INVARIANT AND STATIONARY MEASURES FOR THE SL(2, $\mathbb{R}$) ACTION ON MODULI SPACE

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ABSTRACT

We prove some ergodic-theoretic rigidity properties of the action of SL(2, $\mathbb{R}$) on moduli space. In particular, we show that any ergodic measure invariant under the action of the upper triangular subgroup of SL(2, $\mathbb{R}$) is supported on an invariant affine submanifold.

The main theorems are inspired by the results of several authors on unipotent flows on homogeneous spaces, and in particular by Ratner's seminal work.

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1. Introduction

Suppose $g \geq 1$, and let $\alpha = (\alpha_1, \dots, \alpha_n)$ be a partition of $2g-2$, and let $\mathcal{H}(\alpha)$ be a stratum of Abelian differentials, i.e. the space of pairs (M, ω) where M is a Riemann surface and ω is a holomorphic 1-form on M whose zeroes have multiplicities $\alpha_1 \dots \alpha_n$. The form ω defines a canonical flat metric on M with conical singularities at the zeros of ω. Thus we refer to points of $\mathcal{H}(\alpha)$ as *flat surfaces* or *translation surfaces*. For an introduction to this subject, see the survey [Zo].

The space $\mathcal{H}(\alpha)$ admits an action of the group $SL(2,\mathbb{R})$ which generalizes the action of $SL(2,\mathbb{R})$ on the space $GL(2,\mathbb{R})/SL(2,\mathbb{Z})$ of flat tori. In this paper we prove ergodic-theoretic rigidity properties of this action.

In what follows, we always replace $\mathcal{H}(\alpha)$ by a finite cover X_0 which is a manifold. Such a cover can be found by e.g. considering a level 3 structure (see Section 3). However, in the introduction, we suppress this from the notation.

Let $\Sigma \subset M$ denote the set of zeroes of ω. Let $\{\gamma_1, \dots, \gamma_k\}$ denote a symplectic $\mathbb{Z}$-basis for the relative homology group $H_1(M, \Sigma, \mathbb{Z})$. We can define a map $\Phi : \mathcal{H}(\alpha) \to \mathbb{C}^k$ by

$$\Phi(M, \omega) = \left(\int_{\gamma_1} \omega, \dots, \int_{\gamma_k} \omega \right).$$

The map Φ (which depends on a choice of the basis $\{\gamma_1, \dots, \gamma_k\}$) is a local coordinate system on (M, ω). Alternatively, we may think of the cohomology class $[\omega] \in H^1(M, \Sigma, \mathbb{C})$ as a local coordinate on the stratum $\mathcal{H}(\alpha)$. We will call these coordinates *period coordinates*.

We can consider the measure λ on $\mathcal{H}(\alpha)$ which is given by the pullback of the Lebesgue measure on $H^1(M, \Sigma, \mathbb{C}) \approx \mathbb{C}^k$. The measure λ is independent of the choice of basis $\{\gamma_1, \ldots, \gamma_k\}$, and is easily seen to be $SL(2, \mathbb{R})$-invariant. We call λ the *Lebesgue* or the *Masur-Veech* measure on $\mathcal{H}(\alpha)$.

The area of a translation surface is given by

$$a(M, \omega) = \frac{i}{2} \int_M \omega \wedge \bar{\omega}.$$

A "unit hyperboloid" $\mathcal{H}_1(\alpha)$ is defined as a subset of translation surfaces in $\mathcal{H}(\alpha)$ of area one. The $SL(2, \mathbb{R})$-invariant Lebesgue measure $\lambda_{(1)}$ on $\mathcal{H}_1(\alpha)$ is defined by disintegration of the Lebesgue measure λ on $\mathcal{H}_1(\alpha)$, namely

$$d\lambda = c d\lambda_{(1)} da,$$

where c is a constant. A fundamental result of Masur [Mas1] and Veech [Ve1] is that $\lambda_{(1)}(\mathcal{H}_1(\alpha)) < \infty$. In this paper, we normalize $\lambda_{(1)}$ so that $\lambda_{(1)}(\mathcal{H}_1(\alpha)) = 1$ (and so $\lambda_{(1)}$ is a probability measure).

For a subset $\mathcal{M}_1 \subset \mathcal{H}_1(\alpha)$ we write

$$\mathbb{R}\mathcal{M}_1 = \left\{(M, t\omega) \mid (M, \omega) \in \mathcal{M}_1,\ t \in \mathbb{R} \setminus \{0\}\right\} \subset \mathcal{H}(\alpha).$$

Definition **1.1.** — *An ergodic* $SL(2, \mathbb{R})$*-invariant probability measure* ν_1 *on* $\mathcal{H}_1(\alpha)$ *is called* affine *if the following conditions hold:*

(i) *The support* $\mathcal{M}_1$ *of* ν_1 *is an* immersed submanifold *of* $\mathcal{H}_1(\alpha)$, *i.e. there exists a manifold* $\mathcal{N}$ *and a proper continuous map* $f : \mathcal{N} \to \mathcal{H}_1(\alpha)$ *so that* $\mathcal{M}_1 = f(\mathcal{N})$. *The self-intersection set of* $\mathcal{M}_1$, *i.e. the set of points of* $\mathcal{M}_1$ *which do not have a unique preimage under* f, *is a closed subset of* $\mathcal{M}_1$ *of* ν_1*-measure* 0. *Furthermore, each point in* $\mathcal{N}$ *has a neighborhood* U *such that locally* $\mathbb{R}f(U)$ *is given by a complex linear subspace defined over* $\mathbb{R}$ *in the period coordinates.*

(ii) *Let* ν *be the measure supported on* $\mathcal{M} = \mathbb{R}\mathcal{M}_1$ *so that* $d\nu = d\nu_1 da$. *Then each point in* $\mathcal{N}$ *has a neighborhood* U *such that the restriction of* ν *to* $\mathbb{R}f(U)$ *is an affine linear measure in the period coordinates on* $\mathbb{R}f(U)$, *i.e. it is (up to normalization) the induced measure of the Lebesgue measure* λ *to the subspace* $\mathbb{R}f(U)$.

Definition **1.2.** — *We say that any suborbifold* $\mathcal{M}_1$ *for which there exists a measure* ν_1 *such that the pair* $(\mathcal{M}_1, \nu_1)$ *satisfies (i) and (ii) is an* affine invariant submanifold.

We also consider the entire stratum $\mathcal{H}(\alpha)$ to be an (improper) affine invariant submanifold. It follows from [EMiMo, Theorem 2.2] that the self-intersection set of an affine invariant manifold is itself a finite union of affine invariant manifolds of lower dimension.

For many applications we need the following:

Proposition **1.3.** — *Any stratum* $\mathcal{H}_1(\alpha)$ *contains at most countably many affine invariant submanifolds.*

Proposition 1.3 is deduced as a consequence of some isolation theorems in [EMiMo]. This argument relies on adapting some ideas of G. A. Margulis to the Teichmüller space setting. Another proof is given by A. Wright in [Wr1], where it is proven that affine invariant submanifolds are always defined over a number field.

The classification of the affine invariant submanifolds is complete in genus 2 by the work of McMullen [Mc1] [Mc2] [Mc3] [Mc4] [Mc5] and Calta [Ca]. In genus 3 or greater it is an important open problem. See [Mö1], [Mö2], [Mö3], [Mö4], [BoM], [BaM], [HLM], [LN1], [LN2], [LN3], [Wr1], [Wr2], [MW], [NW], [ANW], [Fi1] and [Fi2] for some results in this direction.

1.1. *The main theorems.* — Let

$$\mathrm{N} = \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix},\ t \in \mathbb{R} \right\}, \qquad \mathrm{A} = \left\{ \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix},\ t \in \mathbb{R} \right\},$$

$$\bar{\mathrm{N}} = \left\{ \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix},\ t \in \mathbb{R} \right\}$$

Let $r_\theta = \left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right)$, and let $\mathrm{SO}(2) = \{r_\theta \mid \theta \in [0, 2\pi)\}$. Then N, $\bar{\mathrm{N}}$, A and SO(2) are subgroups of $\mathrm{SL}(2,\mathbb{R})$. Let $\mathrm{P} = \mathrm{AN}$ denote the set of upper triangular matrices of determinant 1, which is a subgroup of $\mathrm{SL}(2,\mathbb{R})$.

Theorem **1.4.** — *Let ν be any ergodic* P*-invariant probability measure on $\mathcal{H}_1(\alpha)$. Then ν is* $\mathrm{SL}(2,\mathbb{R})$*-invariant and affine.*

The following (which uses Theorem 1.4) is joint work with A. Mohammadi and is proved in [EMiMo]:

Theorem **1.5.** — *Suppose* $\mathrm{S} \in \mathcal{H}_1(\alpha)$*. Then, the orbit closure* $\overline{\mathrm{PS}} = \overline{\mathrm{SL}(2,\mathbb{R})\mathrm{S}}$ *is an affine invariant submanifold of* $\mathcal{H}_1(\alpha)$.

For the case of strata in genus 2, the $\mathrm{SL}(2,\mathbb{R})$ part of Theorems 1.4 and 1.5 were proved using a different method by Curt McMullen [Mc6].

The proof of Theorem 1.4 uses extensively entropy and conditional measure techniques developed in the context of homogeneous spaces (Margulis-Tomanov [MaT], Einsiedler-Katok-Lindenstrauss [EKL]). Some of the ideas came from discussions with Amir Mohammadi. But the main strategy is to replace polynomial divergence by the "exponential drift" idea of Benoist-Quint [BQ].

Stationary measures. — Let μ be an SO(2)-invariant compactly supported measure on $\mathrm{SL}(2,\mathbb{R})$ which is absolutely continuous with respect to Lebesgue measure. A measure ν on $\mathcal{H}_1(\alpha)$ is called *μ-stationary* if $\mu * \nu = \nu$, where

$$\mu * \nu = \int_{\mathrm{SL}(2,\mathbb{R})} (g_* \nu)\, d\mu(g).$$

Recall that by a theorem of Furstenberg [F1], [F2], restated as [NZ, Theorem 1.4], there exists a probability measure ρ on $\mathrm{SL}(2,\mathbb{R})$ such that $\nu \to \rho * \nu$ is a bijection between ergodic P-invariant measures and ergodic μ-stationary measures. Therefore, Theorem 1.4 implies the following:

Theorem **1.6.** — *Any ergodic μ-stationary measure on $\mathcal{H}_1(\alpha)$ is $\mathrm{SL}(2,\mathbb{R})$-invariant and affine.*

Counting periodic trajectories in rational billiards. — Let Q be a rational polygon, and let N(Q, T) denote the number of cylinders of periodic trajectories of length at most T for the billiard flow on Q. By a theorem of H. Masur [Mas2] [Mas3], there exist c_1 and c_2 depending on Q such that for all $t > 1$,

$$c_1 e^{2t} \leq \mathrm{N}\big(\mathrm{Q}, e^t\big) \leq c_2 e^{2t}.$$

Theorem 1.4 and Proposition 1.3 together with some extra work (done in [EMiMo]) imply the following "weak asymptotic formula" (cf. [AEZ]):

Theorem **1.7.** — *For any rational polygon Q, there exists a constant $c = c(\mathrm{Q})$ such that*

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \mathrm{N}\big(\mathrm{Q}, e^s\big) e^{-2s}\, ds = c.$$

The constant c in Theorem 1.7 is the Siegel-Veech constant (see [Ve2], [EMZ]) associated to the affine invariant submanifold $\mathcal{M} = \overline{\mathrm{SL}(2,\mathbb{R})\mathrm{S}}$ where S is the flat surface obtained by unfolding Q.

It is natural to conjecture that the extra averaging on Theorem 1.7 is not necessary, and one has $\lim_{t\to\infty} \mathrm{N}(\mathrm{Q}, e^t)e^{-2t} = c$. This can perhaps be shown if one obtains a classification of the measures invariant under the subgroup N of $\mathrm{SL}(2,\mathbb{R})$. Such a result is in general beyond the reach of the current methods. However it is known in a few very special cases, see [EMS], [EMM], [CW] and [Ba].

Other applications to rational billiards. — All the above theorems apply also to the moduli spaces of flat surfaces with marked points. Thus one should expect applications to the "visibility" and "finite blocking" problems in rational polygons as in [HST]. It is likely that many other applications are possible.

2. Outline of the paper

2.1. *Some notes on the proofs.* — The theorems of Section 1.1 are inspired by the results of several authors on unipotent flows on homogeneous spaces, and in particular by

Ratner's seminal work. In particular, the analogues of Theorems 1.4 and 1.5 in homogeneous dynamics are due to Ratner [Ra4], [Ra5], [Ra6], [Ra7]. (For an introduction to these ideas, and also to the proof by Margulis and Tomanov [MaT] see the book [Mor]. See also the papers [Dan1], [Dan2], [Dan3], [Dan4], [DM1], [DM2], [DM3], [DM4], [Mar1], [Mar2], [Mar3], [Mar4], [Ra1], [Ra2], [Ra3], [MoSh]. The homogeneous analogue of the fact that P-invariant measures are $SL(2,\mathbb{R})$-invariant is due to Mozes [Moz] and is based on Ratner's work. All of these results are based in part on the "polynomial divergence" of the unipotent flow on homogeneous spaces.

However, in our setting, the dynamics of the unipotent flow (i.e. the action of N) on $\mathcal{H}_1(\alpha)$ is poorly understood, and plays no role in our proofs. The main strategy is to replace the "polynomial divergence" of unipotents by the "exponential drift" idea in the recent breakthrough paper by Benoist and Quint [BQ].

One major difficulty is that we have no apriori control over the Lyapunov spectrum of the geodesic flow (i.e. the action of A). By [AV1] the Lyapunov spectrum is simple for the case of Lebesgue (i.e. Masur-Veech) measure, but for the case of an arbitrary P-invariant measure this is not always true, see e.g. [Fo2], [FoM].

In order to use the Benoist-Quint exponential drift argument, we must show that the Zariski closure (or more precisely the algebraic hull, as defined by Zimmer [Zi2]) of the Kontsevich-Zorich cocycle is semisimple. The proof proceeds in the following steps:

Step 1. — We use an entropy argument inspired by the "low entropy method" of [EKL] (using [MaT] together with some ideas from [BQ]) to show that any P-invariant measure ν on $\mathcal{H}_1(\alpha)$ is in fact $SL(2,\mathbb{R})$ invariant. We also prove Theorem 2.1 which gives control over the conditional measures of ν. This argument occupies Sections 3–13 and is outlined in more detail in Section 2.3.

Step 2. — By some results of Forni (see Appendix A), for an $SL(2,\mathbb{R})$-invariant measure ν, the absolute cohomology part of the Kontsevich-Zorich cocycle $A : SL(2,\mathbb{R}) \times \mathcal{H}_1(\alpha) \to Sp(2g,\mathbb{Z})$ is semisimple, i.e. has semisimple algebraic hull. For an exact statement see Theorem A.6.

Step 3. — We pick an $SO(2)$-invariant compactly supported measure μ on $SL(2,\mathbb{R})$ which is absolutely continuous with respect to Lebesgue measure, and work in the random walk setting as in [F1] [F2] and [BQ]. Let B denote the space of infinite sequences $g_0, g_1, \dots$, where $g_i \in SL(2,\mathbb{R})$. We then have a skew product shift map $T : B \times \mathcal{H}_1(\alpha) \to B \times \mathcal{H}_1(\alpha)$ as in [BQ], so that $T(g_0, g_1, \dots; x) = (g_1, g_2, \dots; g_0^{-1}x)$. Then, we use (in Appendix C) a modification of the arguments by Guivarc'h and Raugi [GR1], [GR2], as presented by Goldsheid and Margulis in [GM, §4–5], and an argument of Zimmer (see [Zi1] or [Zi2]) to prove Theorem C.5 which states that the Lyapunov spectrum of T is always "semisimple", which means that for each $SL(2,\mathbb{R})$-irreducible component of the cocycle, there is a T-equivariant non-degenerate inner product on the

Lyapunov subspaces of T (or more precisely on the successive quotients of the Lyapunov flag of T). This statement is trivially true if the Lyapunov spectrum of T is simple.

Step 4. — We can now use the Benoist-Quint exponential drift method to show that the measure ν is affine. This is done in Sections 14–16. At one point, to avoid a problem with relative homology, we need to use a result, Theorem 14.3 about the isometric (Forni) subspace of the cocycle, which is proved in joint work with A. Avila and M. Möller [AEM].

Finally, we note that the proof relies heavily on various recurrence to compact sets results for the $SL(2, \mathbb{R})$ action, such as those of [EMa] and [Ath]. All of these results originate in the ideas of Margulis and Dani [Mar1], [Dan1], [EMM1], [EMM2].

2.2. *Notational conventions.* — For $t \in \mathbb{R}$, let

$$g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, \qquad u_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

Let $A = \{g_t \,:\, t \in \mathbb{R}\}$, $N = \{u_t \,:\, t \in \mathbb{R}\}$. Let $P = AN$.

Let X_0 denote a finite cover of the stratum $\mathcal{H}_1(\alpha)$ which is a manifold (see Section 3). Let $\tilde{X}_0$ denote the universal cover of X_0. Let $\pi : \tilde{X}_0 \to X_0$ denote the natural projection map.

We will need at some point to consider a certain measurable finite cover X of X_0. This cover will be constructed in Section 4.6 below. Let $\tilde{X}$ denote the "universal cover" of X, see Section 4.6 for the exact definition. We abuse notation by denoting the covering map from $\tilde{X}$ to X also by the letter π.

If f is a function on X_0 or X we sometimes abuse notation by denoting $f \circ \pi$ by f and write $f(x)$ instead of $f(\pi(x))$. A point of $\mathcal{H}(\alpha)$ is a pair (M, ω), where M is a compact Riemann surface, and ω is a holomorphic 1-form on M. Let Σ denote the set of zeroes of ω. The cohomology class of ω in the relative cohomology group $H^1(M, \Sigma, \mathbb{C}) \cong H^1(M, \Sigma, \mathbb{R}^2)$ is a local coordinate on $\mathcal{H}(\alpha)$ (see [Fo]). For $x \in \tilde{X}_0$, let $V(x)$ denote a subspace of $H^1(M, \Sigma, \mathbb{R}^2)$. Then we denote by the image of $V(x)$ under the affine exponential map, i.e.

$$V[x] = \left\{ y \in \tilde{X}_0 \,:\, y - x \in V(x) \right\}.$$

(For some subspaces V, we can define $V[x]$ for $x \in \tilde{X}$ as well. This will be explained in Section 4.6. Also, depending on the context, we sometimes consider $V[x]$ to be a subset of X or X_0.)

Let $p : H^1(M, \Sigma, \mathbb{R}) \to H^1(M, \mathbb{R})$ denote the natural map. Let

$$H^1_\perp(x) = \left\{ v \in H^1(M, \Sigma, \mathbb{R}) \,:\, p(\operatorname{Re} x) \wedge p(v) = p(\operatorname{Im} x) \wedge p(v) = 0 \right\}, \tag{2.1}$$

where we are considering the "real part map" Re and the "imaginary part map" Im as maps from $H^1(M, \Sigma, \mathbb{C}) \cong H^1(M, \Sigma, \mathbb{R}^2)$ to $H^1(M, \Sigma, \mathbb{R})$. Let

$$W(x) = \mathbb{R}(\operatorname{Im} x) \oplus H^1_\perp(x) \subset H^1(M, \Sigma, \mathbb{R}),$$

so that

$$W(x) = \{v \in H^1(M, \Sigma, \mathbb{R}) \ : \ p(\operatorname{Im} x) \wedge p(v) = 0\}.$$

Let $\pi_x^- : W(x) \to H^1(M, \Sigma, \mathbb{R})$ denote the map (defined for a.e. $x \in \tilde{X}_0$)

$$\pi_x^-(c \operatorname{Im} x + v) = c \operatorname{Re} x + v \quad c \in \mathbb{R}, \ v \in H^1_\perp(x), \tag{2.2}$$

so that

$$\pi_x^-\big(W(x)\big) = \{v \in H^1(M, \Sigma, \mathbb{R}) \ : \ p(\operatorname{Re} x) \wedge p(v) = 0\}.$$

We have $H^1(M, \Sigma, \mathbb{R}^2) = \mathbb{R}^2 \otimes H^1(M, \Sigma, \mathbb{R})$. For a subspace $V(x) \subset W(x)$, we write

$$V^+(x) = (1, 0) \otimes V(x), \qquad V^-(x) = (0, 1) \otimes \pi_x^-\big(V(x)\big).$$

Then $W^+[x]$ and $W^-[x]$ play the role of the unstable and stable foliations for the action of g_t on X_0 for $t > 0$, see Lemma 3.5.

Starred subsections. — Some technical proofs are relegated to subsections marked with a star. These subsections can be skipped on first reading. The general rule is that no statement from a starred subsection is used in subsequent sections.

2.3. *Outline of the proof of Step 1.* — The general strategy is based on the idea of additional invariance which was used in the proofs of Ratner [Ra4], [Ra5], [Ra6], [Ra7] and Margulis-Tomanov [MaT].

The aim of Step 1 is to prove the following:

Theorem **2.1.** — *Let ν be an ergodic P-invariant measure on X_0. Then ν is $SL(2, \mathbb{R})$-invariant. In addition, there exists an $SL(2, \mathbb{R})$-equivariant system of subspaces $\mathcal{L}(x) \subset W(x)$ such that for almost all x, the conditional measures of ν along $W^+[x]$ are the Lebesgue measures along $\mathcal{L}^+[x]$, and the conditional measures of ν along $W^-[x]$ are the Lebesgue measures along $\mathcal{L}^-[x]$.*

In the sequel, we will often refer to a (generalized) subspace $U^+[x] \subset W^+[x]$ on which we already proved that the conditional measure of ν is Lebesgue. The proof of Theorem 2.1 will be by induction, and in the beginning of the induction, $U^+[x] = Nx$. (Note: generalized subspaces are defined in Section 6.)

In this introductory subsection, let $U^+(x) \subset W^+(x)$ denote the subspace $\{y - x \ : \ y \in U^+[x]\}$. (This definition has to be modified when we are dealing with generalized subspaces, see Section 6.)

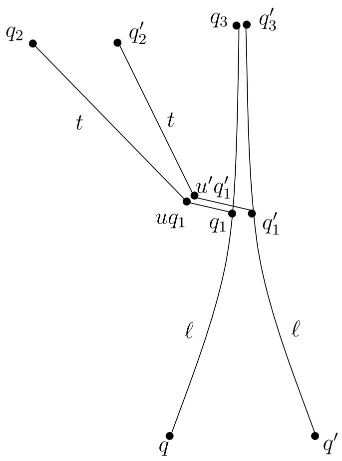

Fig. 1. — Outline of the proof of Theorem 2.1

Outline of the proof of Theorem 2.1. — Let ν be an ergodic P-invariant probability measure on X_0. Since ν is N-invariant, the conditional measure ν_{W^+} of ν along W^+ is non-trivial. This implies that the entropy of A is positive, and thus the conditional measure ν_{W^-} of ν along W^- is non-trivial (see e.g. [EL]). This implies that on a set of almost full measure, we can pick points q and q' in the support of ν such that q and q' are in the same leaf of W^- and $d(q, q') \approx 1/100$, see Figure 1.

Let $\ell > 0$ be a large parameter. Let $q_1 = g_\ell q$ and let $q_1' = g_\ell q'$. Then q_1 and q_1' are very close together. We pick $u \in U^+(q_1)$ with $\|u\| \approx 1/100$, and pick (as described below) $u' \in U^+(q_1')$. Consider the points uq_1 and $u'q_1'$. With our choice of u', the points uq_1 and $u'q_1'$ will be close, but they are no longer in the same leaf of W^-, and we expect them to diverge under the action of g_t as $t \to +\infty$. Let t be chosen so that $q_2 = g_t u q_1$ and $q_2' = g_t u' q_1'$ be such that $d(q_2, q_2') \approx \epsilon$, where $\epsilon > 0$ is fixed.

Consider the bundle (which we will denote for short H^1) whose fiber above $x \in \mathcal{H}(\alpha)$ is $H^1(M, \Sigma, \mathbb{R})$. The presence of the integer lattice $H^1(M, \Sigma, \mathbb{Z})$ in $H^1(M, \Sigma, \mathbb{R})$ allows one to identify the fibers at nearby points. This defines a flat connection, called the Gauss-Manin connection on this bundle.

The action of $SL(2, \mathbb{R})$ and in particular the geodesic flow g_t on $\mathcal{H}(\alpha)$, extends to an action on the bundle H^1, where the action on the fibers is by parallel transport with respect to the Gauss-Manin connection. The action on the bundle takes the form

$$g_t(x, v) = \big(g_t x, A(g_t, v)\big),$$

where $A : SL(2, \mathbb{R}) \times \mathcal{H}_1(\alpha) \to GL(H^1(M, \Sigma, \mathbb{R}))$ is the Kontsevich-Zorich cocycle. It is continuous (in fact locally constant) and log-integrable. Thus the multiplicative ergodic theorem can be applied.

Let

$$1 = \lambda_1\big(H^1\big) > \lambda_2\big(H^1\big) \geq \cdots \geq \lambda_{k-1}\big(H^1\big) > \lambda_k\big(H^1\big) = -1$$

denote the Lyapunov spectrum of the Kontsevich-Zorich cocycle. (The fact that $\lambda_2 < 1$ is due to Veech [Ve1] and Forni [Fo].) We have

$$H^1(M, \Sigma, \mathbb{R}) = \bigoplus_{i=1}^{k} \mathcal{V}_i(H^1)(x)$$

where $\mathcal{V}_i(H^1)(x)$ is the Lyapunov subspace corresponding to $\lambda_i(H^1)$ (see Section 4). Note that $\mathcal{V}_1(H^1)(x)$ corresponds to the unipotent direction inside the $SL(2, \mathbb{R})$ orbit. In the first step of the induction, $U^+(x) = \mathcal{V}_1(H^1)(x)$.

In general, for $y \in U^+[x]$, if we identify H^1 at x and y using the Gauss-Manin connection, we have (see Lemma 4.1),

$$\mathcal{V}_i(H^1)(y) \subset \bigoplus_{j \le i} \mathcal{V}_j(H^1)(x). \tag{2.3}$$

We say that the Lyapunov exponent $\lambda_i(H^1)$ is U^+-inert if for a.e. x, $\mathcal{V}_i(H^1)(x) \not\subset U^+(x)$ and also, for a.e. $y \in U^+[x]$,

$$\mathcal{V}_i(H^1)(y) \subset U^+(x) + \mathcal{V}_i(H^1)(x).$$

(In other words, $\mathcal{V}_i(H^1)(x)$ is constant (modulo U^+) along $U^+[x]$.) Note that in view of (2.3), $\lambda_2(H^1)$ is always U^+-inert. We now assume for simplicity that $\lambda_2(H^1)$ is the only U^+-inert exponent.

We may write

$$u'q_1' - uq_1 = w_+ + g_s(uq_1) + w_-$$

where $w_+ \in W^+(uq_1)$, $w_- \in W^-(uq_1)$, and $s \in \mathbb{R}$. Furthermore, due to the assumption that λ_2 is the only inert exponent, after possibly making a small change to u and u' (see Section 6), we may write

$$w_+ = \sum_{i=2}^{n} v_i$$

where $v_i \in \mathcal{V}_i(H^1)(uq_1)$, and furthermore, $\|v_2\|/\|u'q_1' - uq_1\|$ is bounded from below. Then, $q_2' - q_2$ will be approximately in the direction of $\mathcal{V}_2(H^1)(q_2)$, see Section 8 for the details.

Let $f_2(x)$ denote the conditional measure of ν along $(\mathcal{V}_1 + \mathcal{V}_2)(H^1)[x]$. (This conditional measure can be defined since ν is U^+-invariant.) Let $q_3 = g_s q_1$ and $q_3' = g_s q_1'$ where $s > 0$ is such that the amount of expansion along $\mathcal{V}_2(H^1)$ from q_1 to q_3 is equal to the amount of expansion along $\mathcal{V}_2(H^1)$ from uq_1 to q_2. Then, as in [BQ],

$$f_2(q_2) = A_* f_2(q_3), \quad \text{and} \quad f_2(q_2') = A'_* f_2(q_3'), \tag{2.4}$$

where A and A′ are essentially the same bounded linear map. But q_3 and q_3' approach each other, so that

$$f_2(q_3) \approx f_2(q_3').$$

Hence

$$f_2(q_2) \approx f_2(q_2'). \tag{2.5}$$

Taking a limit as $\ell \to \infty$ of the points q_2 and q_2' we obtain points $\tilde{q}_2$ and $\tilde{q}_2'$ in the same leaf of $(\mathcal{V}_1 + \mathcal{V}_2)(\mathrm{H}^1)$ and distance ϵ apart such that

$$f_2(\tilde{q}_2) = f_2(\tilde{q}_2'). \tag{2.6}$$

This means that the conditional measure $f_2(\tilde{q}_2)$ is invariant under a shift of size approximately ϵ. Repeating this argument with $\epsilon \to 0$ we obtain a point p such that $f_2(p)$ is invariant under arbitrarily small shifts. This implies that the conditional measure $f_2(p)$ restricts to Lebesgue measure on some subspace U_{new} of $(\mathcal{V}_1 + \mathcal{V}_2)(\mathrm{H}^1)$, which is distinct from the orbit of N. Thus, we can enlarge U^+ to be $\mathrm{U}^+ \oplus \mathrm{U}_{new}$.

Technical Problem #1. — The argument requires that all eight points q, q', q_1, q_1', q_2, q_2', q_3, q_3' belong to some "nice" set K of almost full measure. We will give a very rough outline of the solution to this problem here; a more detailed outline is given at the beginning of Section 5.

We have the following elementary statement:

Lemma **2.2.** — *If* ν_{W^-} *is non-trivial, then for any* $\delta > 0$ *there exist constants* $c(\delta) > 0$ *and* $\rho(\delta) > 0$ *such that for any compact* $\mathrm{K} \subset \mathrm{X}_0$ *with* $\nu(\mathrm{K}) > 1 - \delta$ *there exists a compact subset* $\mathrm{K}' \subset \mathrm{K}$ *with* $\nu(\mathrm{K}') > 1 - c(\delta)$ *so that for any* $q \in \mathrm{K}'$ *there exists* $q' \in \mathrm{K} \cap \mathrm{W}^-[q]$ *with*

$$\rho(\delta) < d(q, q') < 1/100.$$

Furthermore, $c(\delta) \to 0$ *as* $\delta \to 0$.

In other words, there is a set $\mathrm{K}' \subset \mathrm{K}$ of almost full measure such that every point $q \in \mathrm{K}'$ has a "friend" $q' \in \mathrm{W}^-[q]$, with q' also in the "nice" set K, such that

$$d(q, q') \approx 1/100.$$

Thus, q can be chosen essentially anywhere in X_0. (In fact we use a variant of Lemma 2.2, namely Proposition 5.3 in Section 5.)

We also note the following trivial statement:

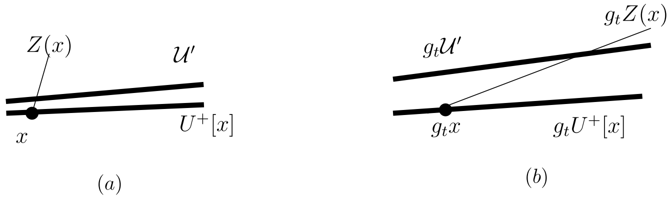

FIG. 2. — (**a**) We keep track of the relative position of the subspaces $U^+[x]$ and $\mathcal{U}'$ in part by picking a transversal $Z(x)$ to $U^+[x]$, and noting the distance between $U^+[x]$ and $\mathcal{U}'$ along $Z[x]$. (**b**) If we apply the flow g_t to the entire picture in (**a**), we see that the transversal $g_tZ[x]$ can get almost parallel to $g_tU^+[x]$. Then, the distance between $g_tU^+[x]$ and $g_t\mathcal{U}'$ along $g_tZ[x]$ may be much larger then the distance between $g_tx \in g_tU^+[x]$ and the closest point in $g_t\mathcal{U}'$

Lemma **2.3.** — *Suppose ν is a measure on X_0 invariant under the flow g_t. Let $\hat{\tau} : X_0 \times \mathbb{R} \to \mathbb{R}$ be a function such that there exists $\kappa > 1$ so that for all $x \in X_0$ and for $t > s$,*

$$\kappa^{-1}(t-s) \le \hat{\tau}(x,t) - \hat{\tau}(x,s) \le \kappa(t-s). \tag{2.7}$$

Let $\psi_t : X_0 \to X_0$ be given by $\psi_t(x) = g_{\hat{\tau}(x,t)}x$. Then, for any $K^c \subset X_0$ and any $\delta > 0$, there exists a subset $E \subset \mathbb{R}$ of density at least $(1-\delta)$ such that for $t \in E$,

$$\nu\big(\psi_t^{-1}(K^c)\big) \le \big(\kappa^2/\delta\big)\nu\big(K^c\big).$$

(We remark that the maps ψ_t are not a flow, since ψ_{t+s} is not in general $\psi_t \circ \psi_s$. However, Lemma 2.3 still holds.)

In Section 7 we show that roughly, $q_2 = \psi_t(q)$, where ψ_t is as in Lemma 2.3. (A more precise statement, and the strategy for dealing with this problem is given at the beginning of Section 5.) Then, to make sure that q_2 avoids a "bad set" K^c of small measure, we make sure that $q \in \psi_t^{-1}(K)$ which by Lemma 2.3 has almost full measure. Combining this with Lemma 2.2, we can see that we can choose q, q' and q_2 all in an a priori prescribed subset K of almost full measure. A similar argument can be done for all eight points, see Section 12, where the precise arguments are assembled.

Technical Problem #2. — Beyond the first step of the induction, the subspace $U^+(x)$ may not be locally constant as x varies along $W^+(x)$. This complication has a ripple effect on the proof. In particular, instead of dealing with the divergence of the points g_tuq_1 and $g_tu'q_1'$ we need to deal with the divergence of the affine subspaces $U^+[g_tuq_1]$ and $U^+[g_tu'q_1']$. As a first step, we project $U^+[g_tu'q_1']$ to the leaf of W^+ containing $U^+[g_tuq_1]$, to get a new affine subspace $\mathcal{U}'$. One way to keep track of the relative location of $U^+ = U^+[g_tu'q_1']$ and $\mathcal{U}'$ is (besides keeping track of the linear parts of U^+ and $\mathcal{U}'$) to pick a transversal $Z(x)$ to $U^+[x]$, and to keep track of the intersection of $\mathcal{U}'$ and $Z(x)$, see Figure 2.

However, since we do not know at this point that the cocycle is semisimple, we cannot pick Z in a way which is invariant under the flow. Thus, we have no choice except to pick some transversal $Z(x)$ to $U^+(x)$ at ν-almost every point $x \in X_0$, and then deal with the need to change transversal.

It turns out that the formula for computing how $\mathcal{U}' \cap Z$ changes when Z changes is non-linear (it involves inverting a certain matrix). However, we would really like to work with linear maps. This is done in two steps: first we show that we can choose the approximation $\mathcal{U}'$ and the transversals $Z(x)$ in such a way that changing transversals involves inverting a unipotent matrix. This makes the formula for changing transversals polynomial. In the second step, we embed the space of parameters of affine subspaces near $U^+[x]$ into a certain tensor power space $\mathbf{H}(x)$ so that on the level of $\mathbf{H}(x)$ the change of transversal map becomes linear. The details of this construction are in Section 6.

Technical Problem #3. — There may be more than one U^+-inert Lyapunov exponent. In that case, we do not have precise control over how q_2 and q_2' diverge. In particular the assumption that $q_2 - q_2'$ is nearly in the direction of $\mathcal{V}_2(H^1)(q_2)$ is not justified. Also we really need to work with $U^+[q_2]$ and $U^+[q_2']$. So let $\mathbf{v} \in \mathbf{H}(q_2)$ denote the vector corresponding to (the projection to $W^+(q_2)$ of) the affine subspace $U^+[q_2']$. (This vector $\mathbf{v}$ takes on the role of $q_2 - q_2'$.) We have no a-priori control over the direction of $\mathbf{v}$ (even though we know that $\|\mathbf{v}\| \approx \epsilon$, and we know that $\mathbf{v}$ is almost contained in $\mathbf{E}(q_2) \subset \mathbf{H}(q_2)$, where $\mathbf{E}(x)$ is defined in Section 8 as the union of the Lyapunov subspaces of $\mathbf{H}(x)$ corresponding to the U^+-inert Lyapunov exponents).

The idea is to vary u (while keeping q_1, q_1', ℓ fixed). To make this work, we need to define a *finite* collection of subspaces $\mathbf{E}_{[ij],bdd}(x)$ of $\mathbf{H}(x)$ (which actually only make sense on a certain finite measurable cover X of X_0) such that

(a) By varying u (while keeping q_1, q_1', ℓ fixed) we can make sure that the vector $\mathbf{v}$ becomes close to one of the subspaces $\mathbf{E}_{[ij],bdd}$, and
(b) For a suitable choice of point $q_3 = q_{3,ij} = g_{s_{ij}} q_1$, the map

$$(g_t u g_{-s_{ij}})_* \mathbf{E}_{[ij],bdd}(q_3) \to \mathbf{E}_{[ij],bdd}(q_2)$$

is a linear map whose norm is bounded independently of the parameters.
(c) Also, for a suitable choice of point $q_3' = q_{3,ij}' = g_{s_{ij}'} q_1$, the map

$$(g_t u g_{-s_{ij}'})_* \mathbf{E}_{[ij],bdd}\big(q_3'\big) \to \mathbf{E}_{[ij],bdd}\big(q_2'\big)$$

is a linear map whose norm is bounded independently of the parameters.

For the precise conditions see Proposition 10.1 and Proposition 10.2. This construction is done in detail in Section 10. The general idea is as follows: Suppose $\mathbf{v} \in \mathbf{E}_i(x) \oplus \mathbf{E}_j(x)$ where $\mathbf{E}_i(x)$ and $\mathbf{E}_j(x)$ are the Lyapunov subspaces corresponding to the U^+-inert (simple) Lyapunov exponents λ_i and λ_j. Then, if while varying u, the vector $\mathbf{v}$ does not swing towards either $\mathbf{E}_i$ or $\mathbf{E}_j$, we say that λ_i and λ_j are "synchronized". In that case, we consider the subspace $\mathbf{E}_{[i]}(x) = \mathbf{E}_i(x) \oplus \mathbf{E}_j(x)$ and show that (b) and (c) hold.

The conditions (b) and (c) allow us to define in Section 11 conditional measures f_{ij} on $W^+(x)$ which are associated to each subspace $\mathbf{E}_{[ij],bdd}$. In fact the measures are supported on the points $y \in W^+[x]$ such that the affine subspace $U^+[y]$ maps to a vector in $\mathbf{E}_{[ij],bdd}(x) \subset \mathbf{H}(x)$.

Technical Problem #4. — More careful analysis (see the discussion following the statement of Proposition 11.4) shows that the maps A and A′ of (2.4) are not exactly the same. Then, when one passes to the limit $\ell \to \infty$ one gets, instead of (2.6),

$$f_{ij}(\tilde{q}_2) = P^+(\tilde{q}_2, \tilde{q}_2')_* f_{ij}(\tilde{q}_2')$$

where $P^+ : W^+(\tilde{q}_2) \to W^+(\tilde{q}_2')$ is a certain unipotent map (defined in Section 4.2). Thus the conditional measure $f_{ij}(\tilde{q}_2)$ is invariant under the composition of a translation of size ϵ and a unipotent map. Repeating the argument with $\epsilon \to 0$ we obtain a point p such that the conditional measure at p is invariant under arbitrarily small combinations of (translation + unipotent map). This does *not* imply that the conditional measure $f_{ij}(p)$ restricts to Lebesgue measure on some subspace of W^+, but it does imply that it is in the Lebesgue measure class along some polynomial curve in W^+. More precisely, for ν-a.e $x \in X$ there is a subgroup $U_{new} = U_{new}(x)$ of the affine group of $W^+(x)$ such that the conditional measure of $f_{ij}(x)$ on the polynomial curve $U_{new}[x] \subset W^+[x]$ is induced from the Haar measure on U_{new}. (We call such a set a "generalized subspace".) The exact definition is given in Section 6.

Thus, during the induction steps, we need to deal with generalized subspaces. This is not a very serious complication since the general machinery developed in Section 6 can deal with generalized subspaces as well as with ordinary affine subspaces.

Completion of the proof of Theorem 2.1. — Let $\mathcal{L}(x) \subset H^1(M, \Sigma, \mathbb{R})$ be the smallest subspace such that $\nu_{W^-(x)}$ is supported on $\mathcal{L}^-(x)$. Roughly, the above argument can be iterated until we know the conditional measure $\nu_{W^+(x)}$ is Lebesgue on a subspace $\mathcal{U}^+[x]$, where $\mathcal{U}(x) \subset H^1(M, \Sigma, \mathbb{R})$ contains $\mathcal{L}(x)$. (The precise condition for when the induction stops is given by Lemma 6.15 and Proposition 6.16.) Then a Margulis-Tomanov style entropy comparison argument (see Section 13) shows that $\mathcal{U}(x) = \mathcal{L}(x)$, and the conditional measures along $\mathcal{L}^-(x)$ are Lebesgue. Since $\mathcal{U}^+(x)$ contains the orbit of the unipotent direction N, this implies that $\mathcal{L}^-(x)$ contains the orbit of the opposite unipotent direction $\bar{N} \subset SL(2, \mathbb{R})$. Thus, the conditional measure along the orbit of $\bar{N}$ is Lebesgue, which means that ν is $\bar{N}$-invariant. This, together with the assumption that ν is $P = AN$-invariant implies that ν is $SL(2, \mathbb{R})$-invariant, completing the proof of Theorem 2.1.

3. Hyperbolic properties of the geodesic flow

The spaces X_0 *and* $\tilde{X}_0$. — Let X_0 be a finite cover of the stratum $\mathcal{H}_1(\alpha)$ which is a manifold. (Such a cover may be obtained by choosing a level 3 structure, i.e. a basis for the mod 3 homology of the surface.) Let $\tilde{X}_0$ be the universal cover of X_0. Then the fundamental group $\pi_1(X_0)$ acts properly discontinuously on $\tilde{X}_0$. Let ν be a P-invariant ergodic probability measure on X_0.

We recall the following standard fact:

Lemma **3.1** *(Mautner phenomenon). — Let* ν *be an ergodic* P*-invariant measure on a space* Z*. Then* ν *is* A*-ergodic.*

Proof. — See e.g. [Moz]. □

Lemma **3.2.** — *For almost all* $x \in X_0$*, the affine exponential map from* $W^+(x)$ *to* $W^+[x]$ *is globally defined and is bijective, endowing* $W^+[x]$ *with a global affine structure. The same holds for* $W^-[x]$.

Proof. — Since W^- and W^+ play the role of the stable and unstable foliations for the action of $g_t \in A$ (cf. Lemma 3.5), this follows from the Poincaré recurrence theorem. □

The bundle H^1. — Let H^1 denote the bundle whose fiber above $x \in X_0$ is $H^1(M, \Sigma, \mathbb{R})$. We denote the fiber above the point $x \in X_0$ by $H^1(x)$.

The geodesic flow acts on H^1 by parallel transport using the Gauss-Manin connection (see Section 2.3).

The bundles H^1_+ *and* H^1_-. — Let H^1_+ denote the same bundle as H^1 except that the action of g_t on H^1_+ includes an extra multiplication by e^t on the fiber. (In other words, if $h_t(x, v) = (x, e^t v)$ and $i : H^1 \to H^1_+$ is the identity map, then $g_t \circ i(x, v) = h_t \circ i \circ g_t(x, v)$.) Similarly, let H^1_- denote the same bundle as H^1 except that the action of g_t includes an extra multiplication by e^{-t} on the fiber.

We use the notation $H^1_+(x)$ and $H^1_-(x)$ to refer to the fiber of the corresponding bundle above the point $x \in X_0$.

