is a stationary generalized Ornstein–Uhlenbeck process characterized formally by the stochastic differential equation

$$d\xi(t) = N\xi(t) dt + B dW_t,$$

with  $BB^* = -2NC$ , where *C* is the compressibility matrix, *N* is the second order differential operator defined below and  $BW_t$  is a mean zero Gaussian field. In particular, this proves the fluctuation-dissipation relation  $-2NC = BB^*$  for this model. Denoting by  $\widehat{M}$  the Fourier transform of a 5 × 5 matrix *M* whose entries are differential operators, we can characterize *N* as follows:

$$\widehat{N} = \prod_{\widehat{F}} \widehat{\mathcal{D}}_{\widehat{F}}$$

 $\Pi_A$  denoting the projection on the space of the operators commuting with A. To conclude, we want to stress that this procedure of subtracting the Euler modes works in this case because the equations for the equilibrium fluctuations are linear.

The central limit theorem for equilibrium fluctuations is a well investigated topics [15, 13]. A standard procedure is to establish first the tightness of the sequence of fluctuation fields. Then, the study of the martingale problem allows to identify the unique weak limit as a generalized Ornstein–Uhlenbeck process by the use of Holley–Stroock theory. It is crucial to evaluate some expression in the martingale problem in terms of the

fluctuations field. This step, called Boltzmann-Gibbs principle, was first achieved in [2] for symmetric zero range process. The alternative method by [6] and [3] was extended to non-gradient systems by [14] and [4]. The extension is based on a suitable modification of the fluctuation field by adding lower order terms, determined by identifying the diffusion coefficient in the hydrodynamic equations. In [5] this approach has been extended to a non-symmetric case by proving a stronger tightness result and as consequence a stronger Boltzmann–Gibbs theorem. We extend the results on tightness and Boltzmann-Gibbs theorem in [5] to the present model. One important ingredient in the proof of our theorem is the equivalence of canonical and grand canonical measures. which is standard for Bernoulli measures. Our dynamics can be seen as the dynamics of  $|\mathcal{V}|$  exclusion processes which are not independent because of the collisions. The invariant measures are a family of Bernoulli measures parametrized by 5 parameters (chemical potentials), which are conjugate to the conserved quantities. It is easy to prove by standard arguments the equivalence of ensembles for  $|\mathcal{V}|$  independent exclusion processes. It is absolutely not obvious that the analogous result would be valid in this case. We prove that this is the case in Lemma 5.1.

Finally, we prove the convergence of the time averages of the form appearing in the martingale problem by implementing in the present context, in a non trivial way, some ideas in [7] which studied the convergence of solutions to the linearized Navier–Stokes equations of solutions to the linearized Boltzmann equation.

As final remark, we think that the extension of our results to the non equilibrium fluctuations should present be easier than in the usual cases since the hydrodynamic fluctuations here are fluctuations with respect the local equilibrium with density, velocity field and energy which are small perturbation of constant profiles. We conjecture that the limiting fluctuation field should be a non-stationary process formal solution of a linear SPDE of the previous form with the same B but different N.

The paper is organized as follows. In Section 2 we define the model and recall the previous results on the hydrodynamic limit that we will need in the sequel. In Section 3 we define the fluctuation field and state the results. In Section 4 we identify the limiting distribution by using Holley–Stroock characterization of Ornstein–Uhlenbeck processes with martingales. The Boltzmann–Gibbs principle is proved in Section 5 together with the tightness of the process and the equivalence of ensembles. The theorems stating the existence of the time averages are in Appendix A.

# 2. Model and hydrodynamic limit

We consider the following model introduced in [1], which is a generalization of the model in [9]: given a finite set of velocities  $\mathcal{V} \subset \mathbb{R}^3$ , particles with velocity  $v \in \mathcal{V}$  evolve on the sub-lattice  $\Lambda_L = \{-L, \ldots, L\}^3$ , with periodic boundary conditions, according to an exclusion process. Collisions between two particles can also occur provided that the momentum and the kinetic energy are conserved. The set  $\mathcal{V}$  is chosen in the following way:

$$\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2,$$

where  $\mathcal{V}_1$  is made of 8 velocities given by

$$\mathcal{V}_1 = \{(\pm 1, \pm 1, \pm 1)\}$$

and  $\mathcal{V}_2$  contains 24 velocities, given up to permutation by

$$(\pm \varpi, \pm 1, \pm 1),$$
 (2.1)

where  $\varpi$  is some irrational number suitably chosen.

Formally, if we denote by  $\eta(x, v) \in \{0, 1\}$  the number of particles on site  $x \in \Lambda_L$  with velocity  $v \in \mathcal{V}$ , then the infinitesimal generator of the dynamics on the space  $\Omega_L = \{\eta = (\eta(x, v), x \in \Lambda_L, v \in \mathcal{V})\}$  is defined as

$$\mathcal{L} = \mathcal{L}^{\mathrm{ex}} + \mathcal{L}^{\mathrm{c}},$$

where  $\mathcal{L}^{ex}$  is the generator of the nearest neighbor exclusion process with different colors (velocities) and  $\mathcal{L}^{c}$  the generator of the collision process. For a local function f on  $\Omega_L$ ,  $\mathcal{L}^{ex}$  is given by

$$\mathcal{L}^{\mathrm{ex}}f(\eta) = \sum_{v \in \mathcal{V}} \sum_{|e|=1} \sum_{x \in \Lambda_L} \left( \chi + \frac{1}{2} e \cdot v \right) \eta(x, v) \left[ f\left(\eta^{x, x+e, v}\right) - f(\eta) \right],$$

where *e* is a unitary vector of  $\mathbb{Z}^3$  ( $e_\alpha$ ,  $\alpha = 1, 2, 3$ , will denote the unitary vectors with positive coordinates),  $\eta^{x,x+e,v}$  is the configuration obtained after exchanging the values of  $\eta(x, v)$  and  $\eta(x + e, v)$  and  $\chi$  is some positive constant large enough such that the jump rate is positive. Note that it is chosen so that the drift of the particles with velocity *v* is exactly *v*.

The collisions generator  $\mathcal{L}^{c}$  is given by

$$\mathcal{L}^{c}f(\eta) = \sum_{x \in \Lambda_{L}} \sum_{q \in \mathcal{Q}} [f(\eta^{x,q}) - f(\eta)],$$

where Q is the set of admissible collisions, namely the set of velocity quadruples  $q = (v, w, v', w') \in \mathcal{V}^4$  such that v + w = v' + w' and  $|v|^2 + |w|^2 = |v'|^2 + |w'|^2$ , and  $\eta^{x,q}$  is the configuration obtained after the collision on site x between two particles with incoming velocities v, w and outgoing velocities v', w'. Notice that in order to preserve the exclusion rule, we take  $\eta^{x,q}$  unchanged with respect to  $\eta$  if one of the conditions  $\eta(x, v) = 0, \eta(x, w) = 0, \eta(x, v') = 1$  or  $\eta(x, w') = 1$  is fulfilled.

We denote by  $\eta_x = \{\eta(x, v), v \in \mathcal{V}\}$  the particle configuration in  $x \in \Lambda_L$ . For a configuration  $\eta$ , the mass, momentum and kinetic energy in site x are

$$\begin{split} I_0(\eta_x) &= \sum_{v \in \mathcal{V}} \eta(x, v), \\ I_\alpha(\eta_x) &= \sum_{v \in \mathcal{V}} (v \cdot e_\alpha) \eta(x, v), \quad \alpha = 1, 2, 3, \\ I_4(\eta_x) &= \sum_{v \in \mathcal{V}} \frac{1}{2} |v|^2 \eta(x, v). \end{split}$$

It is easy to check that the quantities  $N_{\beta}(\eta) = \sum_{x} I_{\beta}(\eta_{x}), \ \beta = 0, ..., 4$ , are conserved by the full dynamics. It is shown in [1] that, by choosing suitably the parameter  $\varpi$  in (2.1), they are the only conserved quantities, in other words this model has the property of local ergodicity.

As a consequence, the grand canonical measures below are invariant for  $\mathcal{L}$ 

$$\mu_{L,n}(\eta) = Z_{L,n}^{-1} \prod_{x \in \Lambda_L} \exp\left\{\sum_{\beta=0}^4 n_\beta I_\beta(\eta_x)\right\},\tag{2.2}$$

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where  $n = (n_0, ..., n_4) \in \mathbb{R}^5$  are the chemical potentials and  $Z_{L,n}$  is a normalization constant. All these product measures are absolutely continuous with respect to the reference measure  $\mu$  obtained by taking n as  $\bar{n} := (r, 0, 0, 0, \theta)$ . We set  $m_\beta = \mathbb{E}^{\mu}[I_\beta(\eta_0)]$  for  $\beta = 0, ..., 4$  (notice that  $m_\beta = 0$  if  $\beta = 1, 2, 3$ ) and  $\tilde{I}_\beta = I_\beta - m_\beta$ .

In the sequel we call  $\varepsilon = L^{-1}$ . The law of the process  $(\eta_t(x, v))$  with generator  $\varepsilon^{-2}\mathcal{L}$  starting from  $\mu$  is denoted by  $\mathbb{P}^{\mu}_{\varepsilon}$  and the corresponding expectation by  $\mathbb{E}^{\mu}_{\varepsilon}$ . We also call  $f_0(v) = \mathbb{E}^{\mu}[\eta(x, v)]$  the density of particles with velocity  $v \in \mathcal{V}$  with respect to the reference measure  $\mu$ . For any function h on  $\mathcal{V}$ , we put  $\langle h \rangle = \sum_{v \in \mathcal{V}} h(v)$ .

The currents  $w_{x,\alpha}^{\beta}$  of the conserved quantities  $I_{\beta}$ ,  $\beta = 0, ..., 4$ , at site x in direction  $e_{\alpha}$ ,  $\alpha = 1, 2, 3$ , are defined by

$$\mathcal{L}I_{\beta}(\eta_{x}) = \sum_{\alpha=1}^{3} \nabla_{\alpha}^{-} w_{x,\alpha}^{\beta},$$

where, if g is a function on  $\Lambda_L$ ,

$$\nabla_{\alpha}^{-}g(x) = (\nabla_{\alpha}g)(x - e_{\alpha})$$
 and  $\nabla_{\alpha}g(x) = g(x + e_{\alpha}) - g(x)$ 

Since the local quantities  $I_{\beta}(\eta_x)$  are conserved by the collision generator, we have  $\mathcal{L}I_{\beta}(\eta_x) = \mathcal{L}^{\text{ex}}I_{\beta}(\eta_x)$  and the currents can be written as the sum of a symmetric and an antisymmetric parts

$$w_{x,\alpha}^{\beta} = \chi \nabla_{\alpha} I_{\beta}(\eta_x) + w_{x,\alpha}^{(a),\beta}$$

and

$$w_{x,\alpha}^{(a),0} = \langle v_{\alpha} b_{x,\alpha}(v) \rangle, \qquad w_{x,\alpha}^{(a),\beta} = \langle v_{\alpha} v_{\beta} b_{x,\alpha}(v) \rangle, \quad \beta = 1, 2, 3,$$
$$w_{x,\alpha}^{(a),4} = \frac{1}{2} \langle v_{\alpha} | v |^2 b_{x,\alpha}(v) \rangle,$$

with

$$b_{x,\alpha}(v) = \eta(x+e_{\alpha},v)\eta(x,v) - \frac{1}{2}\big(\eta(x+e_{\alpha},v)+\eta(x,v)\big).$$

Let  $\mathcal{G}$  be the space of local functions h on  $\Lambda_L$  such that

$$\mathbb{E}^{\mu}[h] = 0 \quad \text{and} \quad \frac{\partial \mathbb{E}^{\mu_{L,n}}[h]}{\partial m_{\beta}(n)} \Big|_{n=\bar{n}} = 0, \quad \beta = 0, \dots, 4,$$
(2.3)

where  $m_{\beta}(n) = \mathbb{E}^{\mu_{L,n}}[I_{\beta}]$ . In view of the application of the Boltzmann–Gibbs principle, it is important to modify the currents  $w_{x,\alpha}^{(a),\beta}$  so that they are in the space  $\mathcal{G}$ . It is enough to subtract suitable combinations of the conserved quantities and we now get their explicit expressions.

