# LASOTA-YORKE MAPS WITH HOLES: CONDITIONALLY INVARIANT PROBABILITY MEASURES AND INVARIANT PROBABILITY MEASURES ON THE SURVIVOR SET 

# APPLICATIONS DE TYPE LASOTA-YORKE À TROU : MESURE DE PROBABILITÉ CONDITIONELLEMENT INVARIANTE ET MESURE DE PROBABILITÉ INVARIANTE SUR L'ENSEMBLE DES SURVIVANTS 

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#### Abstract

Let $T: I \rightarrow I$ be a Lasota-Yorke map on the interval $I$, let $Y$ be a nontrivial subinterval of $I$ and $g^{0}: I \rightarrow \mathbb{R}^{+}$, be a strictly positive potential which belongs to $B V$ and admits a conformal measure $m$. We give constructive conditions on $Y$ ensuring the existence of absolutely continuous (w.r.t. $m$ ) conditionally invariant probability measures to nonabsorption in $Y$. These conditions imply also existence of an invariant probability measure on the set $X_{\infty}$ of points which never fall into $Y$. Our conditions allow rather "large" holes.


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Résumé. - Soient $T: I \rightarrow I$ une application de type Lasota-Yorke sur l'intervalle $I, Y$ un sous intervalle non trivial et $g^{0}: I \rightarrow \mathbb{R}^{+}$un potentiel strictement positif qui admet une mesure conforme $m$. Nous donnons des conditions constructives sur $Y$ qui assurent l'existence d'une mesure de probabilité absolument continue (par rapport à $m$ ), invariante conditionellement à la nonabsorption dans $Y$. Ces conditions impliquent aussi l'existence d'une mesure de probabilité, invariante par $T$ et supportée dans l'ensemble $X_{\infty}$ des points qui ne tombent pas dans le trou. Nos conditions autorisent des trous relativement gros.
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MSC: 37A05; 28D05

[^0]
## Introduction

The notion of conditionally invariant probability measures c.i.p.m. was introduced for countable state Markov chains with an absorbing state in [19]. More precisely, if $\left(U_{n}\right)$ is a Markov chain with law $\mathbb{P}$ and taking values in a countable set $E \cup \partial$, if $\tau_{\partial}=$ $\inf \left\{n \geqslant 0: U_{n}=\partial\right\}$ is the hitting time of $\partial$, then a probability measure $v$ concentrated on $E$, is called a c.i.p.m. (conditioned to stay in $E$ ) if $\mathbb{P}_{\nu}\left\{U_{n} \in A \mid \tau_{\partial}>n\right\}=v(A)$ for every $A \subset E$ and $n \geqslant 0$. It was proven in [10] that geometric absorption was a necessary and sufficient condition for the existence of c.i.p.m. for a wide class of Markov chains. In [17] and later in [6-8] the existence of such measures was investigated for topological Markov chains and Markov expanding dynamical systems with holes. More recently, these questions were also studied for Anosov systems in [4] where small holes are considered and the existence of a c.i.p.m. is obtained by a perturbative argument. General conditions ensuring existence of c.i.p.m. have been given in [9] where it has been proven that $\Phi$-mixing systems satisfying the Gibbs property for some invariant measure $\mu$ admits a c.i.p.m. which is absolutely continuous with respect to $\mu$.

In this article, we are concerned with Lasota-Yorke maps (these systems are in general neither $\Phi$-mixing nor Gibbs).

Let $T: I \rightarrow I$ be a Lasota-Yorke map on the interval $I$, let $Y$ be a nontrivial subinterval of $I$ and $g^{0}: I \rightarrow \mathbb{R}^{+}, \inf g^{0}>0$, be a potential which belongs to BV and admits a conformal measure $m$ (see definition and assumptions below). ${ }^{1}$

Some results have been obtained for such maps with holes, limited to the case in which the potential is given by the Jacobian of the map, in [5] and [1] for very small holes and under some additional geometrical assumption on the holes. Our goal here is on the one hand to find constructive conditions allowing not necessarily small holes and on the other to show that a smallness condition alone suffices.

The plan of the paper is as follows. In Section 1 some general facts are recalled and the main theorems proved in the paper are stated. Section 2 is devoted to obtaining a special type of Lasota-Yorke like inequality that will be the basis for future arguments. Section 3 uses the previous results to establish that the transfer operator is a contraction in an appropriate (projective) metric. From this results the required statistical properties readily follows as is shown in Section 4. Section 5 investigates the Hausdorff dimension of the set of the points that never visit the hole. In Section 6 we investigate many concrete examples and show that the theory so far developed does apply to maps with fairly large holes even in the absence of a Markov structure. Finally, Section 7 points out that if one is concerned only with pertubative results (i.e. rather small holes) then results of the type obtained in the previous sections follow under much more general hypothesis. It should be remarked that, although we do not investigate this explicitly, the size of the holes for which the latter result applies can be (at least in principle) explicitly computed since the perturbation theory we use is constructive.

[^1]
## 1. Statements and results

Let us fix some notation. Let $I \subset \mathbb{R}$ be an interval and let $T: I \rightarrow I$ be a Lasota-Yorke map, i.e. there exists a partition $\mathcal{Z}$ (mod. a finite number of points) of $I$ on subintervals such that $T$ is $C^{1}$ on each $\bar{Z}, Z \in \mathcal{Z}$ and monotonic. Let $\mathcal{Z}^{(n)}$ be the monotonicity partition of $T^{n}$.

Recall that if $f$ is a measurable function on $I$ the variation of $f$ is defined by:

$$
\bigvee f=\sup \left\{\sum_{i=0}^{n-1}\left|f\left(x_{i+1}\right)-f\left(x_{i}\right)\right|\right\}
$$

where the sup is taken over all finite subdivisions $x_{0}<x_{1}<\cdots<x_{n}$ of $I$. If $\bigvee f<\infty$ we say that $f$ has bounded variation and $B V$ denotes the space of functions of bounded variation.

Let $g^{0}: I \rightarrow \mathbb{R}^{+}$, be a strictly positive potential which belongs to $B V$ and admits a conformal measure $m$. By $\mathcal{L}_{0}$ we designate the usual Perron-Frobenius operator (or transfer operator) associated to the dynamics and $g^{0}$. The operator $\mathcal{L}_{0}$ acts on $L^{1}(m)$ and $B V$ :

$$
\begin{equation*}
\mathcal{L}_{0} f(x)=\sum_{T y=x} f(y) g^{0}(y) \tag{1.1}
\end{equation*}
$$

Recall that a measure $m$ is called $g^{0}$-conformal if it satisfies:

$$
\mathcal{L}_{0}^{*} m=c m \quad \text { where } c:=e^{P\left(g^{0}\right)}
$$

and

$$
P\left(g^{0}\right)=\lim _{n \rightarrow \infty} \frac{1}{n} \log \sum_{Z \in \mathcal{Z}^{(n)}} \sup _{Z} g_{n}^{0}
$$

with $g_{n}^{0}(x)=g^{0}(x) \times \cdots \times g^{0}\left(T^{n-1} x\right)$.
Define also $\Theta\left(g^{0}\right)$ to be such that $\log \Theta\left(g^{0}\right):=\lim _{n \rightarrow \infty} \frac{1}{n} \log \sup _{I} g_{n}^{0}$. Our standing assumptions on $g^{0}$ will be the following:

## Condition 0.-

- $\inf g^{0}>0$,
- the potential $g^{0}$ is contracting, i.e., $\Theta\left(g^{0}\right)<P\left(g^{0}\right)$,
- the potential $g^{0}$ belongs to the space $B V$ of functions of bounded variation,
- there exists a $g^{0}$-conformal probability measure $m$.

Remark 1.1. - It is known (see $[12,3,16]$ ) that if $g^{0}$ belongs to $B V$ then so does $g_{n}^{0}$ for all $n \in \mathbb{N}$ and that this together with the contracting condition are sufficient to ensure the existence of a $g^{0}$-conformal nonatomic probability measure provided the partition is generating.

Next, consider a sub-interval $Y \subset I$, the hole. To avoid trivial considerations, we assume $m(Y) m\left(Y^{c}\right) \neq 0, X_{0}$ denotes the complement of the hole: $X_{0}=I \backslash Y . X_{n}$ will
denote the set of points that have not fallen into the hole at time $n: X_{n}=\bigcap_{i=0}^{n} T^{-i} X_{0}$. We will also denote by $g=g^{0} \mathbf{1}_{X_{0}}, g_{n}(x)=g(x) \times \cdots \times g\left(T^{n-1} x\right)$ and $\Theta=\Theta(g)$.

Conditionally invariant probability measures (c.i.p.m. for short) are probability measures $v$ satisfying:

$$
\begin{equation*}
\forall A \in \mathcal{B} \forall n \in \mathbb{Z}_{+} \quad v\left(T^{-n} A \cap X_{n}\right)=v(A) v\left(X_{n}\right), \tag{*}
\end{equation*}
$$

where $\mathcal{B}$ is the Borel $\sigma$-algebra. Condition ( $*$ ) implies that $v$ must be supported in $X_{0}$ and, if $\left.\left.v\left(T^{-1} X_{0}\right)=: \alpha \in\right] 0,1\right]$, that $v\left(X_{n}\right)=\alpha^{n}$, i.e. with respect to $v$, the entrance time into $Y$ has exponential law.

Of course, we are not interested in all c.i.p.m., but on those that have some reasonable properties with respect to the potential $g^{0}$. We will consider only absolutely continuous with respect to $m$ c.i.p.m. (a.c.c.i.p.m. for short). To this end, an useful tool is the transfer operator $\mathcal{L}$ defined by

$$
\begin{equation*}
\mathcal{L}(f)=\mathcal{L}_{0}\left(f \mathbf{1}_{X_{0}}\right) \tag{1.2}
\end{equation*}
$$

The usefulness of $\mathcal{L}$ is readily clarified.

## LEMMA 1.1. - The following two assertions hold true.

(1) Let $v=\mathbf{1}_{X_{0}} h \cdot m$ be a probability measure absolutely continuous with respect to $m$. Then, $v$ is an a.c.c.i.p.m. if and only if $\mathcal{L} h=c \alpha h$ for some $\alpha \in] 0,1]$.
(2) Let $\alpha \in] 0,1]$ and $h \in L^{1}(m)$ be such that $\mathcal{L} h=c \alpha h$, let $\mu$ be a probability measure on I such that $\mathcal{L}^{*} \mu=c \alpha \mu$. Then $\mu$ is supported in $X_{\infty}$ and $\lambda=h \mu$ is $T$-invariant.

Proof. - (1) Let $v=\left(\mathbf{1}_{X_{0}} h\right) m$ and assume $\mathcal{L} h=c \alpha h$. We will make extensive use of the following two easily obtained properties on the iterates of $\mathcal{L}$ :

$$
\begin{align*}
& \forall f \in L^{1}(m), \forall n \in \mathbb{Z}_{+} \quad \mathcal{L}^{n}(f)=\mathcal{L}_{0}^{n}\left(f \mathbf{1}_{X_{n-1}}\right)  \tag{1.3}\\
& \forall f, \varphi \in L^{1}(m), \forall n \in \mathbb{Z}_{+} \quad \int_{X_{0}} \varphi \mathcal{L}^{n} f d m=c^{n} \int_{X_{n}} \varphi \circ T^{n} \cdot f d m \tag{1.4}
\end{align*}
$$

Let $A \in \mathcal{B}$, (1.3), (1.4) give:

$$
\begin{aligned}
v\left(T^{-n} A \cap X_{n}\right) & =\int \mathbf{1}_{A} \circ T^{n} \cdot \mathbf{1}_{X_{n}} \cdot h d m \\
& =\frac{1}{c^{n}} \int_{X_{0}} \mathbf{1}_{A}\left(\mathcal{L}^{n} h\right) d m=\alpha^{n} v(A)
\end{aligned}
$$

In particular, for $A=I$, we get $v\left(X_{n}\right)=\alpha^{n}$ thus, for any $A \in \mathcal{B}, v\left(T^{-n} A \cap X_{n}\right)=$ $v(A) v\left(X_{n}\right)$.

Conversely, assume $v=\left(\mathbf{1}_{X_{0}} h\right) m$ is a a.c.c.i.p.m. Then, by definition of c.i.p.m., there exists $\alpha \in] 0,1]$ such that, for any $A \in \mathcal{B}, v\left(T^{-n} A \cap X_{n}\right)=\alpha^{n} v(A)$. So,

$$
\forall A \in \mathcal{B}, \int_{X_{0}} \mathbf{1}_{A} \cdot \frac{\mathcal{L}^{n} h}{c^{n}} d m=\alpha^{n} \int_{X_{0}} \mathbf{1}_{A} \cdot h d m
$$

we deduce that $\mathcal{L}^{n} h=(c \alpha)^{n} h$.
(2) Let $\mu$ be a probability measure on $I$, assume that $\left.\left.\mathcal{L}^{*} \mu=c \alpha \mu, \alpha \in\right] 0,1\right]$, then

$$
\forall n \in \mathbb{Z}_{+}, \forall f \in L^{1}(m) \quad(c \alpha)^{n} \mu(f)=\int_{I} \mathcal{L}^{n} f d \mu
$$

Assume that $f$ is zero on $X_{n-1}$. Then

$$
(c \alpha)^{n} \mu(f)=\mu\left(\mathcal{L}^{n} f\right)=\mu\left(\mathcal{L}_{0}^{n}\left(\mathbf{1}_{X_{n-1}} f\right)\right)=0
$$

thus $\mu(f)=0$. We deduce that $\mu$ has its support contained in $X_{\infty}$.
The fact that for $h$ such that $\mathcal{L} h=c \alpha h$ the measure $\lambda=h \mu$ is $T$-invariant is a direct computation.

In the next section we will introduce two conditions on the holes (see Condition 1 and Condition 2) under which the following statements hold.