The bundles H_{big}, $H^{(+)}_{big}$, $H^{(-)}_{big}$, $H^{(++)}_{big}$ *and* $H^{(--)}_{big}$. — In this paper, we will need to deal with several bundles derived from the Hodge bundle H^1. It is convenient to introduce a bundle H_{big} so that every bundle we will need will be a subbundle of H_{big}. Let $d \in \mathbb{N}$ be a large integer chosen later (it will be chosen in Section 6 and will depend only on the Lyapunov spectrum of the Kontsevich-Zorich cocycle). Let

$$\hat{H}_{big}(x) = \bigoplus_{k=1}^{d} \bigoplus_{j=1}^{k} \left(\bigotimes_{i=1}^{j} H^1(x) \otimes \bigotimes_{l=1}^{k-j} (H^1(x))^* \right),$$

$$\hat{H}^{(+)}_{big}(x) = \bigoplus_{k=1}^{d} \bigoplus_{j=1}^{k} \left(\bigotimes_{i=1}^{j} H^1_+(x) \otimes \bigotimes_{l=1}^{k-j} (H^1_+(x))^* \right),$$

$$\hat{H}^{(-)}_{big}(x) = \bigoplus_{k=1}^{d} \bigoplus_{j=1}^{k} \left(\bigotimes_{i=1}^{j} H^1_-(x) \otimes \bigotimes_{l=1}^{k-j} (H^1_-(x))^* \right),$$

and let

$$\tilde{H}_{big}(x) = \hat{H}_{big}(x) \oplus \hat{H}^{(+)}_{big}(x) \oplus \hat{H}^{(-)}_{big}(x).$$

Suppose $L_1 \subset L_2 \subset \tilde{H}_{big}$ are g_t-invariant subbundles. We say that L_2/L_1 is an admissible quotient if the cocycle on L_2/L_1 is measurably conjugate to a conformal cocycle (see Lemma 4.3), and also L_2/L_1 is maximal in the sense that if $L_2' \supset L_2$ and $L_1' \subset L_1$ are g_t-invariant subbundles with the cocycle L_2'/L_1' measurably conjugate to a conformal cocycle, then $L_2' = L_2$ and $L_1' = L_1$. We then let Δ_{big} denote the set of all admissible quotients of $\tilde{H}_{big}$ and let

$$H_{big}(x) = \bigoplus_{Q \in \Delta_{big}} Q(x).$$

(We apply a similar operation to the bundles $\hat{H}^{(+)}_{big}$ and $\hat{H}^{(-)}_{big}$ to get bundles $H^{(+)}_{big}$ and $H^{(-)}_{big}$.)

The flow g_t acts on the bundle H_{big} in the natural way. We denote the action on the fibers by $(g_t)_*$. Let $H^{(++)}_{big}(x)$ denote the direct sum of the positive Lyapunov subspaces of $H_{big}(x)$. Similarly, let $H^{(--)}_{big}(x)$ denote the direct sum of the negative Lyapunov subspaces of $H_{big}(x)$.

Lemma **3.3.** — *The subspaces* $H^{(++)}_{big}(x)$ *are locally constant along* $W^+[x]$, *i.e. for almost all* $x \in \tilde{X}_0$ *and almost all* $y \in W^+[x]$ *close to* x *we have* $H^{(++)}_{big}(y) = H^{(++)}_{big}(x)$. *Similarly, the subspaces* $H^{(--)}_{big}(x)$ *are locally constant along* $W^-[x]$.

Proof. — Note that

$$H^{(++)}_{big}(x) = \left\{ v \in H_{big}(x) \ : \ \lim_{t\to\infty} \frac{1}{t} \log \frac{\|(g_{-t})_* v\|}{\|v\|} < 0 \right\}$$

Therefore, the subspace $H^{(++)}_{big}(x)$ depends only on the trajectory $g_{-t}x$ as $t \to \infty$. However, if $y \in W^+[x]$ then $g_{-t}y$ will for large t be close to $g_{-t}x$, and so in view of the affine structure, $(g_{-t})_*$ will be the same linear map on $H_{big}(x)$ and $H_{big}(y)$. This implies that $H^{(++)}_{big}(x) = H^{(++)}_{big}(y)$. □

The Avila-Gouëzel-Yoccoz norm. — The Avila-Gouëzel-Yoccoz norm on the relative cohomology group $H^1(M, \Sigma, \mathbb{R})$ is described in Appendix A. This then induces a norm which we will denote by $\|\cdot\|_Y$ and then, as the projective cross norm, also on H_{big}. We also use the notation $\|\cdot\|_{Y,x}$ to denote the AGY norm at $x \in X_0$.

The distance $d^+(x, y)$. — Since the tangent space to $W^+[x]$ is included in $H^1(M, \Sigma, \mathbb{R})$, the AGY norm on $H^1(M, \Sigma, \mathbb{R})$ defines a distance on $W^+[x]$. We denote this distance by $d^+(\cdot, \cdot)$. (Thus, for $y, z \in W^+[x]$, $d^+(y, z)$ is the length of the shortest path in $W^+[x]$ connecting y and z, where lengths of paths are measured using the AGY norm.)

The ball $B^+(x, r)$. — Let $B^+(x, r) \subset W^+[x]$ denote the ball of radius r centered at x, in the metric $d^+(\cdot, \cdot)$.

The following is a rephrasing of [AG, Proposition 5.3]:

Proposition **3.4.** — *For all* $x \in X_0$, $x + v$ *is well defined for* $v \in W^+(x)$ *with* $\|v\|_Y \leq 1/2$. *Also, for all* $y, z \in B^+(x, 1/50)$, *we have*

$$\frac{1}{2}\|y - z\|_{Y,y} \leq \|y - z\|_{Y,z} \leq 2\|y - z\|_{Y,y},$$

and

$$\frac{1}{2}\|y - z\|_{Y,y} \leq d^+(y, z) \leq 2\|y - z\|_{Y,y}.$$

Note that we have a similar distance $d^-(\cdot, \cdot)$ on $W^-[x]$, and the analogue of Proposition 3.4 holds.

The "distance" $d^{X_0}(\cdot, \cdot)$. — Suppose $x, y \in \tilde{X}_0$ are not far apart. Then, there exist unique $z \in W^+[x]$ and $t \in \mathbb{R}$ such that $g_t z \in W^-[y]$. We then define

$$d^{X_0}(x, y) = d^+(x, z) + |t| + d^-(g_t z, y).$$

Thus, if $y \in W^+[x]$ then $d^{X_0}(x, y) = d^+(x, y)$, and if $y \in W^-[x]$, then $d^{X_0}(x, y) = d^-(x, y)$.

We sometimes abuse notation by using the notation $d^{X_0}(x, y)$ where $x, y \in X_0$. By this we mean $d^{X_0}(\tilde{x}, \tilde{y})$ where $\tilde{x}$ and $\tilde{y}$ are appropriate lifts of x and y.

Choose a compact subset $K'_{thick} \subset X_0$ with $\nu(K'_{thick}) \geq 5/6$. Let $K_{thick} = \{x \in X_0 : d^{X_0}(x, K'_{thick}) \leq 1/100\}$.

Lemma **3.5.** — *There exists* $\alpha > 0$ *such that the following holds:*

(a) *Suppose* $x \in X_0$ *and* $t > 0$ *are such that the geodesic segment from* x *to* $g_t x$ *spends at least half the time in* K_{thick}. *Then, for all* $v \in W^-(x)$,

$$\|(g_t)_* v\|_Y \leq e^{-\alpha t}\|v\|_Y.$$

(b) *Suppose* $x \in X_0$ *and* $t > 0$ *are as in (a). Then, for all* $v \in W^+(x)$,

$$\|(g_t)_* v\|_Y \geq e^{\alpha t}\|v\|_Y.$$

(c) *For every* $\epsilon > 0$ *there exist a compact subset* $K''_{thick} \subset X_0$ *with* $\nu(K''_{thick}) > 1 - \epsilon$ *and* $t_0 > 0$ *such that for* $x \in K''_{thick}$, $t > t_0$ *and all* $v \in H^{(++)}_{big}(x)$,

$$\|(g_t)_* v\|_Y \geq e^{\alpha t}\|v\|_Y.$$

(d) *For all* $v \in W^+(x)$, *all* $x \in X_0$ *and all* $t > 0$,

$$\|(g_t)_* v\|_Y \geq \|v\|_Y.$$

Proof. — Parts (a), (b) and (d) follow from Theorem A.2. Part (c) follows immediately from the Osceledets multiplicative ergodic theorem. □

We also have the following simpler statement:

Lemma **3.6.** — *There exists* $N > 0$ *such that for all* $x \in X_0$, *all* $t \in \mathbb{R}$, *and all* $v \in H_{big}(x)$,

$$e^{-N|t|}\|v\|_Y \leq \|(g_t)_* v\|_Y \leq e^{N|t|}\|v\|_Y.$$

For $v \in W^+[x]$, *we can take* $N = 2$.

Proof. — This follows immediately from Theorem A.2. □

Proposition **3.7.** — *Suppose* $\mathcal{C} \subset X_0$ *is a set with* $\nu(\mathcal{C}) > 0$, *and* $T_0 : \mathcal{C} \to \mathbb{R}^+$ *is a measurable function which is finite a.e. Then we can find* $x_0 \in \tilde{X}_0$, *a subset* $\mathcal{C}_1 \subset W^-[x_0] \cap \pi^{-1}(\mathcal{C})$ *and for each* $c \in \mathcal{C}_1$ *a subset* $E^+[c] \subset W^+[c]$ *of diameter in the AGY metric at most* $1/200$ *and a number* $t(c) > 0$ *such that if we let*

$$J_c = \bigcup_{0 \leq t < t(c)} g_{-t} E^+[c],$$

then the following holds:

(a) $E^+[c]$ *is relatively open in* $W^+[c]$.
(b) $\pi(J_c) \cap \pi(J_{c'}) = \emptyset$ *if* $c \neq c'$.
(c) $\pi(J_c)$ *is embedded in* X_0, *i.e. if* $\pi(g_{-t}x) = \pi(g_{-t'}x')$ *where* $x, x' \in E^+[c]$ *and* $0 \leq t < t(c)$, $0 \leq t' < t(c)$ *then* $x = x'$ *and* $t = t'$.
(d) $\bigcup_{c \in \mathcal{C}_1} \pi(J_c)$ *is conull in* X_0.
(e) *For every* $c \in \mathcal{C}_1$ *there exists* $c' \in \mathcal{C}_1$ *such that* $\pi(g_{-t(c)}E^+[c]) \subset \pi(E^+[c'])$.
(f) $t(c) > T_0(c)$ *for all* $c \in \mathcal{C}_1$.

Remark. — All the construction in Section 3 will depend on the choice of $\mathcal{C}$ and T_0, but we will suppress this from the notation. The set $\mathcal{C}$ and the function T_0 will be finally chosen in Lemma 4.14.

The proof of Proposition 3.7 relies on the following:

Lemma **3.8.** — *Suppose* $\mathcal{C} \subset X_0$ *is a set with* $\nu(\mathcal{C}) > 0$, *and* $T_0 : \mathcal{C} \to \mathbb{R}^+$ *is a measurable function which is finite a.e. Then we can find* $x_0 \in \tilde{X}_0$, *a subset* $\mathcal{C}_1 \subset W^-[x_0] \cap \pi^{-1}(\mathcal{C})$ *and for each* $c \in \mathcal{C}_1$ *a subset* $E^+[c] \subset W^+[c]$ *of diameter in the AGY metric at most* $1/200$ *so that the following hold:*

(0) $E^+[c]$ *is a relatively open subset of* $W^+[c]$.
(1) *The set* $E = \pi(\bigcup_{c\in\mathcal{C}_1} E^+[c])$ *is embedded in* X_0, *i.e. if* $\pi(x) = \pi(x')$ *where* $x \in E^+[c]$ *and* $x' \in E^+[c']$, *then* $x = x'$ *and* $c = c'$.
(2) *For some* $\epsilon > 0$, $\nu(\bigcup_{t\in(0,\epsilon)} g_t E) > 0$.
(3) *If* $t > 0$ *and* $c \in \mathcal{C}_1$ *is such that* $\pi(g_{-t}E^+[c]) \cap E \neq \emptyset$, *then* $\pi(g_{-t}E^+[c]) \subset \pi(E^+[c'])$ *for some* $c' \in \mathcal{C}_1$.
(4) *Suppose* t, c, c' *are as in* (3). *Then* $t > T_0(c)$.

Proof. — This proof is essentially identical to the proof of Lemma B.1, except that we need to take care that (4) is satisfied. In this proof, for $x \in \mathcal{C}$, we denote by $\nu_{W^\pm[x]}$ the conditional measure of ν along $W^\pm[x] \cap \mathcal{C}$.

Choose $T_1 > 0$ so that if we let $\mathcal{C}_4 = \{x \in \mathcal{C} \;:\; T_0(x) < T_1\}$ then $\nu(\mathcal{C}_4) > \nu(\mathcal{C})/2$. Let X_{per} denote the union of the periodic orbits of g_t. By the P-invariance of ν and the ergodicity of g_t, $\nu(X_{per}) = 0$, and the same is true of the set $X'_{per} = \bigcup_{x\in X_{per}} W^-[x]$. Therefore there exists $x_0 \in \pi^{-1}(\mathcal{C}_4)$ and a compact subset $\mathcal{C}_3 \subset W^-[x_0] \cap \pi^{-1}(\mathcal{C}_4)$ with $\nu_{W^-[x_0]}(\mathcal{C}_3) > 0$ such that for $x \in \mathcal{C}_3$ and $0 < t < T_1$, $\pi(g_{-t}x) \notin \pi(\mathcal{C}_3)$. Then, since $\mathcal{C}_3$ is compact, we can find a small neighborhood $V^+ \subset W^+$ of the origin such that the set $\pi(\bigcup_{c\in\mathcal{C}_3} V^+[c])$ is embedded in X_0 and for $x \in \bigcup_{c\in\mathcal{C}_3} V^+[c]$ and $0 < t < T_1$, $\pi(g_{-t}x) \notin \pi(\bigcup_{c\in\mathcal{C}_3} V^+[c])$.

There exists $\mathcal{C}_2 \subset \mathcal{C}_3$ with $\nu_{W^-[x_0]}(\mathcal{C}_2) > 0$ and $N > T_1$ such that for all $c \in \mathcal{C}_2$ and all $T > N$,

$$\left|\left\{t \in [0, T] \;:\; \pi(g_{-t}c) \in K'_{thick}\right\}\right| \geq T/2.$$

Then, for $c \in \mathcal{C}_2$, $T > N$ and $x \in V^+[c]$,

$$\left|\left\{t \in [0, T] \;:\; \pi(g_{-t}x) \in K_{thick}\right\}\right| \geq T/2.$$

Let

$$M = \sup\left\{\frac{\|v\|_{Y,x}}{\|v\|_{Y,y}} \;:\; x \in V^+[c],\; y \in V^+[c],\; c \in \mathcal{C}_2,\; v \in W^+(x)\right\}$$

Let $\alpha > 0$ be as in Lemma 3.5, and choose $N_1 > N$ such that $M^2 e^{-\alpha N_1} < 1/10$. Then, for $c \in \mathcal{C}_2$, $x, y \in \pi(V^+[c])$ and $t > N_1$ such that $g_{-t}x \in \pi(\bigcup_{c\in\mathcal{C}_2} V^+[c])$, in view of Lemma 3.5 and Proposition 3.4,

$$d^{X_0}(g_{-t}x, g_{-t}y) \leq \frac{1}{10} d^{X_0}(x, y).$$

Now choose $\mathcal{C}_1 \subset \mathcal{C}_2$ with $\nu_{W^-[x_0]}(\mathcal{C}_1) > 0$ so that if we let $Y = \pi(\bigcup_{c\in\mathcal{C}_1} V^+[x])$ then $g_{-t}Y \cap Y = \emptyset$ for $0 < t < \max(T_1, N_1)$, in other words, the first return time to Y is at least $\max(T_1, N_1)$. (This can be done e.g. by Rokhlin's Lemma.) Condition (4) now follows since $T_0(c) < T_1$ for all $c \in \mathcal{C}_1$. The rest of the proof is essentially the same as the proof of Lemma B.1, applied to the first return map of g_{-t} to Y. □

Proof of Proposition 3.7. — For $x \in E$, let $t(x) \in \mathbb{R}^+$ be the smallest such that $g_{-t(x)}x \in E$. By property (3), the function $t(x)$ is constant on each set of the form $\pi(E^+[c])$. Let $F_t = \{x \in E : t(x) = t\}$. (We have $F_t = \emptyset$ if $t < N_1$.) By property (2) and the ergodicity of g_{-t}, up to a null set,

$$X_0 = \bigsqcup_{t>0} \bigsqcup_{s<t} g_{-s} F_t.$$

Then properties (a)–(f) are easily verified. □

Notation. — For $x \in X_0$, let $J[x]$ denote the set $\pi(J_c)$ containing x. For $x \in \tilde{X}_0$, let $J[x]$ denote γJ_c where $\gamma \in \pi_1(X_0)$ is such that $\gamma^{-1}x \in J_c$.

Lemma **3.9.** — *Suppose $x \in \tilde{X}_0$, $y \in W^+[x] \cap J[x]$. Then for any $t > 0$,*

$$g_{-t}y \in J[g_{-t}x] \cap W^+[g_{-t}x].$$

Proof. — This follows immediately from property (e) of Proposition 3.7. □

Notation. — For $x \in X_0$, let

$$\mathfrak{B}_t[x] = \pi\left(g_{-t}\left(J[g_t\tilde{x}] \cap W^+[g_t\tilde{x}]\right)\right), \quad \text{where } \tilde{x} \text{ is any element of } \pi^{-1}(x).$$

Lemma **3.10.**

(a) *For $t' > t \geq 0$, $\mathfrak{B}_{t'}[x] \subset \mathfrak{B}_t[x]$.*
(b) *Suppose $t \geq 0$, $t' \geq 0$, $x \in X_0$ and $x' \in X_0$ are such that $\mathfrak{B}_t[x] \cap \mathfrak{B}_{t'}[x'] \neq \emptyset$. Then either $\mathfrak{B}_t[x] \supseteq \mathfrak{B}_{t'}[x']$ or $\mathfrak{B}_{t'}[x'] \supseteq \mathfrak{B}_t[x]$ (or both).*

Proof. — Part (a) is a restatement of Lemma 3.9. For (b), without loss of generality, we may assume that $t' \geq t$. Then, by (a), we have $\mathfrak{B}_t[x] \cap \mathfrak{B}_t[x'] \neq \emptyset$.

Suppose $y \in \mathfrak{B}_t[x] \cap \mathfrak{B}_t[x']$. Then $g_t y \in \mathfrak{B}_0[g_t x]$ and $g_t y \in \mathfrak{B}_0[g_t x']$. Since the sets $\mathfrak{B}_0[z]$, $z \in X_0$ form a partition, we must have $\mathfrak{B}_0[g_t x] = \mathfrak{B}_0[g_t x']$. Therefore, $\mathfrak{B}_t[x] = \mathfrak{B}_t[x']$, and thus, by (a),

$$\mathfrak{B}_{t'}\left[x'\right] \subset \mathfrak{B}_t\left[x'\right] = \mathfrak{B}_t[x].$$ □

By construction, the sets $\mathfrak{B}_0[x]$ are the atoms of a measurable partition of X_0 subordinate to W^+ (see Definition B.4). Then, let $\nu_{W^+[x]}$ denote the conditional measure of ν along the atom of the partition containing x. For notational simplicity, for $E \subset W^+[x]$, we sometimes write $\nu_{W^+}(E)$ instead of $\nu_{W^+[x]}(E)$.

Lemma **3.11.** — *Suppose* $\delta > 0$ *and* $K \subset X_0$ *is such that* $\nu(K) > 1 - \delta$. *Then there exists a subset* $K^* \subset K$ *with* $\nu(K^*) > 1 - \delta^{1/2}$ *such that for any* $x \in K^*$, *and any* $t > 0$,

$$\nu_{W^+}\big(K \cap \mathfrak{B}_t[x]\big) \geq \big(1 - \delta^{1/2}\big)\nu_{W^+}\big(\mathfrak{B}_t[x]\big).$$

Proof. — Let $E = K^c$, so $\nu(E) \leq \delta$. Let E^* denote the set of $x \in X_0$ such that there exists some $\tau \geq 0$ with

$$\nu_{W^+}\big(E \cap \mathfrak{B}_\tau[x]\big) \geq \delta^{1/2}\nu_{W^+}\big(\mathfrak{B}_\tau[x]\big). \tag{3.1}$$

It is enough to show that $\nu(E^*) \leq \delta^{1/2}$. Let $\tau(x)$ be the smallest $\tau > 0$ so that (3.1) holds for x. Then the (distinct) sets $\{\mathfrak{B}_{\tau(x)}[x]\}_{x\in E^*}$ cover E^* and are pairwise disjoint by Lemma 3.10(b). Let

$$F = \bigcup_{x\in E^*} \mathfrak{B}_{\tau(x)}[x].$$

Then $E^* \subset F$. For every set of the form $\mathfrak{B}_0[y]$, let $\Delta(y)$ denote the set of distinct sets $\mathfrak{B}_{\tau(x)}[x]$ where x varies over $\mathfrak{B}_0[y]$. Then, by (3.1)

$$\begin{aligned}\nu_{W^+}\big(F \cap \mathfrak{B}_0[y]\big) &= \sum_{\Delta(y)} \nu_{W^+}(\mathfrak{B}_{\tau(x)}) \\ &\leq \delta^{-1/2} \sum_{\Delta(y)} \nu_{W^+}\big(E \cap \mathfrak{B}_{\tau(x)}[x]\big) \leq \delta^{-1/2}\nu_{W^+}\big(E \cap \mathfrak{B}_0[y]\big).\end{aligned}$$

Integrating over y, we get $\nu(F) \leq \delta^{-1/2}\nu(E)$. Hence,

$$\nu\big(E^*\big) \leq \nu(F) \leq \delta^{-1/2}\nu(E) \leq \delta^{1/2}.$$

□

4. General cocycle lemmas

4.1. *Lyapunov subspaces and flags.* — Let $\mathcal{V}_i(H^1)(x)$, $1 \leq i \leq k$ denote the Lyapunov subspaces of the Kontsevich-Zorich cocycle under the action of the geodesic flow g_t, and let $\lambda_i(H^1)$, $1 \leq i \leq k$ denote the (distinct) Lyapunov exponents. Then we have for almost all $x \in X_0$,

$$H^1(M, \Sigma, \mathbb{R}) = \bigoplus_{i=1}^{k} \mathcal{V}_i\big(H^1\big)(x)$$

and for all non-zero $v \in \mathcal{V}_i(H^1)(x)$,

$$\lim_{t\to\pm\infty} \frac{1}{t} \log \frac{\|(g_t)_* v\|}{\|v\|} = \lambda_i\big(H^1\big),$$

where $\|\cdot\|$ is any reasonable norm on $H^1(M, \Sigma, \mathbb{R})$ for example the Hodge norm or the AGY norm defined in Section A.1. By the notation $(g_t)_* v$ we mean the action of the geodesic flow (i.e. parallel transport using the Gauss-Manin connection) on the Hodge bundle $H^1(M, \Sigma, \mathbb{R})$. We note that the Lyapunov exponents of the geodesic flow (viewed as a diffeomorphism of X_0) are in fact $1 + \lambda_i$, $1 \le i \le k$ and $-1 + \lambda_i$, $1 < i \le k$.

We have

$$1 = \lambda_1\big(H^1\big) > \lambda_2\big(H^1\big) > \cdots > \lambda_k\big(H^1\big) = -1.$$

It is a standard fact that $\dim \mathcal{V}_1(H^1) = \dim \mathcal{V}_k(H^1) = 1$, $\mathcal{V}_1(H^1)$ corresponds to the direction of the unipotent N and $\mathcal{V}_k(H^1)$ corresponds to the direction of $\bar{N}$. Let $p : H^1(M, \Sigma, \mathbb{R}) \to H^1(M, \mathbb{R})$ denote the natural map. Recall that if $x \in X_0$ denotes the pair (M, ω), then

$$H^1_\perp(x) = \big\{\alpha \in H^1(M, \Sigma, \mathbb{R}) \ : \ p(\alpha) \wedge \mathrm{Re}(\omega) = p(\alpha) \wedge \mathrm{Im}(\omega) = 0\big\}.$$

Then

$$H^1_\perp(x) = \bigoplus_{i=2}^{k-1} \mathcal{V}_i\big(H^1\big)(x).$$

We note that the subspaces $H^1_\perp(x)$ are equivariant under the $SL(2, \mathbb{R})$ action on X_0 (since so is the subspace spanned by $\mathrm{Re}\,\omega$ and $\mathrm{Im}\,\omega$). Since the cocycle preserves the symplectic form on $p(H^1_\perp)$, we have

$$\lambda_{k+1-i}\big(H^1\big) = -\lambda_i\big(H^1\big), \quad 1 \le i \le k.$$

Let

$$\mathcal{V}_{\le i}\big(H^1\big)(x) = \bigoplus_{j=1}^{i} \mathcal{V}_j\big(H^1\big)(x), \qquad \mathcal{V}_{\ge i}\big(H^1\big)(x) = \bigoplus_{j=i}^{k} \mathcal{V}_j\big(H^1\big)(x).$$

Then we have the Lyapunov flags

$$\{0\} = \mathcal{V}_{\le 0}\big(H^1\big)(x) \subset \mathcal{V}_{\le 1}\big(H^1\big)(x) \subset \cdots \subset \mathcal{V}_{\le k}\big(H^1\big)(x) = H^1(M, \Sigma, \mathbb{R})$$

and

$$\{0\} = \mathcal{V}_{>k}\big(H^1\big)(x) \subset \mathcal{V}_{>k-1}\big(H^1\big)(x) \subset \cdots \subset \mathcal{V}_{>0}\big(H^1\big)(x) = H^1(M, \Sigma, \mathbb{R}).$$

We record some simple properties of the Lyapunov flags:

Lemma **4.1.**

(a) *The subspaces* $\mathcal{V}_{\le i}(H^1)(x)$ *are locally constant along* $W^+[x]$*, i.e. for almost all* $x \in X_0$*, for almost all* $y \in W^+[x]$ *close to x we have* $\mathcal{V}_{\le i}(H^1)(y) = \mathcal{V}_{\le i}(H^1)(x)$ *for all* $1 \le i \le k$*. (Here and in (b) we identify* $H^1(x)$ *with* $H^1(y)$ *using the Gauss-Manin connection.)*

(b) *The subspaces* $\mathcal{V}_{\ge i}(H^1)(x)$ *are locally constant along* $W^-[x]$*, i.e. for almost all* $x \in X_0$ *and for almost all* $y \in W^-[x]$ *close to x we have* $\mathcal{V}_{\ge i}(H^1)(y) = \mathcal{V}_{\ge i}(H^1)(x)$ *for all* $1 \le i \le k$*.*

Proof. — To prove (a), note that

$$\mathcal{V}_{\le i}(H^1)(x) = \left\{ v \in H^1(M, \Sigma, \mathbb{R}) \ : \ \lim_{t\to\infty} \frac{1}{t} \log \frac{\|(g_{-t})_* v\|}{\|v\|} \le -\lambda_i \right\}.$$

Therefore, the subspace $\mathcal{V}_{\le i}(H^1)(x)$ depends only on the trajectory $g_{-t}x$ as $t \to \infty$. However, if $y \in W^+[x]$ then $g_{-t}y$ will for large t be close to $g_{-t}x$, and so in view of the affine structure, $(g_{-t})_*$ will be the same linear map on $H^1(M, \Sigma, \mathbb{R})$ at x and y, as in Section 3. This implies that $\mathcal{V}_{\le i}(H^1)(x) = \mathcal{V}_{\le i}(H^1)(y)$. The proof of property (b) is identical. □

The action on H^1_+ *and* H^1_-. — Recall that the bundles H^1_+ and H^1_- were defined in Section 3. All of the results of Section 4.1 also apply to these bundles. Also,

$$\lambda_i(H^1_+) = 1 + \lambda_i(H^1), \qquad \lambda_i(H^1_-) = -1 + \lambda_i(H^1).$$

Furthermore, under the natural identification by the identity map, for all $x \in X_0$,

$$\mathcal{V}_i(H^1_+)(x) = \mathcal{V}_i(H^1_-)(x) = \mathcal{V}_i(H^1)(x).$$

4.2. *Equivariant measurable flat connections.* — Let L be a subbundle of $H^{(++)}_{big}$. Recall that by Lemma 3.2, typical leaves of W^+ are simply connected. By an equivariant measurable flat W^+-connection on L we mean a measurable collection of linear "parallel transport" maps:

$$F(x, y) : L(x) \to L(y)$$

defined for ν-almost all $x \in X_0$ and $\nu_{W^+[x]}$ almost all $y \in W^+[x]$ such that

$$F(y, z)F(x, y) = F(x, z), \tag{4.1}$$

and

$$(g_t)_* \circ F(x, y) = F(g_t x, g_t y) \circ (g_t)_*. \tag{4.2}$$

For example, if $L = W^+(x)$, then the Gauss-Manin connection (which in period local coordinates is the identity map) is an equivariant measurable flat W^+ connection on H^1. However, there is another important equivariant measurable flat W^+-connection on H^1 which we describe below.

The maps $P^+(x,y)$ *and* $P^-(x,y)$. — Recall that $\mathcal{V}_i(H^1)(x) \subset H^1(x)$ are the Lyapunov subspaces for the flow g_t. Recall that the $\mathcal{V}_i(H^1)(x)$ are not locally constant along leaves of W^+, but by Lemma 4.1, the subspaces $\mathcal{V}_{\le i}(H^1)(x) = \sum_{j=1}^{i} \mathcal{V}_j(W^+)(x)$ are locally constant along the leaves of W^+. Now suppose $y \in W^+[x]$. Any vector $v \in \mathcal{V}_i(H^1)(x)$ can be written uniquely as

$$v = v' + v'' \quad v' \in \mathcal{V}_i\big(H^1\big)(y), \quad v'' \in \mathcal{V}_{<i}\big(H^1\big)(y).$$

Let $P_i^+(x,y) : \mathcal{V}_i(H^1)(x) \to \mathcal{V}_i(H^1)(y)$ be the linear map sending v to v'. Let $P^+(x,y)$ be the unique linear map which restricts to $P_i^+(x,y)$ on each of the subspaces $\mathcal{V}_i(H^1)(x)$. We call $P^+(x,y)$ the "parallel transport" from x to y. The following is immediate from the definition:

Lemma **4.2.** — *Suppose* $x, y \in W^+[z]$. *Then*

(a) $P^+(x,y)\mathcal{V}_i(H^1)(x) = \mathcal{V}_i(H^1)(y)$.
(b) $P^+(g_t x, g_t y) = (g_t)_* \circ P^+(x,y) \circ (g_t^{-1})_*$.
(c) $P^+(x,y)\mathcal{V}_{\le i}(H^1)(x) = \mathcal{V}_{\le i}(H^1)(y)$. *If we identify* $H^1(x)$ *with* $H^1(y)$ *using the Gauss-Manin connection, then the map* $P^+(x,y)$ *is unipotent.*
(d) $P^+(x,z) = P^+(y,z) \circ P^+(x,y)$.

Note that the map P^+ on H^1_+ is the same as on H^1, provided we identify H^1_+ with H^1 via the identity map.

The statements (b) and (d) imply that the maps $P^+(x,y)$ define an equivariant measurable flat W^+-connection on H^1. This connection is in general different from the Gauss-Manin connection, and is only measurable.

If $y \in W^-[x]$, then we can define a similar map which we denote by $P^-(x,y)$. This yields an equivariant measurable flat W^--connection on H^1.

Clearly the connection $P^+(x,y)$ induces an equivariant measurable flat W^+-connection on $H^{(++)}_{big}$. This connection preserves the Lyapunov subspaces of the g_t-action on $H^{(++)}_{big}$, as in Lemma 4.2(a). In view of Proposition 4.12 below, the connection $P^+(x,y)$ also induces an equivariant measurable flat W^+-connection on any g_t-equivariant subbundle of $H^{(++)}_{big}$.

Equivariant measurable flat U^+*-connections.* — Suppose $U^+[x] \subset W^+[x]$ is a g_t-equivariant family of algebraic subsets, with $U^+[y] = U^+[x]$ for $y \in U^+[x]$. In fact, we will only consider families compatible with ν as defined in Definition 6.2. We denote the conditional measure of ν along $U^+[x]$ by $\nu_{U^+[x]}$. In the cases we will consider, these measures are well defined a.e. and are in the Lebesgue measure class, see Section 6.

By an equivariant measurable flat U^+-connection on a bundle $L \subset H^{(++)}_{big}$ we mean a measurable collection of linear maps $F(x,y) : L(x) \to L(y)$ satisfying (4.1) and (4.2), defined for ν-almost all $x \in X_0$ and $\nu_{U^+[x]}$-almost all $y \in U^+[x]$.

4.3. *The Jordan canonical form of a cocycle.*

Zimmer's amenable reduction. — The following is a general fact about linear cocycles over an action of $\mathbb{R}$ or $\mathbb{Z}$. It is often called "Zimmer's amenable reduction". We state it only for the cases which will be used.

Lemma **4.3.** — *Suppose* L_i *is a* g_t*-equivariant subbundle of* $H^{(++)}_{big}$. *(For example, we could have* $L_i(x) = \mathcal{V}_i(H^1_+)(x)$*.) Then, there exists a measurable finite cover* $\sigma_{L_i} : X_{L_i} \to X_0$ *such that for* $\sigma_{L_i}^{-1}(\nu)$*-a.e* $x \in X_{L_i}$ *there exists an invariant flag*

$$\{0\} = L_{i,0}(x) \subset L_{i,1}(x) \subset \cdots \subset L_{i,n_i}(x) = L_i(x), \tag{4.3}$$

and on each $L_{ij}(x)/L_{i,j-1}(x)$ *there exists a nondegenerate quadratic form* $\langle\cdot,\cdot\rangle_{ij,x}$ *and a cocycle* $\lambda_{ij} : X_{L_i} \times \mathbb{R} \to \mathbb{R}$ *such that for all* $u, v \in L_{ij}(x)/L_{i,j-1}(x)$,

$$\big\langle (g_t)_* u, (g_t)_* v \big\rangle_{ij, g_t x} = e^{\lambda_{ij}(x,t)} \langle u, v\rangle_{ij,x}.$$

(Note: For each i*, the pullback measures* $\sigma_{L_i}^{-1}(\nu)$ *is uniquely defined by the condition that for almost all* $x_0 \in X_0$*, the conditional of* $\sigma_{L_i}^{-1}(\nu)$ *on the (finite) set* $\sigma_{L_i}^{-1}(x_0)$ *is the normalized counting measure.)*

Remark. — The statement of Lemma 4.3 is the assertion that on the finite cover X_{L_i} one can make a change of basis at each $x \in X_{L_i}$ so that in the new basis, the matrix of the cocycle restricted to L_i is of the form

$$\begin{pmatrix} C_{i,1} & * & \dots & * \\ 0 & C_{i,2} & \dots & * \\ \vdots & \vdots & \ddots & * \\ 0 & 0 & \dots & C_{i,n_i} \end{pmatrix}, \tag{4.4}$$

where each $C_{i,j}$ is a conformal matrix (i.e. is the composition of an orthogonal matrix and a scaling factor λ_{ij}).

We call a cocycle *block-conformal* if all the off-diagonal entries labeled $*$ in (4.4) are 0.

Proof of Lemma 4.3. — See [ACO] (which uses many of the ideas of Zimmer). The statement differs slightly from that of [ACO, Theorem 5.6] in that we want the cocycle in each block to be conformal (and not just block-conformal). However, our statement is in fact equivalent because we are willing to replace the original space X_0 by a finite cover X_{L_i}. □

4.4. *Covariantly constant subspaces.* — The main result of this subsection is the following:

Proposition **4.4.** — *Suppose* L *is a* g_t*-equivariant subbundle over the base* X_0. *We can write*

$$L(x) = \bigoplus_i L_i(x),$$

where $L_i(x) \equiv \mathcal{V}_i(L)(x)$ *is the Lyapunov subspace corresponding to the Lyapunov exponent* λ_i. *Suppose there exists an equivariant flat measurable* W^+*-connection* F *on* L, *such that*

$$F(x, y)L_i(x) = L_i(y). \tag{4.5}$$

Suppose that $\mathcal{M}$ *is a finite collection of subspaces of* L *which is* g_t*-equivariant. Then, for almost all* $x \in X_0$ *and almost all* $y \in \mathfrak{B}_0[x]$,

$$F(x, y)\mathcal{M}(x) = \mathcal{M}(y),$$

i.e. the collection of subspaces $\mathcal{M}$ *is locally covariantly constant with respect to the connection* F.

Remark. — The same result holds if F is only assumed to be a measurable U^+-connection, and $\mathfrak{B}_0[x]$ is replaced by $\mathcal{B}[x]$.

The following is a generalization of Lemma 4.1:

Corollary **4.5.** — *Suppose* $M \subset H^1(M, \Sigma, \mathbb{R})$ *is a* g_t*-equivariant subbundle over the base* X_0. *Suppose also for a.e* $x \in X_0$, $\mathcal{V}_{<i}(x) \subset M(x) \subset \mathcal{V}_{\le i}(x)$. *Then (up to a set of measure* 0*),* $M(x)$ *is locally constant along* $W^+(x)$.

Proof of Corollary 4.5. — By Lemma 4.1, $L(x) \equiv \mathcal{V}_{\le i}(x)/\mathcal{V}_{<i}(x)$ is locally constant along $W^+[x]$. Let $F(x, y)$ denote the Gauss-Manin connection (i.e. the identity map) on $L(x)$. Note that the action of g_t on $L(x)$ has only one Lyapunov exponent, namely λ_i. Thus, (4.5) is trivially satisfied. Then, by Proposition 4.4, $M(x)/\mathcal{V}_{<i}(x) \subset L(x)$ is locally constant along $W^+[x]$. Since $\mathcal{V}_{<i}(x)$ is also locally constant (by Lemma 4.1), this implies that $M(x)$ is locally constant. □

Remark. — Our proof of Proposition 4.4 is essentially by reference to [L, Theorem 1]. It is given in Section 4.9* and can be skipped on first reading. For similar results in a partially hyperbolic setting see [AV2], [ASV], [KS].

4.5. *Some estimates on Lyapunov subspaces.* — Let $(V, \|\cdot\|_Y)$ be a normed vector space. By a splitting $E = (E_1, \dots, E_n)$ of V we mean a direct sum decomposition

$$V = E_1 \oplus \cdots \oplus E_n$$

Suppose $E = (E_1, \dots, E_n)$ and $E' = (E'_1, \dots, E'_n)$ are two splittings of V, with $\dim E_i = \dim E'_i$ for $1 \le i \le n$.

We define

$$D^+(E, E') = \max_{1 \le i \le n} \sup_{v \in \bigoplus_{j \le i} E_j \setminus \{0\}} \inf \left\{ \frac{\|w\|_Y}{\|v\|_Y} \ : \ v + w \in \bigoplus_{j \le i} E'_j, \text{ and } w \in \bigoplus_{j > i} E_j \right\},$$

and

$$D^-(E, E') = \max_{1 \le i \le n} \sup_{v \in \bigoplus_{j \ge i} E_j \setminus \{0\}} \inf \left\{ \frac{\|w\|_Y}{\|v\|_Y} \ : \ v + w \in \bigoplus_{j \ge i} E'_j, \text{ and } w \in \bigoplus_{j < i} E_j \right\}.$$

Note that $D^+(E, E')$ depends on E' only via the flag $\bigoplus_{j \le i} E'_j$, $1 \le i \le n$. Similarly, $D^-(E, E')$ depends on E' only via the flag $\bigoplus_{j \ge i} E'_j$, $1 \le i \le n$. Also $D^+(E, E') = D^-(E, E') = 0$ if $E = E'$, and $D^+(E, E') = \infty$ if some $\bigoplus_{j \le i} E'_j$ has non-trivial intersection with $\bigoplus_{j > i} E_j$.

In this subsection, we write $\mathcal{V}_i(x)$ for $\mathcal{V}_i(H^1)(x)$, etc. For almost all x in $\tilde{X}_0$, we have the splitting

$$H^1(x) = \mathcal{V}_1(x) \oplus \cdots \oplus \mathcal{V}_n(x).$$

For $x, y \in \tilde{X}_0$, we have the Gauss-Manin connection $P^{GM}(x, y)$, which is a linear map from $H^1(x)$ to $H^1(y)$ (see Section 2.3). Let

$$D^+(x, y) = D^+\big((\mathcal{V}_1(x), \ldots, \mathcal{V}_n(x)), (P^{GM}(y, x)\mathcal{V}_1(y), \ldots, P^{GM}(y, x)\mathcal{V}_n(y))\big).$$
$$D^-(x, y) = D^-\big((\mathcal{V}_1(x), \ldots, \mathcal{V}_n(x)), (P^{GM}(y, x)\mathcal{V}_1(y), \ldots, P^{GM}(y, x)\mathcal{V}_n(y))\big).$$

Distance between subspaces. — For a subspace V of $H^1(x)$, let SV denote the intersection of V with the unit ball in the AGY norm.

For subspaces V_1, V_2 of $H^1(x)$, we define

(**4.6**) $$d_Y(V_1, V_2) = \text{The Hausdorff distance between } SV_1 \text{ and } SV_2$$

measured with respect to the AGY norm at x.

Lemma **4.6.** — *There exists a continuous function* $C_0 : X_0 \to \mathbb{R}^+$ *such that for subspaces* V_1, V_2 *of* $H^1(x)$ *of the same dimension,*

$$C_0(x)^{-1} d_Y(V_1, V_2) \le \delta_Y(V_1, V_2) \le d_Y(V_1, V_2),$$

where

$$\delta_Y(V_1, V_2) = \max_{v_1 \in SV_1} \min_{v_2 \in SV_2} \|v_1 - v_2\|_Y.$$

Proof. — Since $d_Y(V_1, V_2) = \max(\delta_Y(V_1, V_2), \delta_Y(V_2, V_1))$, the inequality on the left follows immediately from the definition of the Hausdorff distance. To prove the inequality on the right it is enough to show that for some continuous function $C_0 : X_0 \to \mathbb{R}^+$,

$$C_0(x)^{-1}\delta_Y(V_2, V_1) \le \delta_Y(V_1, V_2). \tag{4.7}$$

To prove (4.7), pick some arbitrary inner product $\langle\cdot,\cdot\rangle_0$ on $H^1(M, \Sigma, \mathbb{R})$, and let $\|\cdot\|_0$ be the associated norm. Then, there exists a continuous function $C_1 : X_0 \to \mathbb{R}^+$ such that for all $v \in H^1(x)$,

$$C_1(x)^{-1}\|v\|_0 \le \|v\|_Y \le C_1(x)\|v\|_0.$$

Let $\delta_0(\cdot,\cdot)$ and $d_0(\cdot,\cdot)$ be the analogues of $\delta_Y(\cdot,\cdot)$ and $d_Y(\cdot,\cdot)$ for the norm $\|\cdot\|_0$. Then, it is enough to prove that there exists a constant $c_2 > 0$ depending only on the dimension such that for subspaces V_1, V_2 of equal dimension,

$$c_2\delta_0(V_2, V_1) \le \delta_0(V_1, V_2). \tag{4.8}$$

For subspaces U, V of equal dimension n, let $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ be orthonormal bases for U and V respectively. Then, we have

$$\left(\sum_{i=1}^{n} \inf_{v\in V}\|u_i - v\|_0^2\right)^{1/2} = \left(n - \sum_{i=1}^{n}\sum_{j=1}^{n}\langle u_i, v_j\rangle_0^2\right)^{1/2} \tag{4.9}$$

Note that the expression on the left in (4.9) is independent of the basis for V, and the expression on the right of (4.9) is symmetric in U and V. Thus, the expression in (4.9) is independent of the basis for U as well, and thus defines a function $d_H(U, V)$. (This function is called the Frobenius or chordal distance between subspaces, see e.g. [De], [WWF].)

From the expression on the left of (4.9) it is clear that there exists a constant c_3 depending only on the dimension so that

$$c_3 d_H(V_1, V_2) \le d_0(V_1, V_2) \le c_3^{-1} d_H(V_1, V_2).$$

Since $d_H(V_1, V_2) = d_H(V_2, V_1)$, (4.8) follows. □

Lemma **4.7.** — *There exists $\alpha > 0$ depending only on the Lyapunov spectrum, and a function $C : X_0 \to \mathbb{R}^+$ finite almost everywhere such that the following holds:*

(a) *For all $t > 0$, and all $x \in \tilde{X}_0$, and all $y \in \tilde{X}_0$ such that $d^{X_0}(g_s x, g_s y) \le 1/100$ for $0 \le s \le t$, we have, for all $1 \le i \le n$,*

$$d_Y\big(\mathcal{V}_{\le i}(g_t x), P^{GM}(g_t y, g_t x)\mathcal{V}_{\le i}(g_t y)\big) \le \min_{0\le s\le t} C(g_s x)\big(1 + D^+(x, y)\big)e^{-\alpha t}.$$

(b) *For all* $t > 0$, *and all* $x \in \tilde{X}_0$, *and all* $y \in \tilde{X}_0$ *such that* $d^{X_0}(g_{-s}x, g_{-s}y) \leq 1/100$ *for* $0 \leq s \leq t$, *we have, for all* $1 \leq i \leq n$,

$$d_Y\big(\mathcal{V}_{\geq i}(g_{-t}x), P^{GM}(g_{-t}y, g_{-t}x)\mathcal{V}_{\geq i}(g_{-t}y)\big) \leq \min_{0\leq s\leq t} C(g_{-s}x)\big(1 + D^-(x,y)\big)e^{-\alpha t}.$$

The proof of Lemma 4.7 is a straightforward but tedious argument using the Osceledets multiplicative ergodic theorem. It is done in Section 4.8*.