Let *n* be the chemical potential  $n = \bar{n} + \delta n = (r + \delta n_0, \delta n_1, \delta n_2, \delta n_3, \theta + \delta n_4)$ , then

$$\begin{split} & \mathbb{E}^{\mu_{L,n}} \left[ w_{0,\alpha}^{(a),0} \right] - c_{\alpha}^{0} = \frac{1}{3} \langle |v|^{2} h_{1} \rangle \delta n_{\alpha} + o(|\delta n|), \\ & \mathbb{E}^{\mu_{L,n}} \left[ w_{0,\alpha}^{(a),\beta} \right] - c_{\alpha}^{\beta} = \delta_{\alpha,\beta} \left[ \frac{1}{3} \langle |v|^{2} h_{1} \rangle \delta n_{0} + \frac{1}{6} \langle |v|^{4} h_{1} \rangle \delta n_{4} \right] + o(|\delta n|), \quad \beta = 1, 2, 3, \\ & \mathbb{E}^{\mu_{L,n}} \left[ w_{0,\alpha}^{(a),4} \right] - c_{\alpha}^{4} = \frac{1}{6} \langle |v|^{4} h_{1} \rangle \delta n_{\alpha} + o(|\delta n|), \end{split}$$

where  $h_0 = f_0(1 - f_0)$ ,  $h_1 = h_0(1 - 2f_0)$  and

$$c_{\alpha}^{\beta} = \mathbb{E}^{\mu} \left[ w_{0,\alpha}^{(a),\beta} \right]. \tag{2.4}$$

If we denote by  $\delta m_{\beta} = \mathbb{E}^{\mu_{L,n}}[I_{\beta}(\eta_0)] - m_{\beta}$ , we get

$$\begin{split} \delta n_0 &= \frac{1}{\Phi} \left( \langle |v|^4 h_0 \rangle \delta m_0 - 2 \langle |v|^2 h_0 \rangle \delta m_4 \right) \\ \delta n_\alpha &= \frac{3}{\langle |v|^2 h_0 \rangle} \delta m_\alpha, \\ \delta n_4 &= \frac{2}{\Phi} \left( 2 \langle h_0 \rangle \delta m_4 - \langle |v|^2 h_0 \rangle \delta m_0 \right), \end{split}$$

where

$$\Phi = \langle |v|^4 h_0 \rangle \langle h_0 \rangle - \langle |v|^2 h_0 \rangle^2 > 0.$$

So, defining for  $\beta$ ,  $\nu = 0, \ldots, 4$  and  $\alpha = 1, 2, 3$ ,

$$d_{\alpha}^{\beta,\nu} = \frac{\partial \mathbb{E}^{\mu_{L,n}}[w_{x,\alpha}^{(a),\beta}]}{\partial m_{\nu}(n)}\Big|_{n=\bar{n}},$$

we obtain

$$d_{\alpha}^{\beta,\nu} = b_0 \delta_{\beta,0} \delta_{\alpha,\nu} + b_4 \delta_{\beta,4} \delta_{\alpha,\nu} + \mathbb{1}_{\{1,2,3\}}(\beta) \delta_{\alpha,\beta} \big[ a_0 \delta_{\nu,0} + a_4 \delta_{\nu,4} \big],$$
(2.5)

with

$$b_0 = \frac{\Phi_2}{3\Phi}, \quad b_4 = 2\frac{\Phi_1}{3\Phi}, \quad a_0 = \frac{\langle |v|^2 h_1 \rangle}{\langle |v|^2 h_0 \rangle}, \quad a_4 = \frac{\langle |v|^4 h_1 \rangle}{\langle |v|^2 h_0 \rangle}, \tag{2.6}$$

$$\Phi_{1} = \langle h_{1} | v |^{4} \rangle \langle h_{0} \rangle - \langle h_{1} | v |^{2} \rangle \langle h_{0} | v |^{2} \rangle,$$
  
$$\Phi_{2} = \langle h_{0} | v |^{4} \rangle \langle h_{1} | v |^{2} \rangle - \langle h_{1} | v |^{4} \rangle \langle h_{0} | v |^{2} \rangle.$$

Therefore the local function

$$g_{\alpha}^{\beta}(\eta) = w_{0,\alpha}^{(a),\beta} - c_{\alpha}^{\beta} - \frac{1}{2} \sum_{\nu=0}^{4} d_{\alpha}^{\beta,\nu} \left( \tilde{I}_{\nu}(\eta_{0}) + \tilde{I}_{\nu}(\eta_{e_{\alpha}}) \right)$$
(2.7)

belongs to  $\mathcal{G}$ .

### 2.1. Slow-fast modes decomposition of the currents

We denote by  $\bar{I}_{\ell}^+ = (\bar{I}_{0,\ell}, \dots, \bar{I}_{4,\ell})$  the empirical averages of the conserved quantities over the block  $\Lambda_{\ell}$  of length  $\ell$ :

$$\bar{I}_{\beta,\ell} = \frac{1}{(2\ell+1)^3} \sum_{|y| \leq \ell} I_{\beta}(\eta_y), \quad \beta = 0, \dots, 4.$$

The measure  $\hat{\mu}_{\ell,m}$ ,  $m \in \mathbb{R}^5$  is defined as the canonical Gibbs state of  $(2\ell + 1)^3$  sites with parameters such that  $\bar{I}_{\ell}^+ = m$ . It is the uniform probability on the set  $\Omega_{\ell,m}$  of configurations on the block  $\Lambda_{\ell}$  such that  $\bar{I}_{\ell}^+ = m$ . We denote by  $\alpha_{\ell}(g)$  the conditional expectation of g given the averages  $\bar{I}_{\ell}^+$ 

$$\alpha_{\ell}(g) = \mathbb{E}^{\mu} \big[ g | \bar{I}_{\ell}^+ \big].$$

We call  $\mathcal{L}_{s,\ell}$  the symmetric part of the generator  $\mathcal{L}$  restricted to the block  $\Lambda_{\ell}$ . Since the measures  $\hat{\mu}_{\ell,m}$  are the only extremal invariant measures for  $\mathcal{L}_{s,\ell}$ , we can define  $\mathcal{L}_{s,\ell}^{-1}g$  for any function g such that  $\alpha_{\ell}(g) = 0$ . Given any local function g on  $\Omega_{\ell}$ , the finite volume "variance"  $V_{\ell}(g, n)$  is

$$V_{\ell}(g,n) = \frac{1}{(2\ell_1 + 1)^3} \mathbb{E}^{\mu_{\ell,n}} \bigg[ \bigg( \sum_{|x| \leq \ell_1} (\tau_x g - \alpha_{\ell}(g)) \bigg) (-\mathcal{L}_{s,\ell})^{-1} \bigg( \sum_{|x| \leq \ell_1} (\tau_x g - \alpha_{\ell}(g)) \bigg) \bigg],$$

where  $\tau$  is the translation operator on  $\Omega_L$ ,  $\tau_x g(\eta) = g(\tau_x \eta)$ ,  $\ell_1 = \ell - \ell^{1/9}$ ,  $\ell$  large enough. The "variance" V(G, n) of G is given by

$$V(G, n) = \limsup_{\ell \to \infty} V_{\ell}(G, n).$$
(2.8)

With an abuse of notation, we denote  $V_{\ell}(G, n)$  by  $V_{\ell}(G, r, \theta)$  when *n* is the chemical potential  $\bar{n} = (r, 0, 0, 0, \theta)$ .

We state here the results in [1].

THEOREM 2.1. – There exists a tensor  $\overline{D} = (\overline{D}_{\alpha,\gamma}^{\beta,\nu})$  ( $\overline{D}^{\beta,\nu}$  positive definite matrix) and a sequence of local functions  $h^{(q)} = (h_{\alpha}^{(q),\beta}, \alpha = 1, 2, 3, \beta = 0, ..., 4)$  in  $\mathcal{G}$  such that, setting

$$u_{\alpha}^{(q),\beta}(\eta) = g_{\alpha}^{\beta}(\eta) - \sum_{\gamma=1}^{3} \sum_{\nu=0}^{4} \overline{D}_{\alpha,\gamma}^{\beta,\nu} \nabla_{\gamma} \tilde{I}_{\nu}(\eta(0)) - \mathcal{L}h_{\alpha}^{(q),\beta}(\eta),$$
(2.9)

where  $g^{\beta}_{\alpha}(\eta)$  is defined in (2.7), it results

$$\lim_{q\to\infty}\sum_{\alpha=1}^3\sum_{\beta=0}^4 V(u_{\alpha}^{(q),\beta},r,\theta)=0.$$

Above theorem actually holds for any function in  $\mathcal{G}$ .

LEMMA 2.2. – The tensor  $\overline{D}$  satisfies

$$a \cdot (\overline{D}C)a = \lim_{q \to \infty} \mathbb{E}^{\mu} \big[ \Gamma \big( a \cdot h^{(q)} \big) (-\mathcal{L}_s) \big( a \cdot h^{(q)} \big) \big].$$
(2.10)

In this formula,  $a \cdot b = \sum_{\alpha=1}^{3} \sum_{\beta=0}^{4} a_{\alpha}^{\beta} b_{\alpha}^{\beta}$ ,  $\Gamma(g) = \sum_{x} \tau_{x}g$ ,  $\mathcal{L}_{s}$  is the symmetric part of  $\mathcal{L}$  in  $L^{2}(\mu)$ , C is the 5 × 5 compressibility matrix (see (3.3) below for an explicit expression) and  $\overline{D}C$  is the tensor  $(\overline{D}C)_{\alpha,\gamma}^{\beta,\nu} = (\overline{D}_{\alpha,\gamma}C)^{\beta,\nu}$ . We define

$$D = \overline{D} + \chi \mathbb{I} \tag{2.11}$$

where  $\mathbb{I}_{\alpha,\gamma}^{\beta,\nu} = \delta_{\alpha,\gamma}\delta_{\beta,\nu}$ .

## 2.2. Hydrodynamic limit

Given functions  $n_{\beta}(x)$ ,  $\beta = 0, ..., 4$ , we consider the Gibbs states with chemical potential  $n(x) = (n_0(x), ..., n_4(x))$ 

$$\mu_{L,n(\cdot)}(\eta) = Z_{L,n(\cdot)}^{-1} \prod_{x \in \Lambda_L} \exp\left\{\sum_{\beta=0}^4 n_\beta(x) I_\beta(\eta_x)\right\}.$$
(2.12)

Now, assume that the initial distribution of the particles is  $\mu_{L,n(\cdot)}$  with  $n = (n_{\beta})$  the slowly varying chemical potentials given by

$$n_{\beta}(x) = \lambda_{\beta}^{(0)} + \varepsilon \lambda_{\beta}^{(1)}(\varepsilon x) + \varepsilon^2 \lambda_{\beta}^{(2)}(\varepsilon x), \qquad (2.13)$$

where  $\lambda^{(0)} = (\lambda^{(0)}_{\beta}) = \bar{n}$  and  $\lambda^{i}_{\beta}$  are smooth functions on the 3-d torus  $\mathbb{T}_{3}$ . We define the local equilibrium measure as the Gibbs states  $\mu_{L,n(\cdot,t)}$  with  $n(\cdot,t)$  the chemical potential given by

$$n_{\beta}(x,t) = \lambda_{\beta}^{(0)} + \varepsilon \lambda_{\beta}^{(1)}(\varepsilon x,t) + \varepsilon^2 \lambda_{\beta}^{(2)}(\varepsilon x,t).$$
(2.14)

Furthermore, we assume

div 
$$\underline{\lambda}^{(1)} = 0$$
,  $\langle h_1 | v |^2 \rangle \lambda_0^{(1)} + \frac{1}{2} \langle h_1 | v |^4 \rangle \lambda_4^{(1)} = 0$ .

Then in [1] (see also [9]) it has been proved that the law of the process at time t > 0 is well approximated by the local equilibrium in the sense that the relative entropy per unit volume of the non-equilibrium measure with respect to the local equilibrium times  $\varepsilon^{-2}$  vanishes in the limit  $\varepsilon \to 0$ .

We can now state the result proved in [1] on the hydrodynamic limit. Let u(z, t),  $z \in \mathbb{T}_3$ ,  $t \in [0, t_0]$ ,  $t_0 > 0$ , be the classical smooth solutions of the following Navier–Stokes equation

div 
$$u = 0$$
,  
 $\partial_t u_\beta + \partial_\beta p + K u \cdot \underline{\partial} u_\beta = \sum_{\alpha=1}^3 D^{\beta,\beta}_{\alpha,\alpha} \partial^2_\alpha u_\beta, \quad \beta = 1, 2, 3,$ 
(2.15)

with initial condition  $u_{\alpha}(z) = \mathbb{E}^{\mu_{L,n(\varepsilon^{-1}z)}}[I_{\alpha}(\eta_0)]$  and let  $\mathcal{E}(z,t)$  be the solution of the energy equation

$$\frac{\partial}{\partial t}\mathcal{E} + Hu \cdot \underline{\partial} \mathcal{E} = \sum_{\alpha=1}^{3} \mathcal{K}_{\alpha} (\partial_{\alpha}^{2} \mathcal{E}), \qquad (2.16)$$

with initial condition  $\mathcal{E}(z) = \mathbb{E}^{\mu_{L,n(\varepsilon^{-1}z)}}[I_4(\eta_0)]$ . The constants appearing in (2.15) and (2.16) are given by

$$K = 18 \frac{\langle v_1^2 v_2^2 h_2 \rangle}{\langle h_0 | v |^2 \rangle^2},$$

with  $h_2 = \frac{1}{2}h_1(1 - 6f_0(1 - f_0))$  and

$$H = \frac{1}{\langle h_0 | v |^2 \rangle} \frac{\Psi_1 - 2C\Psi_2}{\Phi_2 + C\Phi_1}, \quad C = \frac{1}{2} \frac{\langle h_1 | v |^4 \rangle}{\langle h_1 | v |^2 \rangle},$$

where

$$\begin{split} \Psi_1 &= \langle h_2 | v |^6 \rangle \langle h_1 | v |^2 \rangle - \langle h_2 | v |^4 \rangle \langle h_1 | v |^4 \rangle, \\ \Psi_2 &= \langle h_2 | v |^4 \rangle \langle h_1 | v |^2 \rangle - \langle h_2 | v |^2 \rangle \langle h_1 | v |^4 \rangle. \end{split}$$

Let  $\mathbb{P}_{\varepsilon}^{\mu_{L,n(\cdot)}}$  be the law of the process  $\eta_t(x, v)$  with generator  $\varepsilon^{-2}\mathcal{L}$  starting from the measure  $\mu_{L,n(\cdot)}$  defined in (2.12), with chemical potentials  $n_{\alpha}(x)$  of the form (2.13). The density  $(v_0^{\varepsilon}(t, z))$ , the momentum  $((v_{\beta}^{\varepsilon}(t, z))_{\beta=1,2,3})$  and energy  $(v_4^{\varepsilon}(t, z))$  empirical fields are defined as

$$\nu_{\beta}^{\varepsilon}(z,t) = \varepsilon^2 \sum_{x \in \Lambda_L} \delta(z - \varepsilon x) \tilde{I}_{\beta}(\eta_t(x)),$$

where  $\tilde{I}_{\beta}(\eta_x) = I_{\beta}(\eta_x) - m_{\beta}, m_{\beta} = \mathbb{E}^{\mu}[I_{\beta}(\eta_0)] \text{ and } \eta_t(x) = \{\eta_t(x, v), v \in \mathcal{V}\}.$ 

THEOREM 2.3. – The density, momentum and energy empirical fields converge, for  $t \leq t_0$ , weakly (in space) in  $\mathbb{P}_{\varepsilon}^{\mu_{L,n(\cdot)}}$  probability, to  $\rho(z,t) dz$ , u(z,t) dz and  $\mathcal{E}(z,t) dz$ , where  $a\rho + b\mathcal{E} = c$  for suitable a, b, c.