Our main result is the following.
Theorem A. - Assume that Conditions 0, 1 and 2 are satisfied. Then there exists a unique conditionally invariant probability measure $v=h m$ which is absolutely continuous with respect to $m$. There exists a unique probability measure $\mu$ supported in $X_{\infty}$ and which satisfies $\mu(\mathcal{L} f)=\rho \mu(f)$, with $\rho \leqslant c$, for any bounded function $f$. The measure $\lambda=h \mu$ is the only $T$ invariant measure supported on $X_{\infty}$ and absolutely continuous with respect to $\mu$. Moreover, there exists $\kappa<1$ such that for any $f \in B V$ and any $A \in \mathcal{B}$ :

$$
\begin{aligned}
& \left\|\frac{\mathcal{L}^{n} f}{\rho^{n}}-h \mu(f)\right\|_{\infty} \leqslant C t \kappa^{n}\|f\|_{B V}, \\
& \left|m\left(T^{-n} A \mid X_{n-1}\right)-v(A)\right| \leqslant C t \kappa^{n}, \quad \text { and } \quad\left|v\left(A \mid X_{n-1}\right)-\lambda(A)\right| \leqslant C t \kappa^{n} .
\end{aligned}
$$

A subproduct of our Theorem A will be the following result on the Hausdorff dimension of the set $X_{\infty}$ of survivors. For any $0 \leqslant t \leqslant 1$, define

$$
\mathcal{L}_{t} f(x)=\sum_{T y=x}\left(g^{0}\right)^{t}(y) \mathbf{1}_{X_{0}}(y) f(y)
$$

and by $\Theta_{t}, \rho_{t}$ and $P(t)$ the number corresponding to $\Theta, \rho, P$ in the case $t=1$ (see Definition 2.1 for the definition of $\rho$ ).

We will say that $g^{0}$ has the Bounded Distortion property if there exists $C>1$ such that for all $n \in \mathbb{N}, Z \in \mathcal{Z}^{(n)}$ and $x, y \in Z$,

$$
\begin{equation*}
\frac{g_{n}^{0}(x)}{g_{n}^{0}(y)} \leqslant C \tag{1.5}
\end{equation*}
$$

We will say that $T$ has large images if

$$
\begin{equation*}
\inf _{n \in \mathbb{N}} \inf _{Z \in \mathcal{Z}^{(n)}} m\left(T^{n} Z\right)>0 \tag{1.6}
\end{equation*}
$$

We will say that $T$ has large images with respect to $Y$ if for all $n \in \mathbb{N}$, for all $Z \in \mathcal{Z}^{(n)}$, $Z \cap X_{\infty} \neq \emptyset, T^{n}\left(Z \cap X_{n-1}\right) \supset X_{\infty}$.

THEOREM B. - Let $g^{0}=\frac{1}{T^{\prime}}$. Assume that for all $0 \leqslant t \leqslant 1$, Conditions 0,1 and 2 are satisfied. Then, there exists a unique $0<t_{0} \leqslant 1$ such that for $0 \leqslant t<t_{0}, \rho_{t}>1$ and for $1 \geqslant t>t_{0}, \rho_{t}<1$. If $T$ has large images and large images with respect to $Y$ then, $H D\left(X_{\infty}\right)=t_{0}$.

The two theorems above will follow from Theorems 4.4 and 5.1.
As we will see in Section 6, Theorems A, B apply to maps with fairly large holes, in fact this is the case in which they are of interest. If, on the contrary, one is willing to settle for small holes, then it is possible to apply a perturbative approach which yields the following stronger result. ${ }^{2}$

Theorem C. - Assume $g^{0}$ is satisfies Condition 0. If the Lasota-Yorke map $T: I \rightarrow I$ has a unique invariant measure $\mu_{0}$ absolutely continuous with respect the conformal measure $m$, and the systems $\left(I, T, \mu_{0}\right)$ is mixing, then there exists $\varepsilon>0$ such that, for each hole $Y, m(Y) \leqslant \varepsilon$, the conclusions of Theorem A apply.

Theorem C is proven in Section 7. In view of Lemma 1.1, we are led to start our investigation by constructing eigenvalues and eigenfunctions for $\mathcal{L}$. As usually in these topics, a Lasota-Yorke inequality is useful.

## 2. Transfer operator and Lasota-Yorke inequalities with holes

As already mentioned, our point of view is to consider the Transfer operator $\mathcal{L}$ as associated to the potential $g=g^{0} \mathbf{1}_{X_{0}}$, that is a positive, but not strictly positive, weight. Weights of such type, and more general, have been studied in quite some detail. In particular the existence of a quasi-invariant and an invariant measure is proven in [3] under very mild technical assumptions plus the hypothesis that the standard bound $\Theta$ for the essential spectral radius of $\mathcal{L}$ be strictly less than the spectral radius of $\mathcal{L}$. Yet, the arguments used there are nonconstructive (quasi-compactness) and both the problem of when such a condition is satisfied and the problem of the uniqueness of the above measure are not addressed. Here we will restrict ourselves to a slightly less general setting and use a different, constructive, approach patterned after some previous results for strictly positive weights (see [16]). The present approach will allow us, in the following sections, to find explicit conditions for the existence and the properties of the quasi-invariant and invariant probability measures.

First of all we need to impose a condition on our system that insures that all the points would not fall into the hole.

Condition 1. - Let $D_{n}:=\left\{x \in I \mid \mathcal{L}^{n} 1(x) \neq 0\right\}$. We will consider only systems that satisfy

C1: $D_{\infty}:=\bigcap_{n \in \mathbb{N}} D_{n} \neq \emptyset$.
Notice that if $x \notin D_{n}$ then $\mathcal{L}^{n} f(x)=0$ for each $f \in L^{\infty}([0,1])$ since

$$
\left|\mathcal{L}^{n} f(x)\right| \leqslant \mathcal{L}^{n}|f|(x) \leqslant\|f\|_{\infty} \mathcal{L}^{n} 1(x)=0 .
$$

[^2]Accordingly, for each $n \in \mathbb{N}$ holds

$$
\begin{equation*}
\mathcal{L}^{n} f=\mathbf{1}_{D_{n}} \mathcal{L}^{n} f \tag{2.1}
\end{equation*}
$$

Eq. (2.1) in particular means that if $x \notin D_{n}$, then

$$
\mathcal{L}^{n+1} 1(x)=\mathcal{L}^{n}(\mathcal{L} 1)(x)=0
$$

hence $x \notin D_{n+1}$, that is $D_{n+1} \subset D_{n}$.
We can now define the functional

$$
\begin{equation*}
\Lambda(f):=\lim _{n \rightarrow \infty} \inf _{x \in D_{n}} \frac{\mathcal{L}^{n} f(x)}{\mathcal{L}^{n} 1(x)} \tag{2.2}
\end{equation*}
$$

The above definition needs a few comments to convince the reader that it is well posed. To start with notice that Condition 1 implies that the ratio is well defined. Second the existence of the limit is assured by the fact that the sequence is increasing and bounded, indeed

$$
\begin{align*}
\inf _{x \in D_{n+1}} \frac{\mathcal{L}^{n+1} f(x)}{\mathcal{L}^{n+1} 1(x)} & =\inf _{x \in D_{n+1}} \frac{\mathcal{L} \mathbf{1}_{D_{n}}\left[\mathcal{L}^{n} 1 \frac{\mathcal{L}^{n} f}{\mathcal{L}^{n} 1}\right]}{\mathcal{L}^{n+1} 1} \\
& \geqslant \inf _{x \in D_{n}} \frac{\mathcal{L}^{n} f(x)}{\mathcal{L}^{n} 1(x)} \inf _{x \in D_{n+1}} \frac{\mathcal{L} \mathbf{1}_{D_{n}}\left[\mathcal{L}^{n} 1\right]}{\mathcal{L}^{n+1} 1} \\
& =\inf _{x \in D_{n}} \frac{\mathcal{L}^{n} f(x)}{\mathcal{L}^{n} 1(x)} \tag{2.3}
\end{align*}
$$

and

$$
-\|f\|_{\infty} \leqslant \inf _{x \in D_{n}} \frac{\mathcal{L}^{n} f(x)}{\mathcal{L}^{n} 1(x)} \leqslant\|f\|_{\infty}
$$

The relevant properties of the above functional are the following: ${ }^{3}$

- $\Lambda(1)=1$;
- $\Lambda$ is continuous in the $L^{\infty}$ norm;
- $f \geqslant g$ implies $\Lambda(f) \geqslant \Lambda(g)$ (monotonicity);
- $\Lambda(\lambda f)=\lambda \Lambda(f)$ (homogeneity);
- $\Lambda(f+g) \geqslant \Lambda(f)+\Lambda(g)$ (super-additivity);
- $\forall b \in \mathbb{R}, \Lambda(f+b)=\Lambda(f)+b$;
- if for $p \subset I$ there exists $n \in \mathbb{N}$ such that $p \cap X_{n}=\emptyset$, then $\Lambda\left(\mathbf{1}_{p}\right)=0 .{ }^{4}$ All the above follows immediately from the definition.

Remark 2.1. - Note that, at the moment, it is not clear if the functional is linear or not, yet homogeneity and super-additivity imply at least convexity.

Definition 2.1. - Set $\rho=\Lambda(\mathcal{L} 1)$.

[^3]Lemma 2.2. - Under condition C 1 we have $\rho \leqslant c$.
Proof. - Let

$$
\begin{equation*}
\rho_{n}:=\inf _{x \in D_{n}} \frac{\mathcal{L}^{n+1} 1(x)}{\mathcal{L}^{n} 1(x)} \tag{2.4}
\end{equation*}
$$

then by (2.2) $\lim _{n \rightarrow \infty} \rho_{n}=\rho$. Accordingly,

$$
\mathbf{1}_{D_{n}} \mathcal{L}^{n} 1 \rho_{n} \leqslant \mathbf{1}_{D_{n+1}} \mathcal{L}^{n+1} 1
$$

Integrating the above equation with respect to $m$, and remembering (2.1), yields

$$
e^{-P} \rho_{n} \leqslant \frac{m\left(X_{n}\right)}{m\left(X_{n-1}\right)} \leqslant 1
$$

which produces the wanted result by taking the limit $n \rightarrow \infty$.
To continue we need to impose one extra condition on the system. To do so we need some notation. Let $\mathcal{Z}^{(n)}$ be the partition of smoothness (or monotonicity) intervals of $T^{n}$. Next let $\mathcal{A}_{n}$ be the set of finite partitions in intervals $A=\left\{A_{i}\right\}$ such that $\bigvee_{A_{i}} g_{n} \leqslant 2\left\|g_{n}\right\|_{\infty} .{ }^{5}$ Given $n \in \mathbb{N}$ and $A \in \mathcal{A}_{i}$ let $\widehat{\mathcal{Z}}^{(n)}$ be the coarsest partition in intervals among all the ones finer than both $A$ and $\mathcal{Z}^{(n)}$ and enjoying the property that the elements of the partition are either disjoint or contained in $X_{n-1}$. Finally, let

$$
\begin{gathered}
\mathcal{Z}_{*}^{(n)}=\left\{Z \in \widehat{\mathcal{Z}}^{(n)} \mid Z \subset X_{n-1}\right\} \\
\mathcal{Z}_{b}^{(n)}=\left\{Z \in \widehat{\mathcal{Z}}^{(n)} \mid Z \subset X_{n-1} \text { and } \Lambda\left(\mathbf{1}_{Z}\right)=0\right\} \\
\text { and } \quad \mathcal{Z}_{g}^{(n)}=\left\{Z \in \widehat{\mathcal{Z}}^{(n)} \mid Z \subset X_{n-1} \text { and } \Lambda\left(\mathbf{1}_{Z}\right)>0\right\} .
\end{gathered}
$$

As we will see in the proof of Lemma 2.5, the elements of $\mathcal{Z}_{b}^{(n)}$ are the problematic ones and those of $\mathcal{Z}_{g}^{(n)}$ are the good ones. We allow $\mathcal{Z}_{b}^{(n)}$ to be nonempty provided it satisfies the following condition C 2 .

DEFINITION 2.3. - We will call contiguous two elements of $\mathcal{Z}_{*}^{(n)}$ that are either contiguous, in the usual sense, or separated by a connected component of $Y_{n}:=$ $\bigcup_{i=0}^{n-1} T^{-i} Y$.

Condition 2. - We will consider only systems that satisfy the following condition:
C2: There exists constants $K \geqslant 0$, and $\xi \geqslant 1$, such that for each $n \in \mathbb{N}$ there exists $A \in \mathcal{A}_{n}$ such that at most $K \xi^{n}$ elements of $\mathcal{Z}_{b}^{(n)}$ are contiguous. In addition, $\xi \Theta<\rho$.
Note that this implies, in particular $\Theta<\rho$.
Remark 2.2. - Note that condition C 2 implies that there exists $\bar{n} \in \mathbb{N}$ such that $D_{n}=D_{\bar{n}}$ for all $n \geqslant \bar{n}$, since if the latter were false it would follow $\rho=0$.

The following is yet another simple consequence of C 2 .