Lemma **4.8.** — *There exists a function* $C_3 : X_0 \to \mathbb{R}^+$ *finite almost everywhere, such that for all* $x \in \tilde{X}_0$, *all* $y \in W^-[x]$ *with* $d^{X_0}(x,y) < 1/100$ *we have* $D^+(x,y) \leq C_3(x)C_3(y)$. *Similarly, for all* $x \in \tilde{X}_0$, *all* $y \in W^+[x]$ *with* $d^{X_0}(x,y) < 1/100$ *we have* $D^-(x,y) \leq C_3(x)C_3(y)$.

Proof of Lemma 4.8. — For $\epsilon > 0$, let $K_\epsilon \subset X_0$ be a compact set with measure at least $1-\epsilon$ on which the functions $x \to \mathcal{V}_i(x)$ are continuous. Then there exists $\rho = \rho(\epsilon)$ such that if $x' \in \pi^{-1}(K_\epsilon)$, $y' \in W^-[x] \cap \pi^{-1}(K_\epsilon)$ and $d^{X_0}(x',y') < \rho$ then $D^+(x',y') < 1$. Then, by the Birkhoff ergodic theorem and Lemma 3.5, there exists a compact $K'_\epsilon \subset X_0$ with $\nu(K'_\epsilon) > 1 - 2\epsilon$ and $C_2 = C_2(\epsilon)$ such that for all $x \in \pi^{-1}(K'_\epsilon)$, all $y \in W^-[x] \cap \pi^{-1}(K'_\epsilon)$ with $d^{X_0}(x,y) < 1/100$ there exists $C_2(\epsilon) < t' < 2C_2(\epsilon)$ with $g_{t'}x \in K_\epsilon$, $g_{t'}y \in K_\epsilon$ and $d^{X_0}(x,y) < \rho(\epsilon)$. Thus, $D^+(g_{t'}x, g_{t'}y) < 1$, which implies that $D^+(x,y) < C'_2 = C'_2(\epsilon)$. Without loss of generality, we may assume that $C'_2 \geq 1$ and that K'_ϵ and $C'_2(\epsilon)$ both decrease as functions of ϵ. Now for $x \in X_0$, let $\Upsilon(x) = \{\epsilon \ : \ x \in K'_\epsilon\}$, and let

$$C_3(x) = \inf\big\{C'_2(\epsilon) \ : \ \epsilon \in \Upsilon(x)\big\}.$$

The proof of the second assertion is identical. □

Corollary **4.9.** — *There exists a measurable function* $C_1 : X_0 \to \mathbb{R}^+$ *finite a.e such that if* $x \in X_0$, $y \in W^-[x]$ *with* $d^{X_0}(x,y) < 1/100$, *we have for all* $t > 0$,

$$\|P^-(g_tx, g_ty)P^{GM}(g_ty, g_tx) - I\|_Y \leq C_1(x)C_1(y)e^{-\alpha t}, \tag{4.10}$$

where $\alpha > 0$ *depends only on the Lyapunov spectrum. Consequently, for almost all* $x \in X_0$, *and almost all* $y \in W^-[x]$,

$$\lim_{t\to\infty} \|P^-(g_tx, g_ty)P^{GM}(g_ty, g_tx) - I\|_Y = 0. \tag{4.11}$$

The same assertions hold if W^- *is replaced by* W^+, g_t *by* g_{-t} *and* P^- *by* P^+.

Proof of Corollary 4.9. — Let $C_1(x) = C(x)C_3(x)$, where $C(\cdot)$ is as in Lemma 4.7 and $C_3(\cdot)$ is as in Lemma 4.8. Then, by Lemmas 4.7 and 4.8,

$$d_Y\big(\mathcal{V}_{\leq i}(g_tx), P^{GM}(g_ty, g_tx)\mathcal{V}_{\leq i}(g_ty)\big) \leq C_1(x)C_1(y)e^{-\alpha t}.$$

Since by Lemma 4.1, $\mathcal{V}_{\geq i}(x) = P^{GM}(y, x)\mathcal{V}_{\geq i}(y)$, we get, for $t > 0$,

$$d_Y\big(\mathcal{V}_i(g_t x), P^{GM}(g_t y, g_t x)\mathcal{V}_i(g_t y)\big) \leq C_1(x)C_1(y)e^{-\alpha t}.$$

This, by the definition of $P^-(x, y)$, implies that (4.10) holds as required. Even if we do not assume that $d^{X_0}(x, y) < 1/100$, then for almost all x and almost all $y \in W^-[x]$, for t large enough $d^{X_0}(g_t x, g_t y) < 1/100$, and thus, in view of (4.10), (4.11) holds. □

4.6. *The cover* X. — Let $L = H_{big}$ viewed as a bundle over X_0. Let $L_i = \mathcal{V}_i(L)$. By Lemma 4.3, there exists a measurable finite cover X of X_0 such that Lemma 4.3 holds on X for all the L_i. We always assume that the degree of the covering map $\sigma_0 : X \to X_0$ is as small as possible.

The set $\Delta(x_0)$. — For $x_0 \in X_0$, let $\Delta_i(x_0)$ denote the set of flags

$$\Delta_i(x_0) = \big\{\{0\} = L_{i,0}(x) \subset L_{i,1}(x) \subset \cdots \subset L_{i,n_i}(x) = L_i(x) \ : \ x \in \sigma_0^{-1}(x_0)\big\}.$$

Let $\Delta(x_0)$ denote the Cartesian product of the $\Delta_i(x_0)$. Then, we can think of a point $x \in X$ as a pair $(x_0, \mathfrak{F})$ where $\mathfrak{F} \in \Delta(x_0)$.

The measure ν *on* X. — We can use σ_0 to define a pullback of the invariant measure ν on X_0 to X, by requiring that the pushforward of the pullback measure by σ_0 is ν, and that the conditionals of the pullback measure on the fibers of σ_0 are the (normalized) counting measure. We abuse notation by denoting the pullback measure also by ν.

Lemma **4.10.** — *The measure* ν *is ergodic for the action of* g_t *on* X.

Proof. — Suppose E is a g_t-invariant set of X with $\nu(E) > 0$. Then by the ergodicity of the action of g_t on X_0, $\sigma(E)$ is conull. Let $N(x_0)$ denote the cardinality of $\sigma_0^{-1}(x_0) \cap E$. Then, again by the ergodicity of g_t, $N(x_0)$ is constant almost everywhere. If E does not have full measure, then we have that $N(x_0)$ is smaller than the degree of the cover σ_0. Then, we could replace X by E, contradicting the assumption that the degree of the covering map σ_0 is as small as possible. □

The space $\tilde{X}$. — Recall that $\tilde{X}_0$ is the universal cover of X_0. Let $\tilde{X}$ denote the cover of $\tilde{X}_0$ corresponding to the cover $\sigma_0 : X \to X_0$. More precisely,

$$\tilde{X} = \big\{(x_0, \mathfrak{F}) \ : \ x_0 \in \tilde{X}_0, \ \mathfrak{F} \in \Delta(x_0)\big\}.$$

We denote the covering map from $\tilde{X}$ to $\tilde{X}_0$ again by σ_0.

Stable and unstable manifolds for X *and* $\tilde{X}$. — Suppose $x = (x_0, \mathfrak{F}) \in \tilde{X}$. We define

$$W^+[x] = \{(y_0, \mathfrak{F}') \in \tilde{X} \ : \ y_0 \in W^+[x_0], \text{ and } \mathfrak{F}' = P^+(x_0, y_0)\mathfrak{F}\}. \tag{4.12}$$

$$W^-[x] = \{(y_0, \mathfrak{F}') \in \tilde{X} \ : \ y_0 \in W^-[x_0], \text{ and } \mathfrak{F}' = P^-(x_0, y_0)\mathfrak{F}\}. \tag{4.13}$$

This definitions make sense, since by Proposition 4.4,

$$P^+(x_0, y_0)\Delta(x_0) = \Delta(y_0) \quad \text{for } y_0 \in W^+[x_0],$$
$$P^-(x_0, y_0)\Delta(x_0) = \Delta(y_0) \quad \text{for } y_0 \in W^-[x_0].$$

Remark. — Even though $\tilde{X}$ itself does not have a manifold structure, for almost all $x \in \tilde{X}$, the sets $W^+[x]$ and $W^-[x]$ have the structure of an affine manifold (intersected with a set of full measure in $\tilde{X}$), see Lemma 3.2. Lemma 4.11 below asserts that these can be interpreted as the strong stable and strong unstable manifolds for the action of g_t on $\tilde{X}$.

Notation. — If $x \in \tilde{X}$ and V is a subspace of $W^+(x)$ or $W^-(x)$ we write

$$V[x] = \{y \in W^{\pm}[x] \ : \ y - x \in V(x)\}.$$

The "distance" $d^X(\cdot, \cdot)$. — For $x = (x_0, \mathfrak{F}) \in \tilde{X}$, and $y = (y_0, \mathfrak{F}') \in \tilde{X}$ and $y \in W^+[x]$ or $W^-[x]$ define

$$d^X(x, y) = d^{X_0}(x_0, y_0) + d_Y(\mathfrak{F}, P^{GM}(y_0, x_0)\mathfrak{F}'), \tag{4.14}$$

where we extend the distance d_Y between subspaces defined in (4.6) to a distance between flags.

Lemma **4.11.** — *For almost all* $x \in \tilde{X}$ *and almost all* $y \in W^+[x]$, $d^X(g_t x, g_t y) \to 0$ *as* $t \to -\infty$. *Similarly, for almost all* $x \in \tilde{X}$ *and almost all* $y \in W^-[x]$, *we have* $d^X(g_t x, g_t y) \to 0$ *as* $t \to \infty$.

Proof. — This follows immediately from Corollary 4.9. □

Notational convention. — If f is an object on X_0, and $x \in X$, we write $f(x)$ instead of $f(\sigma_0(x))$. Thus, we can define $\mathcal{V}_i(H_{big})(x)$ for $x \in X$, $P^+(x, y)$ for $x \in X$ and $y \in W^+[x]$, etc. Also, if $x \in \tilde{X}$, we write $f(x)$ instead of $f(\pi \circ \sigma_0(x))$ etc.

The partitions $\mathfrak{B}_t$ *of* X. — Suppose $x = (x_0, \mathfrak{F}) \in X$. We define

$$\mathfrak{B}_t[x] = \{(y_0, \mathfrak{F}') \ : \ y_0 \in \mathfrak{B}_t[x_0], \ \mathfrak{F}' = P^+(x_0, y_0)\mathfrak{F}\}.$$

Then $\mathfrak{B}_t$ is a measurable partition of X subordinate to W^+. In a similar way, we can define sets $J[x]$ for $x \in X$ and $E^+[c]$ for $c \in \sigma_0^{-1}(\mathcal{C}_1)$, where $\mathcal{C}_1$ is as in Proposition 3.7. Proposition 3.7 and all subsequent results of Section 3 apply to X as well as X_0.

The following is an alternative version of Proposition 4.4 adapted to the cover X.

Proposition **4.12.** — *Suppose* L *is a* g_t*-equivariant subbundle of* H_{big}. *For almost all* $x \in X$, *we can write*

$$L(x) = \bigoplus_i L_i(x),$$

where $L_i(x)$ *is the Lyapunov subspace corresponding to the Lyapunov exponent* λ_i. *Suppose there exists an equivariant flat measurable* W^+*-connection* F *on* L, *such that*

$$F(x,y)L_i(x) = L_i(y),$$

and that $M \subset L$ *is a* g_t*-equivariant subbundle. Then,*

(a) *For almost all* $y \in \mathfrak{B}_0[x]$,

$$F(x,y)M(x) = M(y),$$

i.e. the subbundle M *is locally covariantly constant with respect to the connection* F.

(b) *For all* i, *the decomposition* (4.3) *of* L_i *is locally covariantly constant along* W^+, *i.e. for* $\nu_{W^+[x]}$*-almost all* $y \in \mathfrak{B}_0[x]$, *for all* $i \in I$ *and for all* $1 \leq j \leq n_i$,

$$L_{ij}(y) = F(x,y)L_{ij}(x). \tag{4.15}$$

Also, up to a scaling factor, the quadratic forms $\langle\cdot,\cdot\rangle_{i,j}$ *are locally covariantly constant along* W^+, *i.e. for almost all* $y \in \mathfrak{B}_0[x]$, *and for* $v, w \in L_{ij}(x)/L_{i,j-1}(x)$,

$$\big\langle F(x,y)v, F(x,y)w\big\rangle_{ij,y} = c(x,y)\langle v, w\rangle_{ij,x}. \tag{4.16}$$

Proposition 4.12 will be proved in Section 4.9*. The proof also shows the following:

Remark **4.13.** — Proposition 4.12 applies also to U^+-connections, provided the measure along $U^+[x]$ is in the Lebesgue measure class, and provided that in the statement, the set $\mathfrak{B}_0[x]$ is replaced by $\mathcal{B}[x] = \mathfrak{B}_0[x] \cap U^+[x]$.

4.7. *Dynamically defined norms.* — In this subsection we work on the cover X. We define a norm on $\|\cdot\|$ on $H_{big}^{(++)}$, which has some advantages over the AGY norm $\|\cdot\|_Y$.

Notation. — In Section 4.7 we let L denote the entire bundle H_{big}^{++}, write L_i for $\mathcal{V}_i(L)$, and for each i, consider the decomposition (4.3).

The function $\Xi(x)$. — For $x \in X$, let

$$\Xi^+(x) = \sup_{ij} \sup\left\{ \langle v, v\rangle_{ij,x}^{1/2} \;:\; v \in L_{ij}(x)/L_{i,j-1}(x),\ \|v\|_{Y,x} = 1 \right\},$$

and let

$$\Xi^-(x) = \inf_{ij} \inf\left\{ \langle v, v\rangle_{ij,x}^{1/2} \;:\; v \in L_{ij}(x)/L_{i,j-1}(x),\ \|v\|_{Y,x} = 1 \right\}.$$

Let

$$\Xi(x) = \Xi^+(x)/\Xi^-(x).$$

We have $\Xi(x) \geq 1$ for all $x \in X$. For $x_0 \in X_0$, we define $\Xi(x_0)$ to be $\max_{x \in \sigma_0^{-1}(x_0)} \Xi(x)$.

Let $d_Y(\cdot,\cdot)$ be the distance between subspaces defined in (4.6). Let $\mathcal{C}_0 \subset X_0$ with $\nu(\mathcal{C}_0) > 0$ and $M_0 \geq 1$ be chosen later. (We will choose them immediately before Lemma 6.8 in Section 6.)

Lemma **4.14.** — *Fix* $\epsilon > 0$ *smaller than* $\min_i |\lambda_i|$, *and smaller than* $\min_{i \neq j} |\lambda_i - \lambda_j|$, *where the* λ_i *are the Lyapunov exponents of* $H^{(++)}_{big}$. *There exists a compact subset* $\mathcal{C} \subset \mathcal{C}_0 \subset X_0$ *with* $\nu(\mathcal{C}) > 0$ *and a function* $T_0 : \mathcal{C} \to \mathbb{R}^+$ *with* $T_0(x) < \infty$ *for* ν *a.e.* $x \in \mathcal{C}$ *such that the following hold:*

(a) *There exists* $\sigma > 0$ *such that for all* $c \in \mathcal{C}$, *and any subset* S *of the Lyapunov exponents,*

$$d_Y\left(\bigoplus_{i \in S} L_i(c), \bigoplus_{j \notin S} L_j(c) \right) \geq \sigma.$$

(b) *There exists* $M' > 1$ *such that for all* $c \in \mathcal{C}$, $\Xi(c) \leq M'$.

(b′) *There exists a constant* $M'' < \infty$ *such that for all* $x \in \pi^{-1}(\mathcal{C})$, *for all* $y \in \pi^{-1}(\mathcal{C}) \cap W^+[x]$ *with* $d^{X_0}(x, y) < 1/100$, *the Gauss-Manin connection* P^{GM} *satisfies the estimate:*

$$\|P^{GM}(x,y)\|_Y \equiv \sup_{v \neq 0} \frac{\|P^{GM}(x,y)v\|_{Y,y}}{\|v\|_{Y,x}} \leq M''.$$

(c) *For all* $c \in \mathcal{C}$, *for all* $t > T_0(c)$ *and for any subset* S *of the Lyapunov spectrum,*

$$d_Y\left(\bigoplus_{i \in S} L_i(g_{-t}c), \bigoplus_{j \notin S} L_j(g_{-t}c) \right) \geq e^{-\epsilon t}.$$

Hence, for all $c \in \mathcal{C}$ *and all* $t > T_0(c)$ *and all* $c' \in \mathcal{C} \cap W^+[g_{-t}c]$ *with* $d^{X_0}(g_{-t}c, c') < 1/100$,

$$M_0^{-2}\rho_1 e^{-\epsilon t} \leq \left\| P^+\left(g_{-t}c, c'\right) \right\|_Y \equiv \sup_{v \neq 0} \frac{\|P^+(g_{-t}c, c')v\|_{Y,c'}}{\|v\|_{Y,g_{-t}c}} \leq M_0 \rho_1^{-1} e^{\epsilon t}, \tag{4.17}$$

where $\rho_1 = \rho_1(M', \sigma, M'', M_0) > 0$.

(d) *There exists $\rho > 0$ such that for all $c \in \mathcal{C}$, for all $t > T_0(c)$, for all i and all $v \in L_i(c)$,*

$$e^{-(\lambda_i+\epsilon)t}\rho_1\rho^2\|v\|_{Y,c} \leq \|g_{-t}v\|_{Y,g_{-t}c} \leq \rho_1^{-1}\rho^{-2}e^{-(\lambda_i-\epsilon)t}\|v\|_{Y,c}.$$

Proof. — Parts (a) and (b) hold since the inverse of the angle between Lyapunov subspaces and the ratio of the norms are finite a.e., therefore bounded on a set of almost full measure. To see (c), note that by the Osceledets multiplicative ergodic theorem, [KH, Theorem S.2.9 (2)] for ν-a.e. $x \in X_0$,

$$\lim_{t\to\infty} \frac{1}{t} \log \left| \sin \angle \left(\bigoplus_{i\in S} L_i(g_{-t}x), \bigoplus_{j\notin S} L_j(g_{-t}x) \right) \right| = 0.$$

Also, (d) follows immediately from the multiplicative ergodic theorem. □

We now choose the set $\mathcal{C}$ and the function T_0 of Proposition 3.7 and Lemma 3.8 to be as in Lemma 4.14.

The main result of this subsection is the following:

Proposition **4.15.** — *For almost all $x \in X$ there exists an inner product $\langle \cdot, \cdot \rangle_x$ on $H^{(++)}_{big}(x)$ (or on any bundle for which the conclusions of Lemma 4.14 hold) with the following properties:*

(a) *For a.e. $x \in X$, the distinct eigenspaces $L_i(x)$ are orthogonal.*

(b) *Let $L'_{ij}(x)$ denote the orthogonal complement, relative to the inner product $\langle \cdot, \cdot \rangle_x$ of $L_{i,j-1}(x)$ in $L_{ij}(x)$. Then, for a.e. $x \in X$, all $t \in \mathbb{R}$ and all $v \in L'_{ij}(x) \subset H^{(++)}_{big}(x)$,*

$$(g_t)_* v = e^{\lambda_{ij}(x,t)} v' + v'',$$

where $\lambda_{ij}(x, t) \in \mathbb{R}$, $v' \in L'_{ij}(g_t x)$, $v'' \in L_{i,j-1}(g_t x)$, and $\|v'\| = \|v\|$. Hence (since v' and v'' are orthogonal),

$$\|(g_t)_* v\| \geq e^{\lambda_{ij}(x,t)}\|v\|.$$

(c) *There exists a constant $\kappa > 1$ such that for a.e. $x \in X$ and for all $t > 0$,*

$$\kappa^{-1} t \leq \lambda_{ij}(x, t) \leq \kappa t.$$

(d) *There exists a constant $\kappa > 1$ such that for a.e $x \in X$ and for all $v \in H^{(++)}_{big}(x)$, and all $t \geq 0$,*

$$e^{\kappa^{-1}t}\|v\| \leq \|(g_t)_* v\| \leq e^{\kappa t}\|v\|.$$

(e) *For a.e. $x \in X$, and a.e. $y \in \mathfrak{B}_0[x]$ and all $t \leq 0$,*

$$\lambda_{ij}(x, t) = \lambda_{ij}(y, t).$$

(f) *For a.e. $x \in X$, a.e. $y \in \mathfrak{B}_0[x]$, and any $v, w \in H_{big}^{(++)}(x)$,*

$$\left\langle P^+(x,y)v, P^+(x,y)w \right\rangle_y = \langle v, w \rangle_x.$$

We often omit the subscript from $\langle\cdot,\cdot\rangle_x$ and from the associated norm $\|\cdot\|_x$.

The inner product $\langle\cdot,\cdot\rangle_x$ is first defined for $x \in E^+[c]$ for $c \in \sigma_0^{-1}(\mathcal{C}_1)$ (in the notation of Section 3, see also Section 4.6). We then interpolate between $x \in E^+[c]$ and $g_{-t(c)}x$ (again in the notation of Section 3). The details of the proof of Proposition 4.15, which can be skipped on first reading, are given in Section 4.10*.

The dynamical norm $\|\cdot\|$ on X_0. — The dynamical inner product $\langle\cdot,\cdot\rangle_x$ and the dynamical norm $\|\cdot\|_x$ of Proposition 4.15 are defined for $x \in X$. For $x_0 \in X_0$, and $v, w \in H_{big}(x_0)$ we define

$$\langle v, w \rangle_{x_0} = \frac{1}{|\sigma_0^{-1}(x_0)|} \sum_{x \in \sigma_0^{-1}(x_0)} \langle v, w \rangle_x, \qquad \|v\|_{x_0} = \langle v, v \rangle_{x_0}^{1/2}. \tag{4.18}$$

Remark **4.16.** — The inner product and norm on X_0 satisfy properties (a) and (d) of Proposition 4.15.

Lemma **4.17.** — *For every $\delta > 0$ there exists a compact subset $K(\delta) \subset X_0$ with $\nu(K(\delta)) > 1-\delta$ and a number $C_1(\delta) < \infty$ such that for all $x \in K(\delta)$ and all v on $H_{big}^{(++)}(x)$ or $H_{big}^{(--)}(x)$,*

$$C_1(\delta)^{-1} \le \frac{\|v\|_x}{\|v\|_{Y,x}} \le C_1(\delta),$$

where $\|\cdot\|_x$ is the dynamical norm defined in this subsection and $\|\cdot\|_{Y,x}$ is the AGY norm.

Proof. — Since any two norms on a finite dimensional vector space are equivalent, there exists a function $\Xi_0 : X \to \mathbb{R}^+$ finite a.e. such that for all $x \in X$ and all $v \in H_{big}^{(++)}(x)$,

$$\Xi_0(x)^{-1}\|v\|_{Y,x} \le \|v\|_x \le \Xi_0(x)\|v\|_{Y,x}.$$

Since $\bigcup_{N\in\mathbb{N}}\{x : \Xi_0(x) < N\}$ is conull in X, we can choose $K(\delta) \subset X$ and $C_1 = C_1(\delta)$ so that $\Xi_0(x) < C_1(\delta)$ for $x \in K(\delta)$ and $\nu(K(\delta)) \ge (1-\delta)$. □

4.8*. *Proof of Lemma 4.7.* — We first prove (a). Note that the action of g_t commutes with P^{GM}, i.e.

$$P^{GM}(g_t x, g_t y) \circ g_t = g_t \circ P^{GM}(x,y).$$

Let $\alpha_0 = \min_{i\neq j} |\lambda_i - \lambda_j|$, where the $\lambda_i = \lambda_i(H^1)$. We will choose $0 < \epsilon < \alpha_0/100$. For every $\epsilon > 0$ there exists a compact set $K_0 = K_0(\epsilon) \subset X_0$ with $\nu(K_0) > 1 - \epsilon/4$ and

$\sigma = \sigma(\epsilon) > 0$ such that for any subset S of the Lyapunov exponents,

$$d_Y\left(\bigoplus_{i\in S} \mathcal{V}_i(x), \bigoplus_{j\notin S} \mathcal{V}_j(x)\right) > \sigma \quad \text{for all } x \in \pi^{-1}(K_0). \tag{4.19}$$

By the multiplicative ergodic theorem and the Birkhoff ergodic theorem, there exists a set $K = K(\epsilon) \subset K_0$ with $\nu(K) > 1 - \epsilon/2$ and a constant $C = C(\epsilon)$ such that such that for all $z \in \pi^{-1}(K)$, all $s \in \mathbb{R}$ and all $v \in \mathcal{V}_i(z)$,

$$C(\epsilon)^{-1/2}\|v\|_Y e^{\lambda_i s - (\epsilon/6)|s|} \le \|g_s v\|_Y \le C(\epsilon)^{1/2}\|v\|_Y e^{\lambda_i s + (\epsilon/6)|s|}, \tag{4.20}$$

and also for any interval $I \subset \mathbb{R}$ containing the origin of length at least $4\log C(\epsilon)/\alpha_0$, and any $z \in \pi^{-1}(K)$,

$$|\{s \in I \,:\, g_s z \in K_0\}| \ge (1-\epsilon)|I|. \tag{4.21}$$

Suppose the set $\{g_s x \,:\, 0 \le s \le t\}$ intersects K. We will show that for all $y \in \tilde{X}_0$ such that $d^{X_0}(g_s x, g_s y) \le 1/100$ for $0 \le s \le t$,

$$d_Y\big(\mathcal{V}_{\le i}(g_t x), P^{GM}(g_t y, g_t x)\mathcal{V}_{\le i}(g_t y)\big) \le C_0(x)C(\sigma)C(\epsilon)^2\big(1 + D^+(x,y)\big)e^{-\alpha t}, \tag{4.22}$$

where $C_0(x)$ is as in Lemma 4.6. Let $\Upsilon(x) = \{\epsilon \,:\, x \in K(\epsilon)\}$ and let

$$C(x) = C_0(x)\inf\big\{C(\sigma)C(\epsilon)^2 \,:\, \epsilon \in \Upsilon(x)\big\}.$$

Since the union as $\epsilon \to 0$ of the sets $K = K(\epsilon)$ is conull, (4.22) implies part (a) of the lemma.

We now prove (4.22). We may assume that $t > 4\log C(\epsilon)/\alpha_0$, otherwise (4.22) trivially holds. Then, by (4.21), there exists $(1-\epsilon)t < t' \le t$ with $g_{t'}x \in K_0$. In view of Lemma 3.6 the inequality (4.22) for t' implies the inequality (4.22) for t (after replacing α by $\alpha - 4\epsilon$). Thus, we may assume without loss of generality that $g_t x \in K_0$.

By assumption, there exists $0 < s < t$ such that $g_s x \in K$. Let $z = g_s x$. Then, applying (4.20) twice at z, we get, for all $v \in \mathcal{V}_i(x)$,

$$C(\epsilon)^{-1}\|v\|_Y e^{\lambda_i t - (\epsilon/3)t} \le \|g_t v\|_Y \le C(\epsilon)\|v\|_Y e^{\lambda_i t + (\epsilon/3)t}. \tag{4.23}$$

Let $v' \in P^{GM}(g_t y, g_t x)\mathcal{V}_{\le i}(g_t y)$ be such that $\|v'\|_Y = 1$ and

$$d_Y\big(v', \mathcal{V}_{\le i}(g_t x)\big) = \delta_Y\big(P^{GM}(g_t y, g_t x)\mathcal{V}_{\le i}(g_t y), \mathcal{V}_{\le i}(g_t x)\big),$$

where $\delta_Y(\cdot,\cdot)$ is as in Lemma 4.6. Then, $v' = g_t v$ for some $v \in P^{GM}(y,x)\mathcal{V}_{\le i}(y)$. We may write

$$v = v_0 + w, \quad v_0 \in \mathcal{V}_{\le i}(x),\, w \in \mathcal{V}_{>i}(x).$$

We have, by the definition of $D^+(\cdot,\cdot)$,

$$\|w\|_Y \leq D^+(x,y)\|v_0\|_Y.$$

Then, we have

$$v' = g_t v = g_t v_0 + g_t w,$$

and by (4.23),

$$\|g_t v_0\|_Y \geq C(\epsilon)^{-1} e^{(\lambda_i - \epsilon/3)t}\|v_0\|_Y,$$

and

$$\|g_t w\|_Y \leq C(\epsilon) e^{(\lambda_{i+1}+\epsilon/3)t}\|w\|_Y.$$

Thus,

$$\|g_t w\|_Y \leq C(\epsilon)^2 D^+(x,y) e^{-(\alpha_0 - 2\epsilon/3)t}\|g_t v_0\|_Y.$$

Since $g_t v_0 \in \mathcal{V}_{\leq i}(g_t x)$ and $g_t w \in \mathcal{V}_{>i}(g_t x)$, this, together with (4.19) implies

$$d_Y\big(v', \mathcal{V}_{\leq i}(g_t x)\big) \leq C(\sigma)C(\epsilon)^2\big(1 + D^+(x,y)\big)e^{-(\alpha_0 - 2\epsilon/3)t}.$$

This, together with Lemma 4.6, completes the proof of (4.22).

The proof of (b) is identical. □

4.9*. *Proof of Propositions 4.4 and 4.12.* — The proof of Proposition 4.4 will essentially be by reference to [L, Theorem 1]. We recall the setup (in our notation):

Let (X, ν) be a measure space, and let $T : X \to X$ be a measure preserving transformation. Let $\mathfrak{B}$ be a σ-subalgebra of the σ-algebra of Borel sets on X, such that $\mathfrak{B}$ is T-decreasing (i.e. $T^{-1}\mathfrak{B} \subset \mathfrak{B}$). Let $\mathfrak{B}_{-\infty}$ denote the σ-algebra generated by all the σ-algebras $T^n\mathfrak{B}$, $n \in \mathbb{Z}$.

Let V be a vector space, and let $A : X \to GL(V)$ be a log-integrable $\mathfrak{B}$-measurable function. Let

$$A^{(n)}(x) = A\big(T^{n-1}x\big)\dots A(x) \quad \text{for } n > 0$$
$$A^{(0)}(x) = Id$$

and

$$A^{(n)}(x) = A^{-1}\big(T^n x\big)\dots A^{-1}\big(T^{-1}x\big) \quad \text{for } n < 0$$

We have a skew-product map $\hat{T} : X \times V \to X \times V$ given by

$$\hat{T}(x, v) = \big(Tx, A(x)v\big),$$

and then,

$$\hat{T}^n(x, v) = \big(T^n x, A^{(n)}(x)v\big).$$

Let

$$\gamma_+ = \lim_{n\to\infty} \frac{1}{n}\int_X \log \|A^{(n)}(x)\| d\nu(x),$$
$$\gamma_- = \lim_{n\to\infty} -\frac{1}{n}\int_X \log \big\| \big(A^{(n)}(x)\big)^{-1} \big\| d\nu(x),$$

where $\|\cdot\|$ is the operator norm. The limits exist by the subadditive ergodic theorem.

The matrix A also naturally acts on the projective space $\mathbb{P}(V)$. We use the notation $\hat{T}$ to denote also the associated skew-product map $X \times \mathbb{P}(V) \to X \times \mathbb{P}(V)$.

We have the following:

Theorem **4.18** *(Ledrappier, [L, Theorem 1]). — Suppose*

(a) $\gamma_+ = \gamma_-$.
(b) $x \to \nu_x$ *is a family of measures on* $\mathbb{P}(V)$ *defined for almost every* x *such that* $A(x)\nu_x = \nu_{Tx}$ *and such that the map* $x \to \nu_x$ *is* $\mathfrak{B}_{-\infty}$*-measurable.*

Then, $x \to \nu_x$ *is* $\mathfrak{B}$*-measurable.*

Proof of Proposition 4.4. — We first make some preliminary reductions. For $x \in X_0$, write $\mathcal{M}(x) = \{M^1(x), \dots, M^k(x)\}$. Since $\mathcal{M}(x)$ is g_t-equivariant, for $1 \le j \le k$,

$$M^j(x) = \bigoplus M^j_i(x), \qquad M^j_i(x) \subset L_i(x).$$

Let $\mathcal{M}_i(x) = \{M^1_i(x), \dots, M^k_i(x)\}$. Thus, it is enough to show that

$$F(x,y)\mathcal{M}_i(x) = \mathcal{M}_i(y).$$

Without loss of generality, we may assume that for a fixed i, all the M^j_i have the same dimension. Suppose $x \in J_c$, where J_c is as in Proposition 3.7. Then the sets $\{g_{-t}c \,:\, 0 \le t \le t(c)\}$ and $\mathfrak{B}_0[x] = J_c \cap W^+[x]$ intersect at a unique point $x_0 \in X_0$. Then, we can replace the bundle $L(x)$ by $\tilde{L}(x) \equiv F(x, x_0)L(x)$. Then, for $y \in \mathfrak{B}_0[x]$,

$$\tilde{L}(y) = F(y, x_0)L(y) = F(y, x_0)F(x, y)L(x) = F(x, x_0)L(x) = \tilde{L}(x),$$

i.e. $\tilde{L}(x)$ is locally constant along $W^+(x)$. Also, by (4.2), the action of $(g_t)_*$ on $\tilde{L}$ is locally constant. Thus, without loss of generality, we may assume that F is locally constant (or else we replace L by $\tilde{L}$). Thus, it is enough to show that assuming the subspaces $L_i(x)$ are almost everywhere locally constant along W^+, the set of subspaces $\mathcal{M}_i(x)$ is also almost everywhere locally constant along W^+. In other words, we assume that the functions

$x \to L_i(x)$ are $\mathfrak{B}_0$-measurable, and would like to show that the functions $x \to \mathcal{M}_i(x)$ are $\mathfrak{B}_0$-measurable.

Let $T = g_1$ denote the time 1 map of the geodesic flow. Fix i and j, and let $d_i = \dim M_i^1 = \cdots = \dim M_i^k$. Let $V(x) = \bigwedge^{d_i}(L_i(x)/L_{i-1}(x))$. Note that $V(x)$ is $\mathfrak{B}_0$-measurable and g_t-equivariant.

We can write the action of $(g_t)_*$ (for $t = 1$) on the bundle V as

$$(g_1)_*(x, v) = \big(g_1 x, A(x)v\big).$$

Then, $A(x)$ is $\mathfrak{B}_1$-measurable (where $\mathfrak{B}_t$ is as in Section 3). Also, the condition $\gamma_+ = \gamma_-$ follows from the multiplicative ergodic theorem. (In fact, $\gamma_+ = \gamma_- = d_i\lambda_i$, where λ_i is the Lyapunov exponent corresponding to L_i.)

Let ν_x^j denote the Dirac measure on (the line through) $v_1 \wedge \cdots \wedge v_d$, where $\{v_1, \dots, v_d\}$ is any basis for $M_i^j(x)$, and let

$$\nu_x = \frac{1}{k}\sum_{j=1}^{k} \nu_x^j.$$

Then, since the $\mathcal{M}_i(x)$ are g_t-equivariant, the measures ν_x are $\hat{T}$-invariant. Also note that $\mathfrak{B}_{-\infty}$ is the partition into points. Thus, we can apply Theorem 4.18 (with $\mathfrak{B} = \mathfrak{B}_1$). We conclude that the function $x \to \nu_x$ is $\mathfrak{B}_1$-measurable, which implies that the $\mathcal{M}_i(x)$ are locally constant on atoms of $\mathfrak{B}_1$. Since the $\mathcal{M}_i(x)$ are g_t-equivariant, this implies that the $\mathcal{M}_i(x)$ are also locally constant (in particular the function $x \to \mathcal{M}_i(x)$ is $\mathfrak{B}_0$-measurable). □

Proof of Proposition 4.12. — Note that (a) and also (4.15) follow immediately from Proposition 4.4.

We now prove (4.16). After making the same reductions as in the proof of Proposition 4.4, we may assume that the L_{ij} and F are locally constant. Let $K \subset K_\epsilon$ denote a compact subset with $\nu(K) > 0.9$ where $\langle\cdot,\cdot\rangle_{ij}$ is uniformly continuous. Consider the points $g_t x$ and $g_t y$, as $t \to -\infty$. Then $d^{X_0}(g_t x, g_t y) \to 0$. Let

$$v_t = e^{-\lambda_{ij}(x,t)}(g_t)_* v, \qquad w_t = e^{-\lambda_{ij}(x,t)}(g_t)_* w,$$

where $\lambda_{ij}(x, t)$ is as in Lemma 4.3. Then, by Lemma 4.3, we have

$$\textbf{(4.24)}\qquad \langle v_t, w_t\rangle_{ij,g_t x} = \langle v, w\rangle_{ij,x}, \qquad \langle v_t, w_t\rangle_{ij,g_t y} = c(x, y, t)\langle v, w\rangle_{ij,y},$$

where $c(x, y, t) = e^{\lambda_{ij}(x,t)-\lambda_{ij}(y,t)}$.

Now take a sequence $t_k \to \infty$ with $g_{t_k}x \in K$, $g_{t_k}y \in K$ (such a sequence exists for ν-a.e. x and y with $y \in \mathfrak{B}_0[x]$). Then, since the $L_{ij}(x)$ and the connection F are assumed to be locally constant, $c(x, y, t_k)$ is bounded between two constants. Also,

$$\langle v_{t_k}, w_{t_k}\rangle_{ij,g_{t_k}x} - \langle v_{t_k}, w_{t_k}\rangle_{ij,g_{t_k}y} \to 0.$$

Now Equation (4.16) follows from (4.24). □

4.10*. *Proof of Proposition 4.15.* — To simplify notation, we assume that $M_0 = 1$ (where M_0 is as in Lemma 4.14).

The inner products $\langle\cdot,\cdot\rangle_{ij}$ on $E^+[c]$. — Note that the inner products $\langle\cdot,\cdot\rangle_{ij}$ and the $\mathbb{R}$-valued cocycles λ_{ij} of Lemma 4.3 are not unique, since we can always multiply $\langle\cdot,\cdot\rangle_{ij,x}$ by a scalar factor $c(x)$, and then replace $\lambda_{ij}(x,t)$ by $\lambda_{ij}(x,t)+\log c(g_t x)-\log c(x)$. In view of (4.16) in Proposition 4.12(b), we may (and will) use this freedom to make $\langle\cdot,\cdot\rangle_{ij,x}$ constant on each set $E^+[c]$, where $c \in \sigma_0^{-1}(\mathcal{C}_1)$ and $E^+[c]$ is as in Section 3 (see also Section 4.6).

The inner product $\langle\cdot,\cdot\rangle_x$ on $E^+[c]$. — Let

$$\{0\} = \mathcal{V}_{\le 0} \subset \mathcal{V}_{\le 1} \subset \cdots \tag{4.25}$$

be the Lyapunov flag for $H^{(++)}_{big}$, and for each i, let

$$\mathcal{V}_{\le i-1} = \mathcal{V}_{\le i,0} \subset \mathcal{V}_{i,1} \subset \cdots \mathcal{V}_{\le i,n_i} = \mathcal{V}_{\le i} \tag{4.26}$$

be a maximal invariant refinement.

Let $L_i = \mathcal{V}_i(H^{(++)}_{big})$ denote the Lyapunov subspaces for $H^{(++)}_{big}$. Then we have a maximal invariant flag

$$\{0\} = L_{i,0} \subset L_{i,1} \subset \cdots \subset L_{i,n_i} = L_i,$$

where $L_{ij} = L_i \cap \mathcal{V}_{\le i,j}$.

Let $c \in \sigma_0^{-1}(\mathcal{C}_1)$, $E^+[c]$ be as in Section 3 and Section 4.6. By Lemma 4.14(b), we can (and do) rescale the inner products $\langle\cdot,\cdot\rangle_{ij,c}$ so that after the rescaling, for all $v \in L_{ij}(c)/L_{i,j-1}(c)$,

$$\big(M'\big)^{-1}\|v\|_{Y,c} \le \langle v, v\rangle^{1/2}_{ij,c} \le M'\|v\|_{Y,c},$$

where $\|\cdot\|_{Y,c}$ is the AGY norm at $\sigma_0(c)$ and $M' > 1$ is as in Lemma 4.14. We then choose $L'_{ij}(c) \subset L_{ij}(c)$ to be a complementary subspace to $L_{i,j-1}(c)$ in $L_{ij}(c)$, so that for all $v \in L_{i,j-1}(c)$ and all $v' \in L'_{ij}(c)$,

$$\|v+v'\|_{Y,c} \ge \rho'' \max\big(\|v\|_{Y,c}, \|v'\|_{Y,c}\big),$$

and $\rho'' > 0$ depends only on the dimension.

Then,

$$L'_{ij}(c) \cong L_{ij}(c)/L_{i,j-1}(c) \cong \mathcal{V}_{\le i,j}(c)/\mathcal{V}_{\le i,j-1}(c).$$

Let $\pi_{ij} : \mathcal{V}_{\le i,j} \to \mathcal{V}_{\le i,j}/\mathcal{V}_{\le i,j-1}$ be the natural quotient map. Then the restriction of π_{ij} to $L'_{ij}(c)$ is an isomorphism onto $\mathcal{V}_{\le i,j}(c)/\mathcal{V}_{\le i,j-1}(c)$.

We can now define for $u, v \in H^{(++)}_{big}(c)$,

$$\langle u, v\rangle_c \equiv \sum_{ij} \big\langle \pi_{ij}(u_{ij}), \pi_{ij}(v_{ij})\big\rangle_{ij,c},$$

$$\text{where } u = \sum_{ij} u_{ij},\ \ v = \sum_{ij} v_{ij},\ \ u_{ij} \in L'_{ij}(c),\ \ v_{ij} \in L'_{ij}(c).$$

In other words, the distinct $L'_{ij}(c)$ are orthogonal, and the inner product on each $L'_{ij}(c)$ coincides with $\langle\cdot,\cdot\rangle_{ij,c}$ under the identification π_{ij} of $L'_{ij}(c)$ with $\mathcal{V}_{\le i,j}(c)/\mathcal{V}_{\le i,j-1}(c)$.

We now define, for $x \in E^+[c]$, and $u, v \in H^{(++)}_{big}(x)$

$$\langle u, v\rangle_x \equiv \big\langle P^+(x,c)u, P^+(x,c)v\big\rangle_c,$$

where $P^+(\cdot,\cdot)$ is the connection defined in Section 4.2. Then for $x \in E^+[c]$, the inner product $\langle\cdot,\cdot\rangle_x$ induces the inner product $\langle\cdot,\cdot\rangle_{ij,x}$ on $\mathcal{V}_{\le i,j}(x)/\mathcal{V}_{\le i,j-1}(x)$.

Symmetric space interpretation. — We want to define the inner product $\langle\cdot,\cdot\rangle_x$ for any $x \in J[c]$ by interpolating between $\langle\cdot,\cdot\rangle_c$ and $\langle\cdot,\cdot\rangle_{c'}$, where c' is such that $g_{-t(c)}c \in E^+[c']$. To define this interpolation, we recall that the set of inner products on a vector space V is canonically isomorphic to $SO(V)\backslash GL(V)$, where $GL(V)$ is the general linear group of V and $SO(V)$ is the subgroup preserving the inner product on V. In our case, $V = H^{(++)}_{big}(c)$ with the inner product $\langle\cdot,\cdot\rangle_c$.

Let K_c denote the subgroup of $GL(H^{(++)}_{big}(c))$ which preserves the inner product $\langle\cdot,\cdot\rangle_c$. Let $\mathcal{Q}$ denote the parabolic subgroup of $GL(H^{(++)}_{big}(c))$ which preserves the flags (4.25) and (4.26), and on each successive quotient $\mathcal{V}_{\le i,j}(c)/\mathcal{V}_{\le i,j-1}(c)$ preserves $\langle\cdot,\cdot\rangle_{ij,c}$. Let K_cA' denote the point in $K_c\backslash GL(H^{(++)}_{big}(c))$ which represents the inner product $\langle\cdot,\cdot\rangle_{c'}$, i.e.

$$\langle u, v\rangle_{c'} = \big\langle A'u, A'v\big\rangle_c.$$

Then, since $\langle\cdot,\cdot\rangle_{c'}$ induces the inner products $\langle\cdot,\cdot\rangle_{ij,c'}$ on the space $\mathcal{V}_{\le i,j}(c')/\mathcal{V}_{\le i,j-1}(c')$ which is the same as $\mathcal{V}_{\le i,j}(g_{-t(c)}c)/\mathcal{V}_{\le i,j-1}(g_{-t(c)}c)$, we may assume that the matrix product $A'(g_{-t(c)})_*$ is in $\mathcal{Q}$.