Note that the transport coefficients  $D_{\alpha,\beta}$  and  $\mathcal{K}_{\alpha}$  are suitable combinations of the diffusion coefficients  $D_{\alpha,\gamma}^{\beta,\nu}$  in Theorem 2.1. The explicit expressions are given in [1], but we omit them because they do not play any role in this paper.

#### 3. Fluctuation field and results

In this paper, we are interested in the equilibrium fluctuations of the mass, momentum and energy fields. The initial fluctuations, distributed in terms of the measure  $\mu$ , are finite but they may become infinite at later very long times because of the effect of waves moving with velocity  $\varepsilon^{-1}$ , which are the solutions of the linearized (around the equilibrium) Euler equations (linear hyperbolic equations) for this model. To remove the diverging terms we have to modify the usual definition of fluctuation fields not simply by a shift but considering fluctuations which move together with the traveling waves.

We denote by  $U_t^{\varepsilon}$  the operator  $\exp(-\frac{t}{\varepsilon}E^*)$  where E is the linearized Euler operator, a  $5 \times 5$  matrix whose entries are first order differential operators with constant coefficients,

$$E = \begin{pmatrix} 0 & -a_0\underline{\partial} & 0\\ -b_0\underline{\partial} & 0 & -b_4\underline{\partial}\\ 0 & -a_4\underline{\partial} & 0 \end{pmatrix}$$

and \* is the adjoint with respect to the usual scalar product in  $L^2(\mathbb{T}_3, \mathbb{R}^5)$  (the constants  $a_i$  and  $b_i$  are defined in (2.6)).

For any smooth function  $G = (G_{\beta})_{\beta=0,\dots,4} : \mathbb{T}_3 \to \mathbb{R}^5$  consider the (scalar) fluctuation field  $\xi^{\varepsilon}$  on the state space  $(\mathbb{T}_3)^{\otimes 5}$ 

$$\xi^{\varepsilon}(t,G) = \varepsilon^{3/2} \sum_{\beta=0}^{4} \sum_{x} (U_{t}^{\varepsilon}G)_{\beta}(\varepsilon x) \tilde{I}_{\beta}(\eta_{t}(x)).$$
(3.1)

It is equivalent to consider the vector fluctuation field  $(\xi_{\beta}^{\varepsilon})_{\beta=0,...,4}$  on  $\mathbb{T}_3$  whose components  $\xi_0^{\varepsilon}$ ,  $(\xi_{\beta}^{\varepsilon})_{\beta=1,...,3}$  and  $\xi_4^{\varepsilon}$  are respectively the density, momentum and energy fluctuation fields, defined as

$$\xi_{\beta}^{\varepsilon}(t,\varphi) = \xi^{\varepsilon}(t,G^{(\beta)}), \quad \beta = 0,\ldots,4,$$

where  $G^{(\beta)}$  is the vector function with only the  $\beta$  component non vanishing and  $G_{\beta}^{(\beta)} = \varphi$ .

We want to study the evolution of the fluctuation fields in the limit  $\varepsilon \to 0$  when the fields are initially distributed with the equilibrium measure  $\mu$  (defined after (2.2)). We notice that the initial covariance of the limiting fields  $\lim_{\varepsilon \to 0} \xi_{\beta}^{\varepsilon}(0, \varphi) = \xi_{\beta}(0, \varphi)$  is

$$\mathbb{E}^{\mu}\left[\xi_{\beta}(0,\varphi)\xi_{\nu}(0,\psi)\right] = C_{\beta,\nu} \int_{\mathbb{T}_{3}} dx \,\varphi(x)\psi(x), \qquad (3.2)$$

where *C* is the compressibility matrix  $(5 \times 5)$ 

$$C = \begin{pmatrix} \langle h_0 \rangle & 0 & \langle h_0 \frac{|\nu|^2}{2} \rangle \\ 0 & \frac{1}{3} \langle |\nu|^2 h_0 \rangle \mathbb{I}_3 & 0 \\ \langle h_0 \frac{|\nu|^2}{2} \rangle & 0 & \langle h_0 \frac{|\nu|^4}{4} \rangle \end{pmatrix},$$
(3.3)

with  $\mathbb{I}_3$  the 3 × 3 identity matrix,  $h_0$  defined in the paragraph before (2.4) and  $\langle \cdot \rangle$  in the paragraph after (2.2).

Remark that E is not anti-hermitian in  $L^2(\mathbb{T}_3, \mathbb{R}^5)$ , since  $a_0 \neq b_0$  and  $a_4 \neq b_4$ . However a straightforward computation shows that EC satisfies  $EC + CE^* = 0$ .

We want to show that the fluctuation field converges to a stationary Gaussian vector process with a given covariance. The equal time covariance is exactly (3.2) because of the stationarity of the limiting process.

To state the results we need some extra notation. We introduce the Hilbert spaces  $\mathcal{H}_k, k \in \mathbb{Z}$  defined by the scalar product

$$\langle G, H \rangle_k = \langle G, L^k H \rangle_0,$$

where  $L = I - \Delta$ ,  $\Delta$  the Laplacian operator and  $\langle ., . \rangle_0$  is the usual inner product of  $L^2(\mathbb{T}_3, \mathbb{R}^5)$ :

$$\langle G, H \rangle_0 = \sum_{\beta=0}^4 \int_{\mathbb{T}_3} dx \, G_\beta(x) H_\beta(x). \tag{3.4}$$

Denote by  $\|\cdot\|_k$  the norm of  $\mathcal{H}_k$  and by  $\mathcal{H}_{-k}$  the dual of  $\mathcal{H}_k$  with respect to the inner product of  $L^2(\mathbb{T}_3, \mathbb{R}^5)$ . The fluctuation field  $(\xi^{\varepsilon}(t))_{t\geq 0}$  is a distribution valued stochastic process taking values in the Sobolev space  $\mathcal{H}_{-k_0}$  for some suitable  $k_0$ . Its path space is  $D([0, T], \mathcal{H}_{-k_0})$  (T > 0), the space of functions with values in  $\mathcal{H}_{-k_0}$ , right continuous with left limits, endowed with the uniform (in time) weak (in space) topology. We call  $Q^{\varepsilon}$  the probability measure on  $D([0, T], \mathcal{H}_{-k_0})$  induced by the fluctuation field and the equilibrium measure  $\mu$  and by  $\mathbb{P}^{\mu}_{\varepsilon}$  the law of the process  $\eta_{\varepsilon^{-2}t}$  when the process is initially distributed according to the equilibrium measure  $\mu$ . We denote by  $\mathbb{E}^{\mu}_{\varepsilon}$  the expectation with respect  $\mathbb{P}^{\mu}_{\varepsilon}$ .

By analogy with (3.4), we define for local functions  $g = (g_0, \ldots, g_4)$  on  $\Omega_L$  and smooth functions  $G = (G_0, \ldots, G_4)$  on  $\mathbb{T}_3$ 

$$\langle G, g \rangle_{0,L} = \varepsilon^{3/2} \sum_{\beta=0}^{4} \sum_{x \in \Lambda_L} G_{\beta}(\varepsilon x) \tau_x g_{\beta}.$$
(3.5)

So the fluctuation field (3.1) can be rewritten as

$$\xi^{\varepsilon}(t,G) = \left\langle U_t^{\varepsilon}G, \tilde{I}(\eta_t(0)) \right\rangle_{0,L}.$$

We recall the definition of the linearized Navier–Stokes operator  $\mathcal{D}$ 

$$\mathcal{D}G = \sum_{\alpha,\gamma=1}^{3} D_{\alpha,\gamma} \partial_{\alpha} \partial_{\gamma} G,$$

 $D_{\alpha,\beta}$  being, for any  $\alpha, \beta = 1, ..., 3, 5 \times 5$  matrices, whose elements  $D_{\alpha,\beta}^{\gamma,\nu}$  are defined in (2.10).

We call *N* the operator defined as the limit in  $L^2(\mathbb{T}_3, \mathbb{R}^5)$ 

$$\lim_{\varepsilon \to 0} \frac{1}{T} \int_{0}^{T} dt \, \exp\left(-\frac{t}{\varepsilon}E\right) \mathcal{D} \exp\left(\frac{t}{\varepsilon}E\right) G$$

and for general differential operators  $\mathbb{A}$  and  $\mathbb{D}$  we use the notation

$$\lim_{\varepsilon \to 0} \frac{1}{T} \int_{0}^{T} dt \, \exp\left(-\frac{t}{\varepsilon} \mathbb{A}\right) \mathbb{D} \exp\left(\frac{t}{\varepsilon} \mathbb{A}\right) G = \pi_{\mathbb{A}}(\mathbb{D}) G.$$

Then  $N = \pi_E(\mathcal{D})$  and  $\pi_E(\mathcal{D})^*$  is the adjoint of N with respect to the inner product  $\langle \cdot, \cdot \rangle_0$ in  $L^2(\mathbb{T}_3, \mathbb{R}^5)$ . The main result of this paper is

THEOREM 3.1. – The probability measures  $(Q^{\varepsilon})$  converge weakly in  $D([0, T], \mathcal{H}_{-k_0})$ , for integer  $k_0$  such that  $k_0 > 2 + d/2$ , to the law Q of the stationary generalized Ornstein–Uhlenbeck process  $\xi$  with mean 0 and covariance

$$\mathbb{E}^{Q}\left[\xi_{\beta}(s,\phi)\xi_{\nu}(t,\psi)\right] = \int_{\mathbb{T}_{3}} dx \left( (CS_{|t-s|})_{\beta,\nu}\phi \right)(x)\psi(x),$$

where  $(S_t)_{t\geq 0}$  is the semi-group in  $L^2(\mathbb{T}_3, \mathbb{R}^5)$  associated to  $\pi_E(\mathcal{D})^*$  and C the compressibility matrix. It is formally characterized by the SPDE

$$d\xi(t) = N\xi(t) dt + B dW_t,$$
  
$$BB^* = -2NC.$$

One of the main ingredients needed while studying the equilibrium fluctuations is the so-called Boltzmann–Gibbs principle which states that the non conserved quantities arising in the conservation laws may be replaced by linear combinations of the conserved ones. In the context of a non gradient system, the usual statement is not valid and some corrections to the fluctuation field have to be introduced (see [3,4,14]). The situation in the case of an asymmetric system is more delicate since the usual Boltzmann–Gibbs estimate is not sharp enough and one has to prove a stronger result [5]. We need to generalize such a result to the present setup. Indeed we prove the following

THEOREM 3.2 (Boltzmann–Gibbs principle). – Assume that  $h \in \mathcal{G}$  (see (2.3)). Then, for any smooth function  $G : \mathbb{R}_+ \times \mathbb{T}_3 \to \mathbb{R}$ , the following estimate holds

$$\limsup_{\varepsilon \to 0} \mathbb{E}_{\varepsilon}^{\mu} \left[ \sup_{0 \leq t \leq T} \left( \varepsilon^{3/2 - 1} \int_{0}^{t} \sum_{x} G(s, \varepsilon x) \tau_{x} h(\eta_{s}) \, ds \right)^{2} \right] \leq c V(h; r, \theta) \int_{0}^{T} ds \left\| G(s, \cdot) \right\|_{0}^{2},$$

where V is the infinite volume variance defined in (2.8).

## 4. Limiting distribution of the fluctuation field

The theory of Holley–Stroock [12] characterizes the law Q of the Ornstein–Uhlenbeck process  $\xi$  described in Theorem 3.1 by the following martingale problem:

$$M_{1}(t,G) = \xi(t,G) - \xi(0,G) - \int_{0}^{t} ds \,\xi \left(s, \pi_{E}(\mathcal{D})^{*}G\right),$$

$$M_{2}(t,G) = \left(M_{1}(t,G)\right)^{2} + 2t \left\langle G, C \,\pi_{E}(\mathcal{D})^{*}G \right\rangle_{0}$$
(4.1)

are martingales under Q. In this section, we will prove that any limit law  $\overline{Q}$  of  $Q^{\varepsilon}$  satisfies (4.1). Therefore from the tightness of  $(Q^{\varepsilon})$  (see Theorem 5.5 of Section 5), it has to converge to Q and Theorem 3.1 follows.