[^4]Lemma 2.4. - Condition 2 implies that for all $n \in \mathbb{N}, \mathcal{Z}_{g}^{(n)} \neq \emptyset$.
Proof. - Suppose that $\mathcal{Z}_{g}^{(n)}=\emptyset$ for some $n$, then it must be $\mathcal{Z}_{g}^{(m)}=\emptyset$ for all $m \geqslant n$. Assume that $\mathcal{Z}_{g}^{(n)}=\emptyset$, then $\mathcal{Z}_{*}^{(n)}=\mathcal{Z}_{b}^{(n)}$, thus the number of elements in $\mathcal{Z}_{*}^{(n)}$ is smaller than $K \xi^{n}$ (the elements of $\mathcal{Z}_{b}^{(n)}$ must be all contiguous). Then,

$$
\mathcal{L}^{n} 1(x) \leqslant \sum_{Z \in \mathcal{Z}_{*}^{(n)}} \sup g^{(n)} \leqslant \sup g^{(n)} K \xi^{n}
$$

On the other hand, remembering (2.4), we have, for each $x \in D_{\infty}$,

$$
\left|g^{(n)}\right|_{\infty} K \xi^{n} \geqslant \prod_{i=0}^{n-1} \frac{\mathcal{L}^{i+1} 1(x)}{\mathcal{L}^{i} 1(x)} \geqslant \prod_{i=0}^{n-1} \rho_{i}
$$

Next, taking the logarithm of both sides and the limit for $n \rightarrow \infty$, we get

$$
\ln \xi+\ln \Theta \geqslant \lim _{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln \rho_{i}=\ln \rho
$$

(recall that by definition $\rho=\lim \rho_{i}$ ), contrary to condition C 2 .
Under Condition 2 we will show that the cone

$$
\begin{equation*}
\mathcal{C}_{a}:=\{h \in B V \mid h \not \equiv 0 ; h \geqslant 0 ; \bigvee h \leqslant a \Lambda(h)\} \tag{2.5}
\end{equation*}
$$

is strictly invariant for the Transfer operator $\mathcal{L}$.
The first step is to obtain a suitable Lasota-Yorke type inequality.
Lemma 2.5. - For any $\theta \geqslant \Theta \xi, h \in B V$, we have

$$
\bigvee \mathcal{L}^{n} h \leqslant C_{\theta} \theta^{n} \bigvee h+K_{n} \Lambda(|h|)
$$

where $C_{\theta}$ and $K_{n}$ do not depend on $h$.
Proof. - Notice that, if $Z \in \widehat{\mathcal{Z}}^{(n)} \backslash \mathcal{Z}_{*}^{(n)}$, then $\mathcal{L}^{n}\left(h \mathbf{1}_{Z}\right)=0$ for each $h \in B V$, since $Z \cap X_{n-1}=\emptyset$.

We can then write

$$
\mathcal{L}^{n} h=\sum_{Z \in \mathcal{Z}_{*}^{(n)}} \mathcal{L}^{n}\left(\mathbf{1}_{Z} h\right)=\sum_{Z \in \mathcal{Z}_{*}^{(n)}}\left(\mathbf{1}_{Z} g_{n} h\right) \circ T_{Z}^{-n}
$$

Accordingly,

$$
\bigvee \mathcal{L}^{n} h \leqslant \sum_{Z \in \mathcal{Z}_{*}^{(n)}} \bigvee \mathbf{1}_{T^{n} Z}\left(g_{n} h\right) \circ T_{Z}^{-n}
$$

We will compute separately each term of the sum.

$$
\begin{align*}
\bigvee \mathbf{1}_{T^{n}}\left(g_{n} h\right) \circ T_{Z}^{-n} & \leqslant \bigvee_{Z} h g_{n}+2 \sup _{Z}\left|h \cdot g_{n}\right| \\
& \leqslant 3 \bigvee_{Z} h g_{n}+2 \inf _{Z}\left|h \cdot g_{n}\right| \\
& \leqslant 3\left\|g_{n}\right\|_{\infty} \bigvee_{Z} h+3 \sup _{Z}|h| \bigvee_{Z} g_{n}+2 \inf _{Z}\left|h \cdot g_{n}\right| \\
& \leqslant 3\left\|g_{n}\right\|_{\infty} \bigvee_{Z} h+6\left\|g_{n}\right\|_{\infty} \sup _{Z}|h|+2\left\|g_{n}\right\|_{\infty} \inf _{Z}|h| \\
& \leqslant 9\left\|g_{n}\right\|_{\infty} \bigvee_{Z} h+8\left\|g_{n}\right\|_{\infty} \inf _{Z}|h| . \tag{2.6}
\end{align*}
$$

Next, note that if $Z \in \mathcal{Z}_{g}^{(n)}$, then by definition, there exists $\varepsilon_{n}>0$ such that $\inf _{Z \in \mathcal{Z}_{g}^{(n)}} \Lambda\left(\mathbf{1}_{Z}\right) \geqslant 2 \varepsilon_{n}>0$, it is possible to choose $N_{n} \in \mathbb{N}$ such that, for each $x \in D_{N_{n}}$,

$$
\inf _{Z \in \mathcal{Z}_{g}^{(n)}} \frac{\mathcal{L}^{N_{n}} \mathbf{1}_{Z}(x)}{\mathcal{L}^{N_{n}} 1(x)} \geqslant \varepsilon_{n}
$$

Accordingly, for each $x \in D_{N_{n}}, h \in B V$ and $Z \in \mathcal{Z}_{g}^{(n)}$ holds

$$
\mathcal{L}^{N_{n}}\left(|h| \mathbf{1}_{Z}(x)\right) \geqslant \inf _{Z}|h| \mathcal{L}^{N_{n}} \mathbf{1}_{Z}(x) \geqslant \inf _{Z}|h| \varepsilon_{n} \mathcal{L}^{N_{n}} 1(x) .
$$

To deal with the $Z \in \mathcal{Z}_{b}^{(n)}$ we must use condition $C 2$. Note that the elements of $\mathcal{Z}_{g}^{(n)}$ can be separated by, at most, $K \xi^{n}$ elements of $\mathcal{Z}_{b}^{(n)}$. For each $Z \in \mathcal{Z}_{b}^{(n)}$ let $I_{ \pm}(Z)$ be the union of the contiguous elements of $\mathcal{Z}_{b}^{(n)}$ on the left and on the right of $Z$, respectively. Clearly, for each $Z^{\prime} \subset I_{-}(Z)$ ( or $Z^{\prime} \subset I_{+}(Z)$ ), holds

$$
\inf _{Z^{\prime}}|h| \leqslant \inf _{Z}|h|+\bigvee_{I_{-}(Z)} h
$$

Accordingly,

$$
\sum_{Z \in \mathcal{Z}_{b}^{(n)}} \inf _{Z}|h| \leqslant 2 K \xi^{n}\left[\sum_{Z \in \mathcal{Z}_{g}^{(n)}} \inf _{Z}|h|+\bigvee h\right]
$$

We can then conclude

$$
\begin{aligned}
\bigvee \mathcal{L}^{n} h & \leqslant\left\|g_{n}\right\|_{\infty}\left(9+16 K \xi^{n}\right) \bigvee h+8\left(2 K \xi^{n}+1\right)\left\|g_{n}\right\|_{\infty} \varepsilon_{n}^{-1} \sum_{Z \in \mathcal{Z}_{*}^{(n)}} \frac{\mathcal{L}^{N_{n}}|h| \mathbf{1}_{Z}(x)}{\mathcal{L}^{N_{n}} 1(x)} \\
& \leqslant\left\|g_{n}\right\|_{\infty}\left(9+16 K \xi^{n}\right) \bigvee h+8\left(2 K \xi^{n}+1\right)\left\|g_{n}\right\|_{\infty} \varepsilon_{n}^{-1} \frac{\mathcal{L}^{N_{n}}|h|(x)}{\mathcal{L}^{N_{n}} 1(x)}
\end{aligned}
$$

Taking the inf over $x$ in the previous expression and noticing that, by hypothesis, there must exists $C_{\theta}$ such that $\left(9+16 K \xi^{n}\right)\left\|g_{n}\right\|_{\infty} \leqslant C_{\theta} \theta^{n}$ yields the result.

## 3. Transfer operator and invariant cones

Hilbert metric. In this section, we introduce a theory developed by G. Birkhoff [2], which is a very powerful tool in analyzing of the so called positive operators.

We will apply it to study the Perron-Frobenius operator for our maps. This strategy has been first implemented in [11] to estimate the decay of correlations for some random dynamical systems. Then, this strategy had been used by many authors. Let us mention C. Liverani [14] and M. Viana [20] for Anosov and Axiom A diffeomorphisms. They used Birkhoff cones to obtain exponential decay of correlations. We use this technique in a way very close to [15] and [16].

DEFINITION 3.1. - Let $\mathcal{V}$ be a vector space. We will call convex cone a subset $\mathcal{C} \subset \mathcal{V}$ which enjoys the following properties:
(i) $\mathcal{C} \cap-\mathcal{C}=\emptyset$.
(ii) $\forall \lambda>0 \lambda \mathcal{C}=\mathcal{C}$.
(iii) $\mathcal{C}$ is a convex set.
(iv) $\forall f, g \in \mathcal{C} \forall \alpha_{n} \in \mathbb{R} \alpha_{n} \rightarrow \alpha, g-\alpha_{n} f \in \mathcal{C} \Rightarrow g-\alpha f \in \mathcal{C} \cup\{0\}$.

We now define the Hilbert metric on $\mathcal{C}$ :
DEFINITION 3.2. - The distance $d_{\mathcal{C}}(f, g)$ between two points $f, g$ in $\mathcal{C}$ is given by

$$
\begin{aligned}
\alpha(f, g) & =\sup \{\lambda>0 \mid g-\lambda f \in \mathcal{C}\} \\
\beta(f, g) & =\inf \{\mu>0 \mid \mu f-g \in \mathcal{C}\} \\
d_{\mathcal{C}}(f, g) & =\log \frac{\beta(f, g)}{\alpha(f, g)}
\end{aligned}
$$

where we take $\alpha=0$ or $\beta=\infty$ when the corresponding sets are empty.
The distance $d_{\mathcal{C}}$ is a pseudo-metric, because two elements can be at an infinite distance from each other, and it is a projective metric because any two proportional elements have a null distance.

The next theorem, due to G. Birkhoff [2], will show that every positive linear operator is a contraction, provided that the diameter of the image is finite.

THEOREM 3.3. - Let $\mathcal{V}_{1}$ and $\mathcal{V}_{2}$ be two vector spaces, $\mathcal{C}_{1} \subset \mathcal{V}_{1}$ and $\mathcal{C}_{2} \subset \mathcal{V}_{2}$ two convex cone (see definition above) and $L: \mathcal{V}_{1} \rightarrow \mathcal{V}_{2}$ a positive linear operator (which implies $\left.L\left(\mathcal{C}_{1}\right) \subset \mathcal{C}_{2}\right)$. Let $d_{\mathcal{C}_{i}}$ be the Hilbert metric associated to the cone $\mathcal{C}_{i}$. If we denote

$$
\Delta=\sup _{f, g \in L\left(\mathcal{C}_{1}\right)} d_{\mathcal{C}_{2}}(f, g)
$$

then

$$
d_{\mathcal{C}_{2}}(L f, L g) \leqslant \tanh \left(\frac{\Delta}{4}\right) d_{\mathcal{C}_{1}}(f, g) \quad \forall f, g \in \mathcal{C}_{1}
$$

$(\tanh (\infty)=1)$.
Theorem 3.3 alone is not completely satisfactory: given a cone $\mathcal{C}$ and its metric $d_{\mathcal{C}}$, we do not know if $\left(\mathcal{C}, d_{\mathcal{C}}\right)$ is complete. This aspect is taken care by the following lemma, which allows to link the Hilbert metric to a suitable norm defined on $\mathcal{V}$.

Lemma 3.4 [16]. - Let $\|\cdot\|$ be a norm on $\mathcal{V}$ such that

$$
\forall f, g \in \mathcal{V} \quad g-f, g+f \in \mathcal{C} \Rightarrow\|g\| \leqslant\|f\|
$$

and let $\ell: \mathcal{C} \rightarrow \mathbb{R}^{+}$be a homogeneous and order preserving function, i.e.

$$
\begin{array}{ll}
\forall f \in \mathcal{C}, \forall \lambda \in \mathbb{R}^{+} & \ell(\lambda f)=\lambda \ell(f), \\
\forall f, g \in \mathcal{C} & g-f \in \mathcal{C} \Rightarrow \ell(f) \leqslant \ell(g),
\end{array}
$$

then

$$
\forall f, g \in \mathcal{C} \quad \ell(f)=\ell(g)>0 \Rightarrow\|f-g\| \leqslant\left(e^{d_{\mathcal{C}}(f, g)}-1\right) \min (\|f\|,\|g\|)
$$

Remark 3.1. - In the previous lemma, one can choose $\ell(\cdot)=\|\cdot\|$ which fulfills the hypothesis. An interesting case is also when $\ell$ is a linear functional positive on $\mathcal{C}$. However, we are concerned with the possibly nonlinear $\ell=\Lambda$.