Let $N_{\mathcal{Q}}$ be the normal subgroup of $\mathcal{Q}$ in which all diagonal blocks are the identity, and let $\mathcal{Q}' = \mathcal{Q}/N_{\mathcal{Q}}$. (We may consider $\mathcal{Q}'$ to be the subgroup of $\mathcal{Q}$ in which all off-diagonal blocks are 0.) Let π' denote the natural map $\mathcal{Q} \to \mathcal{Q}'$.

Claim **4.19.** — *We may write*

$$A'(g_{-t(c)})_* = \Lambda A'',$$

where $\Lambda \in \mathcal{Q}'$ *is the diagonal matrix which is scaling by* $e^{-\lambda_i t(c)}$ *on* $L_i(c)$, $A'' \in \mathcal{Q}$ *and* $\|A''\| = O(e^{\epsilon t(c)})$.

Proof of claim. — Suppose $x \in E^+[c]$ and $t = -t(c) < 0$ where $c \in \mathcal{C}_1$ and $t(c)$ is as in Proposition 3.7. By construction, $t(c) > T_0(c)$, where $T_0(c)$ is as in Lemma 4.14. Then, the claim follows from (4.17) and Lemma 4.14(d). □

Interpolation. — We may write $A'' = DA_1$, where D is diagonal, and $\det A_1 = 1$. In view of Claim 4.19, $\|D\| = O(e^{\epsilon t})$ and $\|A_1\| = O(e^{\epsilon t})$.

We now connect $K_c \backslash A_1$ to the identity by the shortest possible path $\Gamma : [-t(c), 0] \to K_c \backslash K_c \mathcal{Q}$, which stays in the subset $K_c \backslash K_c \mathcal{Q}$ of the symmetric space $K_c \backslash SL(V)$. (We parametrize the path so it has constant speed.) This path has length $O(\epsilon t)$ where the implied constant depends only on the symmetric space.

Now for $-t(c) \le t \le 0$, let

$$A(t) = (\Lambda D)^{-t/t(c)} \Gamma(t). \tag{4.27}$$

Then $A(0)$ is the identity map, and $A(-t(c)) = A'(g_{-t(c)})_*$. Then, we define, for $x \in E^+[c]$ and $-t(c) \le t \le 0$,

$$\big\langle (g_t)_* u, (g_t)_* v \big\rangle_{g_t x} = \big\langle A(t) u, A(t) v \big\rangle_x.$$

Proof of Proposition 4.15. — Suppose first that $x = c$, where c and $E^+[c]$ are as in Section 3 and Section 4.6. Then, by construction, (a) and (b) hold. Also, from the construction, it is clear that the inner product $\langle \cdot, \cdot \rangle_c$ induces the inner product $\langle \cdot, \cdot \rangle_{ij,c}$ on $L_{ij}(c)/L_{i,j-1}(c)$.

Now by Proposition 4.12, for $x \in E^+[c]$, $P^+(x, c) L_{ij}(x) = L_{ij}(c)$, and for $\bar{u}, \bar{v} \in L_{ij}(x)/L_{i,j-1}(x)$, $\langle u, v \rangle_{ij,x} = \langle P^+(x, c) u, P^+(x, c) v \rangle_{ij,c}$. Therefore, (a), (b), (e) and (f) hold for $x \in E^+[c]$, and also for $x \in E^+[c]$, the inner product $\langle \cdot, \cdot \rangle_x$ induces the inner product $\langle \cdot, \cdot \rangle_{ij,x}$ on $L_{ij}(x)/L_{i,j-1}(x)$. Now, (a), (b), (e) and (f) hold for arbitrary $x \in J[c]$ since $A(t) \in \mathcal{Q}$.

Let $\psi_{ij} : \mathcal{Q}' \to \mathbb{R}_+$ denote the homomorphism taking the block-conformal matrix $\mathcal{Q}'$ to the scaling part of block corresponding to $L_{ij}/L_{i,j-1}$. Let $\varphi_{ij} = \psi_{ij} \circ \pi'$; then $\varphi_{ij} : \mathcal{Q} \to \mathbb{R}_+$ is a homomorphism.

From (4.27), we have, for $x \in E^+[c]$ and $-t(c) \le t \le 0$,

$$\lambda_{ij}(x, t) = \log \varphi_{ij}\big(A(t)\big) = t\lambda_i + \gamma_{ij}(x, t),$$

where $t\lambda_i$ is the contribution of $\Lambda^{t/t(c)}$ and $\gamma_{ij}(x, t)$ is the contribution of $D^{t/t(c)}\Gamma(t)$. By Claim 4.19, for all $-t(c) \le t \le 0$,

$$\left| \frac{\partial}{\partial t} \gamma_{ij}(x, t) \right| = O(\epsilon) \tag{4.28}$$

where $\epsilon > 0$ is as in Claim 4.19, and the implied constant depends only on the symmetric space. Without loss of generality, the function $T_0(x)$ in Lemma 4.14 can be chosen large enough so that since $t(c) > T_0(c)$, (c) holds.

The lower bound in (d) now follows immediately from (b) and (c). The upper bound in (d) follows from (4.28). □

5. Conditional measure lemmas

In Sections 5–8 we work on X_0 (and not on X).

Motivation. — We use notation from Section 2.3. Recall that $\mathcal{L}^-(q)$ is the smallest linear subspace of $W^-(q)$ containing the support of the conditional measure $\nu_{W^-(q)}$. For two (generalized) subspaces $\mathcal{U}'$ and $\mathcal{U}''$ and $x \in \tilde{X}_0$ let $hd_x^{X_0}(\mathcal{U}', \mathcal{U}'')$ denote the Hausdorff distance between $\mathcal{U}' \cap B^{X_0}(x, 1/100)$ and $\mathcal{U}'' \cap B^{X_0}(x, 1/100)$, where $B^{X_0}(x, r)$ denotes $\{ y \in \tilde{X}_0 \ : \ d^{X_0}(x, y) < r\}$. For $x \in X_0$, we will sometimes write $hd_x^{X_0}(\mathcal{U}', \mathcal{U}'')$ instead of $hd_{\tilde{x}}^{X_0}(\mathcal{U}', \mathcal{U}'')$ as long as the proper lift $\tilde{x} \in \tilde{X}_0$ of x is clear from the context.

We can write

$$hd_{q_2}^{X_0}\big(U^+[q_2'], U^+[q_2]\big) = Q_t\big(q' - q\big),$$

where $Q_t : \mathcal{L}^-(q) \to \mathbb{R}$ is a map depending on q, u, ℓ, and t. The map Q_t is essentially the composition of flowing forward for time ℓ, shifting by $u \in U^+$ and then flowing forward again for time t. We then adjust t so that $hd_{q_2}^{X_0}(U^+[q_2'], U^+[q_2]) \approx \epsilon$, where $\epsilon > 0$ is a priori fixed.

In order to solve "technical difficulty #1" of Section 2.3, it is crucial to ensure that t does not depend on the precise choice of q' (it can depend on q, u, ℓ). The idea is to use the following trivial:

Lemma **5.1.** — *For any $\rho > 0$ there is a constant $c(\rho)$ with the following property: Let $A : \mathcal{V} \to \mathcal{W}$ be a linear map between Euclidean spaces. Then there exists a proper subspace $\mathcal{M} \subset \mathcal{V}$ such that for any v with $\|v\| = 1$ and $d(v, \mathcal{M}) > \rho$, we have*

$$\|A\| \geq \|Av\| \geq c(\rho)\|A\|.$$

Proof of Lemma 5.1. — The matrix A^tA is symmetric, so it has a complete orthogonal set of eigenspaces $W_1, \dots, W_m$ corresponding to eigenvalues $\mu_1 > \mu_2 > \cdots \mu_m$. Let $\mathcal{M} = W_1^\perp$. □

Now suppose the map $Q_t : \mathcal{L}^-(q) \to \mathbb{R}$ is of the form $Q_t(v) = \|\mathcal{Q}_t(v)\|$ where $\mathcal{Q}_t : \mathcal{L}^-(q) \to \mathbf{H}(q_2)$ is a linear map, and $\mathbf{H}(q_2)$ a vector space. This in fact happens in the first step of the induction where U^+ is the unipotent N (and we can take $\mathbf{H}(q_2) = W^+(q_2)/N$). We can then choose t, depending only on q, u and ℓ, such that the operator norm

$$\|\mathcal{Q}_t\| \equiv \sup_{v \in \mathcal{L}^-(q)} \frac{\|\mathcal{Q}_t(v)\|}{\|v\|} = \epsilon.$$

Then, we need to prove that we can choose $q' \in \mathcal{L}^-[q]$ such that $\|q' - q\| \approx 1/100$, q' avoids an a priori given set of small measure, and also $q' - q$ is at least ρ away from the "bad subspace" $\mathcal{M} = \mathcal{M}_u(q, \ell)$ of Lemma 5.1. (Actually, since we do not want the choice of q' to depend on the choice of u, we want to choose q' such that $q' - q$ avoids most of the subspaces $\mathcal{M}_u$ as $u \in U^+$ varies over a unit box.) Then, for most u,

$$c(\rho)\epsilon \le \big\| \mathcal{Q}_t\big(q_2' - q_2\big) \big\| \le \epsilon,$$

and thus

$$c(\rho)\epsilon \le hd_{q_2}\big(U^+[q_2], U^+\big[q_2'\big]\big) \le \epsilon, \tag{5.1}$$

as desired. In general we do not know that the map $\mathcal{Q}_t$ is linear, because we do not know the dependence of the subspace $U^+(q)$ on q. To handle this problem, we can write

$$\mathcal{Q}_t\big(q' - q\big) = \mathcal{A}_t\big(F\big(q'\big) - F(q)\big)$$

where the map $\mathcal{A}_t : \mathcal{L}_{ext}[q]^{(r)} \to W^+(q_2)$ is linear (and can depend on q, u, ℓ), and the measurable map $F : \mathcal{L}^-[q] \to \mathcal{L}_{ext}[q]^{(r)}$ depends only on q. (See Proposition 6.11 below for a precise statement.) The map F and the space $\mathcal{L}_{ext}[q]^{(r)}$ are defined in this section, and the linear map $\mathcal{A}_t = \mathcal{A}(q, u, \ell, t)$ is defined in Section 6.1.

We then proceed in the same way. We choose $t = \hat{\tau}(q, u, \ell, \epsilon)$ so that $\|\mathcal{A}_t\| = \epsilon$. (A crucial bilipshitz type property of the function $\hat{\tau}$ similar to (2.7) is proved in Section 7.) In this section we prove Proposition 5.3, which roughly states that (for most q) we can choose $q' \in \mathcal{L}^-[q]$ while avoiding an a priori given set of small measure, so that $\|F(q') - F(q)\| \approx 1/100$ and also $F(q') - F(q)$ avoids most of a family of linear subspaces of $\mathcal{L}_{ext}[q]^{(r)}$ (which will be the "bad subspaces" of the linear maps $\mathcal{A}_t$ as u varies over U^+). Then as above, for most u, (5.1) holds. We can then proceed using (a variant of) Lemma 2.3 as outlined in Section 2.3.

In view of the above discussion, we need to keep track of the way $U^+[y]$ varies as y varies over $W^-[x]$. In view of Proposition 4.12(a), all bundles equivariant with respect to the geodesic flow are, when restricted to W^-, equivariant with respect to the connection $P^-(x, y)$ defined in Section 4.2. Thus, it will be enough for us to keep track of the maps $P^-(x, y)$. However, this is a bit awkward, since $P^-(x, y)$ depends on two points x and y. Thus, it is convenient to prove the following:

Lemma **5.2.** — *There exists a subbundle* $\mathcal{Y} \subset H^{(-)}_{big}$, *locally constant under the Gauss-Manin connection along* W^-, *and for almost all* $x \in X_0$ *an invertible linear map* $\mathfrak{P}(x) : X_0 \to$ $\mathrm{H}om(\mathcal{Y}(x), H^1(M, \Sigma, \mathbb{R}))$, *such that for almost all* x, y,

$$P^-(x, y) = \mathfrak{P}(y) \circ \mathfrak{P}(x)^{-1}. \tag{5.2}$$

The proof of Lemma 5.2 is simple, but notationally heavy, and is relegated to Section 5.1*. It may be skipped on first reading.

The spaces $\mathcal{L}^-(x)$ *and* $\mathcal{L}_{ext}(x)$. — Let the subspace $\mathcal{L}^-(x) \subset W^-(x)$ be the smallest such that the conditional measure $\nu_{W^-[x]}$ is supported on $\mathcal{L}^-[x]$. Since ν is invariant under N, the entropy of any $g_t \in A$ is positive. Therefore for ν-almost all $x \in X_0$, $\mathcal{L}^-(x) \neq \{0\}$ (see Proposition B.5).

In the same spirit, let

$$\mathcal{L}_{ext}[x] \subset \mathrm{H}om\big(\mathcal{Y}(x), \mathrm{H}^1(\mathrm{M}, \Sigma, \mathbb{R})\big)$$

denote the smallest affine subspace which for almost every $y \in W^-[x]$ contains the vector $\mathfrak{P}(y)$. (This makes sense since $\mathcal{Y}(x)$ is locally constant along $W^-[x]$.) We also set $\mathcal{L}_{ext}(x)$ to be the vector space spanned by all vectors of the form $\mathfrak{P}(y) - \mathfrak{P}(x)$ as y varies over $W^+[x]$. Then,

$$\mathcal{L}_{ext}(x) = \mathcal{L}_{ext}[x] - \mathfrak{P}(x).$$

Note that for almost all x and almost all $y \in W^-[x]$, $\mathcal{L}_{ext}[y] = \mathcal{L}_{ext}[x]$.

The space $\mathcal{L}_{ext}(x)^{(r)}$ *and the function* F. — For a vector space V we use the notation $V^{\otimes m}$ to denote the m-fold tensor product of V with itself. If $f : V \to W$ is a linear map, we write $f^{\otimes m}$ for the induced linear map from $V^{\otimes m}$ to $W^{\otimes m}$. Let $j^{\otimes m} : V \to V^{\otimes m}$ denote the map $v \to v \otimes \cdots \otimes v$ (m-times).

Let $V^{\uplus m}$ denote $\bigoplus_{k=1}^{m} V^{\otimes k}$. If $f : V \to W$ is a linear map, we write $f^{\uplus m}$ for the induced linear map from $V^{\uplus m}$ to $W^{\uplus m}$ given by

$$f^{\uplus m}(v) = \big(f^{\otimes 1}(v), f^{\otimes 2}(v), \dots f^{\otimes m}(v)\big).$$

Now if V and W are affine spaces, then we can still canonically define $V^{\uplus m}$ and $W^{\uplus m}$, and an affine map $f : V \to W$ induces an affine map $f^{\uplus m} : V^{\uplus m} \to W^{\uplus m}$.

Let r be an integer to be chosen later. Let $F : X_0 \to \mathcal{L}_{ext}[x]^{\uplus r}$ denote the diagonal embedding

$$F(x) = \mathfrak{P}(x)^{\uplus r}.$$

Let

$$\mathcal{L}_{ext}[x]^{(r)} \subset \mathcal{L}_{ext}(x)^{\uplus r}$$

denote the smallest affine subspace which contains the vectors $F(y)$ for almost all $y \in W^-[x]$. We also set

$$\mathcal{L}_{ext}(x)^{(r)} = \mathcal{L}_{ext}[x]^{(r)} - F(x).$$

Note that for $y \in W^-[x]$, $\mathcal{L}_{ext}[y]^{(r)} = \mathcal{L}_{ext}[x]^{(r)}$.

In this section, let $(\mathcal{B}, |\cdot|)$ be a finite measure space. (We will use the following proposition with $\mathcal{B} \subset U^+$ is a "unit box". The precise setup will be given in Section 6.)

To carry out the program outlined at the beginning of Section 5, we need the following:

Proposition **5.3.** — *For every $\delta > 0$ there exist constants $c_1(\delta) > 0$, $\epsilon_1(\delta) > 0$ with $c_1(\delta) \to 0$ and $\epsilon_1(\delta) \to 0$ as $\delta \to 0$, and also constants $\rho(\delta) > 0$, $\rho'(\delta) > 0$, and $C(\delta) < \infty$ such that the following holds:*

For any subset $K' \subset X_0$ with $\nu(K') > 1 - \delta$, there exists a subset $K \subset K'$ with $\nu(K) > 1 - c_1(\delta)$ such that the following holds: suppose for each $x \in X_0$ we have a measurable map from $\mathcal{B}$ to proper subspaces of $\mathcal{L}_{ext}(x)^{(r)}$, written as $u \to \mathcal{M}_u(x)$, where $\mathcal{M}_u(x)$ is a proper subspace of $\mathcal{L}_{ext}(x)^{(r)}$. Then, for any $q \in K$ there exists $q' \in K'$ with

$$\rho'(\delta) \le d^{X_0}\big(q, q'\big) \le 1/100 \tag{5.3}$$

and

$$\rho(\delta) \le \big\| F\big(q'\big) - F(q) \big\|_Y \le C(\delta) \tag{5.4}$$

and so that

$$d_Y\big(F\big(q'\big) - F(q), \mathcal{M}_u(q)\big) > \rho(\delta) \quad \textit{for at least } \big(1 - \epsilon_1(\delta)\big)\textit{-fraction of } u \in \mathcal{B}. \tag{5.5}$$

This proposition is proved in Section 5.2*. The proof uses almost nothing about the maps F or the measure ν, other than the definition of $\mathcal{L}_{ext}(x)$. It may be skipped on first reading.

5.1*. *Proof of Lemma 5.2.* — As in Section 4.1, let $\mathcal{V}_i(x) \equiv \mathcal{V}_i(H^1)(x) \subset H^1(M, \Sigma, \mathbb{R})$ denote the subspace corresponding to the (cocycle) Lyapunov exponent λ_i. Let

$$\mathcal{Y}(x) = \bigoplus_{i=1}^{k} \mathcal{V}_{\ge i}(x)/\mathcal{V}_{>i}(x),$$

where $\mathcal{V}_{\ge j}$ and $\mathcal{V}_{>j}$ are as in Section 4.1. Let $\pi_i : \mathcal{V}_{\ge i}(x) \to \mathcal{V}_{\ge i}(x)/\mathcal{V}_{>i}(x)$ denote the natural projection.

For $x \in X_0$, let $P_{i,x} \in \mathrm{H}om(\mathcal{V}_{\ge i}(x)/\mathcal{V}_{>i}(x), H^1(M, \Sigma, \mathbb{R}))$ denote the unique linear map such that for $\bar{x} \in \mathcal{V}_{\ge i}(x)/\mathcal{V}_{>i}(x)$, $P_{i,x}(\bar{x}) \in \mathcal{V}_i(H^1)(x)$ and $\pi_i(P_{i,x}(\bar{x})) = \bar{x}$. Note that the $P_{i,x}$ satisfy the following:

$$P_{i,g_t x} = g_t \circ P_{i,x} \circ g_t^{-1}, \tag{5.6}$$

and

$$P_{i,x}(\bar{u}) - P_{i,y}(\bar{u}) \in \mathcal{V}_{>i}(x). \tag{5.7}$$

Example. — The space $\mathcal{V}_{\ge 1}/\mathcal{V}_{>1}$ is one dimensional, and corresponds to the Lyapunov exponent $\lambda_1 = 1$. If we identify it with $\mathbb{R}$ in the natural way then $P_{1,x} : \mathbb{R} \to H^1(M, \Sigma, \mathbb{R})$ is given by the formula

$$P_{1,x}(\xi) = (\operatorname{Im} x)\xi \tag{5.8}$$

where for $x = (M, \omega)$, we write $\operatorname{Im} x$ for the imaginary part of ω.

Let

$$\mathfrak{P} : X_0 \to \bigoplus_{i=1}^{k} \mathrm{H}om\big(\mathcal{V}_{\geq i}(x)/\mathcal{V}_{>i}(x), H^1(M, \Sigma, \mathbb{R})\big)$$

be given by

$$\mathfrak{P}(x) = (P_{1,x}, \dots P_{k,x}).$$

Then, we can think of $\mathfrak{P}(x)$ as a map from $\mathcal{Y}(x)$ to $H^1(M, \Sigma, \mathbb{R})$ and (5.2) holds, where $P^-(x, y)$ is as in Section 4.2. □

5.2*. *Proof of Proposition 5.3.*

The measure $\tilde{\nu}_x$. — Let $\tilde{\nu}_x = F_*(\nu_{W^-[x]})$ denote the pushforward of ν_{W^-} under F. Then $\tilde{\nu}_x$ is a measure supported on $\mathcal{L}_{ext}[x]^{(r)}$. (Note that for $y \in W^-[x]$, $\tilde{\nu}_x = \tilde{\nu}_y$.)

Lemma **5.4.** — *For ν-almost all $x \in X_0$, for any $\epsilon > 0$ (which is allowed to depend on x), the restriction of the measure $\tilde{\nu}_x$ to the ball $B(F(x), \epsilon) \subset \mathcal{L}_{ext}[x]^{(r)}$ is not supported on a finite union of proper affine subspaces of $\mathcal{L}_{ext}[x]^{(r)}$.*

Outline of proof. — Suppose not. Let $N(x)$ be the minimal integer N such that for some $\epsilon = \epsilon(x) > 0$, the restriction of $\tilde{\nu}_x$ to $B(F(x), \epsilon)$ is supported on N affine subspaces. Note that in view of (5.6) and (5.7), the induced action on $\mathcal{L}_{ext}$ (and hence on $\mathcal{L}_{ext}^{(r)}$) of g_{-t} for $t \geq 0$ is expanding. Then $N(x)$ is invariant under g_{-t}, $t \geq 0$. This implies that $N(x)$ is constant for ν-almost all x, and also that the only affine subspaces of $\mathcal{L}_{ext}[x]^{(r)}$ which contribute to $N(\cdot)$ pass through $F(x)$. Then, $N(x) > 1$ almost everywhere is impossible. Indeed, suppose $N(x) = k$ a.e., then pick y near x such that $F(y)$ is in one of the affine subspaces through $F(x)$; then there must be exactly k affine subspaces of non-zero measure passing though $F(y)$, but then at most one of them passes through $F(x)$. Thus, the measure restricted to a neighborhood of $F(x)$ gives positive weight to at least $k+1$ subspaces, contradicting our assumption. Thus, we must have $N(x) = 1$ almost everywhere; but then (after flowing by g_{-t} for sufficiently large $t > 0$) we see that for almost all x, $\tilde{\nu}_x$ is supported on a proper subspace of $\mathcal{L}_{ext}[x]^{(r)}$ passing through x, which contradicts the definition of $\mathcal{L}_{ext}(x)^{(r)}$. □

Remark. — Besides Lemma 5.4, the rest of the proof of Proposition 5.3 uses only the measurability of the map F.

The measure $\hat{\nu}_x$. — Let $\mathfrak{B}_0^-$ be the analogue of the partition $\mathfrak{B}_0$ constructed in Section 3 but along the stable leaves W^-. (The only properties we use here is that $\mathfrak{B}_0^-$ is a measurable partition subordinate to W^- with atoms of diameter at most 1/100.) Let $\mathfrak{B}_0^-[x] \subset W^-[x]$ denote the atom of the partition $\mathfrak{B}_0^-$ containing x.

Let $\hat{\nu}_x = F_*(\nu_{W^-[x]}|_{\mathfrak{B}_0^-[x]})$, i.e. $\hat{\nu}_x$ is the pushforward under F of the restriction of $\nu_{W^-[x]}$ to $\mathfrak{B}_0^-[x]$. Then, for $y \in \mathfrak{B}_0^-[x]$, $\hat{\nu}_x = \hat{\nu}_y$. Suppose $\delta > 0$ is given. Since

$$\lim_{C\to\infty} \hat{\nu}_x\big(B\big(F(x), C\big)\big) = \hat{\nu}_x\big(\mathcal{L}_{ext}[x]^{(r)}\big),$$

there exists a function $c(x) > 0$ finite almost everywhere such that for almost all x,

$$\hat{\nu}_x\big(B\big(F(x), c(x)\big)\big) > \big(1 - \delta^{1/2}\big)\hat{\nu}_x\big(\mathcal{L}_{ext}[x]^{(r)}\big).$$

Therefore, we can find $C = C(\delta) > 0$ and a compact set K'_δ with $\nu(K'_\delta) > 1 - \delta^{1/2}$ such that for each $x \in K'_\delta$,

$$\hat{\nu}_x\big(B\big(F(x), C\big)\big) > \big(1 - \delta^{1/2}\big)\hat{\nu}_x\big(\mathcal{L}_{ext}[x]^{(r)}\big) \quad \text{for all } x \in K'_\delta. \tag{5.9}$$

In the rest of Section 5.2*, C will refer to the constant of (5.9).

Lemma **5.5.** — *For every $\eta > 0$ and every $N > 0$ there exists $\beta_1 = \beta_1(\eta, N) > 0$, $\rho_1 = \rho_1(\eta, N) > 0$ and a compact subset $K_{\eta,N}$ of measure at least $1 - \eta$ such that for all $x \in K_{\eta,N}$, and any proper subspaces $\mathcal{M}_1(x), \dots, \mathcal{M}_N(x) \subset \mathcal{L}_{ext}(x)^{(r)}$,*

$$\hat{\nu}_x\left(B\big(F(x), C\big) \setminus \bigcup_{k=1}^{N} \text{Nbhd}\big(\mathcal{M}_k(x), \rho_1\big)\right) \geq \beta_1 \hat{\nu}_x\big(B\big(F(x), C\big)\big). \tag{5.10}$$

Outline of proof. — By Lemma 5.4, there exist $\beta_x = \beta_x(N) > 0$ and $\rho_x = \rho_x(N) > 0$ such that for any subspaces $\mathcal{M}_1(x), \dots \mathcal{M}_N(x) \subset \mathcal{L}_{ext}(x)^{(r)}$,

$$\hat{\nu}_x\left(B\big(F(x), C\big) \setminus \bigcup_{k=1}^{N} \text{Nbhd}\big(\mathcal{M}(x), \rho_x\big)\right) \geq \beta_x \hat{\nu}_x\big(B\big(F(x), C\big)\big). \tag{5.11}$$

Let $E(\rho_1, \beta_1)$ be the set of x such that (5.10) holds. By (5.11),

$$\nu\left(\bigcup_{\substack{\rho_1>0\\ \beta_1>0}} E(\rho_1, \beta_1)\right) = 1.$$

Therefore, we can choose $\rho_1 > 0$ and $\beta_1 > 0$ such that $\nu(E(\rho_1, \beta_1)) > 1 - \eta$. □

Lemma **5.6.** — *For every $\eta > 0$ and every $\epsilon_1 > 0$ there exists $\beta = \beta(\eta, \epsilon_1) > 0$, a compact set $K_\eta = K_\eta(\epsilon_1)$ of measure at least $1 - \eta$, and $\rho = \rho(\eta, \epsilon_1) > 0$ such that the following holds: Suppose for each $u \in \mathcal{B}$ let $\mathcal{M}_u(x)$ be a proper subspace of $\mathcal{L}_{ext}(x)^{(r)}$. Let*

$$E_{good}(x) = \big\{v \in B\big(F(x), C\big) : \textit{for at least } (1 - \epsilon_1)\textit{-fraction of } u \textit{ in } \mathcal{B}, \\ d_Y\big(v - F(x), \mathcal{M}_u(x)\big) > \rho/2\big\}.$$

Then, for $x \in K_\eta$,

$$\hat{\nu}_x\big(E_{good}(x)\big) \geq \beta \hat{\nu}_x\big(B\big(F(x), C\big)\big). \tag{5.12}$$

Proof. — Let $n = \dim \mathcal{L}_{ext}[x]^{(r)}$. By considering determinants, it is easy to show that for any $C > 0$ there exists a constant $c_n = c_n(C) > 0$ depending on n and C such that for any $\eta > 0$ and any points $v_1, \dots, v_n$ in a ball of radius C with the property that for all $1 < i \le n$, v_i is not within η of the subspace spanned by $v_1, \dots, v_{i-1}$, then $v_1, \dots, v_n$ are not within $c_n\eta$ of any $n-1$ dimensional subspace. Let $k_{max} \in \mathbb{N}$ denote the smallest integer greater then $1 + n/\epsilon_1$, and let $N = N(\epsilon_1) = \binom{k_{max}}{n-1}$. Let β_1, ρ_1 and $K_{\eta,N}$ be as in Lemma 5.5. Let $\beta = \beta(\eta, \epsilon_1) = \beta_1(\eta, N(\epsilon_1))$, $\rho = \rho(\eta, \epsilon_1) = \rho_1(\eta, N(\epsilon_1))/c_n$, $K_\eta(\epsilon_1) = K_{\eta,N(\epsilon_1)}$. Let $E_{bad}(x) = B(F(x), C) \setminus E_{good}(x)$. To simplify notation, we choose coordinates so that $F(x) = 0$. We claim that $E_{bad}(x)$ is contained in the union of the ρ_1-neighborhoods of at most N subspaces. Suppose this is not true. Then, for $1 \le k \le k_{max}$ we can inductively pick points $v_1, \dots, v_k \in E_{bad}(x)$ such that v_j is not within ρ_1 of any of the subspaces spanned by $v_{i_1}, \dots, v_{i_{n-1}}$ where $i_1 \le \dots \le i_{n-1} < j$. Then, any n-tuple of points $v_{i_1}, \dots, v_{i_n}$ is not contained within $\rho = c_n\rho_1$ of a single subspace. Now, since $v_i \in E_{bad}(x)$, there exists $U_i \subset \mathcal{B}$ with $|U_i| \ge \epsilon_1|\mathcal{B}|$ such that for all $u \in U_i$, $d_Y(v_i, \mathcal{M}_u) < \rho/2$. We now claim that for any $1 \le i_1 < i_2 < \dots < i_n \le k$,

$$U_{i_1} \cap \dots \cap U_{i_n} = \emptyset. \tag{5.13}$$

Indeed, suppose u belongs to the intersection. Then each of the $v_{i_1}, \dots v_{i_n}$ is within $\rho/2$ of the single subspace $\mathcal{M}_u$, but this contradicts the choice of the v_i. This proves (5.13). Now,

$$\epsilon_1 k_{max}|\mathcal{B}| \le \sum_{i=1}^{k_{max}} |U_i| \le n\left|\bigcup_{i=1}^{k_{max}} U_i\right| \le n|\mathcal{B}|.$$

This is a contradiction, since $k_{max} > 1 + n/\epsilon_1$. This proves the claim. Now (5.10) implies that

$$\begin{aligned}\hat{\nu}_x\big(E_{good}(x)\big) &\ge \hat{\nu}_x\left(B\big(F(x), C\big) \setminus \bigcup_{k=1}^{N} \mathrm{Nbhd}\big(\mathcal{M}_k(x), \rho_1\big)\right) \\ &\ge \beta\hat{\nu}_x\big(B\big(F(x), C\big)\big).\end{aligned}$$

□

Proof of Proposition 5.3. — Let

$$K'' = \left\{x \in X_0 \ : \ \nu_{W^-[x]}\big(K' \cap \mathfrak{B}_0^-[x]\big) \ge \big(1 - \delta^{1/2}\big)\nu_{W^-[x]}\big(\mathfrak{B}_0^-[x]\big)\right\}.$$

By Lemma 3.11, we have $\nu(K'') \ge 1 - \delta^{1/2}$.

We have, for $x \in K''$,

$$\hat{\nu}_x\big(F\big(K' \cap \mathfrak{B}_0^-[x]\big)\big) \ge \big(1 - \delta^{1/2}\big)\hat{\nu}_x\big(\mathcal{L}_{ext}[x]^{(r)}\big). \tag{5.14}$$

Let $\beta(\eta, \epsilon_1)$ be as in Lemma 5.6. Let

$$c(\delta) = \delta + \inf\left\{\big(\eta^2 + \epsilon_1^2\big)^{1/2} \ : \ \beta(\eta, \epsilon_1) \ge 8\delta^{1/2}\right\}.$$

We have $c(\delta) \to 0$ as $\delta \to 0$. By the definition of $c(\delta)$ we can choose $\eta = \eta(\delta) < c(\delta)$ and $\epsilon_1 = \epsilon_1(\delta) < c(\delta)$ so that $\beta(\eta, \epsilon_1) \geq 8\delta^{1/2}$.

Now suppose $x \in K'' \cap K'_\delta$. Then, by (5.9) and (5.14),

$$\hat{\nu}_x\big(F\big(K' \cap \mathfrak{B}_0^-[x]\big) \cap B\big(F(x), C\big)\big) \geq \big(1 - 2\delta^{1/2}\big)\hat{\nu}_x\big(B\big(F(x), C\big)\big). \tag{5.15}$$

By (5.12), for $x \in K_\eta$,

$$\hat{\nu}_x\big(E_{good}(x)\big) \geq 8\delta^{1/2}\hat{\nu}_x\big(B\big(F(x), C\big)\big). \tag{5.16}$$

Let $K = K' \cap K'' \cap K'_\delta \cap K_\eta$. We have $\nu(K) \geq 1 - \delta - 2\delta^{1/2} - c(\delta)$, so $\nu(K) \to 1$ as $\delta \to 0$. Also, if $q \in K$, by (5.15) and (5.16),

$$F\big(K' \cap \mathfrak{B}_0^-[q]\big) \cap E_{good}(q) \cap B\big(F(x), C\big) \neq \emptyset.$$

Thus, we can choose $q' \in K' \cap \mathfrak{B}_0^-[q]$ such that $F(q') \in E_{good}(q) \cap B(F(q), C)$. Then (5.5) holds with $\rho = \rho(\eta(\delta), \epsilon_1(\delta)) > 0$. Also the upper bound in (5.3) holds since $\mathfrak{B}_0^-[q]$ has diameter at most $1/100$, and the upper bound in (5.4) holds since $F(q') \in B(F(q), C)$. Since all $\mathcal{M}_u(q)$ contain the origin q, the lower bound in (5.4) follows from (5.5). Finally, the lower bound in (5.3) follows from lower bound in (5.4) since in view of (5.8), $q - q'$ is essentially a component of $F(q) - F(q')$. □

6. Divergence of generalized subspaces

The groups $\mathcal{G}$, $\mathcal{G}_+$ and $\mathcal{G}_{++}$. — Recall that $H^1(x)$ denotes $H^1(M, \Sigma, \mathbb{R})$. (In fact the dependence on x is superfluous, but we find it useful to consider $H^1(x)$ as the fiber over X_0 of a flat bundle.) Let $\mathcal{G}(x) = (SL(H^1) \ltimes H^1)(x)$ which is isomorphic to the group of affine maps of $H^1(x)$ to itself. We can write $g \in \mathcal{G}(x)$ as a pair (L, v) where $L \in SL(H^1(x))$ and $v \in H^1(x)$. We call L the linear part of g, and v the translational part.

Let $Q_+(x)$ denote the group of linear maps from $H^1(x)$ to itself which preserve the flag $\{0\} \subset \mathcal{V}_{\leq 1}(H^1)(x) \subset \cdots \subset \mathcal{V}_{\leq k}(H^1)(x) = H^1(x)$, and let $Q_{++}(x) \subset Q_+(x)$ denote the unipotent subgroup of maps which are the identity on $\mathcal{V}_{\leq i}(H^1)(x)/\mathcal{V}_{<i}(H^1)(x)$ for all i. Let $\mathcal{G}_+(x)$ denote the subgroup of $\mathcal{G}(x)$ in which the linear part lies in $Q_+(x)$, and let $\mathcal{G}_{++}(x)$ denote the subgroup of $\mathcal{G}_+(x)$ in which the linear part lies in $Q_{++}(x)$. Note that $\mathcal{G}_{++}(x)$ is unipotent. Also, since $W^+(x) = \mathcal{V}_{\leq k-1}(H^1)(x)$, $\mathcal{G}_{++}(x)$ preserves $W^+(x)$.

For y near x, we have the Gauss-Manin connection $P^{GM}(x, y) : H^1(x) \to H^1(y)$. This induces a map $P^{GM}_*(x, y) : \mathcal{G}(x) \to \mathcal{G}(y)$. In view of Lemma 4.1, for $y \in W^+[x]$,

$$P^{GM}_*(y, x)\mathcal{G}_+(y) = \mathcal{G}_+(x), \qquad P^{GM}_*(y, x)Q_+(y) = Q_+(x),$$
$$P^{GM}_*(y, x)Q_{++}(y) = Q_{++}(x) \quad \text{and} \quad P^{GM}_*(y, x)\mathcal{G}_{++}(y) = \mathcal{G}_{++}(x).$$

We may consider elements of $\mathcal{G}_+(x)$ and $\mathcal{G}_{++}(x)$ as affine maps from $W^+[x]$ to $W^+[x]$. More precisely, $g = (L, v) \in \mathcal{G}(x)$ corresponds to the affine map $W^+[x] \to W^+[x]$ given by:

$$z \to x + L(z - x) + v. \tag{6.1}$$

Then, $Q_{++}(x)$ is the stabilizer of x in $\mathcal{G}_{++}(x)$. We denote by $\mathrm{Lie}(\mathcal{G}_{++})(x)$ the Lie algebra of $\mathcal{G}_{++}(x)$, etc.

We will often identify $W^+(x)$ with the translational part of $\mathrm{Lie}(\mathcal{G}_{++})(x)$. Then, we have an exponential map $\exp : W^+(x) \to \mathcal{G}_{++}(x)$, taking $v \in W^+(x)$ to $\exp v \in \mathcal{G}_{++}(x)$. Then, $\exp v : W^+[x] \to W^+[x]$ is translation by v.

The maps $Tr(x, y)$ *and* $tr(x, y)$. — For $h \in \mathcal{G}(x)$, let $Conj(h)$ to be the conjugation map $g \to hgh^{-1}$, and let $Ad(h) : \mathrm{Lie}(\mathcal{G})(x) \to \mathrm{Lie}(\mathcal{G})(x)$ be the adjoint map. Suppose $y \in W^+[x]$. Let $Tr(x, y) : \mathcal{G}(x) \to \mathcal{G}(y)$ and $tr(x, y) : \mathrm{Lie}(\mathcal{G})(x) \to \mathrm{Lie}(\mathcal{G})(y)$ be defined as

$$Tr(x, y) = P_*^{GM}(x, y) \circ Conj\big(\exp(x - y)\big),$$
$$tr(x, y) = P_*^{GM}(x, y) \circ Ad\big(\exp(x - y)\big).$$

The following lemma is clear from the definitions:

Lemma **6.1.** — *Suppose* $y \in W^+[x]$. *Then the elements* $g_x \in \mathcal{G}(x)$ *and* $g_y \in \mathcal{G}(y)$ *correspond to the same affine map of* $W^+[x] = W^+[y]$ *(in the sense of (6.1)) if and only if* $g_y = Tr(x, y) g_x$.

Admissible partitions. — By an admissible measurable partition we mean any partition $\mathfrak{B}_0$ as constructed in Section 3 (with some choice of $\mathcal{C}$ and $T_0(x)$).

Generalized subspaces. — Let $U'(x) \subset \mathcal{G}_{++}(x)$ be a connected Lie subgroup. We write

$$U'[x] = \big\{ux \ : \ u \in U'(x)\big\}$$

and call $U'[x]$ a generalized subspace. We have $U'[x] \subset W^+[x]$.

Definition **6.2.** — *Suppose that for almost all* $x \in X_0$ *we have a distinguished subgroup* $U^+(x)$ *of* $\mathcal{G}_{++}(x)$. *We say that the family of subgroups* $U^+(x)$ *is compatible with* ν *if the following hold:*

(i) *The assignment* $x \to U^+(x)$ *is measurable and* g_t*-equivariant.*
(ii) *For any admissible measurable partition* $\mathfrak{B}'$ *of* X_0, *the sets of the form* $U^+[x] \cap \mathfrak{B}'[x]$ *are a measurable partition of* X_0.
(iii) *For any admissible measurable partition* $\mathfrak{B}'$ *of* X_0, *for almost every* $x \in X_0$, *the conditional measure of* ν *along* $U^+[x] \cap \mathfrak{B}'[x]$ *is a multiple of the unique* $U^+(x)$ *invariant measure on* $U^+[x] \cong U^+(x)/(U^+(x) \cap Q_{++}(x))$. *(Note that both* $U^+(x)$ *and* $U^+(x) \cap Q_{++}(x)$ *are unimodular, since they are unipotent. Hence there is a well-defined Haar measure on the quotient* $U^+(x)/(U^+(x) \cap Q_{++}(x))$.*)*

(iv) *We have, for almost all* $x \in X_0$ *and almost all* $u \in U^+(x)$,

$$U^+(ux) = Tr(x, ux)U^+(x). \tag{6.2}$$

(This is motivated by Lemma 6.1 and the fact that we want $U^+[ux] = U^+[x]$.*) Thus,*

$$\mathrm{Lie}\big(U^+\big)(ux) = tr(x, ux)\,\mathrm{Lie}\big(U^+\big)(x). \tag{6.3}$$

(v) $U^+(x) \supset \exp N(x)$ *where* $N(x) \subset W^+(x)$ *is the direction of the orbit of the unipotent* $N \subset SL(2, \mathbb{R})$.

Standing assumption. — We are assuming that for almost every $x \in X_0$ there is a distinguished subgroup $U^+(x)$ of $\mathcal{G}_{++}(x)$ so that the family of subgroups $U^+(x)$ is compatible with ν in the sense of Definition 6.2. This will be used as an inductive assumption in Section 12.

We emphasize that $U^+(x)$ is defined for $x \in X_0$. Using our notational conventions, for $x \in X$, we write $U^+(x)$ for $U^+(\sigma_0(x))$ etc.

The unipotent N *as a compatible system of measures.* — At the start of the induction we have $U^+(x) = \exp N(x) \subset \mathcal{G}_{++}(x)$. We now verify that $U^+(x) = \exp N(x)$ is a family of subgroups compatible with ν in the sense of Definition 6.2. Note that $N(x) = \mathcal{V}_{\le 1}(H^1)(x) = \mathcal{V}_1(H^1)(x)$. In particular, by Lemma 4.1, for $y \in W^+[x]$,

$$N(y) = P_{GM}(x, y)N(x). \tag{6.4}$$

This implies (i) and (ii) of Definition 6.2.

The subgroup $U^+(x) = \exp N(x) \subset \mathcal{G}_{++}(x)$ consists of pure translations (i.e. $U^+(x) \cap \mathcal{Q}_{++}(x)$ is only the identity map). In particular, $U^+[x] = N[x]$. This, together with the N-invariance of ν implies (iii) of Definition 6.2.

Note that since $U^+(x)$ consists of pure translations, for any $y \in W^+[x]$, $\mathrm{C}\mathit{onj}(\exp(y - x))(U^+(x)) = U^+(x)$. This, together with (6.4) implies (iv) of Definition 6.2.

The sets $\mathcal{B}[x]$, $\mathcal{B}_t[x]$ *and* $\mathcal{B}(x)$. — Recall the partitions $\mathfrak{B}_t[x]$ from Section 3. Let $\mathcal{B}_t[x] = U^+[x] \cap \mathfrak{B}_t[x]$. We will also use the notation $\mathcal{B}[x]$ for $\mathcal{B}_0[x]$.

For notational reasons, we will make the following construction: let

$$\mathcal{B}_t(x) = \big\{u \in U^+(x)/\big(U^+(x) \cap Q_{++}(x)\big)\ :\ ux \in \mathcal{B}_t[x]\big\}.$$

We also write $\mathcal{B}(x)$ for $\mathcal{B}_0(x)$.

The Haar measure. — Let $|\cdot|$ denote the conditional measure of ν on $\mathcal{B}[x]$. (By our assumptions, this measure is $U^+(x)$-invariant where it makes sense.) We also denote the Haar measure (with some normalization) on $\mathcal{B}(x)$ by $|\cdot|$. Unless otherwise specified, all statements will be independent of the choice of normalization.

The same argument as Lemma 3.11 also proves the following:

Lemma **6.3.** — *Suppose $\delta > 0$, $\theta' > 0$ and $K \subset X$, with $\nu(K) > 1 - \delta$. Then there exists a subset $K^* \subset K$ with $\nu(K^*) > 1 - \delta/\theta'$ such that for any $x \in K^*$, and any $t > 0$,*

$$|K \cap \mathcal{B}_t[x]| \geq \left(1 - \theta'\right)|\mathcal{B}_t[x]|,$$

and thus

$$\left|\left\{u \in \mathcal{B}_t(x) \ : \ ux \in K\right\}\right| \geq \left(1 - \theta'\right)|\mathcal{B}_t(x)|.$$

The "ball" $\mathcal{B}(x, r)$. — For notational reasons, for $0 < r \leq 1/50$, and $x \in X_0$ we define

$$\mathcal{B}(x, r) = \left\{u \in U^+(x)/\left(U^+(x) \cap Q_{++}(x)\right) \ : \ d^+(ux, x) < r\right\},$$

where $d^+(\cdot, \cdot)$ is as in Section 3. In view of Proposition 3.4, we will normally use the ball $\mathcal{B}(x, 1/100) \subset U^+(x)/(U^+(x) \cap Q_{++}(x))$.