The processes analogous to (4.1) for  $Q^{\varepsilon}$  are

$$M_1^{\varepsilon}(t,G) = \xi^{\varepsilon}(t,G) - \xi^{\varepsilon}(0,G) - \int_0^t ds \,\xi^{\varepsilon} \big(s,\pi_E(\mathcal{D})^*G\big),$$
$$M_2^{\varepsilon}(t,G) = \big(M_1^{\varepsilon}(t,G)\big)^2 + 2t \big\langle G, C \,\pi_E(\mathcal{D})^*G \big\rangle_0$$

and we want to show that these processes are martingales up to some error terms which vanish as  $\varepsilon$  goes to 0. Given local functions  $h = (h_{\alpha})_{\alpha=1,2,3} = (h_{\alpha}^{\beta})_{\alpha=1,2,3}; _{\beta=0,\dots,4} \in \mathcal{G}$ , we introduce the modified fluctuation field

$$\zeta^{\varepsilon}(t,G,h) = \xi^{\varepsilon}(t,G) - \varepsilon \sum_{\alpha=1}^{3} \langle \partial_{\alpha}(U_{t}^{\varepsilon}G), h_{\alpha} \rangle_{0,L},$$

where  $\langle \cdot, \cdot \rangle_{0,L}$  was defined in (3.5). Actually we will choose for *h* the terms of the sequence  $h^{(q)}$  defined in Theorem 2.1, but we will omit the label *q* for sake of shortness. It is clear that the difference between  $\zeta^{\varepsilon}(t, G, h)$  and  $\xi^{\varepsilon}(t, G)$  vanishes in  $L^2(\mathbb{P}^{\mu}_{\varepsilon})$  with  $\varepsilon$ . Moreover, it is well known that the following processes are martingales with respect to the usual filtration related to the process  $(\eta_t(x, v))$ 

$$M_1^{\varepsilon}(t, G, h) = \zeta^{\varepsilon}(t, G, h) - \zeta^{\varepsilon}(0, G, h) - \int_0^t \gamma_1^{\varepsilon}(s, G, h) \, ds,$$

$$M_2^{\varepsilon}(t, G, h) = \left(M_1^{\varepsilon}(t, G, h)\right)^2 - \int_0^t \gamma_2^{\varepsilon}(s, G, h) \, ds,$$
(4.2)

with

$$\gamma_1^{\varepsilon}(t, G, h) = (\partial_t + \varepsilon^{-2}\mathcal{L})(\zeta^{\varepsilon}(t, G, h)),$$
  

$$\gamma_2^{\varepsilon}(t, G, h) = (\partial_t + \varepsilon^{-2}\mathcal{L})(\zeta^{\varepsilon}(t, G, h)^2) - 2\zeta^{\varepsilon}(t, G, h)(\partial_t + \varepsilon^{-2}\mathcal{L})(\zeta^{\varepsilon}(t, G, h)).$$
  
We first compute the compensator  $\gamma_1^{\varepsilon}$ . Let  $w_{x,\alpha}^{(a)} = (w_{x,\alpha}^{(a),\beta})_{\beta=0,\dots,4}$ . Then

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$$\begin{split} \gamma_{1}^{\varepsilon}(t,G,h) &= \left\langle \partial_{t} \left( U_{t}^{\varepsilon}G \right), \tilde{I} \left( \eta_{t}(0) \right) \right\rangle_{0,L} - \varepsilon^{-1} \sum_{\alpha=1}^{3} \left\langle \partial_{\alpha} \left( U_{t}^{\varepsilon}G \right), \mathcal{L}h_{\alpha}(\eta_{t}) \right\rangle_{0,L} \\ &- \varepsilon^{-2} \sum_{\alpha=1}^{3} \left\langle \nabla_{\alpha} \left( U_{t}^{\varepsilon}G \right), \chi \nabla_{\alpha} \tilde{I} \left( \eta_{t}(0) \right) + w_{0,\alpha}^{(a)}(\eta_{t}) \right\rangle_{0,L} \\ &+ R_{5}(t,G,h), \end{split}$$
(4.3)

where, remembering that  $\partial_t U_t^{\varepsilon} = -\varepsilon^{-1} E^* U_t^{\varepsilon}$ ,

$$R_5(t, G, h) = -\sum_{\alpha=1}^{3} \langle \partial_{\alpha} \left( -E^* U_t^{\varepsilon} \right) G, h_{\alpha}(\eta_t) \rangle_{0,L}$$

Now, given  $\overline{D}_{\alpha,\gamma} = (\overline{D}_{\alpha,\gamma}^{\beta,\nu})_{\beta,\nu=0,\dots,4}, \ \alpha, \gamma = 1, 2, 3$ , we add and subtract the term

$$\sum_{\alpha,\gamma=1}^{3} \langle \overline{D}_{\alpha,\gamma} \partial_{\alpha} (U_{t}^{\varepsilon} G), \nabla_{\gamma} \tilde{I} (\eta_{t}(0)) \rangle_{0,L}$$

in (4.3). Then  $\gamma_1^{\varepsilon}(t, G, h)$  is equal to

$$\left\langle \partial_{t} \left( U_{t}^{\varepsilon} G \right), \tilde{I} \left( \eta_{t}(0) \right) \right\rangle_{0,L} + \left\langle \mathcal{D}^{*} \left( U_{t}^{\varepsilon} G \right), \tilde{I} \left( \eta_{t}(0) \right) \right\rangle_{0,L} - \varepsilon^{-1} \sum_{\alpha=1}^{3} \left\langle \partial_{\alpha} \left( U_{t}^{\varepsilon} G \right), w_{0,\alpha}^{(a)}(\eta_{t}) - c_{\alpha} - \sum_{\gamma=1}^{3} \overline{D}_{\alpha,\gamma} \nabla_{\gamma} \tilde{I} \left( \eta_{t}(0) \right) - \mathcal{L} h_{\alpha}(\eta_{t}) \right\rangle_{0,L} + R_{5}(t, G, h) + R_{1}(t, G) + R_{2}(t, G) + R(t, G),$$

$$(4.4)$$

with  $\mathcal{D}^*$  the adjoint in  $L^2(\mathbb{T}_3, \mathbb{R}^5)$  of the differential operator  $\mathcal{D}$  in (3.5),  $c_{\alpha} = (c_{\alpha}^{\beta})_{\beta=0,\dots,4}$ the equilibrium value of the current  $w_{0,\alpha}^{(a),\beta}$  (see (2.4)) and

$$R_{1}(t,G) = \varepsilon^{-1} \sum_{\alpha=1}^{3} \langle \left(\partial_{\alpha} - \varepsilon^{-1} \nabla_{\alpha}\right) \left(U_{t}^{\varepsilon} G\right), \chi \nabla_{\alpha} \tilde{I}(\eta_{t}(0)) \rangle_{0,L}$$

$$R_{2}(t,G) = \sum_{\alpha,\gamma=1}^{3} \langle \left(\varepsilon^{-1}\nabla_{\gamma}^{-} - \partial_{\gamma}\right)\partial_{\alpha}D_{\alpha,\gamma}\left(U_{t}^{\varepsilon}G\right), \tilde{I}(\eta_{t}(0))\rangle_{0,L},$$
$$R(t,G) = \varepsilon^{-1}\sum_{\alpha=1}^{3} \langle \left(\partial_{\alpha} - \varepsilon^{-1}\nabla_{\alpha}\right)\left(U_{t}^{\varepsilon}G\right), w_{0,\alpha}^{(a)} - c_{\alpha}\rangle_{0,L}.$$

From the definition of  $U_t^{\varepsilon}G$ , the first term of the sum (4.4) can be written as

$$\varepsilon^{-1} \sum_{\alpha=1}^{3} \left\langle \left( -E^* U_t^{\varepsilon} \right) G, \, \tilde{I}\left( \eta_t(0) \right) \right\rangle_{0,L} = \varepsilon^{-1} \sum_{\alpha=1}^{3} \left\langle \partial_\alpha \left( U_t^{\varepsilon} G \right), \, d_\alpha \tilde{I}\left( \eta_t(0) \right) \right\rangle_{0,L}$$

where the coefficients of the matrix  $d_{\alpha} = (d_{\alpha}^{\beta,\nu})_{\beta,\nu=0,\dots,4}$  were defined in (2.5). Recalling the definition of the local functions  $g_{\alpha} = (g_{\alpha}^{\beta})_{\beta=0,\dots,4}$  and  $u_{\alpha} = (u_{\alpha}^{\beta})_{\beta=0,\dots,4}$  in (2.7), (2.9)

(we omit the label q), we obtain

$$\gamma_1^{\varepsilon}(t,G,h) = \left\langle \mathcal{D}^*(U_t^{\varepsilon}G), \tilde{I}(\eta_t(0)) \right\rangle_{0,L} + \sum_{i=1}^4 R_i(t,G) + \sum_{i=5}^6 R_i(t,G,h).$$

where

$$R_{3}(t,G) = \varepsilon^{-1} \sum_{\alpha=1}^{3} \left\langle \left( \partial_{\alpha} - \varepsilon^{-1} \nabla_{\alpha} + \frac{1}{2} \nabla_{e_{\alpha}}^{-} \partial_{\alpha} \right) \left( U_{t}^{\varepsilon} G \right), d_{\alpha} \tilde{I} \left( \eta_{t}(0) \right) \right\rangle_{0,L}$$

$$R_4(t,G) = \varepsilon^{-1} \sum_{\alpha=1}^{3} \left\langle \left( \partial_{\alpha} - \varepsilon^{-1} \nabla_{\alpha} \right) \left( U_t^{\varepsilon} G \right), g_{\alpha}(\eta_t) \right\rangle_{0,L}$$

and

$$R_6(t, G, h) = -\varepsilon^{-1} \sum_{\alpha=1}^3 \langle \partial_\alpha (U_t^{\varepsilon} G), u_\alpha(\eta_t) \rangle_{0,L}$$

To prove that the compensator  $\int_0^t \gamma_1^{\varepsilon}(t, G, h) ds$  is converging, we have to control the remainder terms.

The remainders for i = 1, 2, 3 are easily controlled by the following

LEMMA 4.1. – Let h be a mean zero local function and  $G: \mathbb{R}_+ \times \mathbb{T}_3 \to \mathbb{R}$  a continuous function. Then there exists a constant c depending only on h such that, for all  $T \ge 0$  and all  $\varepsilon > 0$ 

$$\mathbb{E}_{\varepsilon}^{\mu}\left[\sup_{0\leqslant t\leqslant T}\left(\int_{0}^{t}ds\,\varepsilon^{3/2}\sum_{x}G(s,\varepsilon x)\tau_{x}h(\eta_{s})\right)^{2}\right]\leqslant cT\int_{0}^{T}ds\,\left\|G(s,\cdot)\right\|_{\infty}^{2}.$$

The proof is an easy consequence of the Schwarz inequality and the stationarity of  $\mathbb{P}^{\mu}_{\varepsilon}$ . We refer to Lemma 4.1 in [5] for details.

By Taylor expansion and using Lemma 4.1 we immediately obtain

$$\lim_{\varepsilon \to 0} \mathbb{E}_{\varepsilon}^{\mu} \left[ \left( \int_{0}^{t} ds \, R_{i}(s, G) \right)^{2} \right] = 0$$

for i = 1, 2, 3.

The other terms are estimated by using the refined Boltzmann–Gibbs principle (Theorem 3.2) because the functions  $g^{\beta}_{\alpha}$  and  $u^{\beta}_{\alpha}$  are in  $\mathcal{G}$  ( $h^{\beta}_{\alpha} \in \mathcal{G}$  by hypothesis). We get

$$\lim_{\varepsilon \to 0} \mathbb{E}_{\varepsilon}^{\mu} \left[ \left( \int_{0}^{t} ds \, R_{4}(s, G) \right)^{2} \right] = \lim_{\varepsilon \to 0} \mathbb{E}_{\varepsilon}^{\mu} \left[ \left( \int_{0}^{t} ds \, R_{5}(s, G, h) \right)^{2} \right] = 0$$

and

$$\limsup_{\varepsilon \to 0} \mathbb{E}_{\varepsilon}^{\mu} \left[ \left( \int_{0}^{t} ds \, R_{6}(s, G, h) \right)^{2} \right] \leqslant c \max_{\alpha = 1, 2, 3} \int_{0}^{t} ds \left\| \partial_{\alpha} \left( U_{s}^{\varepsilon} G(\cdot) \right) \right\|_{0}^{2} \sum_{\alpha = 1}^{3} \sum_{\beta = 0}^{4} V \left( u_{\alpha}^{\beta}; r, \theta \right).$$

From the definition of the semi-group  $(U_t^{\varepsilon})$ , it is clear that  $\int_0^t ds \|\partial_{\alpha}(U_s^{\varepsilon}G(\cdot))\|_0^2$  is uniformly bounded in  $\varepsilon$ . Moreover the diffusion coefficients  $\overline{D}_{\alpha,\gamma}^{\beta,\nu}$  are chosen in such a way that, since we take for  $h_{\alpha}^{\beta}$  the terms of the sequence  $(h_{\alpha}^{(q),\beta})$  given in Theorem 2.1, we have

$$\lim_{q \to \infty} V(u_{\alpha}^{(q),\beta}; r, \theta) = 0.$$

We have shown so far that there exists a random variable  $R^q_{\varepsilon}$  which converges to 0 in  $L^2(\mathbb{P}^{\mu}_{\varepsilon})$  as  $\varepsilon \to 0$  and then  $q \to \infty$  such that

$$M_1^{\varepsilon}(t,G,h) = \xi^{\varepsilon}(t,G) - \xi^{\varepsilon}(0,G) - \int_0^t ds \left\langle \mathcal{D}^*(U_s^{\varepsilon}G), \tilde{I}(\eta_s(0)) \right\rangle_{0,L} + R_{\varepsilon}^q.$$
(4.5)

We would like to have instead of the third term in (4.5) a term of the form

$$\left\langle U_{s}^{\varepsilon}(HG), \tilde{I}(\eta_{s}(0)) \right\rangle_{0 L}$$

for some suitable operator *H* that we could then rewrite as  $\xi^{\varepsilon}(s, HG)$ , so to identify the limiting martingale problem. We proceed in the following way:

$$\langle \mathcal{D}^*(U_s^\varepsilon G), \tilde{I}(\eta_s(0)) \rangle_{0,L} = \langle U_s^\varepsilon (U_s^\varepsilon)^{-1} \mathcal{D}^* U_s^\varepsilon G, \tilde{I}(\eta_s(0)) \rangle_{0,L}.$$

By Lemmas A.2 and A.3

$$\lim_{\varepsilon\to 0} \mathbb{E}_{\varepsilon}^{\mu} \left[ \left( \int_{0}^{t} ds \left[ \xi^{\varepsilon} \left( s, \left( U_{s}^{\varepsilon} \right)^{-1} \mathcal{D}^{*} U_{s}^{\varepsilon} G \right) - \xi^{\varepsilon} \left( s, \pi_{-E^{*}} (\mathcal{D}^{*}) G \right) \right] \right)^{2} \right] = 0.$$