Invariant cone. From now on, we fix $\theta \in \mathbb{R}$ such that $\Theta \xi \leqslant \theta<\rho$.
PROPOSITION 3.5. - There exists $n_{*} \in \mathbb{N}$ and $a_{0}>0$ such that, for each $n \geqslant n_{*}$, if $a \geqslant a_{0}$, then the cone $\mathcal{C}_{a}$ is not empty and

$$
\mathcal{L}^{n} \mathcal{C}_{a} \subset \mathcal{C}_{a}
$$

with finite diameter.
Before proving the above proposition we need few auxiliary results.
Lemma 3.6. - For each $n \in \mathbb{N}$ we have

$$
\Lambda\left(\mathcal{L}^{n} 1\right) \geqslant \rho^{n}
$$

Proof. - For each $g \in B V, g \geqslant 0$ and $x \in D_{n+1}$, holds

$$
\frac{\mathcal{L}^{n+1} g(x)}{\mathcal{L}^{n} 1(x)} \geqslant \frac{\mathcal{L}\left[\mathbf{1}_{D_{n}}\left(\frac{\mathcal{L}^{n} g}{\mathcal{L}^{n 1}}\right) \mathcal{L}^{n} 1\right](x)}{\mathcal{L}^{n} 1(x)} \geqslant \frac{\mathcal{L}^{n+1} 1(x)}{\mathcal{L}^{n} 1(x)} \inf _{D_{n}} \frac{\mathcal{L}^{n} g}{\mathcal{L}^{n} 1}
$$

and, taking the inf on $x$ and the limit $n \rightarrow \infty$ we have

$$
\begin{equation*}
\Lambda(\mathcal{L} g) \geqslant \Lambda(\mathcal{L} 1) \Lambda(g) \tag{3.1}
\end{equation*}
$$

The lemma follows by iterating (3.1).
LEMmA 3.7. - There exists $n_{0} \in \mathbb{N}$ and $a_{0} \in \mathbb{R}^{+}$such that for all $a \geqslant a_{0}$ we have

$$
\mathcal{L}^{n} \mathcal{C}_{a} \subset \mathcal{C}_{a / 2} \quad \forall n \geqslant n_{0} \quad \text { and } \quad \mathcal{L}^{n} \mathcal{C}_{a} \subset \mathcal{C}_{2 a C_{\theta}} \quad \forall n \geqslant 0
$$

Proof. - First of all, it is obvious that $h \geqslant 0$ implies $\mathcal{L}^{n} h \geqslant 0$. Next we choose $n_{0} \in \mathbb{N}$, such that for all $n \geqslant n_{0}, C_{\theta}^{2} \theta^{n} \rho^{-n} \leqslant \frac{1}{4}$. Let $h \in \mathcal{C}_{a}$ then for each $n \in \mathbb{N}$ we write $n=k n_{0}+m, m<n_{0}$, and (recall Lemma 2.5)

$$
\begin{align*}
\bigvee \mathcal{L}^{n} h & \leqslant C_{\theta} \theta^{n_{0}} \bigvee \mathcal{L}^{(k-1) n_{0}+m} h+K_{n_{0}} \Lambda\left(\mathcal{L}^{(k-1) n_{0}+m} h\right) \\
& \leqslant C_{\theta}^{k+1} \theta^{n} \bigvee h+\sum_{i=0}^{k-1}\left(C_{\theta} \theta^{n_{0}}\right)^{i} K_{n_{0}} \Lambda\left(\mathcal{L}^{(k-i-1) n_{0}+m} h\right)+C_{\theta}^{k} \theta^{k n_{0}} K_{m} \Lambda(h) \tag{3.2}
\end{align*}
$$

Thus, (use (3.1))

$$
\bigvee \mathcal{L}^{n} h \leqslant\left[\left(a+\frac{K_{m}}{C_{\theta} \theta^{m}}\right) \frac{C_{\theta}^{k+1} \theta^{n}}{\rho^{n}}+\sum_{i=0}^{k-1}\left(\frac{C_{\theta} \theta^{n_{0}}}{\rho^{n_{0}}}\right)^{i} \frac{K_{n_{0}}}{\rho^{n_{0}}}\right] \Lambda\left(\mathcal{L}^{n} h\right)
$$

Let $a_{0} \geqslant \max _{i \leqslant n_{0}} \frac{K_{i}}{C_{\theta} \rho^{i}}$, for $k=0$, the following holds

$$
\bigvee \mathcal{L}^{n} h \leqslant 2 a C_{\theta} \Lambda\left(\mathcal{L}^{n} h\right)
$$

When $k>0$ instead

$$
\bigvee \mathcal{L}^{n} h \leqslant\left[\frac{1}{4}\left(a+\frac{K_{m}}{C_{\theta} \theta^{m}}\right)+2 K_{n_{0}} \rho^{-n_{0}}\right] \Lambda\left(\mathcal{L}^{n} h\right)
$$

Hence, for all $n \geqslant n_{0}$ and $a \geqslant 8 K_{n_{0}} \rho^{-n_{0}}+\max _{i \leqslant n_{0}} \frac{K_{i}}{C_{\theta} \rho^{i}}:=a_{0}$,

$$
\bigvee \mathcal{L}^{n} h \leqslant \frac{a}{2} \Lambda\left(\mathcal{L}^{n} h\right)
$$

The above lemma shows the invariance of the cone but has also many other implications the first of which being the following.

Lemma 3.8. - There exists a constant $B>0$ such that, for each $h \in B V, h \geqslant 0$ and $m \in \mathbb{N}$,

$$
\Lambda\left(\mathcal{L}^{m} 1\right) \Lambda(h) \leqslant \Lambda\left(\mathcal{L}^{m} h\right) \leqslant B \Lambda\left(\mathcal{L}^{m} 1\right) \Lambda(h)
$$

Proof. - The first inequality follows trivially by iterating (3.1). For the second, consider $n, m \in \mathbb{N}$ and $x \in D_{n+m}$, then

$$
\frac{\mathcal{L}^{m+n} h(x)}{\mathcal{L}^{n} 1(x)}=\frac{\mathcal{L}^{m+n} h(x)}{\mathcal{L}^{n+m} 1(x)} \frac{\mathcal{L}^{n+m} 1(x)}{\mathcal{L}^{n} 1(x)} \leqslant \frac{\mathcal{L}^{m+n} h(x)}{\mathcal{L}^{n+m} 1(x)}\left\|\mathcal{L}^{m} 1\right\|_{\infty}
$$

which, by taking the inf on $x$ and the limit $n \rightarrow \infty$ yields

$$
\Lambda\left(\mathcal{L}^{m} h\right) \leqslant\left\|\mathcal{L}^{m} 1\right\|_{\infty} \Lambda(h)
$$

Next, since $1 \in \mathcal{C}_{a}$, Lemma 3.7, implies

$$
\bigvee \mathcal{L}^{m} 1 \leqslant 2 a C_{\theta} \Lambda\left(\mathcal{L}^{m} 1\right)
$$

Sublemma 3.9. - For each $f \in B V$ we have: for all $x$

$$
f(x) \leqslant \Lambda(f)+\bigvee f
$$

Proof. - For $x$ and $y$,

$$
f(x) \leqslant f(y)+\bigvee f
$$

fix $x$, using the properties of $\Lambda$ we get:

$$
f(x) \leqslant \Lambda(f)+\bigvee f
$$

Thus

$$
\left\|\mathcal{L}^{m} 1\right\|_{\infty} \leqslant \Lambda\left(\mathcal{L}^{m} 1\right)+\bigvee \mathcal{L}^{m} 1 \leqslant\left(2 a C_{\theta}+1\right) \Lambda\left(\mathcal{L}^{m} 1\right)
$$

from which the result follows with $B:=2 a C_{\theta}+1$.
LEMMA 3.10. - For each $\varepsilon>0$ there exists $n_{0}$ such that for each $n \geqslant n_{0}$, the partition $\mathcal{Z}^{(n)}$ has the property

$$
\sup _{Z \in \mathcal{Z}^{(n)}} \Lambda\left(\mathbf{1}_{Z}\right) \leqslant \varepsilon
$$

Proof. - Choose $n_{0} \in \mathbb{N}$ such that for all $n \geqslant n_{0}, C_{\theta} \theta^{n} \rho^{-n} \leqslant \varepsilon$, this is possible due to condition C2. Then, for $Z \in \mathcal{Z}^{(n)}$

$$
\mathcal{L}^{n} \mathbf{1}_{Z}(x)=\sum_{y \in T^{-n_{x}}} g_{n}(y) \mathbf{1}_{Z}(y) \leqslant\left\|g_{n}\right\|_{\infty} \leqslant C_{\theta} \theta^{n}
$$

Accordingly, for each $x \in D_{n+m} \subset D_{m}$,

$$
\frac{\mathcal{L}^{n+m} \mathbf{1}_{Z}(x)}{\mathcal{L}^{n+m} 1(x)} \leqslant C_{\theta} \theta^{n} \frac{1}{\frac{\mathcal{L}^{m} \mathcal{L}^{n} 1(x)}{\mathcal{L}^{m} 1(x)}} \leqslant C_{\theta} \theta^{n} \frac{1}{\inf _{z \in D_{m}} \frac{\mathcal{L}^{m} \mathcal{L}^{n} 1(z)}{\mathcal{L}^{m} 1(z)}} .
$$

Taking the infimum with respect to $x$ and the limit $m \rightarrow \infty$, the above relations yields

$$
\begin{equation*}
\Lambda\left(\mathbf{1}_{Z}\right) \leqslant C_{\theta} \theta^{n} \frac{1}{\Lambda\left(\mathcal{L}^{n} 1\right)} \leqslant C_{\theta} \theta^{n} \rho^{-n} \leqslant \varepsilon \tag{3.3}
\end{equation*}
$$

where we have used Lemma 3.6.
LEMMA 3.11. - For each $a \geqslant a_{0}$ there exists $n \in \mathbb{N}$ such that, for each $h \in \mathcal{C}_{a}$ there exists $Z \in \mathcal{Z}_{g}^{(n)}$ such that

$$
\inf _{x \in Z} h(x) \geqslant \frac{1}{4} \Lambda(h) .
$$

Proof. - For each $n, m \in \mathbb{N}, n<m$, we can write ${ }^{6}$

$$
\mathcal{L}^{m} h(x)=\sum_{Z \in \widehat{\mathcal{Z}}^{(n)}} \mathcal{L}^{m}\left(h \mathbf{1}_{Z}\right)(x)=\sum_{Z \in \mathcal{Z}_{*}^{(n)}} \mathcal{L}^{m}\left(h \mathbf{1}_{Z}\right)(x)
$$

[^5]We will then prove the lemma arguing by contradiction. Suppose that the lemma is not true then, since by condition C 2 and Lemma $2.4, \mathcal{Z}_{g}^{(n)} \neq \emptyset$, we have

$$
\begin{aligned}
\mathcal{L}^{m} h(x) & =\sum_{Z \in \mathcal{Z}_{g}^{(n)}} \mathcal{L}^{m}\left(h \mathbf{1}_{Z}\right)(x)+\sum_{Z \in \mathcal{Z}_{b}^{(n)}} \mathcal{L}^{m}\left(h \mathbf{1}_{Z}\right)(x) \\
& \leqslant \sum_{Z \in \mathcal{Z}_{g}^{(n)}} \mathcal{L}^{m} \mathbf{1}_{Z}(x) \frac{\Lambda(h)}{4}+\sum_{Z \in \mathcal{Z}_{g}^{(n)}} \mathcal{L}^{m} \mathbf{1}_{Z}(x) \bigvee_{Z} h+\|h\|_{\infty} \sum_{Z \in \mathcal{Z}_{b}^{(n)}} \mathcal{L}^{m} \mathbf{1}_{Z}(x) \\
& \leqslant \mathcal{L}^{m} 1(x) \frac{\Lambda(h)}{4}+\sum_{Z \in \mathcal{Z}_{g}^{(n)}}\left[\Lambda\left(\mathcal{L}^{m} \mathbf{1}_{Z}\right)+\bigvee \mathcal{L}^{m} \mathbf{1}_{Z}\right] \bigvee_{Z} h+\|h\|_{\infty} \sum_{Z \in \mathcal{Z}_{b}^{(n)}} \mathcal{L}^{m} \mathbf{1}_{Z}(x)
\end{aligned}
$$

where we have used Sublemma 3.9. To proceed notice that if $Z \in \mathcal{Z}_{b}^{(n)}$, then Lemma 3.8 implies

$$
\Lambda\left(\mathcal{L}^{m} \mathbf{1}_{Z}\right) \leqslant B \Lambda\left(\mathcal{L}^{m} 1\right) \Lambda\left(\mathbf{1}_{Z}\right)=0
$$

Hence, inequality (3.2) and Sublemma 3.9 imply

$$
\mathcal{L}^{m} \mathbf{1}_{Z} \leqslant \bigvee \mathcal{L}^{m} \mathbf{1}_{Z} \leqslant 2 C_{\theta}^{m / n_{0}+1} \theta^{m} \leqslant 2 C_{\theta}\left(C^{1 / n_{0}} \theta \rho^{-1}\right)^{m} \Lambda\left(\mathcal{L}^{m} 1\right)
$$

On the other hand, if $Z \in \mathcal{Z}_{g}^{(n)}$, by the same arguments we obtain

$$
\begin{aligned}
\bigvee \mathcal{L}^{m} \mathbf{1}_{Z} & \leqslant 2 C_{\theta}^{\left[m / n_{0}\right]+1} \theta^{m}+2 K_{n_{0}} \rho^{-n_{0}} \Lambda\left(\mathcal{L}^{m} \mathbf{1}_{Z}\right) \\
& \leqslant\left[2 C_{\theta}\left(C_{\theta}^{1 / n_{0}} \theta \rho^{-1}\right)^{m}+2 K_{n_{0}} \rho^{-n_{0}} B \Lambda\left(\mathbf{1}_{Z}\right)\right] \Lambda\left(\mathcal{L}^{m} 1\right)
\end{aligned}
$$

where we have used Lemma 3.8.
Accordingly, setting $\sigma:=C_{\theta}^{1 / n_{0}} \theta \rho^{-1} \leqslant 4^{-1 / n_{0}}$,

$$
\Lambda\left(\mathcal{L}^{m} 1\right) \Lambda(h) \leqslant \Lambda\left(\mathcal{L}^{m} h\right)
$$

and

$$
\begin{aligned}
\Lambda\left(\mathcal{L}^{m} h\right) \leqslant & \Lambda\left(\mathcal{L}^{m} 1\right) \frac{\Lambda(h)}{4}+\sum_{Z \in \mathcal{Z}_{g}^{(n)}}\left[B \Lambda\left(\mathbf{1}_{Z}\right)+2 C_{\theta} \sigma^{m}+2 K_{n_{0}} \rho^{-n_{0}} B \Lambda\left(\mathbf{1}_{Z}\right)\right] \bigvee_{Z} h \Lambda\left(\mathcal{L}^{m}\right) \\
& +\sum_{Z \in \mathcal{Z}_{b}^{(n)}} 2 C_{\theta} \sigma^{m}\|h\|_{\infty} \Lambda\left(\mathcal{L}^{m} 1\right)
\end{aligned}
$$

Dividing the above inequalities by $\Lambda\left(\mathcal{L}^{m} 1\right)$ and taking the limit $m \rightarrow \infty$ yields the announced contradiction

$$
\begin{aligned}
\Lambda(h) & \leqslant \frac{\Lambda(h)}{4}+\sum_{Z \in \mathcal{Z}_{g}^{(n)}} B\left(2 K_{n_{0}} \rho^{-n_{0}}+1\right) \Lambda\left(\mathbf{1}_{Z}\right) \bigvee_{Z} h \\
& \leqslant \frac{\Lambda(h)}{4}+B\left(2 K_{n_{0}} \rho^{-n_{0}}+1\right) \bigvee h \sup _{Z \in \mathcal{Z}_{g}^{(n)}} \Lambda\left(\mathbf{1}_{Z}\right) \\
& \leqslant\left[\frac{1}{4}+a B\left(2 K_{n_{0}} \rho^{-n_{0}}+1\right) \sup _{Z \in \mathcal{Z}_{g}^{(n)}} \Lambda\left(\mathbf{1}_{Z}\right)\right] \Lambda(h) \leqslant \frac{1}{2} \Lambda(h)
\end{aligned}
$$

where we have chosen $n$ large enough and we have used Lemma 3.10.
We are now ready to go back to the main result of this section.
Proof of Proposition 3.5. - We start by observing that $h \in \mathcal{C}_{a}$ implies $\Lambda(h)>0$, otherwise $h$ would be constant and such a constant would be zero.