Lyapunov subspaces. — Suppose W is a subbundle of H_{big}. Let $\lambda_1(W) > \lambda_2(W) > \cdots > \lambda_n(W)$ denote the Lyapunov exponents of the action of g_t on W, and for $x \in X_0$ let $\mathcal{V}_i(W)(x)$ denote the corresponding subspaces. Let $\mathcal{V}_{\leq i}(W) = \bigoplus_{j=1}^{i} \mathcal{V}_i(W)$.

Notational convention. — In this subsection, we write $\mathcal{V}_i(x)$, $\mathcal{V}_{\leq i}(x)$ and λ_i instead of $\mathcal{V}_i(\mathrm{Lie}(\mathcal{G}_{++}))(x)$, $\mathcal{V}_{\leq i}(\mathrm{Lie}(\mathcal{G}_{++}))(x)$ and $\lambda_i(\mathrm{Lie}(\mathcal{G}_{++}))$.

Since $\mathrm{Lie}(U^+)(x)$ and $\mathrm{Lie}(Q_{++})(x)$ are equivariant under the g_t action, we have

$$\mathrm{Lie}\left(U^+\right)(x) = \bigoplus_i \mathrm{Lie}\left(U^+\right)(x) \cap \mathcal{V}_i(x),$$

$$\mathrm{Lie}(Q_{++})(x) = \bigoplus_i \mathrm{Lie}(Q_{++})(x) \cap \mathcal{V}_i(x).$$

The spaces $\mathcal{H}_+(x)$ and $\mathcal{H}_{++}(x)$. — Let $\mathcal{H}_+(x) = \mathrm{Hom}(\mathrm{Lie}(U^+)(x), \mathrm{Lie}(\mathcal{G}_{++})(x))$. (Here, Hom means linear maps between vector spaces, not Lie algebra homomorphisms.)

For every $M \in \mathcal{H}_+(x)$, we can write

$$M = \sum_{ij} M_{ij} \quad \text{where } M_{ij} \in \mathrm{Hom}\left(\mathrm{Lie}\left(U^+\right)(x) \cap \mathcal{V}_j(x), \mathrm{Lie}(\mathcal{G}_{++})(x) \cap \mathcal{V}_i(x)\right). \tag{6.5}$$

Let

$$\mathcal{H}_{++}(x) = \{ \mathrm{M} \in \mathcal{H}_+(x) \ : \ \mathrm{M}_{ij} = 0 \text{ if } \lambda_i \leq \lambda_j \}.$$

Then, $\mathcal{H}_{++}$ is the direct sum of all the positive Lyapunov subspaces of the action of g_t on $\mathcal{H}_+$.

Parametrization of generalized subspaces. — Suppose $\mathrm{M} \in \mathcal{H}_+(x)$ is such that $(\mathrm{I} + \mathrm{M})\,\mathrm{Lie}(\mathrm{U}^+)(x)$ is a subalgebra of $\mathrm{Lie}(\mathcal{G}_{++})(x)$. We say that the pair $(\mathrm{M}, v) \in \mathcal{H}_+(x) \times \mathrm{W}^+(x)$ parametrizes the generalized subspace $\mathcal{U}$ if

$$\mathcal{U} = \{ \exp[(\mathrm{I}+\mathrm{M})u](x+v) \ : \ u \in \mathrm{Lie}(\mathrm{U}^+)(x) \}.$$

(Thus, $\mathcal{U}$ is the orbit of the subgroup $\exp[(\mathrm{I}+\mathrm{M})\,\mathrm{Lie}(\mathrm{U}^+)(x)]$ through the point $x + v \in \mathrm{W}^+[x]$.) In this case we write $\mathcal{U} = \mathcal{U}(\mathrm{M}, v)$.

Remark. — In this discussion, $\mathcal{U}$ is a generalized subspace which passes near the point $x \in \mathrm{X}_0$. However, $\mathcal{U}$ need not be $\mathrm{U}^+[x]$, or even $\mathrm{U}^+[y]$ for any $y \in \mathrm{X}_0$.

Remark. — From the definitions, it is clear that any generalized subspace $\mathcal{U} \subset \mathrm{W}^+[x]$ can be parametrized by a pair $(\mathrm{M}, v) \in \mathcal{H}_+(x) \times \mathrm{W}^+(x)$. Also, if $v = v'$ and

$$\mathrm{I} + \mathrm{M} = (\mathrm{I} + \mathrm{M}') \circ \mathrm{J}, \tag{6.6}$$

where $\mathrm{J} : \mathrm{Lie}(\mathrm{U}^+)(x) \to \mathrm{Lie}(\mathrm{U}^+)(x)$ is a linear map, then $(\mathrm{M}, v) \in \mathcal{H}_+(x) \times \mathrm{W}^+(x)$ and $(\mathrm{M}', v') \in \mathcal{H}_+(x) \times \mathrm{W}^+(x)$ are two parameterizations of the same generalized subspace $\mathcal{U}$.

Example **1.** — We give an example of a non-linear generalized subspace. (The example does not satisfy condition (v) of Definition 6.2 but this is not relevant for the discussion.) Suppose for simplicity that W^+ has two Lyapunov exponents $\lambda_1(\mathrm{W}^+)$ and $\lambda_2(\mathrm{W}^+)$ with $\lambda_1(\mathrm{W}^+) = 2\lambda_2(\mathrm{W}^+)$. Let $e_1(x)$ and $e_2(x)$ be unit vectors so that $\mathcal{V}_1(\mathrm{W}^+)(x) = \mathbb{R}e_1(x)$, and $\mathcal{V}_2(\mathrm{W}^+)(x) = \mathbb{R}e_2(x)$.

Let $i : \mathrm{W}^+(x) \to \mathbb{R}^3$ be the map sending $ae_1(x) + be_2(x) \to (a, b, 1) \in \mathbb{R}^3$. We identify $\mathrm{W}^+(x)$ with its image in $\mathbb{R}^3$ under i. Then, we can identify

$$\mathcal{G}_{++}(x) = \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}, \qquad \mathrm{Lie}(\mathcal{G}_{++}(x)) = \begin{pmatrix} 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix}.$$

Suppose

$$\mathrm{U}^+(x) = \left\{ \begin{pmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} \ : \ t \in \mathbb{R} \right\},$$

$$\mathrm{Lie}\big(U^+(x)\big) = \left\{ \begin{pmatrix} 0 & t & 0 \\ 0 & 0 & t \\ 0 & 0 & 0 \end{pmatrix} \;:\; t \in \mathbb{R} \right\}.$$

Then, $U^+[x]$ is the parabola $\{x + te_2(x) + \frac{t^2}{2} e_1(x) \;:\; t \in \mathbb{R}\} \subset W^+[x]$.

Transversals. — Note that we have, as vector spaces,

$$\mathrm{Lie}(\mathcal{G}_{++})(x) = \mathrm{Lie}(Q_{++})(x) \oplus W^+(x)$$

where we identify $W^+(x)$ with the subspace of $\mathrm{Lie}(\mathcal{G}_{++})(x)$ corresponding to pure translations.

For each i, and each $x \in X_0$, let $Z_{i1}(x) \subset W^+(x) \cap \mathcal{V}_i(x) \subset \mathrm{Lie}(\mathcal{G}_{++})(x) \cap \mathcal{V}_i(x)$ be a linear subspace so that

$$\mathrm{Lie}(\mathcal{G}_{++})(x) \cap \mathcal{V}_i(x) = Z_{i1}(x) \oplus \Big(\big(\mathrm{Lie}\big(U^+\big) + \mathrm{Lie}(Q_{++})\big)(x) \cap \mathcal{V}_i(x)\Big).$$

Let $Z_{i2}(x) \subset \mathrm{Lie}(Q_{++})(x) \cap \mathcal{V}_i(x)$ be such that

$$\big(\mathrm{Lie}\big(U^+\big) + \mathrm{Lie}(Q_{++})\big)(x) \cap \mathcal{V}_i(x) = \big(\mathrm{Lie}\big(U^+\big)(x) \cap \mathcal{V}_i(x)\big) \oplus Z_{i2}(x).$$

Let $Z_i(x) = Z_{i1}(x) \oplus Z_{i2}(x)$, and let $Z(x) = \bigoplus_i Z_i(x)$. We always assume that the function $x \to Z(x)$ is measurable. We say that $Z(x) \subset \mathrm{Lie}(\mathcal{G}_{++})(x)$ is a *Lyapunov-admissible transversal* to $\mathrm{Lie}(U^+)(x)$. All of our transversals will be of this type, so we will sometimes simply use the word "transversal".

Note that $Z_{i1}(x) = Z(x) \cap W^+(x) \cap \mathcal{V}_i(x)$.

Example **2.** — Suppose $U^+(x)$ is as in Example 1. Then (since $\lambda_1(W^+) - \lambda_2(W^+) = \lambda_2(W^+)$),

$$\lambda_1 \equiv \lambda_1\big(\mathrm{Lie}(\mathcal{G}_{++})\big) = \lambda_1\big(W^+\big) \qquad \lambda_2 \equiv \lambda_2\big(\mathrm{Lie}(\mathcal{G}_{++})\big) = \lambda_2\big(W^+\big),$$

$$\mathcal{V}_1 \equiv \mathcal{V}_1\big(\mathrm{Lie}(\mathcal{G}_{++})\big) = \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\mathcal{V}_2 \equiv \mathcal{V}_2\big(\mathrm{Lie}(\mathcal{G}_{++})\big)(x) = \begin{pmatrix} 0 & * & 0 \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix},$$

$$\big(\mathrm{Lie}(Q_{++}) \cap \mathcal{V}_2\big)(x) = \begin{pmatrix} 0 & * & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\big(\mathrm{Lie}\big(U^+\big) \cap \mathcal{V}_2\big)(x) = \left\{ \begin{pmatrix} 0 & t & 0 \\ 0 & 0 & t \\ 0 & 0 & 0 \end{pmatrix} \;:\; t \in \mathbb{R} \right\},$$

and $(\mathrm{Lie}(U^+) \cap \mathcal{V}_1)(x) = (\mathrm{Lie}(Q_{++}) \cap \mathcal{V}_1)(x) = \{0\}$. Therefore, $Z_{12}(x) = \{0\}$, and

$$Z_{22}(x) = \begin{pmatrix} 0 & * & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad Z_{11}(x) = \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad Z_{21}(x) = \{0\}.$$

We note that in this example, the transversal Z was uniquely determined (and is in fact invariant under the flow g_t). This is a consequence of the fact that we chose an example with simple Lyapunov spectrum, and would not be true in general.

Parametrization adapted to a transversal. — We say that the parametrization $(M, v) \in \mathcal{H}_+(x) \times W^+(x)$ of a generalized subspace $\mathcal{U} = \mathcal{U}(M, v)$ is adapted to the transversal $Z(x)$ if

$$v \in Z(x) \cap W^+(x)$$

and

$$Mu \in Z(x) \quad \text{for all } u \in \mathrm{Lie}\big(U^+\big)(x).$$

The following lemma implies that adapting a parametrization to a transversal is similar to inverting a nilpotent matrix.

Lemma **6.4.** — *Suppose the pair* $(M', v') \in \mathcal{H}_{++}(x) \times W^+(x)$ *parametrizes a generalized subspace* $\mathcal{U}$. *Let* $Z(x)$ *be a Lyapunov-admissible transversal. Then, there exists a unique pair* $(M, v) \in \mathcal{H}_{++}(x) \times W^+(x)$ *which parametrizes* $\mathcal{U}$ *and is adapted to* $Z(x)$. *If we write*

$$M' = \sum_{ij} M'_{ij}$$

as in (6.5), *and*

$$v' = \sum_j v'_j,$$

where $v'_j \in W^+(x) \cap \mathcal{V}_j(x)$, *then* $M = \sum_{ij} M_{ij}$ *and* $v = \sum_i v_i$ *are given by formulas of the form*

$$v_i = L_i v'_i + p_i\big(v', M'\big) \tag{6.7}$$

$$M_{ij} = L_{ij} M'_{ij} + p_{ij}\big(M'\big) \tag{6.8}$$

where L_i *is a linear map and* p_i *is a polynomial in the* v'_j *and* M'_{jk} *which depends only on the* v'_j *with* $\lambda_j < \lambda_i$ *and the* M'_{jk} *with* $\lambda_j - \lambda_k < \lambda_i$. *Similarly,* L_{ij} *is a linear map, and* p_{ij} *is a polynomial which depends on the* M'_{kl} *with* $\lambda_k - \lambda_l < \lambda_i - \lambda_j$.

If we assume in addition that (M', v') *is adapted to another Lyapunov-admissible transversal* $Z'(x)$, *then* L_i *and* L_{ij} *can be taken to be invertible linear maps (depending only on* $Z(x)$ *and* $Z'(x)$*).*

The proof of Lemma 6.4 is a straightforward but tedious calculation. It is done in Section 6.4*.

The map S_x^Z. — Suppose Z is a Lyapunov-admissible transversal to $U^+(x)$. Then, let $S_x^Z : \mathcal{H}_{++}(x) \times W^+(x) \to \mathcal{H}_{++}(x) \times W^+(x)$ be given by

$$S_x^Z(M', v') = (M, v)$$

where M and v are given by (6.8) and (6.7) respectively. Note that S_x^Z is a polynomial, but is *not* a linear map in the entries of M' and v'. To deal with the non-linearity, we work with certain tensor product spaces defined below.

Tensor products: the spaces $\hat{\mathbf{H}}$, $\tilde{\mathbf{H}}$ *and the maps* $\mathbf{j}$. — As in Section 5, for a vector space V and a map $f : V \to W$ we use the notations $V^{\otimes m}$, $V^{\uplus m}$, $f^{\otimes m}$, $f^{\uplus m}$, $j^{\otimes m}$, $j^{\uplus m}$.

Let m be the number of distinct Lyapunov exponents on $\mathcal{H}_{++}$, and let n be the number of distinct Lyapunov exponents on W^+. Let $(\alpha; \beta) = (\alpha_1, \dots, \alpha_m; \beta_1, \dots, \beta_n)$ be a multi-index, and let

$$\tilde{\mathbf{H}}^{(\alpha;\beta)}(x) = \bigotimes_{i=1}^{m} \big(\mathcal{V}_i(\mathcal{H}_{++})(x)\big)^{\otimes \alpha_i} \otimes \bigotimes_{j=1}^{n} \big(\mathcal{V}_j(W^+)(x)\big)^{\otimes \beta_j}$$

and let

$$\hat{\mathbf{H}}^{(\alpha;\beta)}(x) = \bigotimes_{i=1}^{m} \mathcal{H}_{++}(x)^{\otimes \alpha_i} \otimes \bigotimes_{j=1}^{n} W^+(x)^{\otimes \beta_j}.$$

We have a natural map $\hat{\pi}^{(\alpha;\beta)} : \hat{\mathbf{H}}^{(\alpha;\beta)}(x) \to \tilde{\mathbf{H}}^{(\alpha;\beta)}(x)$ given by

$$\begin{aligned} &\hat{\pi}^{(\alpha;\beta)}\big(Y_1 \otimes \cdots \otimes Y_m \otimes (Y'_1) \otimes \cdots \otimes (Y'_n)\big) \\ &\quad = \pi_1^{\otimes \alpha_1}(Y_1) \otimes \cdots \otimes \pi_m^{\otimes \alpha_m}(Y_m) \otimes (\pi'_1)^{\otimes \beta_1}(Y'_1) \otimes \cdots \otimes (\pi'_n)^{\otimes \beta_n}(Y'_n), \end{aligned}$$

where $\pi_i : \mathcal{H}_{++}(x) \to \mathcal{V}_i(\mathcal{H}_{++})(x)$ and $\pi'_j : W^+(x) \to \mathcal{V}_j(W^+)(x)$ are the natural projections associated to the direct sum decompositions $\mathcal{H}_{++}(x) = \bigoplus_{i=1}^m \mathcal{V}_i(\mathcal{H}_{++})(x)$ and $W^+(x) = \bigoplus_{j=1}^n \mathcal{V}_j(W^+)(x)$.

Let $\mathcal{S}$ be a finite collection of multi-indices (chosen in Lemma 6.6 below). Then, let

$$\tilde{\mathbf{H}}_0(x) = \bigoplus_{(\alpha;\beta)\in\mathcal{S}} \tilde{\mathbf{H}}^{(\alpha;\beta)}, \qquad \hat{\mathbf{H}}_0(x) = \bigoplus_{(\alpha;\beta)\in\mathcal{S}} \hat{\mathbf{H}}^{(\alpha;\beta)} \tag{6.9}$$

Let $\hat{\pi} : \hat{\mathbf{H}}_0(x) \to \tilde{\mathbf{H}}_0(x)$ be the linear map with coincides with $\hat{\pi}^{(\alpha;\beta)}$ on each $\hat{\mathbf{H}}^{(\alpha;\beta)}$.

Let $\hat{\mathbf{j}}^{(\alpha;\beta)} : \mathcal{H}_{++}(x) \times W^+(x) \to \hat{\mathbf{H}}^{(\alpha;\beta)}(x)$ be the "diagonal embedding"

$$\hat{\mathbf{j}}^{(\alpha;\beta)}(M, v) = M \otimes M \cdots \otimes M \otimes v \otimes \cdots \otimes v,$$

and let $\hat{\mathbf{j}} : \mathcal{H}_{++}(x) \times W^+(x) \to \hat{\mathbf{H}}_0(x)$ be the linear map $\bigoplus_{(\alpha;\beta)\in\mathcal{S}} \hat{\mathbf{j}}^{(\alpha;\beta)}$. Let

$$\mathbf{j} : \mathcal{H}_{++}(x) \times W^+(x) \to \tilde{\mathbf{H}}_0(x) \tag{6.10}$$

denote $\hat{\pi} \circ \hat{\mathbf{j}}$. Let $\hat{\mathbf{H}}(x)$ denote the linear span of the image of $\hat{\mathbf{j}}$, and let $\tilde{\mathbf{H}}(x)$ denote the linear span of the image of $\mathbf{j}$.

Induced linear maps on $\hat{\mathbf{H}}(x)$ *and* $\tilde{\mathbf{H}}(x)$. — Suppose $F_t : \mathcal{H}_{++}(x) \to \mathcal{H}_{++}(y)$ and $F'_t : W^+(x) \to W^+(y)$ are linear maps. Let $f_t = (F_t, F'_t)$. Then, f_t induces a linear map $\hat{\mathbf{f}}_t : \hat{\mathbf{H}}(x) \to \hat{\mathbf{H}}(y)$. If F_t sends each $\mathcal{V}_i(\mathcal{H}_{++})(x)$ to each $\mathcal{V}_i(\mathcal{H}_{++})(y)$ and F'_t sends each $\mathcal{V}_j(W^+)(x)$ to $\mathcal{V}_j(W^+)(y)$, then f_t also induces a linear map $\tilde{\mathbf{f}}_t : \tilde{\mathbf{H}}(x) \to \tilde{\mathbf{H}}(y)$.

Note that $\tilde{\mathbf{H}}(x) \subset \hat{\mathbf{H}}(x) \subset H^{(++)}_{big}(x)$ where $H^{(++)}_{big}(x)$ is as in Section 3.

Notation. — For an invertible linear map $A : W^+(x) \to W^+(y)$, let $A_* : \mathrm{Lie}(\mathcal{G}_{++})(x) \to \mathrm{Lie}(\mathcal{G}_{++})(y)$ denote the map

$$A_*(Y) = A \circ Y_1 \circ A^{-1} + A \circ Y_2 \tag{6.11}$$

where for $Y \in \mathrm{Lie}(\mathcal{G}_{++})(x)$, Y_1 is the linear part of Y and Y_2 is the pure translation part.

Lemma **6.5.** — *Suppose $x \in X_0$, $u \in U^+(x)$. Then, there exists a linear map $u_* : \mathcal{H}_{++}(x) \times W^+(x) \to \mathcal{H}_{++}(ux) \times W^+(ux)$ with the following properties:*

(a) *If $(M', v') \in \mathcal{H}_{++}(x) \times W^+(x)$ parametrizes a generalized subspace $\mathcal{U}$, then $(M, v) = u_*(M', v')$ parametrizes the same generalized subspace $\mathcal{U}$.*
(b) *If $(M, v) = u_*(M', v')$, then M and v are given by formulas of the form* (6.7) *and* (6.8).

Proof. — In fact we claim that

$$u_*\big(M', v'\big) = \big(tr(x, ux) \circ M' \circ tr(ux, x), \exp\big(\big(I + M'\big)Y\big)\big(x + v'\big) - \exp(Y)x\big), \tag{6.12}$$

where $Y = \log u$.

This can be verified as follows. Let $\mathcal{U} = \mathcal{U}(M', v')$ denote the generalized subspace parametrized by (M', v'), and let $U' = \exp((I + M')\,\mathrm{Lie}(U^+)(x))$, so that U' is a subgroup of $\mathcal{G}_{++}(x)$. Then, for any $w \in \mathcal{U}$, $\mathcal{U} = U'w$. Then, in view of Lemma 6.1 and (6.1),

$$\mathcal{U} = \mathrm{T}r(x, ux)U'\big(ux + (w - ux)\big).$$

Thus, $(M, v) \in \mathcal{H}_{++}(ux) \times W^+(ux)$ parametrizes $\mathcal{U}$ if

$$\exp\big((I + M)\,\mathrm{Lie}\big(U^+\big)(ux)\big) = \mathrm{T}r(x, ux)U' \tag{6.13}$$

and

$$v = w - ux \quad \text{for some } w \in \mathcal{U}. \tag{6.14}$$

Now let (M, v) be the right-hand-side of (6.12). We claim that (6.13) and (6.14) hold.

Indeed, by (6.3),

$$tr(ux, x)\,\mathrm{Lie}\big(U^+\big)(ux) = \mathrm{Lie}\big(U^+\big)(x),$$

and furthermore, $tr(ux, x)(\mathrm{Lie}(U^+) \cap \mathcal{V}_{\le i})(ux) = (\mathrm{Lie}(U^+) \cap \mathcal{V}_{\le i})(x)$. Now,

$$\begin{aligned} Tr(x, ux)U' &= \exp\big(tr(x, ux)\,\mathrm{Lie}\big(U'\big)\big) = \exp\big(tr(x, ux)\big(I + M'\big)\,\mathrm{Lie}\big(U^+\big)(x)\big) \\ &= \exp\big(tr(x, ux)\big(I + M'\big)tr(ux, x)\,\mathrm{Lie}\big(U^+(ux)\big)\big) \\ &= \exp\big((I + M)\,\mathrm{Lie}\big(U^+\big)(ux)\big), \end{aligned}$$

verifying (6.13). Also, let

$$w = \exp\big(\big(I + M'\big)Y\big)\big(x + v'\big) \in \mathcal{U} = \mathcal{U}\big(M', v'\big).$$

Therefore, since $\exp(Y)x = ux$,

$$w - ux = \big(\exp\big(\big(I + M'\big)Y\big)\big(x + v'\big) - \exp(Y)x\big) = v,$$

and hence (6.14) holds. Thus, $u_*(M', v') \in \mathcal{H}_{++}(ux) \times W^+(ux)$ as defined in (6.12) parametrizes the same generalized subspace $\mathcal{U}$ as $(M', v') \in \mathcal{H}_{++}(x) \times W^+(x)$. This completes the proof of part (a).

It is clear from (6.12) that part (b) of the lemma holds. □

Lemma **6.6.** — *For an appropriate choice of $\mathcal{S}$, the following hold:*

(a) *Let $Z(x)$ be a Lyapunov-admissible transversal to $U^+(x)$. There exists a linear map $\mathbf{S}_x^{Z(x)} : \tilde{\mathbf{H}}(x) \to \tilde{\mathbf{H}}(x)$ such that for all $(M, v) \in \mathcal{H}_{++}(x) \times W^+(x)$,*

$$\big(\mathbf{S}_x^{Z(x)} \circ \mathbf{j}\big)(M, v) = \big(\mathbf{j} \circ S_x^{Z(x)}\big)(M, v).$$

(b) *Suppose $u \in U^+(x)$, and let $Z(ux)$ be a Lyapunov-admissible transversal to $U^+(ux)$. Then, there exists a linear map $(u)_* : \tilde{\mathbf{H}}(x) \to \tilde{\mathbf{H}}(ux)$ such that for all $(M, v) \in \mathcal{H}_{++}(x) \times W^+(x)$,*

$$\big((u)_* \circ \mathbf{j}\big)(M, v) = \big(\mathbf{j} \circ S_{ux}^{Z(ux)} \circ u_*\big)(M, v),$$

where $u_* : \mathcal{H}_{++}(x) \times W^+(x) \to \mathcal{H}_{++}(ux) \times W^+(ux)$ is as in (6.12).

Proof. — Part (a) formally follows from the universal property of the tensor product and the partial ordering in (6.7) and (6.8). We now make a brief outline: see also Example 3 below.

Let $\tilde{\mathbf{H}}_{\mathcal{S}}(x)$ and $\mathbf{j}^{\mathcal{S}}$ be as in (6.9) and (6.10) with the dependence on $\mathcal{S}$ explicit. Let $\mathcal{S}_0$ denote the set of multi-indices of the form $(0,\dots,0,1,0,\dots,0;0,\dots,0)$ or $(0,\dots,0;0,\dots,0,1,0,\dots,0)$. Then $\mathbf{j}^{\mathcal{S}_0}$ is an isomorphism between $\mathcal{H}_{++}(x)\times \mathrm{W}^+(x)$ and $\tilde{\mathbf{H}}_{\mathcal{S}_0}(x)$.

Let $(\mathrm{M}, v) = \mathrm{S}_x^{\mathrm{Z}(x)}(\mathrm{M}', v')$. By (6.7), (6.8) and the universal property of the tensor product, there exists $\mathcal{S}_1 \supset \mathcal{S}_0$ and a linear map $\mathbf{S}_1 : \tilde{\mathbf{H}}_{\mathcal{S}_1}(x) \to \tilde{\mathbf{H}}_{\mathcal{S}_0}(x)$ such that

$$\mathbf{j}^{\mathcal{S}_0}(\mathrm{M}, v) = \mathbf{S}_1 \circ \mathbf{j}^{\mathcal{S}_1}\big(\mathrm{M}', v'\big).$$

We now repeat this procedure to get a sequence $\mathcal{S}_j$ of multi-indices. More precisely, at each stage, for each $(\alpha;\beta)\in\mathcal{S}_j$, we may write, by (6.7), (6.8) and the universal property of the tensor product,

$$\mathbf{j}^{(\alpha;\beta)}(\mathrm{M}, v) = \mathbf{L}^{(\alpha;\beta)}\big(\mathbf{j}^{(\alpha;\beta)}(\mathrm{M}, v)\big) + \mathbf{S}_{j+1}^{(\alpha;\beta)}\Bigg(\bigoplus_{(\alpha';\beta')\in\mathcal{S}(\alpha;\beta)} \mathbf{j}^{(\alpha';\beta')}\big(\mathrm{M}', v'\big)\Bigg),$$

where $\mathbf{L}^{(\alpha;\beta)}$ and $\mathbf{S}_{j+1}^{(\alpha;\beta)}$ are linear maps; we then define $\mathcal{S}_{j+1} = \mathcal{S}_j \cup \bigcup_{(\alpha;\beta)\in\mathcal{S}_j}\mathcal{S}(\alpha;\beta)$. Putting these maps together, we then get a linear map $\mathbf{S}_j$ such that

$$\mathbf{j}^{\mathcal{S}_j}(\mathrm{M}, v) = \mathbf{S}_j \circ \mathbf{j}^{\mathcal{S}_{j+1}}\big(\mathrm{M}', v'\big).$$

Because of the partial order in (6.7) and (6.8), we may assume that $\mathcal{S}(\alpha;\beta)$ consists of multi-indices $(\alpha';\beta')$ where either α' has more zero entries than α or β' has more zero entries than β. Therefore, this procedure eventually terminates, so that $\mathcal{S}_{j+1} = \mathcal{S}_j$ for large enough j. We then define $\mathcal{S}$ to be the eventual common value of the $\mathcal{S}_j$; then part (a) of Lemma 6.6 holds.

To prove part (b) of Lemma 6.6, note that part (b) of Lemma 6.5 and the proof of part (a) of Lemma 6.6 show that there exists a map $\tilde{u}_* : \tilde{\mathbf{H}}(x) \to \tilde{\mathbf{H}}(ux)$ such that $\tilde{u}_* \circ \mathbf{j} = \mathbf{j} \circ u_*$, where u_* is as in (6.12). Now, we can define $(u)_* : \tilde{\mathbf{H}}(x) \to \tilde{\mathbf{H}}(ux)$ to be $\mathbf{S}_{ux}^{\mathrm{Z}(ux)} \circ \tilde{u}_*$, where $\mathbf{S}_{ux}^{\mathrm{Z}(ux)}$ is as in (a). Thus $(u)_*$ denotes the induced action of u on $\mathbf{H}(x)$. □

Example **3.** — Suppose U^+ is as in Examples 1 and 2. Let

$$\mathrm{F} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathrm{E}_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathrm{E}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then, $(\mathrm{Lie}(\mathrm{U}^+)\cap\mathcal{V}_2)(x) = \mathbb{R}\mathrm{F}$, $(\mathrm{Lie}(\mathcal{G}_{++})\cap\mathcal{V}_1)(x) = \mathbb{R}\mathrm{E}_1$. Then, for $\mathrm{M}\in\mathcal{H}_{++}(x)$, the only non-zero component is $\mathrm{M}_{12} \in \mathrm{Hom}((\mathrm{Lie}(\mathrm{U}^+)\cap\mathcal{V}_2)(x), (\mathrm{Lie}(\mathcal{G}_{++})\cap\mathcal{V}_1)(x))$, which is 1-dimensional. Let

$$\Psi \in \mathrm{Hom}\big(\big(\mathrm{Lie}\big(U^+\big) \cap \mathcal{V}_2\big)(x), \big(\mathrm{Lie}(\mathcal{G}_{++}) \cap \mathcal{V}_1\big)(x)\big)$$

denote the element such that $\Psi F = E_1$, so that $\mathcal{H}_{++} = \mathbb{R}\Psi$.

With the choice of transversal Z given in Example 2, Equations (6.7) and (6.8) become:

$$v_1 = -M'_{12}v'_2 + v'_1 - \big(v'_2\big)^2, \quad v_2 = 0, \quad M_{12} = M'_{12}. \tag{6.15}$$

Then we can choose $\mathcal{S} = \{(1; 0, 0), (0; 1, 0), (0; 0, 1), (1; 0, 1), (0; 0, 2)\}$, so that (dropping the (x)),

$$\begin{aligned}\tilde{\mathbf{H}}_0 = \mathcal{H}_{++} \oplus \mathcal{V}_1\big(W^+\big) \oplus \mathcal{V}_2\big(W^+\big) \oplus \big(\mathcal{H}_{++} \otimes \mathcal{V}_2\big(W^+\big)\big)\\ \oplus \big(\mathcal{V}_2\big(W^+\big) \otimes \mathcal{V}_2\big(W^+\big)\big).\end{aligned}$$

(Since for any vector space V, $V^{\otimes 0} = \mathbb{R}$, we have omitted such factors in the above formula.) Let $\mathbf{S} = \mathbf{S}_x^{Z(x)}$. Then, the linear map $\mathbf{S} : \tilde{\mathbf{H}}(x) \to \tilde{\mathbf{H}}(x)$ is given by

$$\begin{aligned}&\mathbf{S}(\Psi) = \Psi, \qquad \mathbf{S}(E_1) = E_1, \qquad \mathbf{S}(E_2) = 0, \qquad \mathbf{S}(\Psi \otimes E_2) = -E_1,\\ &\mathbf{S}(E_2 \otimes E_2) = -E_1.\end{aligned}$$

Example **4.** — We keep all notation from Examples 1–3. Suppose $u = \exp Y$, where $Y = tF$. We now compute the map $(u)_*$.

Note that by Lemma 4.1, we have $e_1(ux) = e_1(x)$. Also note that by Example 1, at x, the tangent vector to $U^+[x]$ coincides with $e_2(x)$. Recall that we are assuming that the foliation whose leaves are $U^+[x]$ is invariant under the geodesic flow. This implies that at the point ux, the tangent vector to the parabola $U^+[x]$ is $e_2(ux)$. Therefore,

$$e_1(ux) = e_1(x), \qquad e_2(ux) = te_1(x) + e_2(x).$$

Therefore,

$$P^+(x, ux)e_1(x) = e_1(ux), \qquad P^+(x, ux)e_2(x) = e_2(ux) = te_1(x) + e_2(x).$$

Suppose $\mathcal{U}$ is parametrized by (M', v'), where $M' = M'_{12}\Psi$, $v' = v'_1e_1(x) + v'_2e_2(x)$. Then

$$\exp\big[\big(I + M'\big)Y\big] = \begin{pmatrix} 1 & t & \frac{1}{2}t^2 + M'_{12}t \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}, \qquad \exp(Y) = \begin{pmatrix} 1 & t & \frac{1}{2}t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}.$$

Therefore,

$$\exp\big[\big(I + M'\big)Y\big]\big(x + v'\big) - \exp(Y)x = \begin{pmatrix} v'_1 + tv'_2 + tM'_{12} \\ v'_2 \\ 0 \end{pmatrix}.$$

Let $\Psi' \in \mathrm{Hom}((\mathrm{Lie}(U^+) \cap \mathcal{V}_2)(ux), (\mathrm{Lie}(\mathcal{G}_{++}) \cap \mathcal{V}_1)(ux))$ be the analogue of Ψ, but at the point ux. Then,

$$\begin{aligned} u_*\big(M', v'\big) &= u_*\big(M'_{12}\Psi, v'_1 e_1(x) + v'_2 e_2(x)\big) \\ &= \big(M'_{12}\Psi', \big(v'_1 + tv'_2 + tM'_{12}\big)e_1(x) + v'_2 e_2(x)\big) \\ &= \big(M'_{12}\Psi', \big(v'_1 + tM'_{12}\big)e_1(ux) + v'_2 e_2(ux)\big) \end{aligned}$$

Then, in view of (6.15), $(S_{ux}^{Z(ux)} \circ u_*)(M', v') = (M_{12}\Psi', v_1 e_1(ux) + v_2 e_2(ux))$, where

$$v_1 = -M'_{12}v'_2 + v'_1 + tM'_{12} - \big(v'_2\big)^2, \quad v_2 = 0, \quad M_{12} = M'_{12}.$$

Then, $(u)_* : \tilde{\mathbf{H}}(x) \to \tilde{\mathbf{H}}(ux)$ is given by

$$\begin{aligned} &(u)_*(\Psi) = \Psi' + tE_1, \qquad (u)_*(E_1) = E_1, \qquad (u)_*(E_2) = 0, \\ &(u)_*(\Psi \otimes E_2) = -E_1, \qquad (u)_*(E_2 \otimes E_2) = -E_1. \end{aligned}$$

The dynamical system G_t. — Suppose we fix some Lyapunov-admissible transversal $Z(x)$ for every $x \in X_0$. Suppose $(M, v) \in \mathcal{H}_{++}(x) \times W^+(x)$ is adapted to $Z(x)$. Let

$$G_t(M, v) = S_{g_t x}^{Z(g_t x)}\big(g_t \circ M \circ g_t^{-1}, (g_t)_* v\big) \in \mathcal{H}_{++}(g_t x) \times W^+(g_t x),$$

where $(g_t)_*$ on the right-hand side is g_t acting on $W^+(x)$, and g_t on the right-hand side is the natural map $\mathrm{Lie}(Q_{++})(x) \to \mathrm{Lie}(Q_{++})(g_t x)$, which maps $\mathrm{Lie}(U^+)(x)$ to $\mathrm{Lie}(U^+)(g_t x)$. Then, if $\mathcal{U}'$ is the generalized subspace parametrized by (M, v) then $(M'', v'') = G_t(M, v) \in \mathcal{H}_{++}(g_t x) \times W^+(g_t x)$ parametrizes $g_t\mathcal{U}'$ and is adapted to $Z(g_t x)$. From the definition, we see that

$$G_{t+s} = G_t \circ G_s.$$

Also, it is easy to see that for $(M, v) \in \mathcal{H}_{++}(x) \times W^+(x)$,

$$G_t(M, v) = \big(g_t \circ M' \circ g_t^{-1}, (g_t)_* v'\big), \quad \text{where } \big(M', v'\big) = S_x^{g_t^{-1}Z(g_t x)}(M, v).$$

The bundle $\mathbf{H}(x)$. — Suppose we are given a Lyapunov adapted transversal $Z(x)$ at each $x \in X_0$. Let

$$\mathbf{H}(x) = \mathbf{S}_x^{Z(x)}\tilde{\mathbf{H}}(x)$$

denote the image of $\tilde{\mathbf{H}}(x)$ under $\mathbf{S}_x^{Z(x)}$. Then, if $(M, v) \in \mathcal{H}_{++}(x) \times W^+(x)$ is adapted to $Z(x)$, then $\mathbf{j}(M, v) \in \mathbf{H}(x)$. We can also consider $(u)_*$ as defined in Lemma 6.6(b) to be a map

$$(u)_* : \mathbf{H}(x) \to \mathbf{H}(ux).$$

The bundle $\mathbf{H}$ *and the flow* g_t. — Let $Z(x)$ be an admissible transversal to $U^+(x)$ for every $x \in X_0$. Let $(g_t)_* : \mathbf{H}(x) \to \mathbf{H}(g_t x)$ be given by

$$(g_t)_* = \mathbf{S}_{g_t x}^{Z(g_t x)} \circ \tilde{\mathbf{f}}_t \quad \text{where } f_t(M, v) = \big(g_t \circ M \circ g_t^{-1}, (g_t)_* v\big), \tag{6.16}$$

$\tilde{\mathbf{f}}_t$ is the map induced by f_t on $\tilde{\mathbf{H}} \supset \mathbf{H}$, $(g_t)_*$ on the right-hand side is g_t acting on $W^+(x)$, g_t on the right-hand side is the natural map $\mathrm{Lie}(U^+)(x) \to \mathrm{Lie}(U^+)(g_t x)$, and $\mathbf{S}_x^Z$ is as in Lemma 6.6. Then $(g_t)_*$ is a linear map, and for $(M, v) \in \mathcal{H}_{++}(x) \times W^+(x)$,

$$(g_t)_*\big(\mathbf{j}(M, v)\big) = \mathbf{j}\big(G_t(M, v)\big). \tag{6.17}$$

Since $G_t \circ G_s = G_{t+s}$, and the linear span of $\mathbf{j}(\mathcal{H}_{++}(x) \times W^+(x))$ is $\tilde{\mathbf{H}}(x) \supset \mathbf{H}(x)$, it follows from (6.17) that $(g_t)_* \circ (g_s)_* = (g_{t+s})_*$.

Lemma **6.7.**

(a) *Suppose* $u'x = ux \in U^+[x]$ *and* $\mathbf{v} \in \mathbf{H}(x)$. *Then* $(u)_*\mathbf{v} = (u')_*\mathbf{v}$.

(b) *Suppose* $u \in U^+(g_t x)$. *Then there exists* $u' \in U^+(x)$ *such that* $g_t u' x = u g_t x$. *Furthermore, for any choice of* u' *satisfying* $g_t u' x = u g_t x$ *and any* $\mathbf{v} \in \mathbf{H}(x)$, *we have* $(u)_*(g_t)_*\mathbf{v} = (g_t)_*(u')_*\mathbf{v}$.

Proof. — It is enough to prove (a) for $\mathbf{v} = \mathbf{j}(M, v)$ where $(M, v) \in \mathcal{H}_{++}(x) \times W^+(x)$. Let $\mathcal{U}$ be the generalized subspace parametrized by (M, v). Then, $(u)_*\mathbf{v} = \mathbf{j}(M', v')$ where $(M', v') \in \mathcal{H}_{++}(ux) \times W^+(ux)$ is the (unique) parametrization of $\mathcal{U}$ adapted to $Z(ux)$. But then $(u')_*\mathbf{v}$ is also a parametrization of $\mathcal{U}$ adapted to $Z(ux)$. Therefore $(u')_*\mathbf{v} = (u)_*\mathbf{v}$.

The proof of (b) is essentially the same. □

Choosing M_0 *and* $\mathcal{C}_0$. — For a.e. $x \in X$, let $M^+(x) = \|\mathbf{S}_x^{Z(x)}\|$, and let

$$M^-(x) = \sup_{\mathbf{w} \in \mathbf{S}_x^{Z(x)}(\tilde{\mathbf{H}}(x))} \frac{1}{\|\mathbf{w}\|} \inf\big\{\|\mathbf{v}\| \;:\; \mathbf{v} \in \tilde{\mathbf{H}}(x),\ \mathbf{S}_x^{Z(x)}(\mathbf{v}) = \mathbf{w}\big\}.$$

Choose $M_0 > 1$ sufficiently large so that $\mathcal{C}_0 \equiv \{x \in X_0 \;:\; \max(M^+(x), M^-(x)) < M_0\}$ has positive measure. Let $\mathcal{C} \subset \mathcal{C}_0$ and $T_0 : \mathcal{C} \to \mathbb{R}$ be as in Lemma 4.14 (with this choice of $M_0, \mathcal{C}_0$).

Adjusting the transversal $Z(x)$. — For $c \in \mathcal{C}$, let $E^+[c]$, $t(c)$ and J_c be as in Proposition 3.7. For $x \in E^+[c]$ we define $Z(x) = P^+(c, x)_* Z(c)$, and for $0 \le t < t(c)$, we define $Z(g_{-t}x) = g_{-t}Z(x)$. This defines $Z(y)$ for $y \in J_c$. From now on, we assume that the transversal Z is obtained via this construction.

Lemma **6.8.** — *Let* $(g_t)_* : \mathbf{H}(x) \to \mathbf{H}(g_t x)$ *and* $\tilde{\mathbf{f}}_t : \tilde{\mathbf{H}}(x) \to \tilde{\mathbf{H}}(g_t x)$ *be as in* (6.16). *Then the Lyapunov subspaces for* $(g_t)_*$ *at* x *are the image under* $\mathbf{S}_x^{Z(x)}$ *of the Lyapunov subspaces of* $\tilde{\mathbf{f}}_t$ *at* x, *and the Lyapunov exponents of* g_t *are those Lyapunov exponents of* $\tilde{\mathbf{f}}_t$ *whose Lyapunov subspace at a generic point* x *is not contained in the kernel of* $\mathbf{S}_x^{Z(x)}$.

Proof. — Let $\mathcal{V}_i(\tilde{\mathbf{H}})(x)$ and $\mathcal{V}_i(\mathbf{H})(x)$ denote the Lyapunov subspaces of the flow $\tilde{\mathbf{f}}_t$ and g_t respectively, and let $\lambda_i(\tilde{\mathbf{H}})$ and $\lambda_i(\mathbf{H})$ denote the corresponding Lyapunov exponents. Then, for $\mathbf{v} \in \mathcal{V}_i(\tilde{\mathbf{H}})$, by the multiplicative ergodic theorem, for every $\epsilon > 0$,

$$\|g_t \mathbf{S}_x^{Z(x)} \mathbf{v}\| = \|\mathbf{S}_{g_t x}^{Z(g_t x)} \tilde{\mathbf{f}}_t \mathbf{v}\|_Y \leq \|\mathbf{S}_{g_t x}^{Z(g_t x)}\| \|\tilde{\mathbf{f}}_t \mathbf{v}\| \leq C_\epsilon(x) C_1(g_t x) e^{\lambda_i(\tilde{\mathbf{H}})t + \epsilon|t|}.$$

Taking $t \to \infty$ and $t \to -\infty$ we see that $\lambda_i(\mathbf{H}) = \lambda_i(\tilde{\mathbf{H}})$ and $\mathbf{S}_x^{Z(x)} \mathbf{v} \in \mathcal{V}_i(\mathbf{H})(x)$. □

The measurable flat connection $\mathbf{P}^+(x, y)$. — Recall that the measurable flat g_t-equivariant W^+-connection map P^+ on H^1 induces a measurable flat g_t-equivariant connection on $H_{big}^{(++)}$, and thus on $\tilde{\mathbf{H}}$. We will call this connection $\tilde{\mathbf{P}}^+(x, y)$. Then, we can define a measurable flat W^+-connection $\mathbf{P}^+(x, y) : \mathbf{H}(x) \to \mathbf{H}(y)$ by

$$\mathbf{P}^+(x, y) = \mathbf{S}_y^{Z(y)} \circ \tilde{\mathbf{P}}^+(x, y), \quad y \in W^+[x]. \tag{6.18}$$

Without loss of generality, we may assume that Lemma 4.3 applies to subbundles of $\mathbf{H}$ as well as subbundles of $H_{big}^{(++)}$ (or else we can replace X by a measurable finite cover). Then, Proposition 4.12 applies to $\mathbf{P}^+$.