Hence, noticing that  $\pi_{-E^*}(\mathcal{D}^*) = \pi_E(\mathcal{D})^*$ , we have proved that there exists a random variable  $C^q_{\varepsilon}$  which converges to 0 in  $L^2(\mathbb{P}^{\mu}_{\varepsilon})$  when  $\varepsilon \to 0$  and then  $q \to \infty$  such that

$$M_1^{\varepsilon}(t,G,h) = \xi^{\varepsilon}(t,G) - \xi^{\varepsilon}(0,G) - \int_0^t ds \,\xi^{\varepsilon}(s,\pi_E(\mathcal{D})^*G) + C_{\varepsilon}^q.$$

We now compute the compensator  $\gamma_2^{\varepsilon}$  in (4.2). We first remark that

$$\gamma_2^{\varepsilon}(t,G,h) = \left(\varepsilon^{-2}\mathcal{L}\right)\left(\zeta^{\varepsilon}(t,G,h)^2\right) - 2\zeta^{\varepsilon}(t,G,h)\left(\varepsilon^{-2}\mathcal{L}\right)\zeta^{\varepsilon}(t,G,h).$$

We introduce the operator  $\mathcal{L}^{(2)} = \mathcal{L}^{\text{ex},(2)} + \mathcal{L}^{\text{c},(2)}$  for local functions f and g as

$$\mathcal{L}^{\mathrm{ex},(2)}(f;g) = \mathcal{L}^{\mathrm{ex}}(fg) - f\mathcal{L}^{\mathrm{ex}}g - g\mathcal{L}^{\mathrm{ex}}f,$$
$$\mathcal{L}^{\mathrm{c},(2)}(f;g) = \mathcal{L}^{\mathrm{c}}(fg) - f\mathcal{L}^{\mathrm{c}}g - g\mathcal{L}^{\mathrm{c}}f.$$

Then we obtain

$$\gamma_2^{\varepsilon}(t,G,h) = Y_1^{\varepsilon}(t,G) + Y_2^{\varepsilon}(t,G,h) + Y_3^{\varepsilon}(t,G,h),$$

where

$$\begin{split} Y_{1}^{\varepsilon}(t,G) &= \varepsilon \sum_{\beta,\nu=0}^{4} \sum_{x,y} \left( U_{t}^{\varepsilon}G \right)_{\beta}(\varepsilon x) \left( U_{t}^{\varepsilon}G \right)_{\nu}(\varepsilon y) \mathcal{L}^{\mathrm{ex},(2)} \left( \tilde{I}_{\beta}(\eta_{x}); \tilde{I}_{\nu}(\eta_{y}) \right), \\ Y_{2}^{\varepsilon}(t,G,h) &= -2\varepsilon^{2} \sum_{\alpha=1}^{3} \sum_{\beta,\nu=0}^{4} \sum_{x,y} \left( U_{t}^{\varepsilon}G \right)_{\beta}(\varepsilon x) \partial_{\alpha} \left( U_{t}^{\varepsilon}G \right)_{\nu}(\varepsilon y) \mathcal{L}^{\mathrm{ex},(2)} \left( \tilde{I}_{\beta}(\eta_{x}); \tau_{y} h_{\alpha}^{\nu} \right), \\ Y_{3}^{\varepsilon}(t,G,h) &= \varepsilon^{3} \sum_{\alpha,\gamma=1}^{3} \sum_{\beta,\nu=0}^{4} \sum_{x,y} \partial_{\alpha} \left( U_{t}^{\varepsilon}G \right)_{\beta}(\varepsilon x) \partial_{\gamma} \left( U_{t}^{\varepsilon}G \right)_{\nu}(\varepsilon y) \mathcal{L}^{(2)} \left( \tau_{x} h_{\alpha}^{\beta}; \tau_{y} h_{\gamma}^{\nu} \right). \end{split}$$

From the explicit formulas (4.7), (4.8) and (4.9) that we will get below for  $Y_i(t, G)$ , i = 1, 2, 3 and the use of Lemma 4.1 it is easy to see that

$$\int_{0}^{t} ds \left( \gamma_{2}^{\varepsilon}(s, G, h) - \mathbb{E}_{\varepsilon}^{\mu} \left[ \gamma_{2}^{\varepsilon}(s, G, h) \right] \right)$$

is converging to 0 in  $L^2(\mathbb{P}^{\mu}_{\varepsilon})$ . So, all we need to compute is  $\mathbb{E}^{\mu}_{\varepsilon}[\gamma_2^{\varepsilon}(t, G, h)]$ .

Notice that

$$\mathcal{L}^{\mathrm{ex},(2)}(f;g) = \sum_{x,e,v} b(x,x+e,v) \nabla_{x,x+e,v} f \nabla_{x,x+e,v} g,$$
  
$$\mathcal{L}^{\mathrm{c},(2)}(f;g) = \sum_{x,q} \nabla_{x,q} f \nabla_{x,q} g,$$
  
(4.6)

with  $\nabla_{x,x+e,v}f = f(\eta^{x,x+e,v}) - f(\eta), \nabla_{x,q}f = f(\eta^{x,q}) - f(\eta)$  and

$$b(x, y, v) = \left(\chi + \frac{1}{2}v \cdot (y - x)\right)\eta(x, v)\left(1 - \eta(y, v)\right).$$

So, if we let  $\phi_0(v) = 1$ ,  $\phi_\beta(v) = v_\beta$  for  $\beta = 1, 2, 3$  and  $\phi_4(v) = \frac{1}{2}|v|^2$ , a straightforward computation leads to the following

$$Y_{1}^{\varepsilon}(t,G) = \varepsilon^{3} \sum_{\alpha=1}^{3} \sum_{\beta,\nu=0}^{4} \sum_{x} \left( \partial_{\alpha} \left( U_{t}^{\varepsilon} G \right)_{\beta} \partial_{\alpha} \left( U_{t}^{\varepsilon} G \right)_{\nu} \right) (\varepsilon x) \\ \times \sum_{v} \left[ b(x,x+e_{\alpha},v) + b(x+e_{\alpha},x,v) \right] \phi_{\beta}(v) \phi_{\nu}(v) + \mathcal{O}(\varepsilon).$$
(4.7)

Therefore

$$\begin{split} \mathbb{E}_{\varepsilon}^{\mu} \big[ Y_{1}^{\varepsilon}(t,G) \big] &= \varepsilon^{3} 2\chi \sum_{\alpha=1}^{3} \sum_{\beta,\nu=0}^{4} \sum_{x} C_{\beta,\nu} \big( \partial_{\alpha} \big( U_{t}^{\varepsilon}G \big)_{\beta} \partial_{\alpha} \big( U_{t}^{\varepsilon}G \big)_{\nu} \big) (\varepsilon x) + \mathcal{O}(\varepsilon) \\ &= -2\chi \big\langle \big( U_{t}^{\varepsilon}G \big), \Delta C \big( U_{t}^{\varepsilon}G \big) \big\rangle_{0} + \mathcal{O}(\varepsilon), \end{split}$$

where  $\Delta$  is the vectorial Laplacian operator defined as  $(\Delta G)_{\beta} = \Delta G_{\beta}$ . Observe that

$$\langle (U_t^{\varepsilon}G), \Delta C(U_t^{\varepsilon}G) \rangle_0 = \langle G, e^{-\frac{t}{\varepsilon}E}Ce^{-\frac{t}{\varepsilon}E^*}\Delta G \rangle_0 = \langle G, Ce^{\frac{t}{\varepsilon}E^*}e^{-\frac{t}{\varepsilon}E^*}\Delta G \rangle_0$$

where we have used that  $EC = -CE^*$ . In conclusion,  $Y_1^{\varepsilon}(t, G)$  converges in  $L^2(\mathbb{P}^{\mu}_{\varepsilon})$  to  $-2\chi \langle G, C\Delta G \rangle_0$ .

We get in the same way

$$Y_{2}^{\varepsilon}(t,G,h) = 2\varepsilon^{3} \sum_{\alpha,\gamma=1}^{3} \sum_{\beta,\nu=0}^{4} \sum_{x} \left( \partial_{\alpha} \left( U_{t}^{\varepsilon}G \right)_{\beta} \partial_{\gamma} \left( U_{t}^{\varepsilon}G \right)_{\nu} \right) (\varepsilon x) \\ \times \sum_{v} \left[ b(x,x+e_{\alpha},v) - b(x+e_{\alpha},x,v) \right] \\ \times \phi_{\beta}(v) \nabla_{x,x+e_{\alpha},v} \Gamma(h_{\gamma}^{\nu}) + O(\varepsilon),$$
(4.8)

where  $\Gamma(h_{\gamma}^{\nu}) = \sum_{x} \tau_{x} h_{\gamma}^{\nu}$ . Since  $\mu$  is invariant for the jump generator of particles with a given velocity, it is easy to check that

$$\mathbb{E}^{\mu}\left[\eta(x,v)\left(1-\eta(x+e_{\alpha},v)\right)\nabla_{x,x+e_{\alpha},v}\Gamma\left(h_{\nu}^{\nu}\right)\right]=0,$$

which implies that the time integral of  $Y_2^{\varepsilon}(t, G, h)$  converges to 0 in  $L^2(\mathbb{P}^{\mu}_{\varepsilon})$  by Lemma 4.1.

The last term  $Y_3^{\varepsilon}(t, G, h)$  is given by

$$\varepsilon^{3} \sum_{\alpha,\gamma=1}^{3} \sum_{\beta,\nu=0}^{4} \sum_{x} (\partial_{\alpha} (U_{t}^{\varepsilon}G)_{\beta} \partial_{\gamma} (U_{t}^{\varepsilon}G)_{\nu}) (\varepsilon x) \\ \times \left[ \sum_{v;|e|=1} b(x, x+e, v) \nabla_{x,x+e,\nu} \Gamma(h_{\alpha}^{\beta}) \nabla_{x,x+e,\nu} \Gamma(h_{\gamma}^{\nu}) + \sum_{q} \nabla_{x,q} \Gamma(h_{\alpha}^{\beta}) \nabla_{x,q} \Gamma(h_{\gamma}^{\nu}) \right].$$

$$(4.9)$$

By using again Lemma 4.1 it is immediate to show that the time integral of  $Y_3^{\varepsilon}$  converges in  $L^2(\mathbb{P}^{\mu}_{\varepsilon})$  to its average that we are going to compute.

Let  $\mathcal{L}_s^{\text{ex}}$  be the symmetric part of  $\mathcal{L}^{\text{ex}}$  in  $L^2(\mu)$ . It is easy to check that for any local functions f and g

$$\sum_{v;|e|=1} \mathbb{E}^{\mu} \big[ \eta(0,v) \big( 1 - \eta(e,v) \big) \nabla_{0,e,v} \Gamma(f) \nabla_{0,e,v} \Gamma(g) \big] = 2 \mathbb{E}^{\mu} \big[ \Gamma(f) \big( -\mathcal{L}_s^{\text{ex}} \big) g \big]$$

and

$$\sum_{q} \mathbb{E}^{\mu} \left[ \nabla_{0,q} \Gamma(f) \nabla_{0,q} \Gamma(g) \right] = 2 \mathbb{E}^{\mu} \left[ \Gamma(f) \left( -\mathcal{L}^{\mathsf{c}} \right) g \right].$$

Therefore

$$\mathbb{E}_{\varepsilon}^{\mu} \left[ Y_{3}^{\varepsilon}(t,G) \right] = 2\varepsilon^{3} \sum_{x} \mathbb{E}_{\varepsilon}^{\mu} \left[ \Gamma \left( \partial \left( U_{t}^{\varepsilon}G \right)(\varepsilon x) \cdot h \right) (-\mathcal{L}_{s}) \left( \partial \left( U_{t}^{\varepsilon}G \right)(\varepsilon x) \cdot h \right) \right]$$

where  $\mathcal{L}_s$  and  $a \cdot b$  were defined after (2.10). Remember that the functions  $h = (h_{\alpha}^{\beta})$  are chosen as the terms of the sequence  $(h_{\alpha}^{(q),\beta})$  in Theorem 2.1. Lemma 2.2 asserts that

$$\lim_{q\to\infty} \mathbb{E}^{\mu} \big[ \Gamma \big( a \cdot h^{(q)} \big) (-\mathcal{L}_s) \big( a \cdot h^{(q)} \big) \big] = a \cdot (\overline{D}C) a.$$

Hence,

$$\mathbb{E}_{\varepsilon}^{\mu} [Y_{3}^{\varepsilon}(t, G, h)] = 2\varepsilon^{3} \sum_{x} \underline{\partial} (U_{t}^{\varepsilon}G) \cdot (\overline{D}C) \underline{\partial} (U_{t}^{\varepsilon}G) + o_{q}(1)$$
$$= -2 \langle U_{t}^{\varepsilon}G, (\overline{D}C) (U_{t}^{\varepsilon}G) \rangle_{0} + o_{q}(1) + O(\varepsilon),$$

where, denoting by  $\overline{D}_{\alpha,\gamma}$  the matrix  $(\overline{D}_{\alpha,\gamma}^{\beta,\nu})_{\beta,\nu=0,\dots,4}$ ,

$$\overline{\mathcal{D}}G = \sum_{\alpha,\gamma=1}^{3} \overline{D}_{\alpha,\gamma} \partial_{\alpha} \partial_{\gamma} G.$$

With the property  $\overline{\mathcal{D}}C = C\overline{\mathcal{D}}^*$ , we get

$$\left\langle U_t^{\varepsilon}G, (\overline{\mathcal{D}}C) \left( U_t^{\varepsilon}G \right) \right\rangle_0 = \left\langle e^{-\frac{t}{\varepsilon}E^*}G, (C\overline{\mathcal{D}}^*)e^{-\frac{t}{\varepsilon}E^*}G \right\rangle_0 = \left\langle G, Ce^{\frac{t}{\varepsilon}E^*}\overline{\mathcal{D}}^*e^{-\frac{t}{\varepsilon}E^*}G \right\rangle_0$$

and by Lemma A.2

$$\lim_{\varepsilon \to 0} \int_{0}^{t} ds \, \exp\left(\frac{s}{\varepsilon}E^{*}\right) \overline{\mathcal{D}}^{*} \exp\left(-\frac{s}{\varepsilon}E^{*}\right) = t \, \pi_{-E^{*}}(\overline{\mathcal{D}}^{*}) = t \pi_{E}(\overline{\mathcal{D}})^{*},$$

so

$$\lim_{\varepsilon \to 0} \mathbb{E}_{\varepsilon}^{\mu} \left[ \int_{0}^{t} ds \, Y_{3}^{\varepsilon}(s, G, h) \right] = -2t \langle G, C \, \pi_{E}(\overline{\mathcal{D}})^{*}G \rangle_{0}.$$

To summarize, we have proved that there exists a random variable  $R^q_{\varepsilon}$  vanishing in  $L^2(\mathbb{P}^{\mu}_{\varepsilon})$  in the limits  $\varepsilon \to 0$  and then  $q \to \infty$  such that

$$M_{2}^{\varepsilon}(t,G,h) = \left(M_{1}^{\varepsilon}(t,G,h)\right)^{2} + 2t\langle G, C\pi_{E}(\overline{\mathcal{D}})^{*}G\rangle_{0} + 2t\chi\langle G, C\Delta G\rangle_{0} + R_{\varepsilon}^{q}$$
$$= \left(M_{1}^{\varepsilon}(t,G,h)\right)^{2} + 2t\langle G, C\pi_{E}(\mathcal{D})^{*}G\rangle_{0} + R_{\varepsilon}^{q}.$$

This completes the proof of Theorem 3.1, once the Boltzmann–Gibbs principle and Lemmas A.2 and A.3 are proved.