Secondly note that, if $\mathcal{K}_{\varepsilon}:=\left\{h \in B V \mid\|h\|_{\infty}<\varepsilon, \bigvee h<\varepsilon\right\}$, then $1+\mathcal{K}_{\varepsilon} \subset \mathcal{C}_{a}$, for $\varepsilon$ sufficiently small. That is, $\mathcal{C}_{a}$ contains an open set and thus has nonempty interior.

Invariance has been already proved in Lemma 3.7, to obtain finite diameter choose $n \in \mathbb{N}$ so that Lemma 3.11 applies. For each $h \in \mathcal{C}_{a}$ there exists $Z \in \mathcal{Z}_{g}^{(n)}$ such that, for each $x \in D_{m}$,

$$
\mathcal{L}^{m} h(x) \geqslant \frac{1}{4} \Lambda(h) \inf _{D_{m}} \frac{\mathcal{L}^{m} \mathbf{1}_{Z}}{\mathcal{L}^{m} 1} \inf _{D_{m}} \mathcal{L}^{m} 1
$$

To conclude just chose $m$ so large that, for each $Z \in \mathcal{Z}_{g}^{(n)}$ holds

$$
\inf _{D_{m}} \frac{\mathcal{L}^{m} \mathbf{1}_{Z}}{\mathcal{L}^{m} 1} \geqslant \frac{\Lambda\left(\mathbf{1}_{Z}\right)}{2}
$$

and notice that $\inf _{D_{m}} \mathcal{L}^{m} 1>0$ since $\inf g>0$. We get for $m$ large enough,

$$
\inf \mathcal{L}^{m} h \geqslant \frac{\Lambda(h)}{4} \cdot \inf _{Z \in \mathcal{Z}_{g}^{(n)}} \frac{\Lambda\left(\mathbf{1}_{Z}\right)}{2} \cdot \inf \mathcal{L}^{m} 1
$$

and, using Sublemma 3.9 and Lemma 3.8,

$$
\sup \mathcal{L}^{m} h \leqslant \Lambda\left(\mathcal{L}^{m} h\right)+\bigvee \mathcal{L}^{m} h \leqslant\left[B \Lambda\left(\mathcal{L}^{m} 1\right)+\frac{a}{2}\right] \Lambda(h)
$$

Set $\inf _{Z \in \mathcal{Z}_{g}^{(n)}} \frac{\Lambda\left(\mathbf{1}_{Z}\right)}{2}:=A$. We get (see [15], Lemma 3.5 for the details) that:

$$
\operatorname{diam}_{\mathcal{C}_{a}} \mathcal{L}^{m}\left(\mathcal{C}_{a}\right) \leqslant 2 \log \left[\frac{\max \left(\frac{3}{2}, B \Lambda\left(\mathcal{L}^{m} 1\right)+\frac{a}{2}\right)}{\min \left(\frac{1}{2}, \frac{A \inf \mathcal{L}^{m} 1}{4}\right)}\right]<\infty
$$

## 4. Escape rates and invariant measure

LEMMA 4.1. - There exists a unique $h_{*} \in \mathcal{C}_{a}$ and $\lambda \geqslant \rho$, such that $\mathcal{L} h_{*}=\lambda h_{*}$, moreover $\operatorname{supp}\left(h_{*}\right)=D_{\infty}$.

Proof. - By standard arguments it follows from Theorem 3.3, Lemma 3.4 and Proposition 3.5 that, for each $g \in \mathcal{C}_{a}, \frac{\mathcal{L}^{n} g}{\Lambda\left(\mathcal{L}^{n} g\right)}$ is a Cauchy sequence in $L^{\infty}$. This means that for each $g \in \mathcal{C}_{a}$ there exists $h_{g} \in \mathcal{C}_{a}$ such that $\frac{\mathcal{L}^{n} g}{\Lambda\left(\mathcal{L}^{n} g\right)} \rightarrow h_{g}$. In addition, there must exist $\lambda_{g}>0$ such that $\mathcal{L} h_{g}=\lambda_{g} h_{g}$. In fact, since $\frac{\Lambda\left(\mathcal{L}^{n+1} g\right)}{\Lambda\left(\mathcal{L}^{n} g\right)} \in[\rho, B \rho]$, by Lemma 3.8, there exists a convergent subsequence $\left\{n_{j}\right\}$, let $\lambda_{g}:=\lim _{j \rightarrow \infty} \frac{\Lambda\left(\mathcal{L}^{n_{j}+1} g\right)}{\Lambda\left(\mathcal{L}^{n} g\right)}$. Thus

$$
\mathcal{L} h_{g}=\lim _{j \rightarrow \infty} \frac{\mathcal{L}^{n_{j}+1} g}{\Lambda\left(\mathcal{L}^{n_{j}} g\right)}=\lim _{j \rightarrow \infty} \frac{\mathcal{L}^{n_{j}+1} g}{\Lambda\left(\mathcal{L}^{n_{j}+1} g\right)} \lim _{j \rightarrow \infty} \frac{\Lambda\left(\mathcal{L}^{n_{j}+1} g\right)}{\Lambda\left(\mathcal{L}^{n_{j}} g\right)}=\lambda_{g} h_{g}
$$

We will show now that given $f, g \in \mathcal{C}_{a}$ we have $h_{f}=h_{g}=h_{*}$

$$
\begin{aligned}
\left\|h_{f}-h_{g}\right\|_{\infty} & \leqslant\left(e^{d_{\mathcal{C}_{a}}\left(h_{f}, h_{g}\right)}-1\right)\left\|h_{f}\right\|_{\infty} \\
& \leqslant\left(e^{d_{\mathcal{C}_{a}}\left(\mathcal{L}^{n} h_{f}, \mathcal{L}^{n} h_{g}\right)}-1\right)\left\|h_{f}\right\|_{\infty}
\end{aligned}
$$

which goes to zero when $n$ goes to infinity. This implies $\lambda_{g}=\lambda_{h}:=\lambda$ and $\mathcal{L}\left(h_{*}\right)=\lambda h_{*}$, as well. The claimed relation from $\rho$ and $\lambda$ follows from the following chain of inequalities

$$
\begin{aligned}
\Lambda\left(\mathcal{L} h_{*}\right) & =\lim _{n \rightarrow \infty} \inf _{D_{n}} \frac{\mathcal{L}^{n+1} h_{*}}{\mathcal{L}^{n} 1}=\lim _{n \rightarrow \infty} \inf _{D_{n+1}} \frac{\mathcal{L}^{n+1} h_{*}}{\mathcal{L}^{n} 1} \\
& \geqslant \lim _{n \rightarrow \infty} \inf _{D_{n+1}} \frac{\mathcal{L}^{n+1} h_{*}}{\mathcal{L}^{n+1} 1} \inf _{D_{n}} \frac{\mathcal{L}^{n+1} 1}{\mathcal{L}^{n} 1}=\Lambda\left(h_{*}\right) \rho,
\end{aligned}
$$

where we have used twice Remark 2.2. Finally, since $\Lambda\left(h_{*}\right)>0$, it follows that $\left.\mathcal{L}^{n} h_{*}\right|_{D_{\infty}}>0$ which implies $\left.h_{*}\right|_{D_{\infty}}>0$.

Lemma 4.2. - The functional $\Lambda$ (restricted to $B V$ ) is linear, positive, and enjoys the property $\Lambda(\mathcal{L} f)=\lambda \Lambda(f)$ for all $f \in B V$. Moreover, $\lambda=\rho$.

Proof. - Let $f \in \mathcal{C}_{a}$. For all integers $n, k$ and $x \in D_{\infty}$

$$
\begin{array}{cccc}
\frac{\mathcal{L}^{n+k} f(x)}{\mathcal{L}^{n} f(x)} & =\frac{\mathcal{L}^{n+k} f(x)}{\Lambda\left(\mathcal{L}^{n+k} f\right)} & \frac{\Lambda\left(\mathcal{L}^{n+k} f\right)}{\Lambda\left(\mathcal{L}^{n} f\right)} & \frac{\Lambda\left(\mathcal{L}^{n} f\right)}{\mathcal{L}^{n} f(x)} \\
\downarrow & \downarrow & \downarrow & \downarrow \\
\lim _{n \rightarrow \infty} \frac{\mathcal{L}^{n+k} f(x)}{\mathcal{L}^{n} f(x)} & = & h_{*}(x) & \lambda^{k} \\
& & h_{*}(x)^{-1}
\end{array}
$$

So

$$
\lim _{n \rightarrow \infty} \sup _{D_{\infty}}\left|\frac{\mathcal{L}^{n+k} f(x)}{\mathcal{L}^{n} f(x)}-\lambda^{k}\right|=0
$$

But

$$
\begin{aligned}
\sup _{D \infty}\left|\frac{\mathcal{L}^{n} f}{\mathcal{L}^{n} 1}-\frac{\mathcal{L}^{n+k} f}{\mathcal{L}^{n+k} 1}\right| & \leqslant \sup _{D \infty} \frac{\mathcal{L}^{n+k} f}{\mathcal{L}^{n+k} 1}\left|\frac{\mathcal{L}^{n} f}{\mathcal{L}^{n+k} f} \frac{\mathcal{L}^{n+k} 1}{\mathcal{L}^{n} 1}-1\right| \\
& \leqslant\|f\|_{\infty} \sup _{D_{\infty}}\left|\frac{\mathcal{L}^{n} f}{\mathcal{L}^{n+k} f} \frac{\mathcal{L}^{n+k} 1}{\mathcal{L}^{n} 1}-1\right|
\end{aligned}
$$

and since the sequences $\frac{\mathcal{L}^{n+k} f}{\mathcal{L}^{n} f}$ and $\frac{\mathcal{L}^{n+k} 1}{\mathcal{L}^{n} 1}$ have the same limit $\lambda^{k},\left.\frac{\mathcal{L}^{n} f}{\mathcal{L}^{n} 1}\right|_{D_{\infty}}$ is a Cauchy sequence, hence converges to a function $v_{f}$. Moreover, if we take two points $x, y \in D_{\infty}$, we have

$$
\begin{aligned}
\left|v_{f}(x)-v_{f}(y)\right| & =\lim _{n \rightarrow \infty}\left|\frac{\mathcal{L}^{n} f}{\mathcal{L}^{n} 1}(x)-\frac{\mathcal{L}^{n} f}{\mathcal{L}^{n} 1}(y)\right| \\
& =\lim _{n \rightarrow \infty}\left|\frac{\mathcal{L}^{n} f}{\mathcal{L}^{n} 1}(y)\right| \cdot\left|\frac{\mathcal{L}^{n} f(x) \mathcal{L}^{n} 1(y)}{\mathcal{L}^{n} 1(x) \mathcal{L}^{n} f(y)}-1\right| \\
& \leqslant\|f\|_{\infty} \limsup _{n \rightarrow \infty}\left(e^{d_{\mathcal{C}_{+}}\left(\mathcal{L}^{n} f, \mathcal{L}^{n} 1\right)}-1\right)
\end{aligned}
$$

$$
\leqslant\|f\|_{\infty} \lim _{n \rightarrow \infty}\left(e^{d_{\mathcal{C}_{a}}\left(\mathcal{L}^{n} f, \mathcal{L}^{n} 1\right)}-1\right)=0,
$$

where $\mathcal{C}_{+}:=\{h \in B V \mid h \geqslant 0\}$. Therefore, $v_{f}(x)=\Lambda(f)$ for all $x \in D_{\infty}$. Hence, $\Lambda(f)=$ $\lim _{n \rightarrow \infty} \frac{\mathcal{L}^{n} f}{\mathcal{L}^{n} 1}$ for all $f \in \mathcal{C}_{a}$. Nevertheless, if $f \in B V$, the function $\left(f+a^{-1} \bigvee f-\inf f\right) \in$ $\mathcal{C}_{a}$, so $\Lambda(f)=\lim \frac{\mathcal{L}^{n} f}{\mathcal{L}^{n_{1}}}$ for all $f \in B V$. Clearly, $\Lambda$ is linear by the linearity of the limit.