The dynamical inner product $\langle \cdot, \cdot \rangle_x$ *and the dynamical norm* $\| \cdot \|_x$ *on* $\mathbf{H}$. — Even though $\mathbf{H}$ is not formally a subbundle of $H_{big}^{(++)}$, $\mathbf{H} \subset \tilde{\mathbf{H}} \subset H_{big}^{(++)}$. Thus, the AGY norm makes sense in $\mathbf{H}$. Note that by our choices of $\mathcal{C}_0$ and M_0, (4.17) holds for $\mathbf{P}^+$ in place of P^+ (and 1 in place of M_0). Then, the proof of Proposition 4.15 goes through. Thus, Proposition 4.15 also applies to $\mathbf{H}$, with a norm which may be different from the norm obtained from thinking of $\mathbf{H}$ as a subset of $H_{big}^{(++)}$.

6.1. *Approximation of generalized subspaces and the map* $\mathcal{A}(\cdot, \cdot, \cdot, \cdot)$.

Hausdorff distance between generalized subspaces. — For $x \in \tilde{X}_0$, and two generalized subspaces $\mathcal{U}'$ and $\mathcal{U}''$, let $hd_x^{X_0}(\mathcal{U}', \mathcal{U}'')$ denote the Hausdorff distance using the metric $d^{X_0}(\cdot, \cdot)$ defined in Section 3 between $\mathcal{U}' \cap B^{X_0}(x, 1/100)$ and $\mathcal{U}'' \cap B^{X_0}(x, 1/100)$. (The balls $B^{X_0}(\cdot, \cdot)$ are defined in Section 5.)

Lemma **6.9.** — *Suppose* $x \in \tilde{X}_0$, $(M, v) \in \mathcal{H}_{++}(x) \times W^+(x)$, *and*

$$hd_x^{X_0}\big(U^+[x], \mathcal{U}(M, v)\big) \leq 1/100.$$

(a) *We have for some absolute constant* $C > 0$,

$$hd_x^{X_0}\big(U^+[x], \mathcal{U}(M, v)\big) \le C \max(\|v\|_Y, \|M\|_Y).$$

Also if $(M, v) \in \mathcal{H}_{++}(x) \times W^+(x)$ *is adapted to* $Z(x)$*, then there exists* $c(x) > 0$ *such that*

$$hd_x^{X_0}\big(U^+[x], \mathcal{U}(M, v)\big) \ge c(x) \max(\|v\|_Y, \|M\|_Y).$$

(b) *For some* $c_1(x) > 0$*, we have, for* $(M, v) \in \mathcal{H}_{++}(x) \times W^+(x)$ *adapted to* $Z(x)$*,*

$$c_1(x)\|\mathbf{j}(M, v)\|_Y \le hd_x^{X_0}\big(U^+[x], \mathcal{U}(M, v)\big) \le c_1(x)^{-1}\|\mathbf{j}(M, v)\|_Y.$$

Proof. — Part (a) is immediate from the definitions and Proposition 3.4. To see (b) note that part (a) implies that $\max(\|M\|_Y, \|v\|_Y) = O(1)$, and thus all the higher order terms in $\mathbf{j}(M, v)$ which are polynomials in M_{ij} and v_j, have size bounded by a constant multiple of the size of the first order terms, i.e. by $\max(\|M\|_Y, \|v\|_Y)$. □

We will be dealing with Hausdorff distances of particularly well-behaved sets (i.e. generalized subspaces parametrized by elements of $\mathcal{H}_{++}(x) \times W^+(x)$). For such subspaces, the following holds:

Lemma **6.10.** — *Suppose* $x \in \tilde{X}_0$*, and* $\mathcal{U}' \subset W^+[x]$ *is a generalized subspace. Then,*

(a) *We have, for* $t \in \mathbb{R}$,

$$e^{-2|t|}hd_x^{X_0}\big(U^+[x], \mathcal{U}'\big) \le hd_{g_t x}^{X_0}\big(U^+[g_t x], (g_t)_*\mathcal{U}'\big) \le e^{2|t|}hd_x^{X_0}\big(U^+[x], \mathcal{U}'\big),$$

provided the quantity on the right is at most $1/100$*. (The first inequality in the above line holds as long as the quantity in the middle is at most* $1/100$*.)*

(b) *Suppose that* $\mathcal{U}'$ *is parametrized by an element of* $\mathcal{H}_{++}(x) \times W^+(x)$*. There exists a function* $C : X_0 \to \mathbb{R}^+$ *finite almost everywhere and* $\beta > 0$ *depending only on the Lyapunov spectrum, such that, for* $t \ge 0$,

$$C(x)^{-1}e^{\beta t}hd_x^{X_0}\big(U^+[x], \mathcal{U}'\big) \le hd_{g_t x}^{X_0}\big(U^+[g_t x], (g_t)_*\mathcal{U}'\big),$$

provided the quantity on the right is at most $1/100$*. Also, for* $t < 0$,

$$hd_{g_t x}^{X_0}\big(U^+[g_t x], (g_t)_*\mathcal{U}'\big) \le C(x)e^{-\beta|t|}hd_x^{X_0}\big(U^+[x], \mathcal{U}'\big),$$

provided the quantity on the right is at most $1/100$.

Proof. — Recall that $B^+(x, r) = B^{X_0}(x, r) \cap W^+[x]$ denotes the ball of radius r in the metric $d^+(\cdot, \cdot)$. Suppose $t \ge 0$. Note that, by Lemma 3.5(d), for $t > 0$,

$$B_t^+[x] \equiv g_t^{-1}B^+(g_t x, 1/100) \subset B^+(x, 1/100).$$

Note that the action of g_t can expand in any direction by at most e^{2t}, see also Lemma 3.6. Therefore,

$$hd^{X_0}_{g_t x}\big((g_t)_*U^+[x],(g_t)_*\mathcal{U}'\big) \le e^{2t} hd^{X_0}_x\big(U^+[x]\cap B^+_t[x],\mathcal{U}'\cap B^+_t[x]\big)$$
$$\le e^{2t} hd^{X_0}_x\big(U^+[x],\mathcal{U}'\big).$$

This completes the proof of the second inequality in (a). The first inequality in (a) follows after renaming x to $g_t x$.

We now begin the proof of (b). We assume $t \ge 0$ (the proof for the case $t < 0$ is identical). It is enough to show that for any $\delta > 0$ there exists $C = C(\delta) < \infty$ and a set $K(\delta)$ with measure at least $1-\delta$ such that for $x \in K(\delta)$ and $t > 0$,

$$C(\delta)^{-1} e^{\beta t} hd^{X_0}_x\big(U^+[x],\mathcal{U}'\big) \le hd^{X_0}_{g_t x}\big(U^+[g_t x],(g_t)_*\mathcal{U}'\big). \tag{6.19}$$

For any $\eta > 0$ let K_η be the set where $c_1(x) > \eta$, where $c_1(x)$ is as in Lemma 6.9. Choose η so that K_η has measure at least $1-\delta/4$. By the Birkhoff ergodic theorem we may find a set K' of measure at least $1-\delta/2$ and $t_1 > 0$ such that for $x \in K'$ and $t > t_1$, there exists $t' \in \mathbb{R}$ with $|t-t'| < \epsilon t$, and $g_{t'}x \in K_\eta$.

Let $\alpha > 0$ be as in Lemma 3.5. Choose $\epsilon < \alpha/2$. By Lemma 3.5(c), we may find a set $K'' \subset K_\eta$ of measure at least $1-\delta/2$, and a constant $t_2 = t_2(\delta)$ such that for all $x \in K''$ all $t > t_2$ and all $\mathbf{v} \in \mathbf{H}(x)$,

$$\|(g_t)_*\mathbf{v}\|_Y \ge e^{\alpha t}\|\mathbf{v}\|_Y. \tag{6.20}$$

Let $K(\delta) = K' \cap K''$, and let $t_0 = \max(t_1, t_2)$. If $0 \le t < (1+\epsilon)t_0$, then (6.19) holds in view of Lemma 6.10(a). Suppose $t > (1+\epsilon)t_0$, and let t' be as in the definition of K'. Since $x \in K_\eta$ and $g_{t'}x \in K_\eta$, by Lemma 6.9 and (6.20),

$$hd^{X_0}_{g_{t'}x}\big(U^+[g_{t'}x],(g_{t'})_*\mathcal{U}'\big) \ge \eta^2 e^{\alpha t} hd^{X_0}_x\big(U^+[x],\mathcal{U}'\big).$$

Then, again using Lemma 6.10(a), we get

$$hd^{X_0}_{g_t x}\big(U^+[g_t x],(g_t)_*\mathcal{U}'\big) \ge e^{-\epsilon t} hd^{X_0}_{g_{t'}x}\big(U^+[g_{t'}x],(g_{t'})_*\mathcal{U}'\big).$$

Now, (6.19) follows, with $\beta = (\alpha - \epsilon)$. □

Motivation. — We work in the universal cover $\tilde{X}_0$. Let q_1, q_1' be as in Section 2.3, so in particular, $q_1' \in W^-[q_1]$. Suppose $u \in \mathcal{B}(q_1, 1/100)$ and $t > 0$. Note that the generalized subspace $U^+[g_t q_1] = U^+[g_t u q_1]$ passes through the point $g_t u q_1$. If t is not too large, the generalized subspace $U^+[g_t q_1']$ will pass near $g_t u q_1$. These subspaces are not on the same leaf of W^+ (even though the leaf $W^+[g_t q_1']$ containing $U^+[g_t q_1']$ gets closer to the leaf $W^+[g_t q_1] = W^+[g_t u q_1]$ containing $U^+[g_t u q_1]$ as $t \to \infty$). It is convenient to find a way to "project" the part of $U^+[g_t q_1']$ near $g_t u q_1$ to $W^+[g_t u q_1]$. In particular, we want the projection to be again a generalized subspace (i.e. an orbit of a subgroup of $\mathcal{G}_{++}(g_t u q_1)$). We also want the projection to be exponentially close, in a ball of radius $1/100$ about $g_t u q_1$, to the original generalized subspace $U^+[g_t q_1']$. Furthermore, in order to carry out the program outlined in the beginning of Section 5, we want the pair (M'', v'') parameterizing the projection to be such that $\mathbf{j}(M'', v'') \in \mathbf{H}(g_t u q_1)$ depends polynomially on $P^-(q_1, q_1')$. Then it will depend linearly on $F(q) - F(q')$ since any fixed degree polynomial in $P^-(q_1, q_1')$ can be expressed as a linear function of $F(q) - F(q')$ as long as r in the definition of $\mathcal{L}_{ext}(q)^{(r)}$ is chosen large enough.

More precisely, we need the following:

Proposition **6.11.** — *Suppose $\alpha_3 > 0$ is a constant. We can choose r sufficiently large (depending only on α_3 and the Lyapunov spectrum) so that there exists a linear map $\mathcal{A}(q_1, u, \ell, t) : \mathcal{L}_{ext}(g_{-\ell} q_1)^{(r)} \to \mathbf{H}(g_t u q_1)$, defined for almost all $q_1 \in \tilde{X}_0$, almost all $u \in U^+[x]$, all $\ell \geq 0$ and all $t \geq 0$, and a constant $\alpha_1 > 0$ depending only on α_3 and the Lyapunov spectrum such that the following hold:*

(i) *We have*

$$\mathcal{A}\big(q_1, u, \ell + \ell', t + t'\big) = g_{t'} \circ \mathcal{A}(q_1, u, \ell, t) \circ g_{\ell'}. \tag{6.21}$$

(ii) *Suppose $\delta > 0$, and ℓ is sufficiently large depending on δ. There exists a set $K = K(\delta)$ with $\nu(K) > 1 - \delta$ and constants $C_1(\delta)$ and $C_2(\delta)$ such that the following holds: Suppose $q_1 \in \pi^{-1}(K)$. Let $q = g_{-\ell} q_1$ (see Figure 1). Suppose $q' \in \pi^{-1}(K) \cap W^-[q]$ satisfies the upper bounds in (5.3) and (5.4) with the same constant δ, and write $q_1' = g_\ell q'$. For all $u \in \mathcal{B}(q_1, 1/100)$ such that $uq_1 \in \pi^{-1}(K)$, and any $t > 0$ such that*

$$t \leq \alpha_3 \ell, \tag{6.22}$$

$$d^{X_0}\big(g_t u q_1, U^+\big[g_t q_1'\big]\big) \leq 1/100, \tag{6.23}$$

and also

$$C_1(\delta) e^{-\alpha_1 \ell} \leq hd^{X_0}_{g_t u q_1}\big(U^+[g_t u q_1], U^+\big[g_t q_1'\big]\big), \tag{6.24}$$

we have

$$\begin{aligned} & C(g_t u q_1)^{-1} \big\| \mathcal{A}(q_1, u, \ell, t)\big(F\big(q'\big) - F(q)\big) \big\|_Y \\ & \quad \leq hd^{X_0}_{g_t u q_1}\big(U^+[g_t u q_1], U^+\big[g_t q_1'\big]\big) \\ & \quad \leq C(g_t u q_1) \big\| \mathcal{A}(q_1, u, \ell, t)\big(F\big(q'\big) - F(q)\big) \big\|_Y, \end{aligned} \tag{6.25}$$

where $C : X_0 \to \mathbb{R}^+$ is a measurable function finite almost everywhere.

(iii) *Suppose δ, ℓ, q, u, q', q_1', are as in* (ii)*, and t satisfies* (6.22) *and* (6.23)*. Then, we have*

$$\mathcal{A}(q_1, u, \ell, t)\big(F\big(q'\big) - F(q)\big) = \mathbf{j}\big(M'', v''\big), \tag{6.26}$$

where the pair $(M'', v'') \in \mathcal{H}_{++}(g_t u q_1) \times W^+(g_t u q_1)$ *(which will be chosen in the proof) is adapted to* $Z(g_t u q_1)$ *and parametrizes a generalized subspace* $\mathcal{U}(M'', v'') \subset W^+(g_t u q_1)$ *satisfying*

$$hd^{X_0}_{g_t u q_1}\big(U^+\big[g_t q_1'\big], \mathcal{U}\big(M'', v''\big)\big) \le C_3(\delta) e^{-\alpha_1 \ell}. \tag{6.27}$$

Part (ii) of Proposition 6.11 is key to resolving "Technical Problem #1" of Section 2.3 (see the discussion at the beginning of Section 5). We claim part (ii) of Proposition 6.11 follows easily from part (iii) of Proposition 6.11 and Lemma 6.9(b). Indeed, by the triangle inequality,

$$\begin{aligned} hd^{X_0}_{g_t u q_1}\big(U^+[g_t u q_1], U^+\big[g_t q_1'\big]\big) &= hd^{X_0}_{g_t u q_1}\big(U^+[g_t u q_1], \mathcal{U}\big(M'', v''\big)\big) \\ &\quad + O\big(hd^{X_0}_{g_t u q_1}\big(\mathcal{U}\big(M'', v''\big), U^+\big[g_t q_1'\big]\big)\big). \end{aligned} \tag{6.28}$$

The $O(\cdot)$ term on the right-hand-side of (6.28) is bounded by (6.27), and the size of the first term on the right-hand-side of (6.28) is comparable to $\|\mathbf{j}(M'', v'')\|_Y$ by Lemma 6.9(b). Thus, (6.25) follows from (6.26).

Lemma **6.12.** — *For any* $\delta > 0$, *there exists* $K' = K'(\delta) \subset X_0$ *with* $\nu(K') > 1 - c(\delta)$ *where* $c(\delta) \to 0$ *as* $\delta \to 0$, *and constants* $C_1'(\delta) > 0$, $C_2'(\delta) > 0$ *and* $C_4'(\delta) > 0$ *such that in Proposition 6.11(ii) and (iii), the conditions* (6.23) *and* (6.24) *can be replaced by either*

(a) $g_t u q_1 \in K'$ *and*

$$C_1'(\delta) e^{-\alpha_1 \ell} \le \big\|\mathcal{A}(q_1, u, \ell, t)\big(F\big(q'\big) - F(q)\big)\big\|_Y \le C_2'(\delta), \tag{6.29}$$

or by

(b)

$$C_4'(\delta) e^{-\alpha \ell} \le hd^{X_0}_{g_t u q_1}\big(U^+[g_t u q_1], \mathcal{U}\big(M'', v''\big)\big) \le 1/400, \tag{6.30}$$

where $\mathcal{U}(M'', v'')$ *is as in* (6.26).

Proof of Lemma 6.12. — Let $c_1(x)$ be as in Lemma 6.9(b). There exists a compact $K' \subset X_0$ with $\nu(K') > 1 - c(\delta)$, with $c(\delta) \to 0$ as $\delta \to 0$, and a constant $1 < C'(\delta') < \infty$ with $C'(\delta') \to \infty$ as $\delta \to 0$ such that $c_1(x)^{-1} < C'(\delta')$ for all $x \in K'$. Then, in view of Lemma 6.9(b), there exist $0 < C_1'(\delta) < C_2'(\delta)$ and $C_4'(\delta) > 0$ such that for t such that $g_t u q_1 \in K'$ and (6.29) holds, (6.30) also holds. Thus, it is enough to show that if for some $t > 0$ (6.22) and (6.30) hold, then (6.23) and (6.24) also hold.

Let $t_{max} = \min\{s \in \mathbb{R}^+ \;:\; d^{X_0}(g_s u q_1, U^+[g_s q_1']) \ge 1/100\}$, so that for $0 \le t \le t_{max}$ (6.23) holds. If $t_{max} \ge \alpha_3 \ell$, then for $t \in [0, \alpha_3 \ell)$, (6.23) is automatically satisfied. Now assume $t_{max} < \alpha_3 \ell$. Then, by the definition of t_{max} and Proposition 6.11(iii), (i.e. (6.26) and (6.27)), and assuming ℓ is sufficiently large (depending on δ) we have

$$d^{X_0}\big(g_{t_{max}} u q_1, \mathcal{U}\big(M'', v''\big)\big) \ge 1/200.$$

Let $\mathcal{U}_0 = g_{-t_{max}}\mathcal{U}(M'', v'') \subset W^+[uq_1]$. By Proposition 6.11(iii), for $0 \le t \le t_{max}$, $g_t\mathcal{U}_0$ is parametrized by (M_t, v_t) satisfying (6.26).

Suppose $t > 0$ satisfies (6.22) and (6.30). Let

$$t_1 = \max\{s \in \mathbb{R}^+ \ : \ d^{X_0}(g_s uq_1, g_s\mathcal{U}_0) \le 1/200\}.$$

Since by Lemma 3.5(iv) the function $s \to d^{X_0}(g_s uq_1, g_s\mathcal{U}_0)$ is monotone increasing, we have $t < t_1 \le t_{max}$. Thus, since $t < t_{max}$, (6.23) holds. In particular, Proposition 6.11(iii) applies and then, (6.27) and (6.30) (with a proper choice of $C_4(\delta)$) imply (6.24). □

Corollary **6.13.** — *Suppose δ, ℓ, q, u, q', q'_1, are as in Proposition 6.11*(ii)*, and $s \ge 0$ is such that* (6.22), (6.23), *and* (6.24) *hold for s in place of t. Suppose $t \in \mathbb{R}$ is such that $0 < t + s < \alpha_3\ell$. Then, there exists $C_4(\delta) > 0$ such that*

(a) *We have, for $t \in \mathbb{R}$ such that $0 < t + s < \alpha_3\ell$,*

$$\begin{aligned} &e^{-2|t|} hd^{X_0}_{g_s uq_1}\big(U^+[g_s uq_1], U^+\big[g_s q'_1\big]\big) - C_4(\delta)e^{-\alpha\ell} \\ &\quad\le hd^{X_0}_{g_{s+t} uq_1}\big(U^+[g_{s+t} uq_1], U^+\big[g_{s+t} q'_1\big]\big) \\ &\quad\le e^{2|t|} hd^{X_0}_{g_s uq_1}\big(U^+[g_s uq_1], U^+\big[g_s q'_1\big]\big) + C_4(\delta)e^{-\alpha\ell}, \end{aligned}$$

provided the quantity on the right is at most $1/800$. (The first inequality in the above line holds as long as the quantity in the middle is at most $1/800$.)

(b) *There exists a function $C : X_0 \to \mathbb{R}^+$ finite almost everywhere and $\beta > 0$ depending only on the Lyapunov spectrum, such that, for $t \ge 0$,*

$$\begin{aligned} &C(g_s uq_1)^{-1} e^{\beta t} hd^{X_0}_{g_s uq_1}\big(U^+[g_s uq_1], U^+\big[g_s q'_1\big]\big) - C_4(\delta)e^{-\alpha\ell} \\ &\quad\le hd^{X_0}_{g_{s+t} uq_1}\big(U^+[g_{s+t} uq_1], U^+\big[g_{s+t} q'_1\big]\big), \end{aligned}$$

provided the quantity on the right is at most $1/800$. Also, for $t < 0$,

$$\begin{aligned} &hd^{X_0}_{g_{s+t} uq_1}\big(U^+[g_{s+t} uq_1], U^+\big[g_{s+t} q'_1\big]\big) \\ &\quad\le C(g_s uq_1)e^{-\beta|t|} hd^{X_0}_{g_s uq_1}\big(U^+[g_s uq_1], U^+\big[g_s q'_1\big]\big) + C_4(\delta)e^{-\alpha\ell}, \end{aligned}$$

provided the quantity on the right is at most $1/800$.

Proof. — Suppose $0 \le t \le \alpha_3\ell$, and ℓ is sufficiently large depending on δ. Let $\mathcal{U}_t$ denote the generalized subspace of Proposition 6.11(iii). Then, by Proposition 6.11 if $d^{X_0}(g_t uq_1, \mathcal{U}^+[g_t q'_1]) < 1/200$, then $d^{X_0}(g_t uq_1, \mathcal{U}_t) < 1/100$. Conversely, by (the proof of) Lemma 6.12(b), if $d^{X_0}(g_t uq_1, \mathcal{U}_t) < 1/400$, then $d^{X_0}(g_t uq_1, \mathcal{U}^+[g_t q'_1]) < 1/200$. Also, by Proposition 6.11(iii) and Lemma 6.12(b), if either of these conditions holds, then (6.27) holds. Thus, the corollary follows from Lemma 6.10. □

Proposition 6.11 is proved by constructing a linear map $\tilde{P}_s(uq_1, q_1') : W^+(uq_1) \to W^+(q_1')$ with nice properties; then the approximating subspace $\mathcal{U}(M'', v'')$ is given by $g_t\tilde{P}_s(uq_1, q_1')^{-1}U^+[q_1']$. The construction is technical, and is postponed to Section 6.5*. Then, Proposition 6.11 is proved in Section 6.6*. From the proof, we will also deduce the following lemma (which will be used in Section 12):

Lemma **6.14.** — *For every $\delta > 0$ there exists $\epsilon > 0$ and a compact set $K \subset X_0$ with $\nu(K) > 1 - \delta$ so that the following holds: Suppose $\epsilon_0 < 1/100$. Suppose $q \in \pi^{-1}(K)$, $\ell > 0$ is sufficiently large depending on δ, and suppose $q' \in W^-[q] \cap \pi^{-1}(K)$ is such that (5.3) and (5.4) hold. Let $q_1 = g_\ell q$, $q_1' = g_\ell q'$ (see Figure 1). Fix $u \in \mathcal{B}(q_1, 1/100)$, and suppose $t > 0$ is such that*

$$hd^{X_0}_{g_tuq_1}\left(U^+[g_tuq_1], U^+\left[g_tq_1'\right]\right) \leq \epsilon \ll \epsilon_0.$$

Furthermore, suppose q_1, q_1', uq_1, q_1', and g_tuq_1 all belong to $\pi^{-1}(K)$. Suppose $x \in U^+[g_tuq_1] \cap B^{X_0}(g_tuq_1, 1/100)$. Let

$$A_t = U^+[g_tuq_1] \cap B^{X_0}(x, \epsilon_0),$$
$$A_t' = U^+\left[g_tq_1'\right] \cap B^{X_0}(x, \epsilon_0).$$

Then,

$$\kappa^{-1}\frac{|g_{-t}A_t|}{|U^+[q_1] \cap B^+(q_1, 1/100)|} \leq \frac{|g_{-t}A_t'|}{|U^+[q_1'] \cap B^+(q_1', 1/100)|} \leq \kappa\frac{|g_{-t}A_t|}{|U^+[q_1] \cap B^+(q_1, 1/100)|},$$

where κ depends only on δ and the Lyapunov spectrum, the "Haar measure" $|\cdot|$ is defined at the beginning of Section 6, and the ball $B^+(x, r)$ is defined in Section 3. Also,

$$hd^{X_0}\left(g_{-t}A_t, g_{-t}A_t'\right) \leq e^{-\alpha\ell},$$

where $hd^{X_0}(\cdot, \cdot)$ denotes the Hausdorff distance, and α depends only on the Lyapunov spectrum.

This lemma will also be proved in Section 6.6*.

6.2. *The stopping condition.* — We now state and prove Lemma 6.15 and Proposition 6.16 which tell us when the inductive procedure outlined in Section 2.3 stops.

Recall the notational conventions Section 2.2.

The sets $L^-(q)$ and $L^-[q]$. — For a.e $q \in X_0$, let $L^-[q] \subset W^-[q]$ denote the smallest real-algebraic subset containing, for some $\epsilon > 0$, the intersection of the ball of radius ϵ with the support of the measure $\nu_{W^-[q]}$, which is the conditional measure of ν along $W^-[q]$. Then, $L^-[q]$ is g_t-equivariant. Since the action of g_{-t} is expanding along $W^-[q]$, we see that for almost all q and any $\epsilon > 0$, $L^-[q]$ is the smallest real-algebraic subset of $W^-(q)$ such that $L^-[q]$ contains support$(\nu_{W^-[q]}) \cap B^{X_0}(q, \epsilon)$. Let $L^-(q) = L^-[q] - q$.

The sets $L^+(q)$ *and* $L^+[q]$. — Let $\hat{\pi}^+ : W(x) \to W^+(x)$ and $\hat{\pi}^- : W(x) \to W^-(x)$ denote the maps

$$\hat{\pi}^+_{q_1}(v) = (1, 0) \otimes v, \qquad \hat{\pi}^-_{q_1}(v) = (0, 1) \otimes \pi^-_{q_1}(v),$$

where $\pi^-_{q_1}$ is as in (2.2). Let $L^+(q) = \hat{\pi}^+_q \circ (\hat{\pi}^-_q)^{-1} L^-(q)$, and let $L^+[q] = q + L^+(q)$.

The automorphism h_t *and the set* $S^+[x]$. — Let h_t denote the automorphism of the affine group $\mathcal{G}_{++}(x)$ which is the identity on the linear part and multiplication by e^{2t} on the translational part. For $x \in X_0$, let

$$S^+[x] = \bigcap_{t\in\mathbb{R}} h_t\big(U^+\big)[x].$$

It is clear from the definition that $S^+[x]$ is relatively closed in $W^+[x]$, $S^+[x] \subset U^+[x]$, and also $S^+[x]$ is star-shaped relative to x (so that if $x + v \in S^+[x]$, so is $x + tv$ for all $t > 0$).

Lemma **6.15.** — *The following are equivalent:*

(a) $L^+[x] \subset S^+[x]$ *for almost all* $x \in X_0$.
(b) *There exists* $E \subset X_0$ *with* $\nu(E) > 0$ *such that* $L^+[x] \subset S^+[x]$ *for* $x \in E$.
(c) *There exists* $E \subset X_0$ *with* $\nu(E) > 0$ *such that* $L^+[x] \subset U^+[x]$ *for* $x \in E$.

Proof. — It is immediately clear that (a) implies (b). Also, since $S^+[x] \subset U^+[x]$, (b) immediately implies (c). It remains to prove that (c) implies (a).

Now suppose (c) holds. Let $\Omega \subset X_0$ be the set such that for $q_1 \in \Omega$, $g_t q_1$ spends a positive proportion of the time in E. Then, by the ergodicity of g_t, Ω is conull. For $q_1 \in \Omega$, we have, for a positive fraction of t,

$$L^+[g_t q_1] \subset U^+[g_t q_1].$$

Let $A(x, t)$ denote the Kontsevich-Zorich cocycle. Then g_t acts on W^+ by $e^t A(q_1, t)$ and acts on W^- by $e^{-t} A(q_1, t)$. Therefore, $L^-(g_t q_1) = e^{-t} A(q_1, t) L^-(q_1)$, and thus $L^+(g_t q_1) = e^{-t} A(q_1, t) L^+(q_1)$. Also, we have $U^+(g_t q_1) = e^t A(q_1, t) U^+(q_1)$. Thus, for a positive measure set of t, we have

$$L^+(q_1) \subset e^{2t} U^+(q_1) = h_t\big(U^+\big)(q_1), \tag{6.31}$$

where h_t is as in the statement of Proposition 6.16. Since both sides of (6.31) depend analytically on t, we see that (6.31) holds for all t. Then, $L^+[q_1] \subset S^+[q_1]$. □

Proposition **6.16.** — *Suppose the equivalent conditions of Lemma 6.15 do not hold. Then, there exist constants* $\alpha_1' > 0$, $\alpha_2' > 0$ *and* $\alpha_1'' > 0$ *depending only on the Lyapunov spectrum, such that for any* $\delta > 0$ *and any sufficiently small (depending on* δ*)* $\epsilon > 0$, *there exist* $\ell_0(\delta, \epsilon) > 0$ *and a compact*

$K \subset X_0$ with $\nu(K) > 1-\delta$ such that for $q_1 \in K$ there exists a subset $Q(q_1) \subset \mathcal{B}(q_1, 1/100)$ with $|Q(q_1)| > (1-\delta)|\mathcal{B}(q_1, 1/100)|$, such that for $\ell > \ell_0(\delta, \epsilon)$, for $u \in Q(q_1)$, and for $t > 0$ such that

$$-\alpha_1''\ell \leq \alpha_2' t - \alpha_1'\ell \leq 0, \tag{6.32}$$

we have

$$\|\mathcal{A}(q_1, u, \ell, t)\| \geq e^{-\alpha_1'\ell} e^{\alpha_2' t}. \tag{6.33}$$

Consequently, if $\epsilon > 0$ is sufficiently small depending on δ, $\ell > \ell_0(\delta, \epsilon)$, $q_1 \in K$, $u \in Q(q_1)$, and $t > 0$ is chosen to be as small as possible so that

$$\|\mathcal{A}(q_1, u, \ell, t)\| = \epsilon,$$

then $t < \frac{1}{2}\alpha_3\ell$, where $\alpha_3 = \alpha_1'/\alpha_2'$ depends only on the Lyapunov spectrum.

Remark. — The constant α_3 constructed during the proof of Proposition 6.16 depends only on the Lyapunov spectrum. This value of α_3 is then used in Proposition 6.11 to construct the function $\mathcal{A}(\cdot, \cdot, \cdot, \cdot)$, which is referred to in (6.33).

6.3*. *Proof of Proposition 6.16.*

Lemma **6.17.** — *Suppose $k \in \mathbb{N}$, and $\epsilon > 0$. For every sufficiently small $\delta > 0$, and every compact K' with $\nu(K') > 1-\delta$, there exists a constant $\beta(\epsilon, k, \delta) > 0$ and compact set $K'' = K''(\epsilon, K', k, \delta) \subset K'$ with $\nu(K'') > 1 - c_1(\delta)$ where $c_1(\delta) \to 0$ as $\delta \to 0$ such that the following holds:*

Suppose $q \in \pi^{-1}(K'')$ and $H \subset L^-[q]$ is a connected, degree at most k, $\mathbb{R}$-algebraic set which is a proper subset of $L^-[q]$. Then there exists $q' \in \pi^{-1}(K') \cap L^-[q]$ with $d^{X_0}(q', q) < \epsilon$ and

$$d^{X_0}\big(q', H\big) > \beta.$$

Proof. — This argument is virtually identical to the proof of Lemma 5.4 and of Lemma 5.5. ☐

Lemma **6.18.** — *Suppose $k \in \mathbb{N}$, $m \in \mathbb{N}$, $q_1 \in \tilde{X}_0$, and $\mathcal{U}' \subset W^+[q_1]$ is the image of a polynomial map of degree at most k from $\mathbb{R}^m$ to $W^+[q_1]$. Suppose furthermore that $U^+[q_1]$ is also the image of a polynomial map of degree at most k from $\mathbb{R}^m$ to $W^+[q_1]$, and $\epsilon > 0$ is such that there exists $u \in \mathcal{B}(q_1, 1/100)$ with*

$$d^{X_0}\big(uq_1, \mathcal{U}'\big) = \epsilon.$$

Suppose $\delta > 0$. Then, for at least $(1-\delta)$-fraction of $u \in \mathcal{B}(q_1, 1/100)$,

$$d^{X_0}\big(uq_1, \mathcal{U}'\big) > \beta\epsilon,$$

where $\beta > 0$ depends only on k, m, δ and the dimension.

Proof. — This is a compactness argument. If the lemma was false, we would (after passing to a limit) obtain polynomial maps whose images are Hausdorff distance $\epsilon > 0$ apart, yet coincide on a set of measure at least δ. This leads to a contradiction. □

The following lemma is stated in terms of the distance $d^{X_0}(\cdot,\cdot)$. However, in view of Proposition 3.4, it is equivalent to the analogous statement for the Euclidean distance on $W^+[x]$.

Lemma **6.19.** — *There exists* $C : X_0 \to \mathbb{R}^+$ *finite a.e and* $\alpha > 0$ *depending only on the Lyapunov spectrum such that for all* $q_1 \in \tilde{X}_0$ *and all* $z \in L^+[x]$ *with* $d^{X_0}(z, q_1) < 1/100$,

$$d^{X_0}\big(z, U^+[x]\big) \geq C(x) d^{X_0}\big(z, U^+[x] \cap L^+[x]\big)^{\alpha}.$$

Proof. — By the Łojasiewicz inequality [KuSp, Theorem 2] for any $x \in \tilde{X}_0$ and any k-algebraic sets $U \subset W^+[x]$, $L \subset W^+[x]$, and any z with $d^{X_0}(z, x) < 1/100$,

$$d^{X_0}(z, U) + d^{X_0}(z, L) \geq c(U, L) d^{X_0}(z, U \cap L)^{\alpha},$$

where $c(U, L) > 0$ and $\alpha > 0$ depends only on k and the dimension.

In our case, $U = U^+[x]$. $L = L^+[x]$, and $z \in L^+[x]$. The lemma follows. □

Recall that for x near q_1, $\pi_{W^+(q_1)}(x)$ is the unique point in $W^+[q_1] \cap AW^-[x]$. Let $n_\tau = \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix} \subset N \subset SL(2, \mathbb{R})$.

Lemma **6.20.** — *Suppose* $q_1 \in \tilde{X}_0$, $q_1' \in W^-[q_1]$. *Then, we have*

$$\pi_{W^+(q_1)}\big(n_\tau q_1'\big) = n_{\tau'}\Big(q_1 + (1, 0) \otimes \tau (1 + c\tau)^{-1}\big(\hat{\pi}_{q_1}^-\big)^{-1}\big(q_1' - q_1\big)\Big),$$

where $c = p(v) \wedge p(\operatorname{Im} q_1)$, $q_1' - q_1 = (0, 1) \otimes v$, *and* $\tau' = (1 - c)\tau(1 + c\tau)^{-1}$.

Proof. — Abusing notation, we work in period coordinates. Since $q_1' \in W^-[q_1]$, we can write $q_1' = q_1 + (0, 1) \otimes v$, where $p(v) \wedge p(\operatorname{Re} q_1) = 0$. Then,

$$n_\tau q_1' = (1, 0) \otimes \big(\operatorname{Re} q_1 + \tau(\operatorname{Im} q_1 + v)\big) + (0, 1) \otimes (\operatorname{Im} q_1 + v).$$

Let

$$w = v + c\tau(1 + c\tau)^{-1} \operatorname{Im} q_1.$$

Then, $p(w) \wedge p(\operatorname{Re}(n_\tau q_1')) = 0$, and thus, $(0, 1) \otimes w \in W^-(n_\tau q_1')$. Therefore,

$$\begin{aligned} n_\tau q_1' - (0, 1) \otimes w = (1, 0) &\otimes \big(\operatorname{Re} q_1 + \tau(\operatorname{Im} q_1 + v)\big) \\ &+ (0, 1) \otimes (1 + c\tau)^{-1} \operatorname{Im} q_1 \in W^-\big[n_\tau q_1'\big]. \end{aligned}$$

We have $\begin{pmatrix} (1+c\tau)^{-1} & 0 \\ 0 & 1+c\tau \end{pmatrix} \in A$. Therefore,

$$(1,0)\otimes(1+c\tau)^{-1}\big(\operatorname{Re} q_1 + \tau(\operatorname{Im} q_1 + v)\big) + (0,1)\otimes \operatorname{Im} q_1 \in AW^-\big[n_\tau q_1'\big]. \tag{6.34}$$

It is easy to check that (6.34) is in $W^+[q_1]$. Therefore,

$$\begin{aligned}\pi_{W^+(q_1)}\big(n_\tau q_1'\big) &= (1,0)\otimes(1+c\tau)^{-1}\big(\operatorname{Re} q_1 + \tau(\operatorname{Im} q_1 + v)\big)\\ &\quad + (0,1)\otimes \operatorname{Im} q_1\\ &= q_1 + (1,0)\otimes \tau(1+c\tau)^{-1}\big(\operatorname{Im} q_1 + v'\big),\end{aligned}$$

where $v' \in H^1_\perp$ is such that $v = c\operatorname{Re} q_1 + v'$. Also

$$\begin{aligned}n_{\tau'}^{-1}\pi_{W^+(q_1)}\big(n_\tau q_1'\big) &= \pi_{W^+(q_1)}\big(n_\tau q_1'\big) - (1,0)\otimes\tau' \operatorname{Im} q_1\\ &= \pi_{W^+(q_1)}\big(n_\tau q_1'\big) - (1,0)\otimes(1-c)\tau(1+c\tau)^{-1}\operatorname{Im} q_1.\end{aligned}$$

Therefore,

$$n_{\tau'}^{-1}\pi_{W^+(q_1)}\big(n_\tau q_1'\big) = q_1 + (1,0)\otimes\tau(1+c\tau)^{-1}\big(c\operatorname{Im} q_1 + v'\big).$$

Also,

$$c\operatorname{Im} q_1 + v' = \big(\pi^-_{q_1}\big)^{-1}(v) = \big(\hat\pi^-_{q_1}\big)^{-1}\big(q_1' - q_1\big).$$

This completes the proof of the lemma. □

Proof of Proposition 6.16. — Suppose the equivalent conditions of Lemma 6.15 do not hold. For $x \in X_0$, let $U^-(x) = \hat\pi^-_x \circ (\hat\pi^+_x)^{-1} U^+(x)$, and let $U^-[x] = x + U^-(x)$. Then, for a.e $x \in X_0$, $L^-[x] \not\subset U^-[x]$, and hence $U^-[x]\cap L^-[x]$ is a proper algebraic subset of $L^-[x]$.

By Lemma 6.17, there exists a $K' \subset X_0$ with $\nu(K') > 1-\delta/4$ and $K'' \subset X_0$ with $\nu(K'') > 1-\delta/2$ such that for any $q \in \pi^{-1}(K'')$ and any degree k proper real algebraic subset H of $L^-[q]$, there exists $q' \in L^-[q]$ satisfying the upper bounds in (5.3) and (5.4) such that $d^{X_0}(q', H) > \beta'(\delta)$.

Now assume that $q \equiv g_{-\ell} q_1 \in \pi^{-1}(K'')$. (We will later remove this assumption.) Then, we apply Lemma 6.17 with $H = g_{-\ell}(U^-[q_1]\cap L^-[q_1])$ to get $q' \in L^-[q]\cap\pi^{-1}(K'')$ satisfying the upper bounds in (5.3) and (5.4) and so that

$$d^{X_0}\big(q', g_{-\ell}\big(U^-[q_1]\cap L^-[q_1]\big)\big) \geq \beta'(\delta).$$

In view of Lemma 3.6 and Proposition 3.4, there exists $N > 0$ such that for all $x \in \tilde X_0$ and all $y \in W^-[x]$ with $d^{X_0}(x,y) < 1/100$ and all $t > 1$,

$$d^{X_0}(g_t x, g_t y) > e^{-Nt} d^{X_0}(x,y).$$

Let $q_1' = g_\ell q'$. Then, $q_1' \in L^-[q_1]$, and

$$d^{X_0}\left(q_1', U^-[q_1] \cap L^-[q_1]\right) \geq \beta'(\delta)e^{-N\ell}.$$

Let $z \in L^+[q_1]$ be such that $\hat{\pi}^+_{q_1} \circ (\hat{\pi}^-_{q_1})^{-1}(q_1') = z$. Then, we have

$$d^{X_0}\left(z, U^+[q_1] \cap L^+[q_1]\right) \geq \beta'(\delta)e^{-N\ell},$$

and thus by Lemma 6.19,

$$d^{X_0}\left(z, U^+[q_1]\right) \geq \beta(\delta)\beta'(\delta)e^{-\alpha N\ell}. \tag{6.35}$$

Let $\mathcal{U} = U^+[q_1']$. Then, $\mathcal{U}$ is a generalized subspace, and $q_1' \in \mathcal{U}$. Furthermore, both $\mathcal{U}$ and $U^+[q_1]$ are invariant under the action of $N \subset SL(2, \mathbb{R})$.

Without loss of generality, we may assume that ℓ is large enough so that the constant c in Lemma 6.20 satisfies $c < 1/2$. Now choose τ so that $\tau(1 + c\tau)^{-1} = 1$, and let τ' be as in Lemma 6.20.

Let $\mathcal{U}' = \pi_{W^+(q_1)}(\mathcal{U})$. Then, since $n_\tau q_1' \in \mathcal{U}$, we have, by Lemma 6.20,

$$n_{\tau'} z = \pi_{W^+(q_1)}\left(n_\tau q_1'\right) \in \mathcal{U}'.$$

But, since $U^+[q_1]$ is N-invariant and (6.35) holds, we have

$$d^{X_0}\left(n_{\tau'} z, U^+[q_1]\right) > \beta''(\delta)e^{-\alpha N\ell}.$$

Thus (because of $n_{\tau'} z$ and Lemma 6.18),

$$hd^{X_0}_{q_1}\left(U^+[q_1], \mathcal{U}'\right) > \beta''(\delta)e^{-\alpha N\ell}.$$

Then, by Lemma 6.18, for $(1-\delta)$-fraction of $u \in \mathcal{B}(q_1, 1/100)$,

$$d^{X_0}\left(uq_1, \mathcal{U}'\right) > \beta'''(\delta)e^{-\alpha N\ell}. \tag{6.36}$$

By Lemma 3.5, and Proposition 3.4, there exists a compact set K_2 of measure at least $(1-\delta)$ and λ_{min} depending only on the Lyapunov spectrum such that for $x \in \pi^{-1}(K_2)$ and $y \in W^+[x]$,

$$d^{X_0}(g_t x, g_t y) > c(\delta)e^{\lambda_{min} t} d^{X_0}(x, y),$$

as long as $t > 0$ and $d^{X_0}(g_t x, g_t y) < 1/100$. Let $t_0 > 0$ be the smallest such that $d^{X_0}(g_{t_0} x, g_{t_0} \mathcal{U}') = 1/100$. Therefore, assuming $uq_1 \in \pi^{-1}(K_2)$ in addition to (6.36) we have, for $0 < t < t_0$,

$$d^{X_0}\left(g_t uq_1, g_t \mathcal{U}'\right) > c(\delta)\beta'''(\delta)e^{\lambda_{min} t - \alpha N\ell}.$$

Hence, for $0 < t < t_0$,

$$hd^{X_0}_{g_t uq_1}\left(U^+[g_t uq_1], g_t \mathcal{U}'\right) > c_1(\delta)e^{\lambda_{min} t - \alpha N\ell},$$

and thus, in view of Proposition 6.11(iii) and Lemma 6.12(b),

$$hd^{X_0}_{g_t u q_1}\big(U^+[g_t u q_1], g_t \mathcal{U}\big) > c_2(\delta) e^{\lambda_{min} t - \alpha N \ell}.$$

Let $\alpha_2' = \lambda_{min}/2$, $\alpha_1' = 2\alpha N$, and let $\alpha_3 = \alpha_1'/\alpha_2'$. Let $\alpha_1 > 0$ be as in Proposition 6.11 for this choice of α_3. Then we can choose $\alpha_1'' > 0$ to be smaller than α_1, so that if (6.32) holds and ℓ is sufficiently large then (6.24) holds. Hence, by Proposition 6.11(ii) if (6.32) holds, $0 < t < t_0$, (and assuming that $g_t u q_1 \in \pi^{-1}(K''')$ where K''' is a compact set of measure at least $1 - \delta$),

$$\big\| \mathcal{A}(q_1, u, \ell, t)\big(F(q') - F(q)\big)\big\| \geq c_3(\delta) e^{\lambda_{min} t - \alpha N \ell}$$

Then, for $0 < t < t_0$ satisfying (6.32),

$$\|\mathcal{A}(q_1, u, \ell, t)\| \geq c_4(\delta) e^{\lambda_{min} t - \alpha N \ell}$$

If $t \geq t_0$ satisfies (6.32), then

$$\|\mathcal{A}(q_1, u, \ell, t)\| \geq \|\mathcal{A}(q_1, u, \ell, t_0)\| \geq c_5(\delta) \geq c_5(\delta) e^{\lambda_{min} t - \alpha N \ell}$$

Thus, for all t such that (6.32) holds,

$$\|\mathcal{A}(q_1, u, \ell, t)\| \geq c_6(\delta) e^{\lambda_{min} t - \alpha N \ell}.$$

This implies (6.33), assuming that ℓ is sufficiently large (depending on δ), $q \in \pi^{-1}(K'')$ and $g_t u q_1 \in \pi^{-1}(K''')$.