#### 5. The Boltzmann–Gibbs principle

Since we closely follow the strategy proposed in [5] to prove Theorem 3.2, we will only focus our attention to the points where non trivial changes are necessary.

One of the ingredients in the proof is the equivalence of ensembles, which is classical for Bernoulli product measures but, as far as we know, is not in our case. We state below a weaker statement which will suffice to our purpose.

For a given chemical potential  $n \in \mathbb{R}^5$ , let  $M(n) = (M_0(n), \ldots, M_4(n))$  be defined as  $M_\beta(n) = \mathbb{E}^{\mu_{L,n}}[I_\beta(\eta_0)]$ . If we put  $A = M(\mathbb{R}^5)$ , it is easy to verify that  $n \mapsto M(n)$  is a  $C^1$  diffeomorphism from  $\mathbb{R}^5$  onto A, in particular the inverse function  $M \mapsto n(M)$  is continuous on A. Given a > 0, we introduce the set  $A^a$  of  $M \in A$  such that,  $|n(M) - \bar{n}| \leq a$ , with  $\bar{n} = (r, 0, 0, 0, \theta)$  the equilibrium chemical potentials. We denote by  $\bar{\mu}_{L,M}$  the grand canonical measure  $\mu_{L,n(M)}$  which satisfies therefore  $E^{\bar{\mu}_{L,M}}[I_\beta(\eta_0)] = M_\beta$  for  $\beta = 0, \ldots, 4$ .

Recall that  $\bar{I}_L^+(\eta) = (\bar{I}_{0,L}^+(\eta), \dots, \bar{I}_{4,L}^+(\eta))$  are the empirical averages of the conserved quantities in  $\Lambda_L$ . For any particle configuration  $\eta$  in  $\Omega_L$ , we call  $\overline{N}_L^v(\eta)$ ,  $v \in \mathcal{V}$ , the average number of particles with velocity v in  $\Lambda_L$ .

Also recall the definition of  $\phi_{\beta}(v)$  before (4.6). Given  $k = (k_v)_{v \in \mathcal{V}}$ , we set  $I_{\beta}(k) = \sum_v \phi_{\beta}(v) k_v$  and  $I^+(k) = (I_0(k), \dots, I_4(k))$ .

LEMMA 5.1 (Equivalence of ensembles). – Let h be a local function. Then there exists a constant c = c(h, a) such that

$$\left|\mathbb{E}^{\mu}\left[h \mid \bar{I}_{L}^{+}=M\right]-E^{\bar{\mu}_{L,M}}[h]\right| \leqslant c\varepsilon^{3}$$

uniformly in  $M \in A^a$ .

*Proof.* – Let  $\ell$  be the number of velocities in  $\mathcal{V}$  and denote by  $\nu_{\alpha}$ ,  $\alpha = (\alpha_v)_{v \in \mathcal{V}}$ , the product measure on  $\Omega_L$  of Bernoulli measures with parameters  $\alpha = (\alpha_v)_v$ , i.e.  $E^{\nu_{\alpha}}[\eta(x, v)] = \alpha_v$ . A straightforward extension of the classical strong equivalence of ensembles asserts that for any local function g,

$$\left|E^{\nu_{\alpha}}\left[g \mid \overline{N}_{L}^{v} = k_{v}, v \in \mathcal{V}\right] - E^{\nu_{k}}[h]\right| \leq C(h)\varepsilon^{3}$$

$$(5.1)$$

uniformly in  $k = (k_v)_{v \in \mathcal{V}} \in B_L = \{0, L^{-3}, ..., 1\}^{\ell}$ .

We first compute the term  $\mathbb{E}^{\mu}[h \mid \overline{I}_{L}^{+} = M]$ . Since this expectation does not depend on the chemical potential (here  $\overline{n}$ ), it is equal to  $E^{\nu_{1/2}}[h \mid \overline{I}_{L}^{+} = M]$  with the obvious abuse of notation 1/2 = (1/2, ..., 1/2). Therefore, from (5.1),

$$\mathbb{E}^{\mu}[h \mid \bar{I}_{L}^{+} = M] = \frac{\sum_{k \in B_{L}, \bar{I}_{L}^{+}(k) = M} \nu_{1/2}(\overline{N}_{L}^{v} = k_{v}, v \in \mathcal{V}) E^{\nu_{k}}[h]}{\sum_{k \in B_{L}, \bar{I}_{L}^{+}(k) = M} \nu_{1/2}(\overline{N}_{L}^{v} = k_{v}, v \in \mathcal{V})} + O(\varepsilon^{3}).$$
(5.2)

Since the particles with different velocities are independent

$$\nu_{1/2}(\overline{N}_L^v = k_v, v \in \mathcal{V}) = \prod_{v \in \mathcal{V}} \nu_{1/2}(\overline{N}_L^v = k_v)$$

and the asymptotics of a single term in the product above is given by the Stirling formula

$$\nu_{1/2}(\overline{N}_L^v = k) = \frac{1}{\sqrt{2\pi\varepsilon^{-3}k(1-k)}} \exp\left[-\varepsilon^{-3}(s(k) + \log 2)\right] \\ \times \left(1 + O\left(\frac{\varepsilon^3}{k(1-k)}\right)\right),$$
(5.3)

where  $s(k) = k \log k + (1 - k) \log(1 - k)$  is the entropy. In particular, if  $(k_v)_v$  belongs to  $B_L^{\delta} := B_L \cap [\delta, 1 - \delta]^{\ell}$  for some small  $\delta > 0$ , then

$$\nu_{1/2}(\overline{N}_L^v = k_v, v \in \mathcal{V}) = \frac{1}{\sqrt{(2\pi\varepsilon^{-3})^\ell \prod_v k_v (1 - k_v)}} \\ \times \exp\left[-\varepsilon^{-3} \sum_v (s(k_v) + \log 2)\right] (1 + \mathcal{O}(\varepsilon^3)).$$

The fact that the entropy is convex suggests to use the Laplace method to derive the asymptotics of both terms in the ratio (5.2). This is the aim of Lemma 5.2 below which is stated in the  $\ell = 1$  case without any constraint on k, nevertheless the generalization to higher dimension with constraints is easy because, up to a linear change of variables  $k \mapsto k'$ , the sums over k in (5.2) with constraints can be written as a sum without constraint over k' in a cube of dimension  $\ell - 5$  (5 is the number of linear conditions  $I^+(k) = M$ ). Therefore, we have

$$\sum_{\substack{k \in B_L^{\delta}, \bar{I}^+(k) = M}} \nu_{1/2} (\overline{N}_L^v = k_v, v \in \mathcal{V}) E^{\nu_k}[h]$$
  
= 
$$\frac{T_{\varepsilon}}{\sqrt{(2\pi \varepsilon^{-3})^{\ell} \prod_v k_v^* (1 - k_v^*)}} \exp\left[-\varepsilon^{-3} \sum_v (s(k_v^*) + \log 2)\right] E^{\nu_k^*}[h] (1 + O(\varepsilon^3)),$$

where  $k^*$  is the minimizer of  $\sum_{v} (s(k_v) + \log 2)$  under the constraints  $k \in [\delta, 1 - \delta]^n$  and  $I^+(k) = M$ ,

$$T_{\varepsilon} = \sum_{\substack{k \in B_L^{\delta}, \bar{I}^+(k) = M \\ |k-k^*| \leq \varepsilon^{3\alpha}}} \exp\left[-\frac{\sum_v s''(k_v^*)}{2}\varepsilon^{-3}(k-k^*)^2\right].$$

with  $0 < \alpha < 1/2$ . Notice that this result holds provided that the minimizer  $k^*$  satisfies  $k^* \in ]\delta, 1 - \delta[\ell]$ , that will be shown below. As a consequence, the ratio (5.2) is equal to

$$E^{\nu_k*}[h](1+\mathcal{O}(\varepsilon^3)),$$

provided that the contributions from "bad" configurations are negligible.

Let  $\kappa$  be the minimizer of  $\sum_{v} (s(k_v) + \log 2)$  under the constraints  $k \in [0, 1]^{\ell}$ ,  $I^+(k) = M$ . From Lagrange optimization theorem,  $\kappa$  has to minimize the function

$$\sum_{v} (s(k_v) + \log 2) - \sum_{\beta=0}^{4} \gamma_{\beta} \sum_{v} (\phi_{\beta}(v)k_v - M_{\beta}),$$

where  $\phi_{\beta}$  have been defined in the line before (4.7) and ( $\gamma_{\beta}$ ) are Lagrangian multipliers. So the minimizer satisfies

$$s'(\kappa_v) = \sum_{\beta=0}^4 \gamma_\beta \phi_\beta(v), \quad v \in \mathcal{V}.$$

Since the derivative of the entropy  $s'(\alpha)$  is equal to the associated chemical potential  $\lambda = \log \frac{\alpha}{1-\alpha}$ , we have  $\nu_{\kappa} = \mu_{L,\gamma}$ ,  $\gamma = (\gamma_0, \dots, \gamma_4)$  but the constraint  $I^+(\kappa) = M$  implies that  $\gamma = n(M)$  that is to say  $\nu_{\kappa} = \bar{\mu}_{L,M}$  and in particular  $E^{\nu_{\kappa}}[h] = E^{\bar{\mu}_{L,M}}[h]$ . Moreover, if  $\lambda_v = \log \frac{\kappa_v}{1-\kappa_v}$  is the chemical potential related to  $\kappa_v$ , then we have

$$\lambda_v = \sum_{\beta=0}^4 \phi_\beta(v) n_\beta(M).$$

From the assumption  $M \in A^a$ , the previous equality implies that we can choose  $\delta > 0$  small enough such that  $\kappa \in ]2\delta, 1-2\delta[^{\ell}$  uniformly in  $M \in A^a$ . Such a choice of  $\delta$  implies that  $k^* = \kappa$ .

So the lemma will be proved if we finally show that the contribution of the "bad" k  $(k \in B_L \setminus [\delta, 1 - \delta]^{\ell})$  inside the sums in the numerator and denominator of the ration (5.2) is irrelevant with respect to the leading term. From Stirling formula (5.3), there is c > 0 such that

$$\sum_{k \notin B_L^\delta, \bar{I}^+(k)=M} \nu_{1/2} \big( \overline{N}_L^v = k_v, v \in \mathcal{V} \big) \leqslant \exp \bigg[ -\varepsilon^{-3} \sum_v \big( s(k_v) + \log 2 \big) - c \log \varepsilon \bigg].$$

From the discussion above, there exists b > 0 such that  $\sum_{v} s(k_v) \ge \sum_{v} s(\kappa_v) + b$ , therefore

$$\sum_{\substack{k \notin B_L^{\delta}, \bar{I}^+(k) = M}} v_{1/2} \left( \overline{N}_L^v = k_v, v \in \mathcal{V} \right)$$
$$\leqslant c \exp\left[ -\varepsilon^{-3} \sum_v \left( s(\kappa_v) + \log 2 \right) \right] \exp\left[ -b\varepsilon^{-3}/2 \right]. \qquad \Box$$

LEMMA 5.2. – Let  $\psi$  and  $\phi$  be smooth functions on [0, 1],  $\psi$  concave,  $\phi$  non negative. Assume that the maximizer  $\theta$  of  $\psi$  is in ]0, 1[, then

$$\sum_{i=0}^{N} \phi\left(\frac{i}{N}\right) \exp\left[N\psi\left(\frac{i}{N}\right)\right] = S_N\left(\alpha, \frac{\psi''(\theta)}{2}\right) \phi(\theta) \exp\left[N\psi(\theta)\right] \left(1 + O\left(\frac{1}{N}\right)\right),$$

with  $\theta$  the maximizer of  $\psi$  and

$$S_N(\alpha, a) = \sum_{|i-N\theta| \leqslant N^{1-\alpha}} \exp\left[a\frac{(i-N\theta)^2}{N}\right], \quad 0 < \alpha < \frac{1}{2}$$

*Proof.* – We start by factorizing the leading term  $\exp[N\psi(\theta)]$  in the sum. For simplicity call

$$U_N(i) = \phi\left(\frac{i}{N}\right) \exp N\left[\psi\left(\frac{i}{N}\right) - \psi(\theta)\right].$$