Next, as $\mathcal{L} f \in B V$, we know that

$$
\Lambda(\mathcal{L} f)=\lim _{n \rightarrow \infty} \frac{\mathcal{L}^{n+1} f}{\mathcal{L}^{n} 1}=\lim _{n \rightarrow \infty} \frac{\mathcal{L}^{n+1} f}{\mathcal{L}^{n+1} 1} \frac{\mathcal{L}^{n+1} 1}{\mathcal{L}^{n} 1}=\Lambda(f) \Lambda(\mathcal{L} 1)=\rho \Lambda(f)
$$

But then $\rho=\lambda$ is obtained by taking $f=h_{*}$. Notice that all the convergences take place at an exponential rate.

LEMMA 4.3. - The functional $\Lambda$ can be interpreted as a non-atomic measure $\mu$, i.e.

$$
\Lambda(f)=\int f d \mu \quad \forall f \in B V(I, m)
$$

In addition, $\operatorname{supp} \mu \subset X_{\infty}$ and the measure $h_{*} \mu$ is $T$-invariant.
Proof. - Clearly, $\Lambda$ can be extended to all continuous functions since it is continuous in the sup norm and continuous functions can be uniformly approximated by bounded variation functions. Hence by the Riesz theorem there exists a measure $\mu$ such that $\Lambda(f)=\mu(f)$ on each continuous function. Lemma 3.10 implies immediately that the measure $\mu$ is nonatomic. Moreover it must agree with $\Lambda$ on the characteristic function of each interval. Indeed, let $J$ be an interval, since $\mu$ is a Borel measure, for each $\varepsilon>0$ there exists a larger open interval $\tilde{J}$ such that $\mu(\tilde{J})-\mu(J) \leqslant \varepsilon$ moreover Lemma 3.10 implies that $\tilde{J}$ can be chosen so that $\Lambda\left(\mathbf{1}_{\tilde{J}}-\mathbf{1}_{J}\right) \leqslant \varepsilon$. Thus, choosing a continuous function $f$ such that $\mathbf{1}_{J} \leqslant f \leqslant \mathbf{1}_{\tilde{J}}$, holds ${ }^{7}$

$$
\begin{aligned}
& \Lambda\left(\mathbf{1}_{J}\right)-\mu\left(\mathbf{1}_{J}\right) \leqslant \Lambda(f)-\mu(f)+\Lambda\left(\mathbf{1}_{\tilde{J}}-\mathbf{1}_{J}\right) \leqslant \varepsilon \\
& \mu\left(\mathbf{1}_{J}\right)-\Lambda\left(\mathbf{1}_{J}\right) \leqslant \mu(f)-\Lambda(f)+\mu\left(\mathbf{1}_{\tilde{J}}-\mathbf{1}_{J}\right) \leqslant \varepsilon
\end{aligned}
$$

Since a function in $B V$ can be uniformly approximated by a finite linear combination of characteristic functions of intervals it follows that $\mu(f)=\Lambda(f)$ for each function of bounded variation. The conclusion of the lemma follows from Lemma 1.1.

In conclusion, we have proved the following result.
THEOREM 4.4. - Assume $g^{0}$ is a contracting potential which belongs to BV. Assume that Condition 1 and Condition 2 are satisfied. Then there exists a unique conditionally invariant probability measure $v=h m$ which is absolutely continuous with respect to $m$. There exists a unique probability measure $\mu$ whose support is contained in $X_{\infty}$ and which satisfies $\mu(\mathcal{L} f)=\rho \mu(f)$ for any bounded function $f$. Moreover, there exists $\kappa<1$ such that for any $f \in B V$ and any $A \subset I$ :

$$
\left\|\frac{\mathcal{L}^{n} f}{\rho^{n}}-h \mu(f)\right\|_{\infty} \leqslant C t \kappa^{n}\|f\|_{B V} \quad \text { and } \quad\left|m\left(T^{-n} A \mid X_{n-1}\right)-v(A)\right| \leqslant C t \kappa^{n}
$$

[^6]
## 5. The Hausdorff dimension of $X_{\infty}$

In this section we assume that $T$ is uniformly expanding, i.e. $\inf \left|T^{\prime}\right|>1$. We remark that this implies that the partition $\mathcal{Z}^{(n)}$ is generating. Throughout this section, $g^{0}=1 /\left|T^{\prime}\right|$. For $t \geqslant 0$, let $\mathcal{L}_{t}$ be the transfer operator with hole associated to $\left(g^{0}\right)^{t}$, i.e.:

$$
\mathcal{L}_{t} f(x)=\sum_{T y=x}\left(g^{0}\right)^{t}(y) \mathbf{1}_{X_{0}}(y) f(y),
$$

and let $\Theta_{t}, \rho_{t}$ and $P(t)$ be the numbers corresponding to $\Theta, \rho, P$, in case $t=1$. Recall that in the case $g^{0}=1 /\left|T^{\prime}\right|, P=P\left(g^{0}\right)=0$.

THEOREM 5.1. - Let $g^{0}=1 /\left|T^{\prime}\right|$. Assume that for all $0 \leqslant t \leqslant 1$, Conditions 0,1 and 2 are satisfied. Then, there exists a unique $0<t_{0} \leqslant 1$ such that for $0 \leqslant t<t_{0}, \rho_{t}>1$ and for $1>t>t_{0}, \rho_{t}<1$. If $T$ has large images and large images with respect to $Y$ then, $H D\left(X_{\infty}\right)=t_{0}$.

Proof. - The hypothesis of Theorem 5.1 allow us to apply Theorem A to the operators $\mathcal{L}_{t}$ for all $0 \leqslant t \leqslant 1$. Let us denote by $\mu_{t}$ the conformal measure associated to $g^{t}=$ $\left(g^{0}\right)^{t} \cdot \mathbf{1}_{X_{0}}\left(\right.$ i.e. $\left.\mathcal{L}_{t}^{*} \mu_{t}=\rho_{t} \mu_{t}\right)$.

The application $t \mapsto \rho_{t}$ is strictly decreasing. Indeed, remark that: for all $x \in I$,

$$
\mathcal{L}_{t}^{n} 1(x) \leqslant \sup g_{n}^{t-t^{\prime}} \mathcal{L}_{t^{\prime}}^{n} 1(x)
$$

taking the power $1 / n$ and the limit gives: $\rho_{t} \leqslant \Theta^{t-t^{\prime}} \cdot \rho_{t^{\prime}}$ so that $\rho_{t}<\rho_{t^{\prime}}$ provided $t>t^{\prime}$ (recall that $g<1$ and remark that Theorem A implies: $\lim \left(\mathcal{L}_{t}^{n} 1(x)\right)^{1 / n}=\rho_{t}$ for all $x$ ). Moreover, $\rho_{1} \leqslant e^{P(1)}=1$ (see Lemma 2.2), so there exists a unique number $0 \leqslant t_{0} \leqslant 1$ such that for $1>t>t_{0}, \rho_{t}<1$ and for $0 \leqslant t<t_{0}, \rho_{t}>1$.

The following lemma is a direct consequence of the bounded distortion and large images hypothesis.

Lemma 5.2. - Assume that $g^{0}=1 / T^{\prime}$. Assume that for all $0 \leqslant t \leqslant 1$, Conditions 0 , 1 and 2 are satisfied. For all $0 \leqslant t \leqslant 1$, there exists $K>0$, such that for all $n \in \mathbb{N}$ and $Z \in \mathcal{Z}^{(n)}$, if $\mu_{t}(Z)>0$ then for all $x \in Z$,

$$
\begin{equation*}
K^{-1} \leqslant \frac{\left(g_{n}^{0}\right)^{t}(x)}{\rho_{t}^{n} \mu_{t}(Z)} \quad \text { and } \quad K^{-1} \leqslant \frac{g_{n}^{0}(x)}{m(Z)} \tag{5.1}
\end{equation*}
$$

If moreover $T$ has large images and large images with respect to $Y$ then

$$
\begin{equation*}
\frac{\left(g_{n}^{0}\right)^{t}(x)}{\rho_{t}^{n} \mu_{t}(Z)} \leqslant K \quad \text { and } \quad \frac{g_{n}^{0}(x)}{m(Z)} \leqslant K \tag{5.2}
\end{equation*}
$$

where $m$ is the Lebesgue measure.
Proof. - First of all, we remark that the large images with respect to $Y$ property implies that for all $0 \leqslant t \leqslant 1$, the support of $\mu_{t}$ is $X_{\infty}$. So, $\mu_{t}(Z)>0$ if and only if $Z \cap X_{\infty} \neq \emptyset$. In addition, $Z \in \mathcal{Z}^{(n)}$ with $Z \cap X_{\infty} \neq \emptyset, \mu_{t}\left(T^{n} Z\right)=1$. Now, we compute

$$
\begin{align*}
\mu_{t}(Z) & =\int \mathbf{1}_{Z} d \mu_{t}=\frac{1}{\rho_{t}^{n}} \int \mathcal{L}_{t}^{n} \mathbf{1}_{Z} d \mu_{t}=\frac{1}{\rho_{t}^{n}} \int_{T^{n} Z}\left[\left(g_{n}^{0}\right)^{t} \mathbf{1}_{X_{n-1}}\right] \circ T_{Z}^{-n} d \mu_{t} \\
& =\frac{1}{\rho_{t}^{n}} \int_{T^{n} Z}\left(g_{n}^{0}\right)^{t} \circ T_{Z}^{-n} d \mu_{t} \tag{5.3}
\end{align*}
$$

(recall that we assume $T^{n}\left(Z \cap X_{n-1} \supset X_{\infty}\right)$ ). The bounded distortion property implies, for $x \in Z$,

$$
K^{-1} \mu_{t}\left(T^{n} Z\right)\left(g_{n}^{0}\right)^{t}(x) \leqslant \int_{T^{n} Z}\left(g_{n}^{0}\right)^{t} \circ T_{Z}^{-n} d \mu_{t} \leqslant K \mu_{t}\left(T^{n} Z\right)\left(g_{n}^{0}\right)^{t}(x)
$$

This gives (5.1) for $\mu_{t}$ and (5.2) for $\mu_{t}$ using the large images property (which implies $\left.\mu_{t}\left(T^{n} Z\right)=1\right)$. The computation is the same for $m$, recalling that $m$ is $g^{0}$ conformal with eigenvalue 1 and using the large image property.

Fix $\varepsilon>0$ and $n \in \mathbb{N}$ such that for all $Z \in \mathcal{Z}^{(n)}$, the diameter of $Z$ is less than $\varepsilon$. Let $\mathcal{F}=\left\{Z \in \mathcal{Z}^{(n)} \mid Z \cap X_{\infty} \neq \emptyset\right\}$. It is a cover of $X_{\infty}$ of diameter less than $\varepsilon$. In what follows, $x_{Z}$ denotes any element of $Z$.

$$
\begin{aligned}
\sum_{Z \in \mathcal{F}}(\operatorname{diam} Z)^{t} & \leqslant K^{t} \sum_{Z \in \mathcal{F}}\left(g_{n}^{0}\right)^{t}\left(x_{Z}\right) \quad \text { using (5.1) } \\
& \leqslant K^{2 t} \rho_{t}^{n} \sum_{Z \in \mathcal{F}} \mu_{t}(Z) \quad \text { using (5.2) } \\
& =K^{2 t} \rho_{t}^{n} \mu_{t}\left(X_{\infty}\right)=K^{2 t} \rho_{t}^{n}
\end{aligned}
$$

By our choice of $n$, it is clear that $n \rightarrow \infty$ when $\varepsilon \rightarrow 0$. If $t>t_{0}$ then $\rho_{t}<1$ and the above expression goes to zero. Hence we conclude $H D\left(X_{\infty}\right) \leqslant t_{0}$.

Let us prove the converse inequality. We use the following result of Young.
THEOREM 5.3 [21]. - Let $X$ be a metric space, let $Z \subset X$ assume there exists a probability measure $\mu$ such that $\mu(Z)>0$, for any $x \in Z$, define:

$$
\underline{d_{\mu}}(x)=\liminf _{\varepsilon \rightarrow 0} \frac{\log \mu(B(x, \varepsilon))}{\log \varepsilon},
$$

if for all $x \in Z, \underline{d_{\mu}}(x) \geqslant d$ then $H D(Z) \geqslant d$.
Take $x \in X_{\infty}$ and $\varepsilon>0$, let

$$
n_{0}=\inf \left\{n \in \mathbb{N} \mid \exists y \in B(x, \varepsilon): g_{n}^{0}(y) \leqslant 2 K \varepsilon\right\}-1
$$

Accordingly, there exists $y_{0} \in B(x, \varepsilon)$ such that,

$$
\begin{align*}
g_{n_{0}}^{0}\left(y_{0}\right) g^{0}\left(T^{n_{0}} y_{0}\right) & =g_{n_{0}+1}^{0}\left(y_{0}\right) \leqslant 2 K \varepsilon \quad \text { so } \\
2 K \varepsilon & <g_{n_{0}}^{0}\left(y_{0}\right) \leqslant \frac{2 K \varepsilon}{\inf g^{0}} . \tag{5.4}
\end{align*}
$$

Using Lemma 5.2 and (5.4) we get:

$$
2 \varepsilon \leqslant \operatorname{diam} Z_{n_{0}}\left(y_{0}\right) \leqslant \frac{2 K^{2} \varepsilon}{\inf g^{0}}:=C_{1} \varepsilon
$$

Let $B_{1}=B(x, \varepsilon) \backslash Z_{n_{0}}\left(y_{0}\right)$. If $B_{1} \neq \emptyset$, then let us define:

$$
n_{1}=\inf \left\{n \in \mathbb{N} \mid \exists y \in B_{1}: g_{n}^{0}(y) \leqslant 2 K \varepsilon\right\}-1
$$

Hence, there exists $y_{1} \in B_{1}$ such that

$$
2 \varepsilon \leqslant \operatorname{diam} Z_{n_{1}}\left(y_{1}\right) \leqslant C_{1} \varepsilon
$$

by the same arguments as before. We remark that $n_{1} \geqslant n_{0}$ by construction, so $Z_{n_{0}}\left(y_{0}\right) \cap$ $Z_{n_{1}}\left(y_{1}\right)=\emptyset$. This implies that $Z_{n_{0}}\left(y_{0}\right) \cup Z_{n_{1}}\left(y_{1}\right) \supset B(x, \varepsilon)$. Indeed if it were not the case, we could find $y_{2} \in B_{1} \backslash Z_{n_{1}}\left(y_{1}\right)$ and $n_{2} \geqslant n_{1}$ such that:

$$
2 \varepsilon \leqslant \operatorname{diam} Z_{n_{2}}\left(y_{2}\right) \leqslant C_{1} \varepsilon
$$

By construction, we would obtain three disjoint intervals, with diameter larger than $2 \varepsilon$, all intersecting $B(x, \varepsilon)$, but this is clearly impossible.