For the general case (i.e. without the assumptions that $q \in \pi^{-1}(K'')$ and $g_t u q_1 \in \pi^{-1}(K''')$), note that we can assume that $g_{-\ell} q_1 \in \pi^{-1}(K'')$ for a set of ℓ of density at least $(1 - 2\delta)$, and also $g_t u q_1 \in \pi^{-1}(K''')$ for a set of t of density at least $(1 - 2\delta)$. Now the general case of (6.33) follows from the special case, Proposition 6.11(i) and Lemma 3.6. □

6.4*. *Proof of Lemma 6.4.* — We can choose a subspace $T(x) \subset \mathrm{Lie}(U^+)(x)$, so that

$$\mathrm{Lie}\big(U^+\big)(x) + \mathrm{Lie}(Q_{++})(x) = T(x) \oplus \mathrm{Lie}(Q_{++})(x).$$

(In particular, if $\mathrm{Lie}(U^+)(x) \cap \mathrm{Lie}(Q_{++}) = \{0\}$, $T(x) = \mathrm{Lie}(U^+)(x)$.) Then,

$$\mathrm{Lie}(\mathcal{G}_{++})(x) = \big(Z(x) \cap W^+(x)\big) \oplus T(x) \oplus \mathrm{Lie}(Q_{++})(x).$$

Thus, for any vector $Y \in \mathrm{Lie}(\mathcal{G}_{++})(x)$, we can write

$$Y = \pi_Q(Y) + \pi_Z(Y) + \pi_T(Y), \tag{6.37}$$

where $\pi_Q(Y) \in \mathrm{Lie}(Q_{++})(x)$, $\pi_Z(Y) \in Z(x) \cap W^+(x)$, $\pi_T(Y) \in T(x)$.

Suppose there exists $\tilde{u} \in T(x)$ such that (in $W^+(x)$)

$$x + v \equiv \exp\big[\big(I + M'\big)\tilde{u}\big]\big(x + v'\big) \in x + Z(x) \cap W^+(x). \tag{6.38}$$

Then there exists $q \in \mathrm{Lie}(\mathrm{Q}_{++})(x)$, $z \in \mathrm{Z}(x) \cap \mathrm{W}^+(x)$ such that in $\mathcal{G}_{++}(x)$,

$$\exp\big[(\mathrm{I}+\mathrm{M}')\tilde{u}\big]\exp(v') = \exp(z)\exp(q). \tag{6.39}$$

In this subsection, we write $\mathcal{V}_i(x)$ for $\mathcal{V}_i(\mathrm{Lie}(\mathcal{G}_{++}))(x)$, and λ_i for $\lambda_i(\mathrm{Lie}(\mathcal{G}_{++}))$. We also write $\mathcal{V}_{<i}(x) = \bigoplus_{j=1}^{i-1} \mathcal{V}_j(x)$.

Write $\tilde{u} = \sum_i \tilde{u}_i$, where $\tilde{u}_i \in (\mathrm{Lie}(\mathrm{U}^+) \cap \mathcal{V}_i)(x)$. Also, write $q = \sum_i q_i$, where $q_i \in (\mathrm{Lie}(\mathrm{Q}_{++}) \cap \mathcal{V}_i)(x)$, $v = \sum_i v_i$, where $v_i \in (\mathrm{W}^+ \cap \mathcal{V}_i)(x)$, and $z = \sum_i z_i$ where $z_i \in \mathrm{Z}_{i1}(x) = \mathrm{Z}(x) \cap \mathrm{W}^+(x) \cap \mathcal{V}_i(x)$.

For $h \in \mathcal{G}_{++}(x)$ we may write $h = h_1 h_2$ where $h_1 \in \mathcal{Q}_{++}(x)$, and $h_2 \in \mathrm{W}^+(x)$ is a pure translation. Let $\hat{i}(h)$ denote the element of $\mathrm{Lie}(\mathcal{G}_{++})(x)$ whose linear part is $h_1 - \mathrm{I}$ and whose pure translation part is h_2. Then, $\hat{i} : \mathcal{G}_{++}(x) \to \mathrm{Lie}(\mathcal{G}_{++})(x)$ is a bijective g_t-equivariant map.

Recall that our Lyapunov exponents are numbered so that $\lambda_i > \lambda_j$ for $i < j$. Then, we claim that

$$\begin{aligned}&\hat{i}\big(\exp\big[(\mathrm{I}+\mathrm{M}')\tilde{u}\big]\exp(v')\big) + \mathcal{V}_{<i}(x)\\ &\quad = \tilde{u}_i + v_i' + \hat{i}\left(\exp\left[(\mathrm{I}+\mathrm{M}')\sum_{j>i}\tilde{u}_j\right]\exp\left[\sum_{j>i} v_j'\right]\right) + \mathcal{V}_{<i}(x).\end{aligned} \tag{6.40}$$

Indeed, any term involving $\tilde{u}_j$ or v_j' for $j < i$ would belong to $\mathcal{V}_{<i}(x)$ (since it would lie in a subspace with Lyapunov exponent bigger than λ_i). Also, for the same reason, any terms involving $\tilde{u}_i$ or v_i' other than those written on the left-hand-side of (6.40) would belong to $\mathcal{V}_{<i}(x)$. Similarly,

$$\begin{aligned}&\hat{i}\big(\exp(z)\exp(q)\big) + \mathcal{V}_{<i}(x)\\ &\quad = z_i + q_i + \hat{i}\left(\exp\left(\sum_{j>i} z_j\right)\exp\left(\sum_{j>i} q_j\right)\right) + \mathcal{V}_{<i}(x).\end{aligned} \tag{6.41}$$

We now apply $\hat{i}$ to both sides of (6.39), plug in (6.40) and (6.41), and compare terms in $\mathcal{V}_i(x)$. We get equations of the form

$$\tilde{u}_i + v_i' + p_i = z_i + q_i,$$

where p_i is a polynomial in the $\tilde{u}_j$ and q_j for $\lambda_j < \lambda_i$, and in the M_{jk}' for $\lambda_j - \lambda_k < \lambda_i$. Then, the equation can be solved inductively, starting with the equation with i maximal (and thus λ_i minimal). Thus, Equation (6.38) can indeed be solved for $\tilde{u}$ and we get,

$$\tilde{u}_i = -\pi_{\mathrm{T}}(v_i' + p_i), \quad z_i = \pi_{\mathrm{Z}}(v_i' + p_i), \quad q_i = \pi_{\mathrm{Q}}(v_i' + p_i),$$

where π_{Q}, π_{T} and π_{Z} as in (6.37). This shows that $v = \exp(z)v'$ has the form given in (6.7).

Let $U' = \exp((I + M')\,\mathrm{Lie}(U^+)(x))$. By our assumptions, U' is a subgroup of $\mathcal{G}_{++}$. Therefore, for $\tilde{u}$ as in (6.38),

$$\mathcal{U} = U' \cdot (x + v') = U' \exp\big(-(I + M')\tilde{u}\big) \cdot (x + v) = U' \cdot (x + v).$$

Then, (M', v) is also a parametrization of $\mathcal{U}$. To make M' adapted to $Z(x)$ we proceed as follows:

For $u \in \mathrm{Lie}(\mathcal{G}_{++})(x)$, we can write $u = u'' + z''$, where $u'' \in \mathrm{Lie}(U^+)(x)$ and $z'' \in Z(x)$. Let $\pi^{Z}_{U^+} : \mathrm{Lie}(\mathcal{G}_{++}) \to \mathrm{Lie}(U^+)$ be the linear map sending u to u''.

In view of (6.6), we need to find a linear map $J : \mathrm{Lie}(U^+)(x) \to \mathrm{Lie}(U^+)(x)$, so that if we define M via the formula (6.6), then M is adapted to $Z(x)$. Write $u' = Ju$. Then, $u' \in \mathrm{Lie}(U^+)(x)$ must be such that $u' + M'u' = u + z$, where $z \in Z$. Then,

$$u' + \pi^{Z}_{U^+}\big(M'u'\big) = u,$$

hence $u' = Ju$ must be given by the formula

$$u' = \big(I + \pi^{Z}_{U^+} \circ M'\big)^{-1} u.$$

Thus, in view of (6.6), we define M by

$$M = \big(I + M'\big)\big(I + \pi^{Z}_{U^+} \circ M'\big)^{-1} - I. \tag{6.42}$$

Then for all $u \in \mathrm{Lie}(U^+)(x)$, $Mu = (I + M)u - u = (I + M')u' - u \in Z(x)$. Thus (M, v) is adapted to $Z(x)$. Since $M' \in \mathcal{H}_{++}(x)$,

$$\pi^{Z}_{U^+} \circ M' = \sum_{i<j} \pi^{Z}_{U^+} \circ M'_{ij},$$

where $M'_{ij} \in \mathrm{Hom}(\mathrm{Lie}(U^+) \cap \mathcal{V}_j, \mathrm{Lie}(\mathcal{G}_{++}) \cap \mathcal{V}_i)$. Since $Z(x)$ is a Lyapunov-admissible transversal, $\pi^{Z}_{U^+}$ takes $\mathrm{Lie}(\mathcal{G}_{++}) \cap \mathcal{V}_j$ to $\mathrm{Lie}(\mathcal{U}^+) \cap \mathcal{V}_i$. Therefore,

$$\pi^{Z}_{U^+} \circ M'_{ij} \in \mathrm{Hom}\big(\mathrm{Lie}\big(U^+\big) \cap \mathcal{V}_j, \mathrm{Lie}\big(U^+\big) \cap \mathcal{V}_i\big).$$

Thus, $\pi^{Z}_{U^+} \circ M'$ is nilpotent. Then (6.8) follows from (6.42).

This argument shows the existence of a pair (M, v) which parametrizes $\mathcal{U}$ and is adapted to $Z(x)$. The uniqueness follows from the same argument. Essentially one shows that any (M, v) which parametrizes $\mathcal{U}$ and is adapted to $Z(x)$ must satisfy equations whose unique solution is given by (6.7) and (6.8). □

6.5*. *Construction of the map* $\mathcal{A}(q_1, u, \ell, t)$.

Motivation. — Suppose $q_1 \in \tilde{X}_0$, $q_1' \in W^-[q_1]$, $u \in U^+(q_1)$, so $uq_1 \in W^+[q_1]$. To construct the generalized subspace $\mathcal{U} = \mathcal{U}(M'', v'')$ of Proposition 6.11, we first let $\mathcal{U} = g_t\mathcal{U}_0$ and construct the generalized subspace $\mathcal{U}_0 \subset W^+[uq_1]$. Let $z = \pi_{W^+(q_1')}(uq_1)$, so that z is the unique point in $W^+[q_1'] \cap AW^-[uq_1]$. In particular, $W^+[q_1'] = W^+[z]$. (Note that we are not assuming any ergodic properties of z; in particular the Lyapunov subspaces at z may not be defined.)

We will construct a $\pi_1(X_0)$-equivariant linear map $\tilde{P}_s(uq_1, q_1') : W^+(uq_1) \to W^+(z)$, and let $\mathcal{U}_0 = \tilde{P}_s(uq_1, q_1')^{-1}U^+[q_1']$. (This makes sense since $U^+[q_1'] \subset W^+[q_1'] = W^+[z]$.) We want $\tilde{P}_s(uq_1, q_1')$ to have the following properties:

(P1) $\tilde{P}_s(uq_1, q_1')$ depends only on $W^+[q_1']$, i.e. for $z' \in W^+[q_1']$, we have $\tilde{P}_s(uq_1, z') = \tilde{P}_s(uq_1, q_1')$. In particular, for any $u' \in U^+(q_1')$, $\tilde{P}_s(uq_1, q_1') = \tilde{P}_s(uq_1, u'q_1')$.

(P2) For nearby $x, y \in \tilde{X}_0$, let $P^{GM}(x, y) : H^1(x) \to H^1(y)$ denote the Gauss-Manin connection. For $u \in \mathcal{B}(q_1, 1/100)$, $u' \in \mathcal{B}(q_1', 1/50)$ and $t \geq 0$ with

$$d^{X_0}\left(g_t uq_1, g_t u' q_1'\right) < 1/100, \tag{6.43}$$

let $z' = \pi_{W^+(g_t u' q_1')}(g_t uq_1)$. Then, there exists $\alpha_1 > 0$ depending only on the Lyapunov spectrum such that $\|\tilde{P}_s(g_t uq_1, g_t u' q_1')^{-1}P^{GM}(g_t uq_1, z') - I\|_Y = O(e^{-\alpha_1 \ell})$, for all $t > 0$ such that (6.43) holds. (Also note that the points uq_1 and $u'q_1'$ satisfy $d^{X_0}(g_{-\tau}uq_1, g_{-\tau}u'q_1') = O(1)$ for all $0 \leq \tau \leq \ell$.)

Note that as long as (6.43) holds, $d^{X_0}(g_t uq_1, z') = O(1)$ and $d^{X_0}(z', g_t u' q_1') = O(1)$ so that $P^{GM}(g_t uq_1, z')$ connects nearby points. This would not be the case if we defined $\tilde{P}_s(uq_1, q_1')$ to be a linear map from $W^+(uq_1)$ to $W^+(q_1')$, since $g_t uq_1$ and $g_t q_1'$ would quickly become far apart.

(P3) The (entries of the matrix) $\tilde{P}_s(uq_1, q_1')^{-1}$ are polynomials of degree at most s in (the entries of the matrix) $P^-(q_1, q_1')$.

(P4) The generalized subspace $\mathcal{U} = \tilde{P}_s(uq_1, q_1')^{-1}U^+[q_1']$ can be parametrized by $(M'', v'') \in \mathcal{H}_{++}(uq_1) \times W^+(uq_1)$ (and not by an arbitrary element of $\mathcal{H}_+(uq_1) \times W^+(uq_1)$).

The construction will take place in several steps.

Notation. — In this subsection, $\mathcal{V}_i(x)$ refers to $\mathcal{V}_i(H^1)(x)$.

The map $\hat{P}(x, y)$. — There exists a set K of full measure such that each point x in K is Lyapunov-regular with respect to the bundle W^+, i.e.

$$H^1(x) = \bigoplus_i \mathcal{V}_i(x),$$

where $\mathcal{V}_i(x) = \mathcal{V}_i(\mathrm{H}^1)(x)$ are the Lyapunov subspaces, and the multiplicative ergodic theorem holds. We have the flag

$$\{0\} \subset \mathcal{V}_{\leq 1}(x) \subset \cdots \subset \mathcal{V}_{\leq n}(x) = \mathrm{H}^1(x), \tag{6.44}$$

where $\mathcal{V}_{\leq j}(x) = \bigoplus_{i=1}^{j} \mathcal{V}_i(x)$. Note that $\mathcal{V}_{\leq n-1}(x) = \mathrm{W}^+(x)$. If $y \in \mathrm{W}^+[x]$ is also Lyapunov-regular, then the flag (6.44) at y agrees with the flag at x, provided we identify $\mathrm{H}^1(y)$ with $\mathrm{H}^1(x)$ using the Gauss-Manin connection. Thus, we may define (6.44) at any point x such that $\mathrm{W}^+[x]$ contains a regular point.

Now suppose x and y are restricted to a subset where the $\mathcal{V}_i$ vary continuously. Then, for nearby x and y, we have, for each i,

$$\mathrm{H}^1(x) = \mathcal{V}_{\leq i}(y) \oplus \bigoplus_{j=i+1}^{n} \mathcal{V}_j(x). \tag{6.45}$$

Let $z = \pi_{\mathrm{W}^+(y)}(x)$, and let $\hat{\mathrm{P}}_i : \mathcal{V}_i(x) \to \mathrm{H}^1(z)$ be the map taking $v \in \mathcal{V}_i(x)$ to its $\mathcal{V}_{\leq i}(y)$ component under the decomposition (6.45). Let $\hat{\mathrm{P}}(x, y) : \mathrm{H}^1(x) \to \mathrm{H}^1(z)$ be the linear map which agrees with $\hat{\mathrm{P}}_i$ on each $\mathcal{V}_i(x)$. Note that $\hat{\mathrm{P}}(x, y)$ is defined for all nearby x, y such that (6.45) holds for all i. Let $\hat{\mathrm{P}}[x, y]$ be the affine map from $\mathrm{W}^+[x]$ to $\mathrm{W}^+[y]$ whose linear part is $\hat{\mathrm{P}}(x, y)$ and such that x maps to $z = \pi_{\mathrm{W}^+(y)}(x)$. To simplify notation, we will denote $\hat{\mathrm{P}}[x, y]$ also by $\hat{\mathrm{P}}(x, y)$.

We have

$$\hat{\mathrm{P}}(g_t x, g_t y) = g_t \circ \hat{\mathrm{P}}(x, y) \circ g_{-t},$$

and

$$\hat{\mathrm{P}}(x, y)\mathcal{V}_{\leq i}(x) = \mathrm{P}^{\mathrm{GM}}(y, z)\mathcal{V}_{\leq i}(y) = \mathcal{V}_{\leq i}(z). \tag{6.46}$$

(Since $z \in \mathrm{W}^+[y]$, we can define $\mathcal{V}_{\leq i}(z)$ to be $\mathrm{P}^{\mathrm{GM}}(y, z)\mathcal{V}_{\leq i}(y)$ even if $\mathcal{V}_i(z)$ were not originally defined.)

The following lemma essentially states that the map $\hat{\mathrm{P}}(uq_1, q_1')$ has properties (P1) and (P2).

Lemma **6.21.** — *Suppose $\delta > 0$, $\alpha_3 > 0$ and ℓ is sufficiently large depending on δ and α_3. Suppose $q \in \tilde{\mathrm{X}}_0$ and $q' \in \mathrm{W}^-[q]$ satisfy the upper bounds in (5.3) and (5.4). Let $q_1 = g_\ell q$ (see Figure 1), and write $q_1' = g_\ell q'$. Then, for almost all $u \in \mathcal{B}(q_1, 1/100)$ and t with $0 < t < \alpha_3 \ell$ such that*

$$d^{\mathrm{X}_0}\left(g_t u q_1, \mathrm{U}^+\left[g_t q_1'\right]\right) < 1/100,$$

the following holds:

Let $\widehat{\mathcal{U}} = \hat{\mathrm{P}}(uq_1, q_1')^{-1}(\mathrm{U}^+[q_1'])$. *Then* $\widehat{\mathcal{U}} \subset \mathrm{W}^+[q_1]$ *is a generalized subspace, and*

$$hd^{\mathrm{X}_0}_{g_t uq_1}\big(g_t\widehat{\mathcal{U}}, \mathrm{U}^+\big[g_t q_1'\big]\big) \leq \mathrm{C}(q_1)\mathrm{C}(uq_1)e^{-\alpha(t+\ell)},$$

where $\alpha > 0$ *depends only on* α_3 *and the Lyapunov spectrum, and* $\mathrm{C} : \mathrm{X}_0 \to \mathbb{R}^+$ *is finite almost everywhere.*

Proof. — In this proof, we write $\mathcal{V}_i(x)$ for $\mathcal{V}_i(\mathrm{H}^1)(x)$ and $\mathcal{V}_{\leq i}(x)$ for $\mathcal{V}_{\leq i}(\mathrm{H}^1)(x)$. For convenience, we also choose $u' \in \mathcal{B}(q_1', 1/50)$ with

$$d^{\mathrm{X}_0}\big(g_t uq_1, g_t u' q_1'\big) = d^{\mathrm{X}_0}\big(g_t uq_1, \mathrm{U}^+\big[g_t q_1'\big]\big) \leq 1/100.$$

(Nothing in the proof will depend on the choice of u'.)

Let $q_2 = g_t uq_1$, $q_2' = g_t u' q_1'$. We claim that

$$d_{\mathrm{Y}}\big(\mathcal{V}_{\leq i}(q_2), \mathrm{P}^{\mathrm{GM}}\big(q_2', q_2\big)\mathcal{V}_{\leq i}\big(q_2'\big)\big) \leq \mathrm{C}(q_1)\mathrm{C}(uq_1)e^{-\alpha(t+\ell)}, \tag{6.47}$$

where $\alpha > 0$ depends only on the Lyapunov spectrum, and $\mathrm{C} : \mathrm{X}_0 \to \mathbb{R}^+$ (which depends on δ) is finite a.e.

We will apply Lemma 4.7 (with $t + \ell$ in place of t) to the points $x = g_{-(t+\ell)}q_2$ and $y = g_{-(t+\ell)}q_2'$. Thus, we need to bound $\mathrm{D}^+(x, y)$. In the following argument, we identify $\mathrm{H}^1(x)$, $\mathrm{H}^1(y)$, $\mathrm{H}^1(q)$ and $\mathrm{H}^1(q')$ using the Gauss-Manin connection, while suppressing P^{GM} from the notation.

Suppose $v' \in \mathcal{V}_{\leq i}(y)$ realizes the supremum in the definition of $\mathrm{D}^+(x, y)$, i.e. $v' = v + w$ where $v \in \mathcal{V}_{\leq i}(x)$, $w \in \mathcal{V}_{>i}(x)$, and $\mathrm{D}^+(x, y) = \|w\|_{\mathrm{Y}}/\|v\|_{\mathrm{Y}}$.

Note that $\mathcal{V}_{\leq i}(x) = \mathcal{V}_{\leq i}(q)$ and $\mathcal{V}_{\leq i}(y) = \mathcal{V}_{\leq i}(q')$. Thus, $v' \in \mathcal{V}_{\leq i}(q')$. Also note that $\mathcal{V}_{>i}(q') = \mathcal{V}_{>i}(q)$ for all i, $\mathrm{P}^-(q', q)\mathcal{V}_i(q') = \mathcal{V}_i(q)$, and by Lemma 4.2(c), $\mathrm{P}^-(q', q)$ is lower triangular and unipotent. By the upper bound in (5.4), $\|\mathrm{P}^-(q', q)\|_{\mathrm{Y}} \leq \mathrm{C}'(\delta)$. (In particular, we have a lower bound, depending on δ, on the angles between the Lyapunov subspaces $\mathcal{V}_i(q')$.) Hence we can write

$$v' = v'' + w'' \quad v'' \in \mathcal{V}_{\leq i}(q),\ w'' \in \mathcal{V}_{>i}(q),\ \|w''\|_{\mathrm{Y}} \leq \mathrm{C}(\delta)\|v''\|_{\mathrm{Y}}.$$

Since $\mathcal{V}_{\leq i}(x) = \mathcal{V}_{\leq i}(q)$, we have $v'' \in \mathcal{V}_{\leq i}(x)$. By Corollary 4.9 (applied with $x = q_1, y = uq_1$ and $t = \ell$) we can write

$$w'' = v_2 + w_2 \quad v_2 \in \mathcal{V}_{\leq i}(x),\ w_2 \in \mathcal{V}_{>i}(x),$$
$$\text{and } \|v_2\|_{\mathrm{Y}} \leq \mathrm{C}_1(q_1)\mathrm{C}_1(uq_1)e^{-\alpha\ell}\|w''\|_{\mathrm{Y}}.$$

Thus,

$$v = v'' + v_2, \qquad w = w_2.$$

If ℓ is bounded depending on $C_1(q_1)C_1(uq_1)$ and δ, then (in view of the condition $t < \alpha_3\ell$), the desired estimate (6.47) is trivially true. Thus, we may assume that ℓ is sufficiently large so that

$$C_1(q_1)C_1(uq_1)e^{-\alpha\ell} \le 1.$$

Then,

$$\|w_2\|_Y \le \|w''\|_Y + \|v_2\|_Y \le 2\|w''\|_Y \le 2C(\delta)\|v''\|_Y.$$

But,

$$\|v_2\|_Y \le C_1(q_1)C_1(uq_1)e^{-\alpha\ell}\|w_2\|_Y \le 2C(\delta)C_1(q_1)C_1(uq_1)e^{-\alpha\ell}\|v''\|_Y.$$

Arguing as above, we may assume, without loss of generality, that ℓ is sufficiently large so that

$$\|v\|_Y \ge \|v''\|_Y - \|v_2\|_Y \ge (1/2)\|v''\|_Y.$$

Then,

$$D^+(x,y) = \frac{\|w_2\|_Y}{\|v\|_Y} \le 4C(\delta).$$

Hence, by Lemma 4.7, (6.47) follows.

By Lemma 4.14(c), for any $\epsilon > 0$ and any subset S of the Lyapunov exponents,

$$d_Y\Big(\bigoplus_{i\in S}\mathcal{V}_i(q_2), \bigoplus_{j\notin S}\mathcal{V}_j(q_2)\Big) > C_\epsilon(uq_1)e^{-\epsilon t} > C_\epsilon(uq_1)e^{-\epsilon(t+\ell)}. \tag{6.48}$$

Choose $\epsilon < \alpha/2$, where α is as in (6.47). Then, by (6.48), (6.47), and the definition of $\hat{P}(q_2, q_2') = \hat{P}(g_t uq_1, g_t u' q_1')$,

$$\big\|\hat{P}\big(g_t uq_1, g_t u' q_1'\big)^{-1} P^{GM}\big(g_t uq_1, g_t' u' q_1'\big) - I\big\|_Y \le C_\epsilon'(uq_1)C'(q_1)e^{-\alpha'(\ell+t)}, \tag{6.49}$$

where $\alpha' = \alpha - \epsilon$ depends only on the Lyapunov spectrum, and $C'(\cdot)$, $C_\epsilon'(\cdot)$ are finite a.e. Also note that by the upper bound in (5.3) and Lemma 3.5, we have

$$d_Y(uq_1, z) \le C_\epsilon(q_1)e^{-\alpha'\ell},$$

and again by Lemma 3.5,

$$d_Y(g_t uq_1, g_t z) < C_\epsilon(uq_1)e^{-\alpha' t}d_Y(uq_1, z) \le C_\epsilon(q_1)C_\epsilon(uq_1)e^{-\alpha'(t+\ell)}. \tag{6.50}$$

Note that $\widehat{\mathcal{U}}$ is the orbit of a subgroup $\hat{U}$ of $\mathcal{G}(uq_1)$ whose Lie algebra is

$$\hat{P}\big(uq_1, q_1'\big)_*^{-1}\,\mathrm{Lie}\big(U^+\big)\big(q_1'\big)$$

(and we are using the notation (6.11)). By (6.46) and the fact that $\mathrm{Lie}(\mathrm{U}^+)(q_1') \in \mathcal{G}_{++}(q_1')$ we have $\mathrm{Lie}(\hat{\mathrm{U}}) \in \mathcal{G}_{++}(uq_1)$. Thus, $\widehat{\mathcal{U}}$ is a generalized subspace.

Since $\mathrm{U}^+[q_1']$ is a generalized subspace, for all $u' \in \mathrm{U}^+(q_1')$, $\mathrm{U}^+[q_1'] = \mathrm{U}^+[u'q_1']$. We have

$$g_t\widehat{\mathcal{U}} = g_t\hat{\mathrm{P}}\big(uq_1, u'q_1'\big)^{-1}\mathrm{U}^+\big[u'q_1'\big] = \hat{\mathrm{P}}\big(g_tuq_1, g_tu'q_1'\big)^{-1}\mathrm{U}^+\big[g_tu'q_1'\big].$$

Therefore, the lemma follows from (6.49) and (6.50). □

Motivation. — Suppose $q_1 \in \tilde{\mathrm{X}}_0$, $u \in \mathrm{U}^+(q_1)$, $q_1' \in \mathrm{W}^-[q_1]$. In view of Lemma 6.21, $\hat{\mathrm{P}}(uq_1, q_1')$ has properties (P1) and (P2). We claim that it does not in general have the properties (P3) and (P4).

Let $z = \pi_{\mathrm{W}^+(q_1')}(uq_1)$ so in particular $\hat{\mathrm{P}}(uq_1, q_1') = \hat{\mathrm{P}}(uq_1, z)$ and let

$$\hat{\mathrm{Q}}\big(uq_1; q_1'\big) = \hat{\mathrm{P}}(uq_1, z)^{-1}\mathrm{P}^{\mathrm{GM}}\big(q_1', z\big)\mathrm{P}^-\big(q_1, q_1'\big) \circ \mathrm{P}^+(uq_1, q_1), \tag{6.51}$$

so that

$$\hat{\mathrm{P}}(uq_1, z)\hat{\mathrm{Q}}\big(uq_1; q_1'\big) = \mathrm{P}^{\mathrm{GM}}\big(q_1', z\big)\mathrm{P}^-\big(q_1, q_1'\big)\mathrm{P}^+(uq_1, q_1). \tag{6.52}$$

Then, $\hat{\mathrm{Q}}(uq_1; q_1') : \mathrm{H}^1(uq_1) \to \mathrm{H}^1(uq_1)$ and $\hat{\mathrm{Q}}(uq_1; q_1')\mathcal{V}_{\le i}(uq_1) = \mathcal{V}_{\le i}(uq_1)$, hence $\hat{\mathrm{Q}}(uq_1; q_1') \in \mathrm{Q}_+(uq_1)$. In particular $\hat{\mathrm{Q}}(uq_1; q_1')\mathrm{W}^+(uq_1) = \mathrm{W}^+(uq_1)$.

We now show how to compute $\hat{\mathrm{P}}(uq_1, q_1')$ and $\hat{\mathrm{Q}}(uq_1; q_1')$ in terms of $\mathrm{P}^+ = \mathrm{P}^+(uq_1, q_1)$ and $\mathrm{P}^- = \mathrm{P}^-(q_1, q_1')$. In view of Lemma 4.2, P^+ is upper triangular with 1's along the diagonal in terms of a basis adapted to $\mathcal{V}_i(uq_1)$. Also by Lemma 4.2 applied to P^- instead of P^+, P^- is lower triangular with 1's along the diagonal in terms of a basis adapted to $\mathcal{V}_i(q_1)$. Therefore, since P^+ takes $\mathcal{V}_i(uq_1)$ to $\mathcal{V}_i(q_1)$, $(\mathrm{P}^+)^{-1}\mathrm{P}^-\mathrm{P}^+$ is lower triangular with 1's along the diagonal in terms of a basis adapted to $\mathcal{V}_i(uq_1)$.

Let $\hat{\mathrm{P}} = \hat{\mathrm{P}}(uq_1, q_1')$, $\hat{\mathrm{Q}} = \hat{\mathrm{Q}}(uq_1; q_1')$. Then, in view of the definition of $\hat{\mathrm{P}}$, $\hat{\mathrm{P}}$ is lower triangular with 1's along the diagonal in terms of a basis adapted to $\mathcal{V}_i(uq_1)$ (and we identify $\mathrm{H}^1(q_1')$ with $\mathrm{H}^1(uq_1)$ using the Gauss-Manin connection). Also, since $\hat{\mathrm{Q}}$ preserves the flag $\mathcal{V}_{\le i}(uq_1)$, $\hat{\mathrm{Q}}$ is upper triangular in terms of the basis adapted to $\mathcal{V}_i(uq_1)$. Thus, (6.52) can we written as

$$\hat{\mathrm{P}}\hat{\mathrm{Q}} = \mathrm{P}^-\mathrm{P}^+ = \mathrm{P}^+\big(\big(\mathrm{P}^+\big)^{-1}\mathrm{P}^-\mathrm{P}^+\big) \tag{6.53}$$

Recall that the Gaussian elimination algorithm shows that any matrix A in neighborhood of the identity I can be written uniquely as $\mathrm{A} = \mathrm{LU}$ where L is lower triangular with 1's along the diagonal and U is upper triangular. Thus, $\hat{\mathrm{P}} = \hat{\mathrm{P}}(uq_1, q_1')$ and $\hat{\mathrm{Q}} = \hat{\mathrm{Q}}(uq_1; q_1')$ are the L and U parts of the LU decomposition of the matrix $\mathrm{A} = \mathrm{P}^-(q_1, q_1')\mathrm{P}^+(uq_1, q_1)$. (Note that we are given $\mathrm{A} = \mathrm{U}'\mathrm{L}'$ where $\mathrm{U}' = \mathrm{P}^+$ is upper triangular and $\mathrm{L}' = (\mathrm{P}^+)^{-1}\mathrm{P}^-\mathrm{P}^+$ is lower triangular, so we are really solving the equation $\mathrm{LU} = \mathrm{U}'\mathrm{L}'$ for L and U.)

Since the Gaussian elimination algorithm involves division, the entries of $\hat{P}(uq_1, q_1')^{-1}$ are rational functions of the entries of $P^+(uq_1, q_1)$ and $P^-(q_1, q_1')$, but not in general polynomials. This means that $\hat{P}(uq_1, q_1')$ does not in general have property (P3). Also, the diagonal entries of $\hat{Q}(uq_1; q_1')$ are not 1. This eventually translates to the failure of the property (P4). Both problems are addressed below.

The maps $\hat{P}_s(uq_1, q_1')$ *and* $\tilde{P}_s(uq_1, q_1')$. — For $s > 1$, let $\hat{Q}_s(uq_1; q_1')$ be the order s Taylor approximation to $\hat{Q}(uq_1; q_1')$, where the variables are the entries of $P^-(q_1, q_1')$ (and u, q_1 and the entries of $P^+(uq_1, q_1)$ are considered constants). Then, $\hat{Q}_s = \hat{Q}_s(uq_1; q_1') \in Q_+(uq_1)$. We may write

$$\hat{Q}_s = D_s + \tilde{Q}_s,$$

where D_s preserves all the subspaces $\mathcal{V}_i(uq_1)$ and $\tilde{Q}_s = \tilde{Q}_s(uq_1; q_1') \in Q_{++}(uq_1)$. Let $\tilde{P}_s(uq_1, q_1') = \tilde{P}_s(uq_1, z)$ be defined by the relation:

$$\tilde{P}_s\big(uq_1, q_1'\big)^{-1} = \tilde{Q}_s\big(uq_1; q_1'\big)P^+(q_1, uq_1)P^-\big(q_1', q_1\big)P^{GM}\big(z, q_1'\big). \tag{6.54}$$

Motivation. — We will effectively show that for s sufficiently large (chosen at the end of the proof of Proposition 6.11) the map $\tilde{P}_s(uq_1, q_1')$ has the properties (P1), (P2), (P3) and (P4).

We have, by (6.54),

$$\tilde{P}_s\big(uq_1, q_1'\big)^{-1}\mathcal{V}_{\le i}\big(q_1'\big) = \tilde{P}_s\big(uq_1, q_1'\big)^{-1}\mathcal{V}_{\le i}(z) = \mathcal{V}_{\le i}(uq_1).$$

As a consequence,

$$\tilde{P}_s\big(uq_1, q_1'\big)^{-1} \circ Y \circ \tilde{P}_s\big(uq_1, q_1'\big) \in \mathcal{G}_{++}(uq_1) \quad \text{for all } Y \in \mathcal{G}_{++}\big(q_1'\big).$$

Thus, for any subalgebra L of $\mathrm{Lie}(\mathcal{G}_{++})(q_1')$, it follows that $\tilde{P}_s(uq_1, q_1')_*^{-1}(L)$ is a subalgebra of $\mathrm{Lie}(\mathcal{G}_{++})(uq_1)$, where $\tilde{P}_s(uq_1, q_1')_*^{-1} : \mathrm{Lie}(\mathcal{G}_{++})(q_1') \to \mathrm{Lie}(\mathcal{G}_{++})(uq_1)$ is as in (6.11).

The map $i_{u,q_1,s}$.

Motivation. — For $q_1 \in X_0$ and $u \in \mathcal{B}(q_1, 1/100)$, we want $i_{u,q_1,s} : \mathcal{L}_{ext}(q_1) \to \mathcal{H}_{++}(uq_1) \times W^+(uq_1)$ to be such that

$$i_{u,q_1,s}\big(\mathfrak{P}\big(q_1'\big) - \mathfrak{P}(q_1)\big) = (M_s, v_s),$$

where the pair $(M_s, v_s) \in \mathcal{H}_{++}(uq_1) \times W^+(uq_1)$ parametrizes the approximation $\tilde{P}_s(uq_1, q_1')^{-1}U^+[q_1']$ to $U^+[q_1']$ constructed above. Furthermore, we want $i_{u,q_1,s}$ to be a polynomial map of degree at most s in the entries of $\mathfrak{P}(q_1') - \mathfrak{P}(q_1)$.

By Proposition 4.12(a), we have

$$\text{(6.55)} \qquad \mathrm{Lie}\big(\mathrm{U}^+\big)\big(q_1'\big) = \mathrm{P}^-\big(q_1, q_1'\big)_* \circ \mathrm{P}^+(uq_1, q_1)_*\big(\mathrm{Lie}\big(\mathrm{U}^+\big)(uq_1)\big),$$

where we used the notation (6.11). Let $\mathcal{U}_s' = \tilde{\mathrm{P}}_s(uq_1, q_1')^{-1}\mathrm{U}^+[q_1']$. We first find $(\mathrm{M}_s', v_s') \in \mathcal{H}_+(q_1) \times \mathrm{W}^+(q_1)$ which parametrizes $\mathcal{U}_s'$. Let

$$v_s = \tilde{\mathrm{P}}_s\big(uq_1, q_1'\big)^{-1} q_1' \in \mathcal{U}_s' \subset \mathrm{W}^+[q_1] = \mathrm{W}^+[uq_1].$$

By (6.55), $\mathcal{U}_s' = \mathrm{U}_s \cdot v_s$ where the subgroup U_s of $\mathcal{G}_{++}(uq_1)$ is such that

$$\begin{aligned}\mathrm{Lie}(\mathrm{U}_s) = \tilde{\mathrm{P}}_s(uq_1, z)_*^{-1} &\circ \mathrm{P}^{\mathrm{GM}}\big(q_1', z\big)_* \circ \mathrm{P}^-\big(q_1, q_1'\big)_* \\ &\circ \mathrm{P}^+(uq_1, q_1)_*\big(\mathrm{Lie}\big(\mathrm{U}^+\big)(uq_1)\big).\end{aligned}$$

By (6.54),

$$\text{(6.56)} \qquad \mathrm{Lie}(\mathrm{U}_s) = \tilde{\mathrm{Q}}_s\big(uq_1; q_1'\big)_* \mathrm{Lie}\big(\mathrm{U}^+\big)(uq_1).$$

Let

$$\mathrm{M}_s = \tilde{\mathrm{Q}}_s\big(uq_1; q_1'\big)_* - \mathrm{I}.$$

Then (M_s, v_s) parametrizes $\mathcal{U}_s'$. Since $\tilde{\mathrm{Q}}_s(uq_1; q_1') \in \mathrm{Q}_{++}(uq_1)$, $\mathrm{M}_s \in \mathcal{H}_{++}(q_1)$.

Note that by (5.8), we can recover $\mathrm{Im}\, q_1$ from $\mathfrak{P}(q_1)$. Also, since q_1 is considered known and fixed here, knowing $\mathrm{Im}\, q_1'$ is equivalent to knowing q_1' since $\mathrm{Re}\, q_1 = \mathrm{Re}\, q_1'$.

Also, since by Proposition 4.12(a), for $q_1' \in \mathrm{W}^-[q_1]$,

$$\text{(6.57)} \qquad \mathrm{Lie}\big(\mathrm{U}^+\big)\big(q_1'\big) = \mathrm{P}^-\big(q_1, q_1'\big)_* \mathrm{Lie}\big(\mathrm{U}^+\big)(q_1) = \big(\mathfrak{P}\big(q_1'\big) \circ \mathfrak{P}(q_1)^{-1}\big)_* \mathrm{Lie}\big(\mathrm{U}^+\big)(q_1),$$

we can reconstruct $\mathrm{U}^+(q_1')$ if we know $\mathfrak{P}(q_1)$, $\mathrm{U}^+(q_1)$ and $\mathfrak{P}(q_1')$. Now let $i_{u,q_1,s} : \mathcal{L}_{ext}(q_1) \to \mathcal{H}_{++}(uq_1) \times \mathrm{W}^+(uq_1)$ be the map taking $\mathfrak{P}(q_1') - \mathfrak{P}(q_1)$ to (M_s, v_s). In view of (6.56), this is a polynomial map, since $\tilde{\mathrm{Q}}_s$ is a polynomial, and both $\mathrm{Im}\, q_1'$ and $\mathrm{Lie}(\mathrm{U}^+)(q_1')$ can be recovered from $\mathfrak{P}(q_1')$ using (5.8) and (6.57). (Note that q_1 is considered fixed here, so knowing $\mathfrak{P}(q_1') - \mathfrak{P}(q_1)$ is equivalent to knowing $\mathfrak{P}(q_1')$.)

The maps $(i_{u,q_1,s})_*$ *and* $\mathbf{i}_{u,q_1,s}$. — For $a \in \mathbb{N}$, let $\hat{\mathbf{j}}^{\otimes a} : \mathcal{L}_{ext}(x) \to \mathcal{L}_{ext}(x)^{\otimes a}$ be the "diagonal embedding"

$$\hat{\mathbf{j}}^{\otimes a}(v) = v \otimes \cdots \otimes v, \qquad (a \text{ times})$$

and let $\hat{\mathbf{j}}^{\uplus a}$ denote the corresponding map $\mathcal{L}_{ext}(x) \to \mathcal{L}_{ext}(x)^{\uplus a}$.

Since $i_{u,q_1,s} : \mathcal{L}_{ext}(q_1) \to \mathcal{H}_{++}(uq_1) \times \mathrm{W}^+(uq_1)$ is a polynomial map, by the universal property of the tensor product, there exists $a > 0$ and a *linear* map $(i_{u,q_1,s})_* : \mathcal{L}_{ext}(q_1)^{\uplus a} \to \mathcal{H}_{++}(uq_1) \times \mathrm{W}^+(uq_1)$ such that

$$i_{u,q_1,s} = (i_{u,q_1,s})_* \circ \hat{\mathbf{j}}^{\uplus a}.$$

Furthermore, there exists $r > a$ and a linear map $\mathbf{i}_{u,q_1,s} : \mathcal{L}_{ext}(q_1)^{\uplus r} \to \tilde{\mathbf{H}}(uq_1)$ such that

$$\mathbf{j} \circ (i_{u,q_1,s})_* = \mathbf{i}_{u,q_1,s} \circ \hat{\mathbf{j}}^{\uplus r}, \tag{6.58}$$

where $\mathbf{j}$ is as in (6.10). Then $\mathbf{i}_{u,q_1,s}$ takes $F(q_1') - F(q_1) \in \mathcal{L}_{ext}(q_1)^{\uplus r}$ to $\mathbf{j}(M_s, v_s) \in \tilde{\mathbf{H}}(uq_1)$, where (M_s, v_s) is a parametrization of the approximation $\tilde{P}_s(uq_1; q_1')^{-1}U^+[q_1']$ to $U^+[q_1']$.