From the assumption on  $\theta$ , if  $\delta > 0$  there exists a constant  $c(\delta) > 0$  such that

$$\sum_{|i-N\theta|>N\delta} U_N(i) \leqslant \exp\left[-Nc(\delta)\right].$$

Moreover, choosing  $\delta$  small enough ensures that  $\psi(x) - \psi(\theta) \leq -c(x-\theta)^2$  provided that  $|x-\theta| \leq \delta$ , where c > 0 is a constant which will change from line to line. Then, given  $0 < \alpha < \frac{1}{2}$ ,

$$\sum_{N^{1-\alpha} < |i-N\theta| \le N\delta} U_N(i) \le \exp\left[-N^{1-2\alpha}c(\delta)\right].$$
(5.4)

So the main contribution is coming from  $\sum_{|i-N\theta| \leq N^{1-\alpha}} U_N(i)$ . Using Taylor expansion, we see that in this range of *i*'s,

$$U_N(i) = \exp\left[b_0 \frac{j^2}{N}\right] \left(a_0 + a_1 \left(\frac{j}{N}\right) + a_0 b_1 N \left(\frac{j}{N}\right)^3 + O\left(\left(\frac{j}{N}\right)^2\right) + O\left(N \left(\frac{j}{N}\right)^4\right)\right),$$

where  $j = \frac{i}{N} - \theta$ ,  $a_0 = \phi(\theta)$ ,  $a_1 = \phi'(\theta)$ ,  $b_0 = \frac{\psi''(\theta)}{2} < 0$  and  $b_1 = \frac{\psi^{(3)}(\theta)}{6}$ . By the "almost oddness" of j,

$$\sum_{|i-N\theta| \leqslant N^{1-\alpha}} \exp\left[b_0 \frac{j^2}{N}\right] \left(a_1 \left(\frac{j}{N}\right) + a_0 b_1 N \left(\frac{j}{N}\right)^3\right) = \mathcal{O}\left(\frac{1}{N}\right) S_N(\alpha, b_0).$$
(5.5)

We also remark

$$\sum_{|i-N\theta|\leqslant N^{1-\alpha}} \left(\frac{j}{N}\right)^2 \exp\left[b_0 \frac{j^2}{N}\right] \leqslant \frac{c}{N} \sum_{|i-N\theta|\leqslant N^{1-\alpha}} \exp\left[b_0 \frac{j^2}{2N}\right],$$

so that

$$\sum_{|i-N\theta| \leq N^{1-\alpha}} \left(\frac{j}{N}\right)^2 \exp\left[b_0 \frac{j^2}{N}\right] = O\left(\frac{1}{N}\right) S_N\left(\alpha, \frac{b_0}{2}\right).$$
(5.6)

We get in the same way

$$\sum_{|i-N\theta| \leq N^{1-\alpha}} N\left(\frac{j}{N}\right)^4 \exp\left[b_0 \frac{j^2}{N}\right] = O\left(\frac{1}{N}\right) S_N\left(\alpha, \frac{b_0}{2}\right).$$
(5.7)

Finally, comparing  $N^{-1/2}S_N(\alpha, a)$  (a < 0) with the integral of a Gaussian, it is easy to check that  $S_N(\alpha, a) = c(a)\sqrt{N} + O(1)$ , therefore

$$S_N\left(\alpha, \frac{b_0}{2}\right) = \mathcal{O}(1)S_N(\alpha, b_0).$$
(5.8)

Putting together formulas (5.4) to (5.8), the lemma is proved.  $\Box$ 

Even if the equivalence of ensembles that we stated in Lemma 5.1 is weaker than the classical one, it is enough to prove the following result which is actually the only estimate needed in the proof of Boltzmann–Gibbs principle.

COROLLARY 5.3. – If  $h \in \mathcal{G}$  is a local function then

$$\mathbb{E}^{\mu}\big[\big(\mathbb{E}^{\mu}\big[h \mid \bar{I}_{L}^{+} = M\big]\big)^{2}\big] \leqslant c\varepsilon^{6}.$$

*Proof.* – Let  $\hat{h} = \mathbb{E}^{\mu}[h \mid \bar{I}_{L}^{+}], \tilde{h}(M) = E^{\bar{\mu}_{L,M}}[h]$  and consider the decomposition

$$\mathbb{E}^{\mu}[\hat{h}^{2}] \leq 2\mathbb{E}^{\mu}[(\hat{h} - \tilde{h}(\bar{I}_{L}^{+}))^{2}] + 2\mathbb{E}^{\mu}[(\tilde{h}(\bar{I}_{L}^{+}))^{2}].$$

Since h is in  $\mathcal{G}$ , we have

$$\tilde{h}(m) = 0$$
 and  $\frac{\partial \tilde{h}}{\partial M_{\beta}}\Big|_{M=m} = 0,$ 

with  $m_{\beta}$  the equilibrium values of  $I_{\beta}(\eta_0)$ . Therefore

$$\left|\tilde{h}(\bar{I}_{L}^{+})\right| \leq c \sum_{\beta,\nu=0}^{4} \left| \left( \bar{I}_{\beta,L}(\eta) - m_{\beta} \right) \left( \bar{I}_{\nu,L}(\eta) - m_{\nu} \right) \right|.$$

Hence  $\mathbb{E}^{\mu}[(\tilde{h}(\bar{I}_{L}^{+}))^{2}] \leq c\varepsilon^{6}$ .

On the other hand, it results from Lemma 5.1 that for any a > 0

$$\mathbb{E}^{\mu}\left[\left(\hat{h}-\tilde{h}(\bar{I}_{L}^{+})\right)^{2}\right] \leqslant c(a)\varepsilon^{6}+cP^{\mu}\left[I_{L}^{+}\notin A^{a}\right].$$

From the continuity of the function  $M \mapsto n(M)$ , there exists b > 0 such that

$$P^{\mu}[I_L^+ \notin A^a] \leqslant P^{\mu}[|I_L^+ - m| > b].$$

Finally, since  $I_L^+ = \frac{1}{|\Lambda_L|} \sum_x I^+(\eta_x)$  with  $I^+(\eta_x)$  i.i.d. random vectors with finite exponential moments and expectation *m* under  $\mu$ , a large deviation estimate provides

$$P^{\mu}[|I_L^+ - m| > b] \leq \exp(-c\varepsilon^3). \qquad \Box$$

The first result used in [5] (Lemma 4.3) before establishing Boltzmann–Gibbs is a general estimate bounding the equilibrium expectation of the squared time integral of

zero mean functions of Markov processes by their  $\|\cdot\|_{-1}$  norm. More precisely, if *X* is a Markov process on the finite state space  $\mathcal{E}$  with generator *L* and ergodic invariant measure  $\pi$ , then there exists a (universal) constant c > 0 such that for any function  $f:[0, T] \times \mathcal{E} \to \mathbb{R}$  satisfying  $\mathbb{E}^{\pi}[f(t, X_t)] = 0$  for any  $t \in [0, T]$ , we have

$$\mathbb{E}^{\pi}\left[\sup_{0\leqslant t\leqslant T}\left(\int_{0}^{t}ds f(s, X_{s})\right)^{2}\right]\leqslant c\int_{0}^{T}ds\left\|f(s, \cdot)\right\|_{-1}^{2},$$
(5.9)

where

$$\|f\|_{-1}^2 = \sup_g \left\{ 2\langle f, g \rangle + \langle g, L^s g \rangle \right\}$$

and  $L^s$  is the symmetric part of L in  $L^2(\pi)$ .

The next lemma (corresponding to Lemma 4.4 in [5]) is needed to control the remainder terms in the proof of the Boltzmann–Gibbs principle.

LEMMA 5.4. – For any local function  $h \in \mathcal{G}$ , there exists a constant c(h) > 0 such that for any subset B of  $\Lambda_L$ , any smooth function  $G:[0,T] \times \mathbb{T}_3 \to \mathbb{R}$  and  $\varepsilon$  small enough,

$$\mathbb{E}_{\varepsilon}^{\mu} \left[ \sup_{0 \leq t \leq T} \left( \int_{0}^{t} ds \, \varepsilon^{3/2 - 1} \sum_{x \in B} G(s, \varepsilon x) \tau_{x} h(\eta_{s}) \, ds \right)^{2} \right]$$
  
$$\leq c(h)(1 + T) \int_{0}^{T} ds \, \varepsilon^{3} \sum_{x \in B} G^{2}(s, \varepsilon x).$$
(5.10)

*Proof.* – Following [9] (Section 4), we introduce an alternative representation for the particle configuration  $\eta_x = (\eta(x, v))_{v \in \mathcal{V}}$  at site *x*: one can find n(n - 5) numbers  $c_\beta(v)$ ,  $\beta = -n + 5, ..., -1$  and  $v \in \mathcal{V}$ , such that if we put

$$I_{\beta}(\eta_x) = \sum_{v \in \mathcal{V}} c_{\beta}(v) \eta(x, v),$$

then the map  $\eta_x \mapsto (I_\beta(\eta_x))_{\beta=-n+5,...,4}$  is one to one. Moreover the coefficients  $c_\beta(v)$  can be chosen in such a way that the covariances  $\mathbb{E}^{\mu}[I_\beta(\eta_x); I_\nu(\eta_x)], \beta \neq \nu$ , vanish (except  $\beta, \nu \in \{0, 4\}$ ). We also introduce the variables  $\tilde{I}_\beta(\eta_x) = I_\beta(\eta_x) - \mathbb{E}^{\mu}[I_\beta(\eta_x)]$ .

Let  $\mathcal{G}^{\text{ex}}$  be the space of functions *h* such that  $\mathbb{E}^{\mu}[h] = 0$  and  $\sum_{x} \mathbb{E}^{\mu}[h; I_{\beta}(\eta_{x})] = 0$  for any  $\beta = -n + 5, \dots, 4$ . The integration by parts lemma valid for ASEP (Lemma 6.1 in [8]) easily generalizes to a superposition of ASEP.

We now turn to the proof. Fix  $h \in \mathcal{G}$ , we can find coefficients  $(a_{\beta})_{\beta<0}$  such that  $h - \sum_{\beta<0} a_{\beta} \bar{I}_{\beta,L}$  is in  $\mathcal{G}^{\text{ex}}$ , where  $\bar{I}_{\beta,L}(\eta) = |\Lambda_L|^{-1} \sum_x I_{\beta}(\eta_x)$ . Therefore, it is enough to prove the lemma in the case where  $h \in \mathcal{G}^{\text{ex}}$  and in the case where  $h = \bar{I}_{\beta,L}(\eta)$ . The first case is a straightforward generalization of Lemma 4.4 in [5] since the integration by parts formula is valid in  $\mathcal{G}^{\text{ex}}$ . In the second case, denote by  $\hat{I}_{\beta}(\eta_x)$  ( $\beta < 0$  fixed) the conditional expectation of  $I_{\beta}(\eta_x)$  with respect to the empirical averages of the conserved

quantities  $\bar{I}_L^+$ . Then the left hand side of (5.10) is bounded above (up to a factor 2) by the sum of the two terms

$$\mathbb{E}_{\varepsilon}^{\mu} \left[ \sup_{0 \leqslant t \leqslant T} \left( \int_{0}^{t} ds \, \varepsilon^{3/2 - 1} \overline{G}_{s}^{B} \sum_{x} \left( I_{\beta}(\eta_{x}) - \hat{I}_{\beta}(\eta_{x}) \right) ds \right)^{2} \right], \tag{5.11}$$

where  $\overline{G}_s^B := \varepsilon^3 \sum_{x \in B} G(s, \varepsilon x)$ , and

$$\mathbb{E}_{\varepsilon}^{\mu} \left[ \sup_{0 \leq t \leq T} \left( \int_{0}^{t} ds \, \varepsilon^{3/2 - 1} \sum_{x \in B} G(s, \varepsilon x) \hat{I}_{\beta}(\eta_{0}) \, ds \right)^{2} \right].$$
(5.12)

From the inequality (5.9), (5.11) is less than or equal to

$$cV_L(\widetilde{I}_{\beta}(\eta_0), r, \theta) \int_0^T ds \left(\overline{G}_s^B\right)^2$$

and by Corollary 4.6 of [9],  $V(\tilde{I}_{\beta}(\eta_0), r, \theta) = \limsup_L V_L(\tilde{I}_{\beta}(\eta_0), r, \theta) < +\infty$ . So (5.11) is bounded above by

$$c\int_{0}^{T} ds \,\varepsilon^{3} \sum_{x \in B} G^{2}(s, \varepsilon x)$$

(c a positive constant). Finally, by stationarity of  $\mu$ , the term (5.12) is less than

$$\varepsilon^{-5}\mathbb{E}^{\mu}[(\hat{I}_{\beta}(\eta_0))^2]T\int_0^T ds (\overline{G}_s^B)^2.$$

From Corollary 5.3,  $\mathbb{E}^{\mu}[(\hat{I}_{\beta}(\eta_0))^2] \leq c\varepsilon^6$  and (5.12) is going to zero as  $\varepsilon \to 0$ .  $\Box$ 

Finally, Corollary 5.3 and Lemma 5.4 allow to extend straightforwardly the proof of the Boltzmann–Gibbs principle given in Section 4 of [5] to obtain Theorem 3.2.