Therefore, we have shown that $B(x, \varepsilon) \supset Z_{n_{0}} \cup Z_{n_{1}}$, where the second set may be empty. We have

$$
\mu_{t}\left(Z_{n_{i}}\right) \leqslant \frac{K^{t}}{\rho_{t}^{n_{i}}}\left(\operatorname{diam} Z_{n_{i}}\right)^{t}
$$

by (5.3) and (5.2). So,

$$
\begin{aligned}
\frac{\log \mu_{t}(B(x, \varepsilon))}{\log \varepsilon} & \geqslant \frac{\log \left(\mu_{t}\left(Z_{n_{0}}\right)+\mu_{t}\left(Z_{n_{1}}\right)\right)}{\log \varepsilon} \\
& \geqslant t \frac{\log K}{\log \varepsilon}+\frac{\log \left(\rho_{t}^{-n_{0}} \operatorname{diam}\left(Z_{n_{0}}\right)^{t}+\rho_{t}^{-n_{1}} \operatorname{diam}\left(Z_{n_{1}}\right)^{t}\right)}{\log \varepsilon} \\
& \geqslant t \frac{\log K C_{1}}{\log \varepsilon}+\frac{\log \left(\rho_{t}^{-n_{0}}+\rho_{t}^{-n_{1}}\right)}{\log \varepsilon}+t
\end{aligned}
$$

Since, for $\varepsilon$ small enough, $n_{0}$ and $n_{1}$ are arbitrarily large and for $t<t_{0}, \rho_{t}>1$, we can assume $\rho_{t}^{-n_{0}}+\rho_{t}^{-n_{1}}<1$ so, $\frac{\log \left(\rho_{t}^{-n_{0}}+\rho_{t}^{-n_{1}}\right)}{\log \varepsilon}>0$. Therefore, taking the liminf, $\underline{d_{\mu}}(x) \geqslant t$ for all $t<t_{0}$. We conclude that $H D\left(X_{\infty}\right) \geqslant t_{0}$.

## 6. Examples

In this section we give verifiable criteria to ensure conditions $\mathrm{C} 1, \mathrm{C} 2$ in concrete situations and we discuss some explicit examples.

Condition C 1 is rather mild and in most cases can be checked easily (for example the presence of a full branch outside the hole suffices).

Next notice that, setting

$$
\rho_{n}:=\inf _{x \in D_{n}} \frac{\mathcal{L}^{n+1} 1(x)}{\mathcal{L}^{n} 1(x)}
$$

then $\rho \geqslant \rho_{n}$ (see (2.3)), hence one can verify condition C2 by using some $\rho_{n}$ (which is explicitly computable) rather than $\rho$. The main problem is then to control the number of contiguous elements in $\mathcal{Z}_{b}^{(n)}$. This of course is a case by case matter, yet it is possible to make some rather general statements. Let us clarify the situation by looking at few relevant examples.

Markov maps with non-Markov hole. Let us give examples of Markov maps with a non-Markov hole. Recall that $T$ is said to be Markov with respect to the partition $\mathcal{Z}$ if for all $Z \in \mathcal{Z}, T Z$ is exactly a union of some elements of $\mathcal{Z}$. We call $Y$ a Markov hole if $T$ is Markov and $Y \in \mathcal{Z}^{(n)}$ for some $n ;{ }^{8}$ up to replacing $\mathcal{Z}$ by $\mathcal{Z}^{(n)}$, we may always assume that a Markov hole is an element of $\mathcal{Z}$. Let $Y$ be such a Markov hole, let $\widehat{\mathcal{Z}}$ be the set of elements of $\mathcal{Z}$ that are not $Y$. We call $Y$ an aperiodic Markov hole if there exists $N \in \mathbb{N}$ such that for all $n \geqslant N$, for all $Z, Z^{\prime} \in \mathcal{Z}$, there are $Z_{1}, \ldots, Z_{n} \in \widehat{\mathcal{Z}}$ such that the ( $n+1$ )-cylinder $Z \cap T^{-1} Z_{1} \cap \cdots \cap T^{-n} Z_{n} \cap T^{-n-1} Z^{\prime}$ is nonempty. For expanding Markov maps with an aperiodic Markov hole, Theorem A has been proved in [7].

We are now in position to give examples of Markov maps with non Markov hole such that $\mathcal{Z}_{b}^{(n)}=\emptyset$.

Lemma 6.1. - Let $T$ be a Markov map with Lipschitz derivative and let $\widetilde{Y}$ be an aperiodic Markov hole. Let $Y \subset \widetilde{Y}$ be a hole such that there exists $p \in \mathbb{N}$ and $C \in \mathcal{Z}^{(p)}$ such that $C \subset \widetilde{Y} \backslash Y$ and $C \subset X_{p-1}$. Then for the map $T$ with hole $Y$ one can choose $\xi=1$ in condition 2 (indeed, for all $n, \mathcal{Z}_{b}^{(n)}=\emptyset$, hence one can choose $K=0$ as well).

Proof. - First of all, we remark that since $T^{\prime}$ is Lipschitz, there exists a constant $K(T)$ such that for all $Z \in \mathcal{Z}^{(n)}, \bigvee_{Z} g_{n}^{0} \leqslant K(T)\left\|g_{n}^{0}\right\|_{\infty}$ so that we may avoid the use of partitions $\mathcal{A}$ in the definition of $\mathcal{Z}_{*}^{(n)}$ (see Section 2). Take $n \in \mathbb{N}$, we are going to prove that $\Lambda\left(\mathbf{1}_{Z}\right)>0$ for all $Z \in \mathcal{Z}_{*}^{(n)}$, towards this end, it suffices to prove that for some $k, \mathcal{L}^{k} \mathbf{1}_{Z}>0$. In other words, it suffices to prove that for some $k$, every $x \in I$ has a $k$-preimage in $Z \cap X_{k-1}$.

Take $Z \in \mathcal{Z}_{*}^{(n)}$, then according to the definitions,

$$
Z \supset \bigcap_{i=0}^{n-1} T^{-i} C_{i}:=\widetilde{Z}
$$

where $C_{i}$ is either an element of $\widehat{\mathcal{Z}}$ or is equal to $C$ and $\widetilde{Z}$ is a $p^{\prime}$-cylinder with $n \leqslant p^{\prime} \leqslant p n$. Then using the aperiodicity of $\widetilde{Y}$, we have that for all $q \geqslant N$, any $x \in I$ has a $\left(p^{\prime}+q\right)$-preimage in $Z \cap X_{p^{\prime}+q-1}$.

We conclude with a concrete Markov example with a non-Markov hole. Consider a partition of $I$ into two subintervals $Z_{0}$ and $Z_{1}$. Take $T$ uniformly expanding and

[^7]

Fig. 1. Aperiodic Markov hole.
increasing on each $Z_{i}$, with Lipschitz derivative and such that $T Z_{i}=I, i=0,1$. Take $\widetilde{Y}=Z_{0} \cap T^{-1} Z_{0}$, it is clear that $\widetilde{Y}$ is an aperiodic Markov hole (see Fig. 1).

First consider $Y=[0, \alpha] \subset \widetilde{Y}$, there exists $p \in \mathbb{N}$ such that $C=Z_{0} \cap T^{-1} Z_{0} \cap$ $\bigcap_{i=2}^{p-1} T^{-i} Z_{1} \subset \widetilde{Y} \backslash Y$, this $C$ satisfies the hypothesis of Lemma 6.1 so the map $T$ with hole $Y$ satisfies condition C 2 provided it satisfies condition $\Theta<\rho$.

Second, consider $Y=[\varepsilon, \gamma]$ with $\gamma$ such that $\widetilde{Y}=[0, \gamma]$, let $\gamma_{1}$ be such that $Z_{0} \cap T^{-1} Z_{0} \cap T^{-2} Z_{1}=\left[\gamma_{1}, \gamma\right]$. If $\varepsilon \geqslant \gamma_{1}$ then $Y$ satisfies the hypothesis of Lemma 6.1: for $p$ large enough, the cylinder $C:=\bigcap_{i=0}^{p-2} T^{-i} Z_{0} \cap T^{-p+1} Z_{1}$ is a $p$-cylinder included in $X_{p-1} \cap \widetilde{Y} \backslash Y$.

If $\varepsilon<\gamma_{1}$ then it is easy to see that for all $n, \mathcal{Z}_{b}^{(n)}$ satisfies condition C 2 with $K=1$ and $\xi=1$ (the elements of $\mathcal{Z}_{b}^{(n)}$ are those made up with the interval $[0, \varepsilon]$ and they are never contiguous).

In this last example we have seen that some special cases can be easily handled even if $Z_{b}^{(n)} \neq \emptyset$. The next examples go further in this direction.

Non-Markov maps. Let $I=[0,1]$, for $\beta>1$ and consider the $\beta$-map $T(x)=$ $\beta x(\bmod 1)$ and the potential $g^{0}:=D T^{-1}=\beta^{-1}$. If $\beta \notin \mathbb{N}$, then the map it is not Markov. We will consider only such cases and we will designate by $[\beta]$ the integer part of $\beta$. Let $\gamma=\frac{[\beta]}{\beta}$ and $Y=\left[\gamma_{1}, 1\right]$ with $\gamma<\gamma_{1}<1$. Denote the element of $\mathcal{Z}$ by $Z_{1}, \ldots, Z_{[\beta+1]}$, it is clear that for $p$ large enough, $C:=Z_{[\beta]} \cap \bigcap_{i=1}^{p-1} T^{-i} Z_{1}$ is included in $X_{p-1}$, this leads to the conclusion that there are no contiguous elements of $\mathcal{Z}_{b}^{(n)}$. So, condition C2 is satisfied (with $K=\xi=1$ ) provided condition C 1 is. Note that since the behavior of the map inside the hole is completely irrelevant we could modify the map inside the hole to be Markov, accordingly this case bears no difference with the ones discussed in the previous subsection.

On the contrary, if we consider the case $Y=\left[\gamma, \gamma_{1}\right]$ we have a non-Markov map with an hole. In this case then the number of contiguous elements of $\mathcal{Z}_{b}^{(n)}$ is bounded by $2^{n}$ (since the worst case scenario is when the preimages of a contiguous group join the


Fig. 2. Non-Markov $\beta$-map with a hole $\left(\beta=\frac{7}{2}\right)$.
preimages of another group across the hole ${ }^{9}$ - see Lemma 6.3 for a similar discussion in a more general context) so that condition C 2 is satisfied provided $\frac{2}{\beta}<\rho$. We remark that $\rho \geqslant \frac{[\beta]}{\beta}$, so that condition C 2 will be satisfied for $\beta \geqslant 3$. In particular, the map in Fig. 2, with $\beta=\frac{7}{2}$ does satisfy our conditions.

In addition notice that one can consider also bigger holes that encompass more than one element of the dynamical partition. For example if $\beta=\frac{9}{2}$ and the hole is of the form $Y=\left[\frac{4}{9}, 1-\varepsilon\right], \varepsilon \in\left(0, \frac{1}{9}\right)$, then the map has three full branches outside the hole hence $\rho \geqslant \frac{2}{3}$ while the maximal number of contiguous elements in $\mathcal{Z}_{b}^{(n)}$ is still $2^{n}$, thus C 2 is satisfied. Note that in this case we can have holes with size almost $\frac{1}{3}$ which is rather large. In fact, even more dramatic examples can be easily produced.

We have seen that it is possible to insure condition C 2 by using the combinatorial properties of a Markov map or the special behavior of $\beta$-maps. Some of the above discussion can be generalized by requiring the existence of well behaved elements in the partition: let $\mathcal{Z}_{f}^{(n)}$ be the collection of elements in $\mathcal{Z}_{*}^{(n)}$ such that $T^{n} Z=[0,1]$. Call $\mathcal{Z}_{u}^{(n)}$ the collection of the others.

DEFINITION 6.2. - For $\xi>0$, we call a map $\xi$ full branched ( $\xi$-f.b. for short) if there exists $K>0$ such that the number of contiguous elements in $\mathcal{Z}_{u}^{(n)}$ does not exceed $K \xi^{n}$.

Obviously a $\xi$-f.b. map satisfies condition C 2 , provided $\Theta \xi<\rho$, since if $Z \in \mathcal{Z}_{f}^{(n)}$ then $\Lambda\left(\mathbf{1}_{Z}\right)>0$. The point is that it may be easy to verify that a map is $\xi$-f.b. as the next lemma shows.

LEMmA 6.3. - Calling $C_{n}$ the maximal number of contiguous elements in $\mathcal{Z}_{u}^{(n)}$, holds

$$
C_{n} \leqslant 2 \sum_{i=0}^{n-1}\left(C_{1}+2\right)^{i} C_{1}
$$

[^8]Proof. - The proof is by induction on $n$. Clearly it is true for $n=1$. Let us suppose it true for $n$. The elements of the partition $\mathcal{Z}_{*}^{(n+1)}$ are formed by $\left\{T^{-1} Z \cap Z_{1}\right\}$ where $Z \in \mathcal{Z}_{*}^{(n)}$ and $Z_{1} \in \mathcal{Z}_{*}^{(1)}$. Now if $Z_{1} \in \mathcal{Z}_{f}^{(1)}$, the elements maintain the same nature (i.e. if $Z \in \mathcal{Z}_{u}^{(n)}$ then $T^{-1} Z \cap Z_{1} \in \mathcal{Z}_{u}^{(n+1)}$ and if $Z \in \mathcal{Z}_{f}^{(n)}$ then $T^{-1} Z \cap Z_{1} \in \mathcal{Z}_{f}^{(n+1)}$ ). So we have in $Z_{1}$ at most $C_{n}$ contiguous elements of $\mathcal{Z}_{u}^{(n+1)}$. The only problem can arise when a block of contiguous elements ends at the boundary of $Z_{1}$ since in such a case it can still be contiguous to other elements of $\mathcal{Z}_{u}^{(n+1)}$. Yet, if the contiguous elements of $Z_{1}$ are in $\mathcal{Z}_{f}^{(1)}$, then there can be at most a block of length $2 C_{n}$. One must then analyze what can happen if $Z_{1} \in \mathcal{Z}_{u}^{(1)}$. In this case a set of contiguous elements can either have only partial preimage in $Z_{1}$, hence we get a shorter groups of contiguous elements or all the group can have preimage. In this last case the worst case scenario is when the elements contiguous to the groups (that must belong to $\mathcal{Z}_{f}^{(n)}$ ) are cut while taking preimages. This means that at most two new contiguous elements can be generated, but in this case the group must end at the boundary of $Z_{1}$. Since there are at most $C_{1}$ contiguous elements in $\mathcal{Z}_{u}^{(1)}$ in this way we can generate, at most, $C_{1}\left(C_{n}+2\right)$ contiguous elements that, again in the worst case scenario, can be contiguous to two blocks belonging to the neighboring elements in $\mathcal{Z}_{f}^{(1)}$. Accordingly

$$
C_{n+1} \leqslant C_{1}\left(C_{n}+2\right)+2 C_{n}=\left(C_{1}+2\right) C_{n}+2 C_{1} \leqslant 2 \sum_{i=0}^{n}\left(C_{1}+2\right)^{i} C_{1}
$$

where we have used the induction hypothesis.
The lemma says that if $\rho \Theta^{-1}>C_{1}+2$, then the hypothesis C 2 is verified. The interest of this condition is that it applies to general non-Markov maps provided the potential is sufficiently contracting and there are enough full branches outside the hole. ${ }^{10}$

## 7. Small holes

In this section we will see that, if one is interested only in very small holes then results stronger than the one in the previous sections can be readily obtained by regarding the system with holes as a small perturbation of the system without holes.

The basic idea is to consider the transfer operator $\mathcal{L}$ as a small perturbation of the operator $\mathcal{L}_{0}$. Of course, the norm of the difference of the above operators equal 2 both in the $L^{1}$ and $B V$ norm, hence standard perturbation theory does not apply directly (but see [5] for an indirect application), yet they are close as operators from $B V$ to $L^{1}$.

DEFINITION 7.1. - For each operator $\mathcal{L}: B V(I, m) \rightarrow L^{1}(I, m)$ let

$$
\|\mathcal{L}\| \|:=\sup _{\|f\|_{B V} \leqslant 1}|\mathcal{L} f|_{1}
$$

Then the exact statement of the closeness of the two operators is given by the following lemma. Let $\mathcal{L}_{Y}$ be the transfer operator associated to the hole $Y$.

[^9]Lemma 7.2. - If $\mathcal{L}_{0}$ and $\mathcal{L}_{Y}$ are the two operators defined in (1.1) and (1.2), respectively, then

$$
\left\|\mathcal{L}_{0}-\mathcal{L}_{Y}\right\| \leqslant e^{P\left(g^{0}\right)} m(Y)
$$

Proof. - For each $f \in B V$ the following holds

$$
\begin{aligned}
\left|\mathcal{L}_{0}(f)-\mathcal{L}_{Y}(f)\right|_{1} & =\left|\mathcal{L}_{0}\left(\mathbf{1}_{Y} f\right)\right|_{1} \leqslant e^{P\left(g^{0}\right)}\left|\mathbf{1}_{Y} f\right|_{1} \\
& \leqslant e^{P\left(g^{0}\right)}|f|_{\infty} m(Y) \leqslant e^{P\left(g^{0}\right)}\|f\|_{B V} m(Y)
\end{aligned}
$$

from which the lemma follows.
The above notion of closeness is the one employed in [13], it is then natural to try to verify the conditions of the abstract perturbation result contained in such a paper.

For the reader convenience let us summarize the above mention result specialized to the simple case under consideration.

THEOREM 7.3 [13]. - If there exists constants $A, B>0$, independent of $Y$, and $\theta \in\left(\Theta\left(g^{0}\right), e^{P\left(g^{0}\right)}\right)$ such that, for each $f \in B V$,

$$
\begin{aligned}
& \left\|\mathcal{L}_{0}^{n} f\right\|_{B V} \leqslant A \theta^{n}\|f\|_{B V}+B|f|_{1} \\
& \left\|\mathcal{L}_{Y}^{n} f\right\|_{B V} \leqslant A^{n} \theta^{n}\|f\|_{B V}+B|f|_{1}
\end{aligned}
$$

then for each $\theta_{1} \in(\theta, 1)$ and $\delta \in\left(0,1-\theta_{1}\right)$, there exists $\varepsilon_{0}>0$ such that if $\left\|\mathcal{L}_{0}-\mathcal{L}_{Y}\right\|<$ $\varepsilon_{0}$ then the spectrum of $\mathcal{L}_{Y}$ outside the disk $\left\{z \in \mathbb{C}\left||z| \leqslant \theta_{1}\right\}\right.$ is $\delta$-close, with multiplicity, to the one of $\mathcal{L}_{0}$.

Clearly, from Theorem 7.3 Theorem C readily follows. In fact, if the map $T$ has a unique invariant measure $\mu_{0}$ absolutely continuous with respect to $m$, this means that $\mathcal{L}_{0}$ has $e^{P\left(g^{0}\right)}$ as an isolated eigenvalues and, if the systems $\left(I, T, \mu_{0}\right)$ is mixing, this means that there are no other eigenvalues of modulus one, which in turn implies the existence of a spectral gap. Let $\lambda_{1},\left|\lambda_{1}\right|<1$ be the second largest eigenvalue then, in the above theorem, choose $\theta_{1} \geqslant \max \left\{\theta, \lambda_{1}\right\}$ and $\delta=\frac{e^{P\left(g^{0}\right)}-\theta_{1}}{2}$. Theorem 7.3 implies that, for sufficiently small holes, the spectrum of $\mathcal{L}$ outside the disk $\left\{z \in \mathbb{C}\left||z| \leqslant \theta_{1}\right\}\right.$ consists of only one eigenvalue $\lambda_{0}$ (that moves continuously from $e^{P\left(g^{0}\right)}$ as the hole gets larger) of multiplicity one and of modulus larger than $1-\delta .{ }^{11}$ The projector $\Pi$ $\left(\mathcal{L}_{Y} \Pi=\Pi \mathcal{L}_{Y}=\lambda_{0} \Pi\right)$ associated to such an eigenvalue is of the form $\Pi(f)=h \mu(f)$ where $\mathcal{L}_{Y} h=\lambda_{0} h$, gives the quasi invariant measure and $h \mu$ is the invariant measure. ${ }^{12}$

Hence, to conclude we need only verify the hypotheses of Theorem 7.3.
LEmmA 7.4. - For each $\theta \in\left(\Theta\left(g^{0}\right), e^{P\left(g^{0}\right)}\right)$ there exists $A, B>0$, independent of $Y$, such that, for each $f \in B V$,

$$
\begin{aligned}
& \left\|\mathcal{L}_{0}^{n} f\right\|_{B V} \leqslant A \theta^{n}\|f\|_{B V}+B|f|_{1} \\
& \left\|\mathcal{L}_{Y}^{n} f\right\|_{B V} \leqslant A \theta^{n}\|f\|_{B V}+B|f|_{1}
\end{aligned}
$$

[^10]Proof. - The first inequality is nothing else than the usual Lasota-Yorke inequality, the second is proved by a simplified version of Lemma 2.5.

Remember that $\mathcal{A}_{n}$ is the set of finite partitions in intervals $A=\left\{A_{i}\right\}$ such that $\bigvee_{A_{i}} g_{n} \leqslant 2\left\|g_{n}\right\|_{\infty}$. Given $n \in \mathbb{N}$ and $A \in \mathcal{A}_{i}$ let $\widetilde{\mathcal{Z}}^{(n)}$ be the coarsest partition in intervals among all the ones finer than both $A$ and $\mathcal{Z}^{(n)}$. For each $Z \in \mathcal{Z}_{*}^{(n)}$ let $\widetilde{Z} \in \widetilde{\mathcal{Z}}^{(n)}$ be such that $Z \subset \widetilde{Z}$. We have then the following analogous of Eq. (2.6):

$$
\begin{align*}
\bigvee \mathbf{1}_{T^{n} Z}\left(g_{n} h\right) \circ T_{Z}^{-n} & \leqslant \bigvee_{Z} h g_{n}+2 \sup _{Z}\left|h \cdot g_{n}\right| \\
& \leqslant \bigvee_{Z} h g_{n}+2 \inf _{\widetilde{Z}}\left|h \cdot g_{n}\right|+2 \bigvee_{\widetilde{Z}} h g_{n} \\
& \leqslant 9\left\|g_{n}\right\|_{\infty} \bigvee_{\tilde{Z}} h+8\left\|g_{n}\right\|_{\infty}{\underset{\widetilde{Z}}{ }|h|} \quad \text {. } \tag{7.1}
\end{align*}
$$

Sublemma 7.5. - For each $\widetilde{Z} \in \widehat{\mathcal{Z}}^{(n)}, \#\left\{Z \in \mathcal{Z}_{*}^{(n)} \mid Z \subset \widetilde{Z}\right\} \leqslant n+1$.
Proof. - Since, by definition, $\left.T^{i}\right|_{\tilde{Z}}, i \leqslant n$, is invertible, then $T^{-i} Y$ can have at most one preimage in $\widetilde{Z}$. Accordingly, $Y_{n} \cap \widetilde{Z}$ can consist of, at most, $n$ sub-intervals, hence $X_{n}$ can have, at most, $n+1$ connected components which are exactly $\left\{Z \in \mathcal{Z}_{*}^{(n)} \mid Z \subset\right.$ Z̃\}.
By Sublemma 7.5 it follows that we can sum over $Z \in \mathcal{Z}_{*}^{(n)}$ and obtain

$$
\bigvee \mathcal{L}^{n} h \leqslant 9(n+1)\left\|g_{n}\right\|_{\infty} \bigvee h+8(n+1)\left\|g_{n}\right\|_{\infty} \sup _{\widetilde{Z} \in \widehat{\mathcal{Z}}^{(n)}} m(Z)^{-1} \int|h| d m
$$

Since there exists $\bar{n} \in \mathbb{N}: \theta^{\bar{n}}>9(\bar{n}+1)\left\|g_{\bar{n}}\right\|_{\infty}$, the result follows by choosing

$$
\begin{aligned}
& A:=\sup _{n \leqslant \bar{n}} 9(n+1)\|g\|_{\infty}, \\
& B:=2\left(1-\theta^{\bar{n}}\right)^{-1} \sup _{n \leqslant \bar{n}} 8(n+1)\|g\|_{\infty} \sup _{\widetilde{Z} \in \widetilde{\mathcal{Z}}(n)} m(Z)^{-1},
\end{aligned}
$$

and using the same iteration scheme employed in the proof of Lemma 3.7. Notice that, as announced, $A$ and $B$ do not depend on the hole $Y$.

## Acknowledgements

We acknowledge partial support from the ESF Program PRODYN. We would like to thank P. Collet for suggesting the relevance of [13] to the present setting and G. Keller, B. Schmitt and J-P. Thouvenot for useful discussions.

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[^1]:    ${ }^{1}$ In fact, all the following can be easily extended to the case in which $Y$ is a finite collection of subintervals. We choose not to do so to keep the exposition as simple as possible.

[^2]:    ${ }^{2}$ In fact, the hypothesis that $\left(I, T, \mu_{0}\right)$ is mixing is superfluous and here is used only to make an easy comparison with Theorem A which conditions insure that the invariant measure is unique.

[^3]:    ${ }^{3}$ Essentially the properties of $\Lambda$ are similar to the ones of an inner measure. In the following we will see that, under certain conditions, it is indeed a measure.
    ${ }^{4}$ This follows remembering Eq. (1.3).

[^4]:    ${ }^{5}$ Such partitions always exist, if in doubt see [18] Lemma 6.

[^5]:    ${ }^{6}$ See Lemma 2.5 for the definition of $\widehat{\mathcal{Z}}^{(n)}$ and $\mathcal{Z}_{*}^{(n)}$.

[^6]:    ${ }^{7}$ The existence of such a function is insured by Urysohn's Lemma.

[^7]:    ${ }^{8}$ In fact, one could work with $Y=\bigcup Y_{i}$ and $Y_{i} \in \mathcal{Z}^{(n)}, i=1, \ldots, k$.

[^8]:    ${ }^{9}$ Note that this is a general bound, better bounds may be available for specific values of $\beta$.

[^9]:    ${ }^{10}$ Note that if a map does not satisfy immediately such a criteria, some of its powers may.

[^10]:    ${ }^{11}$ In fact, the results in [13] imply that there exist constants $C>0$ such that $e^{P\left(g^{0}\right)}-\lambda_{0} \leqslant C m(Y)$, provided $\delta$ is chosen small enough.
    ${ }^{12}$ See [12] for the proof that $\mu$ is a measure.