We conclude this section by pointing out that the arguments for the proof of tightness (Section 5 of [5]) can be easily adapted to our case. Notice that, up to now, we did not need to have the supremum over time inside the expectation in the Boltzmann–Gibbs statement, however it is used in this part to control some terms arising in martingale compensators. So, we can state

THEOREM 5.5. – The family of probability  $(Q^{\varepsilon})_{\varepsilon>0}$  on  $D([0, T], \mathcal{H}_{-k_0})$ , with  $k_0$  an integer:  $k_0 > 2 + d/2$ , is tight since

$$\lim_{M \to \infty} \limsup_{\varepsilon \to 0} Q^{\varepsilon} \left( \sup_{0 \le t \le T} \left\| \xi_t^{\varepsilon} \right\|_{-k_0} > M \right) = 0$$
(5.13)

and for any a > 0

$$\lim_{\delta \to 0} \limsup_{\varepsilon \to 0} Q^{\varepsilon} \left( \sup_{\substack{|s-t| < \delta \\ 0 \leqslant s, t \leqslant T}} \left\| \xi_t^{\varepsilon} - \xi_s^{\varepsilon} \right\|_{-k_0} > a \right) = 0.$$
(5.14)

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# **Appendix A**

Let A be the space of  $n \times n$  matrices with complex entries. A is a Hilbert space under the scalar product

$$(X, Y) = \sum_{1 \leq k, \ell \leq n} \overline{X}_{k,\ell} Y_{k,\ell}.$$

Given a matrix A in A the projector  $\Pi_A$  is defined as the orthogonal projection onto C(A), the commutator space of A

$$\mathcal{C}(A) = \{ M \in \mathcal{A} : [M, A] = 0 \}, \quad [M, A] := MA - AM.$$

LEMMA A.1. – Let A be a diagonalizable matrix,  $Sp(A) \in i\mathbb{R}$ . Then, for any matrix M and t > 0

$$\lim_{\varepsilon \to 0} \frac{1}{t} \int_{0}^{t} ds \, \exp\left(-\frac{s}{\varepsilon}A\right) M \exp\left(\frac{s}{\varepsilon}A\right) = \Pi_{A}(M).$$

*Proof.* – We follow the proof in [7]. Let *P* be a non-singular matrix and *R* a real diagonal matrix such that  $A = P^{-1}iRP$ . Let  $\{S_j, j = 1, ..., m\}$  be a partition of the integers  $\{j = 1, ..., n\}$  such that

$$R_k = R_\ell$$
 if  $k, \ell \in S_j$  for some  $j$ ,  
 $R_k \neq R_\ell$  otherwise,

where  $R_j$ , j = 1, ..., n, are the eigenvalues of R. We define the bar operation in the following way: let  $K = (K_{k,\ell}) \in A$  be

$$K_{k,\ell} = \begin{cases} 1 & \text{if } k, \ell \in S_j \text{ for some } j, \\ 0 & \text{otherwise.} \end{cases}$$

Then  $\overline{M}$ ,  $M \in \mathcal{A}$ , is defined as

$$M_{k,\ell} = K_{k,\ell} M_{k,\ell}$$

Observe that  $\overline{M}$  is the diagonal part of M in the simple case of R with distinct eigenvalues.

We have that

$$\exp\left(-\frac{s}{\varepsilon}A\right)M\exp\left(\frac{s}{\varepsilon}A\right) = P^{-1}\exp\left(-\frac{is}{\varepsilon}R\right)PMP^{-1}\exp\left(\frac{is}{\varepsilon}R\right)P$$

It is proved in [7] that  $P^{-1}\overline{PMP^{-1}P}$  is a projection onto  $\mathcal{C}(A)$ . Hence  $\overline{M} = \Pi_R(M)$  because *R* is diagonal. Moreover,  $\Pi_R(M) = \Pi_A(M)$  because *R* is diagonal. So it is enough to prove that for any *M* 

$$\lim_{\varepsilon \to 0} \frac{1}{t} \int_{0}^{t} ds \, \exp\left(-\frac{is}{\varepsilon}R\right) M \exp\left(\frac{is}{\varepsilon}R\right) = \overline{M}.$$

In [7] it is also shown that for any matrix M there exists a matrix S such that M can be decomposed as

$$M = \overline{M} + [S, R].$$

Since  $\overline{M}$  commutes with *R* 

$$\exp\left(-\frac{is}{\varepsilon}R\right)M\exp\left(\frac{is}{\varepsilon}R\right) = \overline{M} + \exp\left(-\frac{is}{\varepsilon}R\right)[S, R]\exp\left(\frac{is}{\varepsilon}R\right).$$

The second term on the r.h.s gives

$$\left(\exp\left(-\frac{is}{\varepsilon}R\right)[S,R]\exp\left(\frac{is}{\varepsilon}R\right)\right)_{k,\ell}=S_{k,\ell}(R_{\ell}-R_{k})\exp\left(\frac{is}{\varepsilon}(R_{k}-R_{\ell})\right),$$

where  $S = (S_{k,\ell})$  and  $R = (R_{k,\ell}) = (R_k \delta_{k,\ell})$ . As a consequence,

$$\lim_{\varepsilon \to 0} \frac{1}{t} \int_{0}^{t} ds \, \exp\left(-\frac{is}{\varepsilon}R\right) [S, R] \exp\left(\frac{is}{\varepsilon}R\right) = 0. \qquad \Box$$

LEMMA A.2. – Let  $\mathbb{A}$  be a first order differential operator such that its Fourier transform  $\hat{\mathbb{A}}(k)$  satisfies  $\operatorname{Sp}(\hat{\mathbb{A}}(k)) \in i\mathbb{R}$  for any k and let  $\mathbb{D} = \sum_{\alpha,\gamma=1}^{3} D_{\alpha,\gamma} \partial_{\alpha} \partial_{\gamma}$  be a second order differential operator, where  $D = (D_{\alpha,\gamma}) = (D_{\alpha,\gamma}^{\beta,\nu})$  is a positive definite rank 2 tensor. Then there exists a positive definite second order differential operator  $\pi_{\mathbb{A}}(\mathbb{D})$  such that for any G smooth

$$\lim_{\varepsilon \to 0} \left\| \int_{0}^{t} ds \left[ \exp\left(-\frac{s}{\varepsilon} \mathbb{A}\right) \mathbb{D} \exp\left(\frac{s}{\varepsilon} \mathbb{A}\right) - \pi_{\mathbb{A}}(\mathbb{D}) \right] G \right\|_{0} = 0.$$

*Proof.* – Let  $\widehat{\mathbb{D}}(k)$  be the Fourier transform of  $\mathbb{D}$ 

$$\widehat{\mathbb{D}}(k) = -\sum_{\alpha,\gamma=1}^{3} D_{\alpha,\gamma}^{\beta,\nu} k_{\alpha} k_{\gamma} \widehat{G}(k).$$

It is enough to prove that for any t > 0 and for any G smooth there exists a matrix  $\hat{\pi}_{\mathbb{A}}(\mathbb{D})$  such that

$$\lim_{\varepsilon \to 0} \left\| \int_{0}^{t} ds \left[ \exp\left(-\frac{s}{\varepsilon} \hat{\mathbb{A}}\right) \widehat{\mathbb{D}} \exp\left(\frac{s}{\varepsilon} \hat{\mathbb{A}}\right) - \hat{\pi}_{\mathbb{A}}(\mathbb{D}) \right] G \right\|_{0} = 0$$

where  $\|\cdot\|_0$  is the usual norm in  $L^2(\mathbb{T}_3, \mathbb{R}^5)$ . Choosing  $\hat{\pi}_{\mathbb{A}}(\mathbb{D}) = \pi_{\hat{\mathbb{A}}}(\widehat{\mathbb{D}})$ , that is an easy consequence of Lemma A.1 via dominated convergence theorem since, by assumption,  $\hat{\mathbb{A}}$  is diagonalizable with pure complex eigenvalues which implies  $\|\exp(\frac{s}{\varepsilon}\mathbb{A})\|_0 \leq \text{const.}$ Finally, since  $\widehat{\mathbb{D}}$  is positive definite, the same is true for  $\pi_{\hat{\mathbb{A}}}(\widehat{\mathbb{D}})$ .  $\Box$ 

Notice that Lemma A.2 implies that for any  $0 \le s \le t$ ,

$$\lim_{\varepsilon \to 0} \left\| \int_{s}^{t} du \left[ \exp\left(-\frac{u}{\varepsilon} \mathbb{A}\right) \mathbb{D} \exp\left(\frac{u}{\varepsilon} \mathbb{A}\right) - \pi_{\mathbb{A}}(\mathbb{D}) \right] G \right\|_{0} = 0.$$

LEMMA A.3. – Let  $A^{\varepsilon}(s)$ , A be linear operators from  $\mathcal{H}_{k_0+2}$  to  $\mathcal{H}_{k_0}$  such that

$$\sup_{\varepsilon,0\leqslant s\leqslant t} \left\|A^{\varepsilon}(s)\right\|_{k_0+2\to k_0} < \infty$$

and for any  $G \in \mathcal{H}_{k_0+2}$  and  $0 \leq s < t$ 

$$\lim_{\varepsilon \to 0} \left\| \int_{s}^{t} du \left[ A^{\varepsilon}(u) - A \right] G \right\|_{k_{0}} = 0.$$

Then, for any  $G \in \mathcal{H}_{k_0+2}$ 

$$\lim_{\varepsilon \to 0} E_{\varepsilon}^{\mu} \left[ \left( \int_{0}^{t} ds \, \xi^{\varepsilon} \big( s, \left[ A^{\varepsilon}(s) - A \right] G \big) \right)^{2} \right] = 0,$$

where  $\xi_s^{\varepsilon}$  is the fluctuation field.

*Proof.* – We set  $\langle \xi_t^{\varepsilon}, G \rangle = \xi^{\varepsilon}(t, G)$ . Let  $0 = t_0 < t_1 < \cdots < t_{\ell} = t$  be a subdivision of the interval [0, t] of size  $\delta > 0$ . Then

$$\int_{0}^{t} ds \,\xi^{\varepsilon} \big( s, A^{\varepsilon}(s)G \big) = \sum_{i=0}^{\ell-1} \left\langle \xi_{t_i}^{\varepsilon}, \int_{t_i}^{t_{i+1}} ds \, A^{\varepsilon}(s)G \right\rangle + R_1^{\varepsilon}$$

with

$$R_1^{\varepsilon} = \sum_{i=0}^{\ell-1} \int_{t_i}^{t_{i+1}} ds \left\langle \xi_s^{\varepsilon} - \xi_{t_i}^{\varepsilon}, A^{\varepsilon}(s) G \right\rangle.$$

Since

$$\left|R_{1}^{\varepsilon}\right| \leqslant t \sup_{\substack{|s_{1}-s_{2}| \leqslant \delta\\0\leqslant s_{1},s_{2}\leqslant t}} \left\|\xi_{s_{1}}^{\varepsilon}-\xi_{s_{2}}^{\varepsilon}\right\|_{-k_{0}} \sup_{0\leqslant s\leqslant t} \left\|A^{\varepsilon}(s)\right\|_{k_{0}+2 \to k_{0}} \|G\|_{k_{0}+2},$$

it results from tightness (5.14) that for any a > 0,

$$\lim_{\delta \to 0} \limsup_{\varepsilon \to 0} \mathbb{P}^{\varepsilon}_{\mu} (|R_1^{\varepsilon}| > a) = 0.$$

Moreover

$$\int_{0}^{t} ds \left\langle \xi_{s}^{\varepsilon}, A^{\varepsilon}(s) G \right\rangle = \sum_{i=0}^{\ell-1} \left\langle \xi_{t_{i}}^{\varepsilon}, AG \right\rangle (t_{i+1} - t_{i}) + R_{1}^{\varepsilon} + R_{2}^{\varepsilon},$$

with

$$|R_{2}^{\varepsilon}| = \left|\sum_{i=0}^{\ell-1} \int_{t_{i}}^{t_{i+1}} ds \left\langle \xi_{t_{i}}^{\varepsilon}, \left(A^{\varepsilon}(s) - A\right)G\right\rangle\right|$$
$$\leq t \sup_{0 \leq s \leq t} \left\|\xi_{s}^{\varepsilon}\right\|_{-k_{0}} \max_{i} \left\|\int_{t_{i}}^{t_{i+1}} ds \left[A^{\varepsilon}(s) - A\right]G\right\|_{k_{0}}.$$

By assumption

$$\lim_{\varepsilon \to 0} \max_{i} \left\| \int_{t_{i}}^{t_{i+1}} ds \left[ A^{\varepsilon}(s) - A \right] G \right\|_{k_{0}} = 0.$$

So, using tightness (5.13), we get for M > 0 and  $\varepsilon$  small enough

$$\mathbb{P}^{\varepsilon}_{\mu}(|R_{2}^{\varepsilon}| > a) \leqslant \mathbb{P}^{\varepsilon}_{\mu}\left(\sup_{0 \leqslant s \leqslant t} \left\|\xi_{s}^{\varepsilon}\right\|_{-k_{0}} > \frac{a}{tM}\right)$$

which vanishes in the limit  $M \to 0$  after  $\varepsilon \to 0$ . With the same kind of arguments (using tightness again), we get

$$\sum_{i=0}^{\ell-1} \langle \xi_{t_i}^{\varepsilon}, AG \rangle (t_{i+1} - t_i) = \int_0^t ds \, \langle \xi_s^{\varepsilon}, AG \rangle + R_3^{\varepsilon},$$

where

$$\lim_{\delta \to 0} \limsup_{\varepsilon \to 0} \mathbb{P}^{\varepsilon}_{\mu} (|R_{3}^{\varepsilon}| > a) = 0.$$

We have proved so far that  $\int_0^t ds (\xi^{\varepsilon}(s, A^{\varepsilon}(s)G) - \xi^{\varepsilon}(s, AG))$  converges to 0 in  $\mathbb{P}^{\mu}_{\varepsilon}$  probability. To assert that the convergence occurs in  $L^2(P^{\mu})$  it suffices, e.g., to check that

$$\sup_{\varepsilon} \mathbb{E}_{\varepsilon}^{\mu} \left[ \left( \int_{0}^{t} ds \left\langle \xi_{s}^{\varepsilon}, \left( A^{\varepsilon}(s) - A \right) G \right\rangle \right)^{4} \right] < \infty$$

which is clear from the assumptions on the operator  $A^{\varepsilon}(s)$  and A.  $\Box$ 

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