Construction of the map $\mathcal{A}(q_1, u, \ell, t)$. — Let $s \in \mathbb{N}$ be a sufficiently large integer to be chosen later. (It will be chosen near the end of the proof of Proposition 6.11, depending only on the Lyapunov spectrum.) Let $r \in \mathbb{N}$ be such that (6.58) holds. Suppose $q_1 \in X_0$ and $u \in \mathcal{B}(q_1, 1/100)$. For $\ell > 0$ and $t > 0$, let

$$\mathcal{A}(q_1, u, \ell, t) : \mathcal{L}_{ext}(g_{-\ell} q_1)^{(r)} \to \mathbf{H}(g_t u q_1),$$

be given by

$$\mathcal{A}(q_1, u, \ell, t) = (g_t)_* \circ \mathbf{S}_{uq_1}^{Z(uq_1)} \circ \hat{\pi} \circ \mathbf{i}_{u,q_1,s} \circ (g_\ell)_*^{\uplus r}$$

where $(g_\ell)_* : \mathcal{L}_{ext}(q) \to \mathcal{L}_{ext}(g_\ell q)$ is given by

$$(g_\ell)_*(P) = g_\ell \circ P \circ g_\ell^{-1}.$$

Then $\mathcal{A}(q_1, u, \ell, t)$ is a *linear* map. Unraveling the definitions, we have, for $P \in \mathcal{L}_{ext}(g_{-\ell} q_1)$,

$$\mathcal{A}(q_1, u, \ell, t)\big(\hat{\mathbf{j}}^{\uplus r}(P)\big) = \mathbf{j}\big(G_t^+ \circ S_{uq_1}^{Z(uq_1)} \circ (i_{u,q_1,s}) \circ (g_\ell)_*(P)\big)$$

Thus, for q' satisfying the upper bounds in (5.3) and (5.4),

$$\mathcal{A}(q_1, u, \ell, t)\big(F(q) - F\big(q'\big)\big) = \mathbf{j}\big(M'', v''\big), \tag{6.59}$$

where $(M'', v'') \in \mathcal{H}_{++}(g_t u q_1) \times W^+(uq_1)$ is a parametrization of the approximation

$$g_t \tilde{P}_s\big(uq_1, u'q_1'\big)^{-1} U^+\big[u'q_1'\big]$$

to $U^+[g_t u' q_1']$, where $u'q_1' \in U^+[q_1']$ is such that $d^{X_0}(g_t u q_1, g_t u' q_1') < 1/100$.

6.6*. *Proofs of Proposition 6.11 and Lemma 6.14. Proof of Proposition 6.11.* — Note that Proposition 6.11(i) follows immediately from the definition of $\mathcal{A}(\cdot,\cdot,\cdot,\cdot)$. We now begin the proof of Proposition 6.11(iii). Let $P = \mathfrak{P}(q') - \mathfrak{P}(q) \in \mathcal{L}_{ext}(q)$. Let

$$P_1 = (g_\ell)_*(P) = g_\ell \circ P \circ g_\ell^{-1} \in \mathcal{L}_{ext}(q_1).$$

Let

$$(M_s, v_s) = i_{u,q_1,s}(P_1).$$

Let $\widetilde{\mathcal{U}}_s = \widetilde{\mathcal{U}}_s(\mathrm{M}_s, v_s)$ be the generalized subspace parametrized by (M_s, v_s). Then

$$\widetilde{\mathcal{U}}_s = \tilde{\mathrm{P}}_s(uq_1, q_1')^{-1}\mathrm{U}^+[q_1']. \tag{6.60}$$

Let

$$\widehat{\mathcal{U}} = \hat{\mathrm{P}}(uq_1, q_1')^{-1}\mathrm{U}^+[q_1'], \qquad \widehat{\mathcal{U}}_s = \hat{\mathrm{P}}_s(uq_1, q_1')^{-1}\mathrm{U}^+[q_1]. \tag{6.61}$$

Suppose (6.23) holds. By Lemma 6.21,

$$hd^{\mathrm{X}_0}_{g_tuq_1}(\widehat{\mathcal{U}}, \mathrm{U}^+[g_tu'q_1']) = \mathrm{O}_{uq_1}(e^{-\alpha_1 t}), \tag{6.62}$$

where α_1 depends only on the Lyapunov spectrum. We have, in view of (5.3) and (5.4), for ℓ sufficiently large depending on δ,

$$\|\mathrm{P}^-(q_1, q_1')\mathrm{P}^{\mathrm{GM}}(q_1', q_1) - \mathrm{I}\|_{\mathrm{Y}} = \mathrm{O}_{q_1}(e^{-\alpha_2\ell}) \tag{6.63}$$

where α_2 depends only on the Lyapunov spectrum. Therefore,

$$hd^{\mathrm{X}_0}_{uq_1}(\mathrm{U}^+[uq_1], \mathrm{U}^+[q_1']) = \mathrm{O}_{q_1}(e^{-\alpha_2\ell})$$

To go from $\hat{\mathrm{Q}}$ to $\hat{\mathrm{Q}}_s$ we are doing order s Taylor expansion of the solution to (6.53) in the entries of $\mathrm{P}^-(q_1, q_1')\mathrm{P}^{\mathrm{GM}}(q_1', q_1) - \mathrm{I}$. Thus, by (6.63),

$$\|\hat{\mathrm{Q}}_s(uq_1; q_1') - \hat{\mathrm{Q}}(uq_1; q_1')\|_{\mathrm{Y}} = \mathrm{O}_{q_1, uq_1}(e^{-\alpha_2(s+1)\ell})$$

and thus, by (6.54),

$$\|\hat{\mathrm{P}}_s(uq_1, q_1')^{-1} - \hat{\mathrm{P}}(uq_1, q_1')^{-1}\|_{\mathrm{Y}} = \mathrm{O}_{q_1, uq_1}(e^{-\alpha_2(s+1)\ell}) \tag{6.64}$$

Then, by (6.61),

$$hd^{\mathrm{X}_0}_{uq_1}(\widehat{\mathcal{U}}, \widehat{\mathcal{U}}_s) = \mathrm{O}_{q_1, uq_1}(e^{-\alpha_2(s+1)\ell}).$$

Then, by Lemma 6.10(a),

$$hd^{\mathrm{X}_0}_{g_tuq_1}(g_t\widehat{\mathcal{U}}, g_t\widehat{\mathcal{U}}_s) = \mathrm{O}_{q_1, uq_1}(e^{-\alpha_2(s+1)\ell + 2t}). \tag{6.65}$$

Also, by (6.63), (6.51) and (6.49), we have

$$\|\hat{\mathrm{Q}}(uq_1; q_1') - \mathrm{I}\|_{\mathrm{Y}} = \mathrm{O}_{q_1, uq_1}(e^{-\alpha_2\ell}),$$

and therefore

$$\|\hat{\mathrm{Q}}_s(uq_1; q_1') - \mathrm{I}\|_{\mathrm{Y}} = \mathrm{O}_{q_1, uq_1}(e^{-\alpha_2\ell}).$$

Thus,

$$\|D_s\|_Y = \|\tilde{Q}_s(uq_1; q_1') - \hat{Q}_s(uq_1; q_1')\|_Y = O_{q_1}(e^{-\alpha_2 \ell})$$

Therefore, since D_s preserves all the eigenspaces $\mathcal{V}_i$, and the Osceledets multiplicative ergodic theorem, for sufficiently small $\epsilon > 0$ (depending on the Lyapunov spectrum),

$$\|g_t \circ D_s \circ g_t^{-1}\|_Y \le C_1(q_1) C_2(uq_1, \epsilon) e^{-\alpha_2 \ell + \epsilon t} \le C_1(q_1) C_2'(uq_1) e^{-(\alpha_2/2)\ell}.$$

Thus,

$$\|\tilde{P}_s(g_t uq_1, g_t u' q_1')^{-1} - \hat{P}_s(g_t uq_1, g_t u' q_1')^{-1}\| = O_{uq_1}(e^{-(\alpha_2/2)\ell}) \tag{6.66}$$

and hence by (6.60) and (6.61),

$$hd^{X_0}_{g_t uq_1}(g_t \widehat{\mathcal{U}}_s, g_t \widetilde{\mathcal{U}}_s) = O_{uq_1}(\|g_t \circ D_s \circ g_t^{-1}\|_Y) = O_{uq_1}(e^{-(\alpha_2/2)\ell}). \tag{6.67}$$

We now choose s so that $\alpha_2 \alpha_3 (s+1) - 3 > \alpha_2$. Then, by (6.22), (6.62), (6.65), and (6.67),

$$hd^{X_0}_{g_t uq_1}(g_t \widetilde{\mathcal{U}}_s, U^+[g_t q_1']) \le C(q_1) C(uq_1) e^{-\alpha \ell}, \tag{6.68}$$

where α depends only on the Lyapunov spectrum. In view of (6.59), the pair (M'', v'') parametrizes $g_t \widetilde{\mathcal{U}}_s$. Therefore, (6.26) holds. Finally, (6.27) is an immediate consequence of (6.68). This completes the proof of Proposition 6.11(iii). (Note that is was shown immediately after the statement of Proposition 6.11 that Proposition 6.11(iii) implies Proposition 6.11(ii).) □

Proof of Lemma 6.14. — In the proof of this lemma we normalize the measure $|\cdot|$ on $U^+[q_1]$ so that $|U^+[q_1] \cap B^+(q_1, 1/100)| = 1$ and similarly we normalize the measure $|\cdot|$ on $U^+[q_1']$ so that $|U^+[q_1'] \cap B^+(q_1', 1/100)| = 1$. As in the proof of Lemma 6.21, we choose $u' \in \mathcal{B}(q_1', 1/50)$ with $\mathcal{V}_i(g_t u' q_1')$ and $U^+[g_t u' q_1'] = U^+[g_t q_1']$ defined and

$$d^{X_0}(g_t uq_1, g_t u' q_1') \le hd^{X_0}_{g_t uq_1}(U^+[g_t uq_1], U^+[g_t q_1']) \le \epsilon.$$

(Nothing in the proof will depend on the choice of u'.)

Let $A_0 = g_{-t} A_t$, $A_0' = g_{-t} A_t'$. Let $\tilde{P}_s$ be as in (6.54). Let $\tilde{A}_t = \tilde{P}_s(g_t uq_1, g_t u' q_1')^{-1} A_t'$. Then,

$$\tilde{A}_0 \equiv g_{-t} \tilde{A}_t = \tilde{P}_s(uq_1, u' q_1')^{-1} A_0'.$$

As in the proof of Proposition 6.11 (i.e. by combining (6.49), (6.64) and (6.66)), we have

$$\|\tilde{P}_s(uq_1, u' q_1')^{-1} P^{GM}(uq_1, u' q_1') - I\|_Y = O(e^{-\alpha \ell}).$$

$$\|\tilde{P}_s(g_t uq_1, g_t u' q_1')^{-1} P^{GM}(g_t uq_1, g_t' u' q_1') - I\|_Y = O(e^{-\alpha \ell}).$$

Hence, $|\tilde{A}_t|$ is comparable to $|A'_t|$ and $|\tilde{A}_0|$ is comparable to $|A'_0|$. Thus, it is enough to show that $|\tilde{A}_0|$ is comparable to $|A_0|$.

As in the proof of Proposition 6.11, let (M'', v'') be the pair parameterizing $g_t\tilde{\mathcal{U}}_s = \tilde{P}_s(g_t uq_1, g_t u' q_1)^{-1}U^+[g_t u' q'_1]$. Let $\tilde{f}_t : \mathrm{Lie}(U^+)(g_t uq_1) \to g_t\tilde{\mathcal{U}}_s$ be the "parametrization" map

$$\tilde{f}_t(Y) = \exp\big[\big(I + M''\big)Y\big](g_t uq_1)\big(g_t uq_1 + v''\big).$$

Similarly, let $f_t : \mathrm{Lie}(U^+)(g_t uq_1) \to U^+[g_t uq_1]$ be the exponential map

$$f_t(Y) = \exp(Y) g_t uq_1.$$

Then, provided that ϵ is sufficiently small, we have

$$0.5 f^{-1}(A_t) \subset \tilde{f}_t^{-1}(\tilde{A}_t) \subset 2 f^{-1}(A_t) \tag{6.69}$$

Let $M_0 = g_t^{-1} \circ M'' \circ g_t$, $v_0 = g_t^{-1} v''$. Then, $g_t^{-1} \circ \tilde{f}_t \circ g_t = \tilde{f}_0$, where $\tilde{f}_0 : \mathrm{Lie}(U^+)(uq_1) \to \mathcal{U}_s$ is given by

$$\tilde{f}_0(Y) = \exp\big[(I + M_0)Y\big](g_t uq_1)(g_t uq_1 + v_0).$$

Similarly, $g_t^{-1} \circ f_t \circ g_t = f_0$, where $f_0 : \mathrm{Lie}(U^+)(uq_1) \to U^+[uq_1]$ is given by the exponential map

$$f_0(Y) = \exp(Y) uq_1.$$

Then, it follows from applying g_t^{-1} to (6.69) that

$$0.5 f_0^{-1}(A_0) \subset \tilde{f}_0^{-1}(\tilde{A}_0) \subset 2 f_0^{-1}(A_0) \tag{6.70}$$

Thus, $|\tilde{f}_0^{-1}(\tilde{A}_0)|$ is comparable to $|f_0^{-1}(A_0)| = |A_0|$. But, since $M'' \in \mathcal{H}_{++}(g_t uq_1)$ and $v'' \in W^+(g_t uq_1)$ are $O(\epsilon)$, M_0 and v_0 are exponentially small. Therefore, the map $\tilde{f}_0$ is close to f_0 (and since Y is small, it is close to the identity). Therefore, $|\tilde{f}_0^{-1}(\tilde{A}_0)|$ is comparable to $|\tilde{A}_0|$. The second assertion of the Lemma also follows from (6.70) and the fact that M_0 and v_0 are exponentially small. □

7. Bilipshitz estimates

In this section, we continue working on X_0 (and not X). Let $\|\cdot\|$ be the norm on $H_{big}^{(++)}$ defined in (4.18). Since $\mathbf{H} \subset H_{big}^{(++)}$, $\|\cdot\|$ is also a norm on $\mathbf{H}$. We can also define a norm on $H_{big}^{(--)}$ in an analogous way. Since $\mathcal{L}_{ext}(x)^{(r)} \subset H_{big}^{(--)}(x)$, the norm $\|\cdot\|_x$ is also a norm on $\mathcal{L}_{ext}(x)^{(r)}$. Let $A(q_1, u, \ell, t) = \|\mathcal{A}(q_1, u, \ell, t)\|$ where the operator norm is with respect to the dynamical norms $\|\cdot\|$ at $g_{-\ell}q_1$ and $g_t uq_1$. In the rest of this section we assume that the equivalent conditions of Lemma 6.15 do not hold, and then by Proposition 6.16, (6.33) holds.

For $1/100 > \epsilon > 0$, almost all $q_1 \in X_0$, almost all $u \in \mathcal{B}(q_1, 1/100)$ and $\ell > 0$, let

$$\hat{\tau}_{(\epsilon)}(q_1, u, \ell) = \sup\big\{t \;:\; t > 0 \text{ and } A(q_1, u, \ell, t) \leq \epsilon\big\}.$$

Note that $\hat{\tau}_{(\epsilon)}(q_1, u, 0)$ need not be 0.

For $x \in X_0$, let $\mathcal{A}_+(x, t) : \mathbf{H}(x) \to \mathbf{H}(g_t x)$ denote the action of g_t on $\mathbf{H}$ as in (6.16). Let $\mathcal{A}_-(x, s) : \mathcal{L}^{(r)}_{ext}(x) \to \mathcal{L}^{(r)}_{ext}(g_s x)$ denote the action of g_s on $\mathcal{L}^{(r)}_{ext}(x)$.

Lemma **7.1.** — *There exist absolute constants* $N > 0$, $\alpha > 0$ *such that for almost all* x, *and* $t > 0$,

$$e^{-\alpha t} \geq \|\mathcal{A}_-(x, t)\| \geq e^{-Nt}, \qquad e^{\alpha t} \leq \|\mathcal{A}_+(x, t)\| \leq e^{Nt},$$

and,

$$e^{Nt} \geq \|\mathcal{A}_-(x, -t)\| \geq e^{\alpha t}, \qquad e^{-Nt} \leq \|\mathcal{A}_+(x, -t)\| \leq e^{-\alpha t}.$$

Proof. — This follows immediately from Proposition 4.15. □

Lemma **7.2.** — *Suppose* $0 < \epsilon < 1/100$. *There exists* $\kappa_1 > 1$ *(depending only on the Lyapunov spectrum) with the following property: for almost all* $q_1 \in X_0$, $u \in \mathcal{B}(q_1, 1/100)$, *for all* $\ell > 0$ *and* $s > 0$,

$$\hat{\tau}_{(\epsilon)}(q_1, u, \ell + s) > \hat{\tau}_{(\epsilon)}(q_1, u, \ell) + \kappa_1^{-1} s.$$

Proof. — Note that by (6.21),

$$\mathcal{A}(q_1, u, \ell + s, t + \tau) = \mathcal{A}_+(g_t u q_1, \tau)\mathcal{A}(q_1, u, \ell, t)\mathcal{A}_-(g_{-(\ell+s)} q_1, s).$$

Let $t = \hat{\tau}_{(\epsilon)}(q_1, u, \ell)$, so that $A(q_1, u, \ell, t) = \epsilon$. Therefore,

$$\begin{aligned} A(q_1, u, \ell + s, t + \tau) &\leq \|\mathcal{A}_+(g_t u q_1, \tau)\| A(q_1, u, \ell, t) \|\mathcal{A}_-(g_{-(\ell+s)} q_1, s)\| \\ &\leq \epsilon \|\mathcal{A}_+(q_t u q_1, \tau)\| \|\mathcal{A}_-(g_{-(\ell+s)} q_1, s)\| \leq \epsilon e^{N\tau - \alpha s}, \end{aligned}$$

where we have used the fact that $A(q_1, u, \ell, t) = \epsilon$ and Lemma 7.1. If $t + \tau = \hat{\tau}_{(\epsilon)}(q_1, u, \ell + s)$ then $A(q_1, u, \ell + s, t + \tau) = \epsilon$. It follows that $N\tau - \alpha s \geq 0$, i.e. $\tau \geq (\alpha/N)s$. Hence,

$$\hat{\tau}_{(\epsilon)}(q_1, u, \ell + s) \geq \hat{\tau}_{(\epsilon)}(q_1, u, \ell) + (\alpha/N)s. \qquad \square$$

Lemma **7.3.** — *Suppose* $0 < \epsilon < 1/100$. *There exists* $\kappa_2 > 1$ *(depending only on the Lyapunov spectrum) such that for almost all* $q_1 \in X_0$, *almost all* $u \in \mathcal{B}(q_1, 1/100)$, *all* $\ell > 0$ *and all* $s > 0$,

$$\hat{\tau}_{(\epsilon)}(q_1, u, \ell + s) < \hat{\tau}_{(\epsilon)}(q_1, u, \ell) + \kappa_2 s.$$

Proof. — We have

$$\mathcal{A}(q_1, u, \ell, t) = \mathcal{A}_+(g_{t+\tau} u q_1, -\tau)\mathcal{A}(q_1, u, \ell + s, t + \tau)\mathcal{A}_-(g_{-\ell} q_1, -s).$$

Let $t + \tau = \hat{\tau}_{(\epsilon)}(q_1, u, \ell + s)$. Then, by Lemma 7.1,

$$\begin{aligned} \mathrm{A}(q_1, u, \ell, t) &\le \|\mathcal{A}_+(q_{t+\tau} u q_1, -\tau)\| \mathrm{A}(q_1, u, \ell + s, t + \tau)\|\mathcal{A}_-(g_{-\ell} q_1, -s)\| \\ &\le \epsilon \|\mathcal{A}_+(q_{t+\tau} u q_1, -\tau)\| \|\mathcal{A}_-(g_{-\ell} q_1, -s)\| \le \epsilon e^{-\alpha\tau + \mathrm{N}s}, \end{aligned}$$

where we have used the fact that $\mathrm{A}(q_1, u, \ell + s, t + \tau) = \epsilon$. Since $\mathrm{A}(q_1, u, \ell, t) = \epsilon$, it follows that $-\alpha\tau + \mathrm{N}s > 0$, i.e. $\tau < (\mathrm{N}/\alpha)s$. It follows that

$$\hat{\tau}_{(\epsilon)}(q_1, u, \ell + s) < \hat{\tau}_{(\epsilon)}(q_1, u, \ell) + (\mathrm{N}/\alpha)s \qquad \square$$

Proposition **7.4.** — *There exists $\kappa > 1$ depending only on the Lyapunov spectrum, and such that for almost all $q_1 \in \mathrm{X}_0$, almost all $u \in \mathcal{B}(q_1, 1/100)$, any $\ell > 0$ and any measurable subset $\mathrm{E}_{bad} \subset \mathbb{R}^+$,*

$$\begin{aligned} &\left|\hat{\tau}_{(\epsilon)}(q_1, u, \mathrm{E}_{bad}) \cap \left[\hat{\tau}_{(\epsilon)}(q_1, u, 0), \hat{\tau}_{(\epsilon)}(q_1, u, \ell)\right]\right| \le \kappa |\mathrm{E}_{bad} \cap [0, \ell]| \\ &\left|\left\{t \in [0, \ell]| \ : \ \hat{\tau}_{(\epsilon)}(q_1, u, t) \in \mathrm{E}_{bad}\right\}\right| \le \kappa \left|\mathrm{E}_{bad} \cap \left[\hat{\tau}_{(\epsilon)}(q_1, u, 0), \hat{\tau}_{(\epsilon)}(q_1, u, \ell)\right]\right|. \end{aligned}$$

Proof. — Let $\kappa = \max(\kappa_1^{-1}, \kappa_2)$, where κ_1, κ_2 are as in Lemmas 7.2 and 7.3. Then, for fixed q_1, u, $\hat{\tau}_{(\epsilon)}(q_1, u, \ell)$ is κ-bilipshitz as a function of ℓ. The proposition follows immediately. $\square$

8. Preliminary divergence estimates

In this section, we continue working on X_0 (and not X).

Motivation. — Suppose in the notation of Section 2.3, q_1 and q_1' are fixed, but $u \in \mathcal{B}(q_1, 1/100)$ and $u' \in \mathcal{B}(q_1', 1/100)$ vary. Then, as u and u' vary, so do the points q_2 and q_2', and thus the subspaces $\mathrm{U}^+[q_2]$ and $\mathrm{U}^+[q_2']$. Let $\mathcal{U} = \mathcal{U}(\mathrm{M}''(u), v''(u))$ be the approximation to $\mathrm{U}^+[q_2']$ given by Proposition 6.11, and as in Proposition 6.11, let $\mathbf{v}(u) = \mathbf{j}(\mathrm{M}''(u), v''(u)) \in \mathbf{H}(q_2)$ be the associated vector in $\mathbf{H}(q_2)$.

In this section we define a certain g_t-equivariant and $(u)_*$-equivariant subbundle $\mathbf{E} \subset \mathbf{H}$ such that, for fixed q_1, q_1', for most $u \in \mathrm{U}^+[q_1]$, $\mathbf{v} = \mathbf{v}(u)$ is near $\mathbf{E}(q_2)$ (see Proposition 8.5(a) below for the precise statement). We call $\mathbf{E}$ the U^+-inert subbundle of $\mathbf{H}$. The subbundle $\mathbf{E}$ is the direct sum of subbundles $\mathbf{E}_i$, where $\mathbf{E}_i$ is contained in the i-th Lyapunov subspace of $\mathbf{H}$, and also each $\mathbf{E}_i$ is both g_t-equivariant and $(u)_*$-equivariant.

8.1. *The* U^+*-inert subspaces* $\mathbf{E}(x)$. — We apply the Osceledets multiplicative ergodic theorem to the action on $\mathbf{H}(x)$ (see (6.16)). We often drop the $*$ and denote the action simply by g_t. In this section, λ_i denotes the i-th Lyapunov exponent of the flow g_t on the bundle $\mathbf{H}$.

Let

$$\mathbf{V}_{\le i}(x) = \bigoplus_{j\le i} \mathcal{V}_j(\mathbf{H})(x), \qquad \mathbf{V}_{<i}(x) = \bigoplus_{j<i} \mathcal{V}_j(\mathbf{H})(x),$$
$$\mathbf{V}_{\ge i}(x) = \bigoplus_{j\ge i} \mathcal{V}_j(\mathbf{H})(x), \qquad \mathbf{V}_{>i}(x) = \bigoplus_{j>i} \mathcal{V}_j(\mathbf{H})(x).$$

This means that for almost all $x \in X_0$ and for $\mathbf{v} \in \mathbf{V}_{\le i}(x)$ such that $\mathbf{v} \notin \mathbf{V}_{<i}(x)$,

$$\lim_{t\to -\infty} \frac{1}{t} \log \frac{\|g_t \mathbf{v}\|}{\|\mathbf{v}\|} = \lambda_i, \tag{8.1}$$

and for $\mathbf{v} \in \mathbf{V}_{\ge i}(x)$ such that $\mathbf{v} \notin \mathbf{V}_{>i}(x)$,

$$\lim_{t\to \infty} \frac{1}{t} \log \frac{\|g_t \mathbf{v}\|}{\|\mathbf{v}\|} = \lambda_i. \tag{8.2}$$

By e.g. [GM, Lemma 1.5], we have for a.e. $x \in X_0$,

$$\mathbf{H}(x) = \mathbf{V}_{\le i}(x) \oplus \mathbf{V}_{>i}(x). \tag{8.3}$$

Let

$$\mathbf{F}_{\ge j}(x) = \left\{ \mathbf{v} \in \mathbf{H}(x) \ : \ \text{for almost all } u \in \mathcal{B}(x),\ (u)_* \mathbf{v} \in \mathbf{V}_{\ge j}(ux) \right\}, \tag{8.4}$$

where $(u)_*$ is as in Lemma 6.6. In other words, if $\mathbf{v} \in \mathbf{F}_{\ge j}(x)$, then for almost all $u \in \mathcal{B}(x)$,

$$\limsup_{t\to\infty} \frac{1}{t} \log \|(g_t)_* (u)_* \mathbf{v}\| \le \lambda_j. \tag{8.5}$$

From the definition of $\mathbf{F}_{\ge j}(x)$, we have

$$\{0\} = \mathbf{F}_{\ge n+1}(x) \subset \mathbf{F}_n(x) \subset \mathbf{F}_{\ge n-1}(x) \subset \cdots \mathbf{F}_2(x) \subset \mathbf{F}_1(x) = \mathbf{H}(x). \tag{8.6}$$

Let

$$\mathbf{E}_j(x) = \mathbf{F}_{\ge j}(x) \cap \mathbf{V}_{\le j}(x).$$

In particular, $\mathbf{E}_1(x) = \mathbf{V}_{\le 1}(x) = \mathcal{V}_1(\mathbf{H})(x)$. We may have $\mathbf{E}_j(x) = \{0\}$ if $j \neq 1$.

Lemma **8.1.** — *For almost all $x \in X_0$ the following holds: suppose $\mathbf{v} \in \mathbf{E}_j(x) \setminus \{0\}$. Then for almost all $u \in \mathcal{B}(x)$,*

$$(\mathbf{8.7}) \qquad \lim_{t\to\infty} \frac{1}{t} \log \|(g_t)_*(u)_*(\mathbf{v})\| = \lambda_j.$$

Thus (recalling that $\mathcal{V}_j(\mathbf{H})$ denotes the subspace of $\mathbf{H}$ corresponding to the Lyapunov exponent λ_j), we have for almost all x, using Fubini's theorem,

$$\mathbf{E}_j(x) \subset \mathcal{V}_j(\mathbf{H})(x).$$

In particular, if $i \neq j$, $\mathbf{E}_i(x) \cap \mathbf{E}_j(x) = \{0\}$ for almost all $x \in X_0$.

Proof. — Suppose $\mathbf{v} \in \mathbf{E}_j(x)$. Then $\mathbf{v} \in \mathbf{V}_{\le j}(x)$. Since in view of (8.1), $\mathbf{V}_{\le j}(ux) = (u)_*\mathbf{V}_{\le j}(x)$ for all $u \in U^+(x)$, we have for almost all $u \in \mathcal{B}(x)$, $(u)_*\mathbf{v} \in \mathbf{V}_{\le j}(ux)$. It follows from (8.3) that (outside of a set of measure 0), $(u)_*\mathbf{v} \notin \mathbf{V}_{>j}(ux)$. Now (8.7) follows from (8.2). □

Lemma **8.2.** — *After possibly modifying $\mathbf{E}_j(x)$ and $\mathbf{F}_{\ge j}(x)$ on a subset of measure 0 of X, the following hold:*

(a) *$\mathbf{E}_j(x)$ and $\mathbf{F}_{\ge j}(x)$ are g_t-equivariant, i.e. $(g_t)_*\mathbf{E}_j(x) = \mathbf{E}_j(g_t x)$, and $(g_t)_*\mathbf{F}_{\ge j}(x) = \mathbf{F}_{\ge j}(g_t x)$.*

(b) *For almost all $u \in U^+(x)$, $\mathbf{E}_j(ux) = (u)_*\mathbf{E}_j(x)$, and $\mathbf{F}_{\ge j}(ux) = (u)_*\mathbf{F}_{\ge j}(x)$.*

Proof. — Note that for $t > 0$, $g_t\mathcal{B}[x] \supset \mathcal{B}[g_t x]$. Therefore, (a) for the case $t > 0$ follows immediately from the definitions of $\mathbf{E}_j(x)$ and $\mathbf{F}_{\ge j}(x)$. Since the flow $\{g_t\}_{t>0}$ is ergodic, it follows that almost everywhere (8.4) holds with $\mathcal{B}[x]$ replaced by arbitrary large balls in $U^+[x]$. This implies that almost everywhere,

$$\mathbf{F}_{\ge j}(x) = \left\{\mathbf{v} \in \mathbf{H}(x) \ : \ \text{for almost all } u \in U^+,\ (u)_*\mathbf{v} \in \mathbf{V}_{\ge j}(ux)\right\},$$

where $(u)_*\mathbf{v}$ is as in Lemma 6.6. Therefore (b) holds. Then, (a) for $t < 0$ also holds, as long as both x and $g_t x$ belong to a subset of full measure. By considering a transversal for the flow g_t, it is easy to check that it is possible to modify $\mathbf{E}_j(x)$ and $\mathbf{F}_{\ge j}(x)$ on a subset of measure 0 of X_0 in such a way that (a) holds for x in a subset of full measure and all $t \in \mathbb{R}$. □

Lemma **8.3.** — *For $x \in X_0$, let*

$$Q(\mathbf{v}) = \left\{u \in \mathcal{B}(x) \ : \ (u)_*\mathbf{v} \in \mathbf{V}_{\ge j}(ux)\right\}.$$

Then for almost all x, either $|Q(\mathbf{v})| = 0$, or $|Q(\mathbf{v})| = |\mathcal{B}(x)|$ (and thus $\mathbf{v} \in \mathbf{F}_{\ge j}(x)$).

Proof. — For a subspace $\mathbf{V} \subset \mathbf{H}(x)$, let

$$Q(\mathbf{V}) = \left\{ u \in \mathcal{B}(x) \ : \ (u)_* \mathbf{V} \subset \mathbf{V}_{\geq j}(ux) \right\}.$$

Let d be the maximal number such that there exists $E' \subset X_0$ with $\nu(E') > 0$ such that for $x \in E'$ there exists a subspace $\mathbf{V} \subset \mathbf{H}(x)$ of dimension d with $|Q(\mathbf{V})| > 0$. For a fixed $x \in E'$, let $\mathcal{W}(x)$ denote the set of subspaces $\mathbf{V}$ of dimension d for which $|Q(\mathbf{V})| > 0$. Then, by the maximality of d, if $\mathbf{V}$ and $\mathbf{V}'$ are distinct elements of $\mathcal{W}(x)$ then $Q(\mathbf{V}) \cap Q(\mathbf{V}')$ has measure 0. Let $\mathbf{V}_x \in \mathcal{W}(x)$ be such that $|Q(\mathbf{V}_x)|$ is maximal (among elements of $\mathcal{W}(x)$).

Let $\epsilon > 0$ be arbitrary, and suppose $x \in E'$. By the same Vitali-type argument as in the proof of Lemma 3.11, there exists $t_0 > 0$ and a subset $Q(\mathbf{V}_x)^* \subset Q(\mathbf{V}_x) \subset \mathcal{B}(x)$ such that for all $u \in Q(\mathbf{V}_x)^*$ and all $t > t_0$,

$$|\mathcal{B}_t(ux) \cap Q(\mathbf{V}_x)| \geq (1-\epsilon)|\mathcal{B}_t(ux)|. \tag{8.8}$$

(In other words, $Q(\mathbf{V}_x)^*$ are "points of density" for $Q(\mathbf{V}_x)$, relative to the "balls" $\mathcal{B}_t$.) Let

$$E^* = \left\{ ux \ : \ x \in E', \ u \in Q(\mathbf{V}_x)^* \right\}.$$

Then, $\nu(E^*) > 0$. Let $\Omega = \{x \in X_0 \ : \ g_{-t}x \in E^* \text{ for an unbounded set of } t > 0 \}$. Then $\nu(\Omega) = 1$. Suppose $x \in \Omega$. We can choose $t > t_0$ such that $g_{-t}x \in E^*$. Note that

$$\mathcal{B}[x] = g_t \mathcal{B}_t[g_{-t}x]. \tag{8.9}$$

Let $x' = g_{-t}x$, and let $\mathbf{V}_{t,x} = (g_t)_* \mathbf{V}_{x'}$. Then in view of (8.8) and (8.9),

$$|Q(\mathbf{V}_{t,x})| \geq (1-\epsilon)|\mathcal{B}(x)|.$$

By the maximality of d (and assuming $\epsilon < 1/2$), $\mathbf{V}_{t,x}$ does not depend on t. Hence, for every $x \in \Omega$, there exists $\mathbf{V} \subset \mathbf{H}(x)$ such that $\dim \mathbf{V} = d$ and $|Q(\mathbf{V})| \geq (1-\epsilon)|\mathcal{B}(x)|$. Since $\epsilon > 0$ is arbitrary, for each $x \in \Omega$, there exists $\mathbf{V} \subset \mathbf{H}(x)$ with $\dim \mathbf{V} = d$, and $|Q(\mathbf{V})| = |\mathcal{B}(x)|$. Now the maximality of d implies that if $\mathbf{v} \notin \mathbf{V}$ then $|Q(\mathbf{v})| = 0$. □

By Lemma 8.1, $\mathbf{E}_j(x) \cap \mathbf{E}_k(x) = \{0\}$ if $j \neq k$. Let

$$\Lambda' = \left\{ i \ : \ \mathbf{E}_i(x) \neq \{0\} \text{ for a.e. } x \right\}.$$

Let the U^+-inert subbundle $\mathbf{E}$ be defined by

$$\mathbf{E}(x) = \bigoplus_{i \in \Lambda'} \mathbf{E}_i(x).$$

Then $\mathbf{E}(x) \subset \mathbf{H}(x)$.

In view of (8.5), (8.6) and Lemma 8.1, we have $\mathbf{F}_{\geq j}(x) = \mathbf{F}_{\geq j+1}(x)$ unless $j \in \Lambda'$. Therefore if we write the elements of Λ' in decreasing order as $i_1, \dots, i_m$ we have the flag (consisting of distinct subspaces)

$$\{0\} = \mathbf{F}_{\geq i_{m+1}} \subset \mathbf{F}_{\geq i_m}(x) \subset \mathbf{F}_{\geq i_{m-1}}(x) \subset \cdots \mathbf{F}_{\geq i_2}(x) \subset \mathbf{F}_{\geq i_1}(x) = \mathbf{H}(x). \tag{8.10}$$

For a.e. $x \in X_0$, and $1 \leq r \leq m$, let $\mathbf{F}'_{i_r}(x)$ be the orthogonal complement (using the inner product $\langle \cdot, \cdot \rangle_x$ defined in Section 4.7) to $\mathbf{F}_{\geq i_{r+1}}(x)$ in $\mathbf{F}_{\geq i_r}(x)$.

Lemma **8.4.** — *Given $\delta > 0$ there exists a compact $K_{01} \subset X_0$ with $\nu(K_{01}) > 1 - \delta$, $\beta(\delta) > 0$, $\beta'(\delta) > 0$, and for every $x \in K_{01}$ any $j \in \Lambda'$ any $\mathbf{v}' \in \mathbb{P}(\mathbf{F}'_j)(x)$ a subset $Q_{01} = Q_{01}(x, \mathbf{v}') \subset \mathcal{B}(x)$ with $|Q_{01}| > (1 - \delta)|\mathcal{B}(x)|$ such that for any $j \in \Lambda'$ any $\mathbf{v}' \in \mathbf{F}'_j(x)$ and any $u \in Q_{01}$, we can write*

$$(u)_* \mathbf{v}' = \mathbf{v}_u + \mathbf{w}_u, \qquad \mathbf{v}_u \in \mathbf{E}_j(ux), \qquad \mathbf{w}_u \in \mathbf{V}_{>j}(ux),$$

with $\|\mathbf{v}_u\| \geq \beta(\delta)\|\mathbf{v}'\|$, and $\|\mathbf{v}_u\| > \beta'(\delta)\|\mathbf{w}_u\|$.

Proof. — This is a corollary of Lemma 8.3. Let $\Phi \subset X_0$ be the conull set where (8.3) holds and where $\mathbf{F}_{\geq i}(x) = \mathbf{F}_{\geq i+1}(x)$ for all $i \notin \Lambda'$. Suppose $x \in \Phi$.

Let $\mathbf{F}_{\geq k}(x) \subset \mathbf{F}_{\geq j}(x)$ be the next subspace in the flag (8.10) (i.e. $\mathbf{F}_{\geq k} = \{0\}$ if j is the maximal index in Λ' and otherwise we have $k > j$ be minimal such that $k \in \Lambda'$). Then $\mathbf{F}_{\geq j+1}(x) = \mathbf{F}_{\geq k}(x)$. Since $\mathbf{F}'_j(x)$ is complementary to $\mathbf{F}_{\geq k}(x)$ we have that $\mathbf{F}'_j(x)$ is complementary to $\mathbf{F}_{\geq j+1}(x)$.

By Lemma 8.2, $\mathbf{F}_{\geq j}$ is g_t-equivariant, and therefore, by the multiplicative ergodic theorem applied to $\mathbf{F}_{\geq j}$, $\mathbf{F}_{\geq j}$ is the direct sum of its Lyapunov subspaces. Therefore, in view of (8.3), for almost all $y \in X_0$,

$$\mathbf{F}_{\geq j}(y) = \big(\mathbf{F}_{\geq j}(y) \cap \mathbf{V}_{\leq j}(y)\big) \oplus \big(\mathbf{F}_{\geq j}(y) \cap \mathbf{V}_{>j}(y)\big). \tag{8.11}$$

Since $\mathbf{F}'_j(x) \subset \mathbf{F}_{\geq j}(x)$, we have by Lemma 8.2, $(u)_* \mathbf{v}' \in \mathbf{F}_{\geq j}(ux)$ for almost all $u \in \mathcal{B}(x)$. By the definition of $\mathbf{F}_{\geq j+1}(x)$, since $\mathbf{v}' \notin \mathbf{F}_{\geq j+1}(x)$, for almost all u if we decompose using (8.11),

$$(u)_* \mathbf{v}' = \mathbf{v}_u + \mathbf{w}_u, \qquad \mathbf{v}_u \in \mathbf{F}_{\geq j}(ux) \cap \mathbf{V}_{\leq j}(ux), \qquad \mathbf{w}_u \in \mathbf{F}_{\geq j}(ux) \cap \mathbf{V}_{>j}(ux),$$

then $\mathbf{v}_u \neq 0$. Since by definition $\mathbf{F}_{\geq j}(ux) \cap \mathbf{V}_{\leq j}(ux) = \mathbf{E}_j(ux)$ we have $\mathbf{v}_u \in \mathbf{E}_j(ux)$. Let

$$E_n(x) = \Big\{ \mathbf{v}' \in \mathbb{P}\big(\mathbf{F}'(x)\big) : \left| \left\{ u \in \mathcal{B}(x) \ : \ \|\mathbf{v}_u\| \geq \frac{1}{n}\|\mathbf{v}'\| \right\} \right| > (1 - \delta/2)|\mathcal{B}(x)| \Big\}.$$

Then the $E_n(x)$ are an increasing family of open sets, and $\bigcup_{n=1}^{\infty} E_n(x) = \mathbb{P}(\mathbf{F}'_j(x))$. Since $\mathbb{P}(\mathbf{F}'_j(x))$ is compact, there exists $n(x)$ such that $E_{n(x)}(x) = \mathbb{P}(\mathbf{F}'_j(x))$. We can now choose

$K'_{01} \subset \Phi$ with $\nu(K'_{01}) > 1 - \delta/2$ such that for $x \in K'_{01}$, $n(x) < 1/\beta(\delta)$. This shows that for $x \in K'_{01}$, for any $\mathbf{v}' \in \mathbb{P}(\mathbf{F}'_j(x))$, for $(1-\delta/2)$-fraction of $u \in \mathcal{B}(x)$ we have $\|\mathbf{v}_u\| > \beta(\delta)\|\mathbf{v}'\|$.

To prove the final estimate note that there exists a set K''_{01} with $\nu(K''_{01}) > 1 - \delta/2$ and a constant $C(\delta)$ such that for all $x \in K''_{01}$ and at least $(1-\delta/2)$-fraction of $u \in \mathcal{B}(x)$, we have $\|(u)_* \mathbf{v}'\| \le C(\delta)\|\mathbf{v}'\|$. Let $K_{01} = K'_{01} \cap K''_{01}$. Then, for at least $(1-\delta)$-fraction of $u \in \mathcal{B}(x)$, we have

$$\|\mathbf{w}_u\| \le \|(u)_* \mathbf{v}'\| \le C(\delta)\|\mathbf{v}'\| \le C(\delta)\beta(\delta)^{-1}\|\mathbf{v}_u\|. \qquad \square$$

Proposition **8.5.**

(a) *For every $\delta > 0$ there exists $K \subset X_0$ of measure at least $1-\delta$ and a number $L_2(\delta) > 0$ such that the following holds: Suppose $x \in K$, $\mathbf{v} \in \mathbf{H}(x)$. Then, for any $L' > L_2(\delta)$ there exists $L' < t < 2L'$ such that for at least $(1-\delta)$-fraction of $u \in \mathcal{B}(g_{-t}x)$,*

$$d\left(\frac{(g_s)_*(u)_*(g_{-t})_*\mathbf{v}}{\|(g_s)_*(u)_*(g_{-t})_*\mathbf{v}\|}, \mathbf{E}(g_s u g_{-t}x)\right) \le C(\delta)e^{-\alpha t},$$

where $s > 0$ is such that

$$\|(g_s)_*(u)_*(g_{-t})_*\mathbf{v}\| = \|\mathbf{v}\|, \tag{8.12}$$

and α depends only on the Lyapunov spectrum.

(b) *There exists $\epsilon' > 0$ (depending only on the Lyapunov spectrum) and for every $\delta > 0$ a compact set K'' with $\nu(K'') > 1 - c(\delta)$ where $c(\delta) \to 0$ as $\delta \to 0$ such that the following holds: Suppose there exist arbitrarily large $t > 0$ with $g_{-t}x \in K''$ so that for at least $(1-\delta)$-fraction of $u \in \mathcal{B}(x)$, the number $s > 0$ satisfying (8.12), also satisfies*

$$s \ge \left(1 - \epsilon'\right)t. \tag{8.13}$$

Then $\mathbf{v} \in \mathbf{E}(x)$.

Proof. — Let $\epsilon > 0$ be smaller than one third of the difference between any two Lyapunov exponents for the action on $\mathbf{H}$. By the Osceledets multiplicative ergodic theorem, there exists a compact subset $K_1 \subset X_0$ with $\nu(K_1) > 1 - \delta^2$ and $L > 0$ such that for $x \in K_1$ and all j and all $t > L$,

$$\|(g_t)_*\mathbf{v}\| \le e^{(\lambda_j + \epsilon)t}\|\mathbf{v}\|, \qquad \mathbf{v} \in \mathbf{V}_{\ge j}(x)$$

and

$$\|(g_t)_*\mathbf{v}\| \ge e^{(\lambda_j - \epsilon)t}\|\mathbf{v}\|, \qquad \mathbf{v} \in \mathbf{V}_{\le j}(x).$$

By Fubini's theorem there exists $K_1^* \subset X_0$ with $\nu(K_1^*) > 1 - 2\delta$ such that for $x \in K_1^*$,

$$\left|\left\{u \in \mathcal{B}(x) \;:\; ux \in K_1\right\}\right| \ge (1 - \delta/2)|\mathcal{B}(x)|.$$

Let $K'' = K_{01} \cap K_1^*$, where K_{01} is as in Lemma 8.4 (with δ replaced by $\delta/2$). Let K, $L_2(\delta)$ be such that for all $x \in K$ and all $L' > L_2$, there exists t with $L' < t < 2L'$ and $g_{-t}x \in K''$. Write

$$(\mathbf{8.14}) \qquad (g_{-t})_* \mathbf{v} = \sum_{j \in \Lambda'} \mathbf{v}'_j, \qquad \mathbf{v}'_j \in \mathbf{F}'_j(g_{-t}x).$$

We have $g_{-t}x \in K_{01} \cap K_1^*$. Suppose $u \in Q_{01}(g_{-tx})$ and $ug_{-t}x \in K_1$. Then, by Lemma 8.4, we have

$$(\mathbf{8.15}) \qquad (u)_*(g_{-t})_* \mathbf{v} = \sum_{j \in \Lambda'} (\mathbf{v}_j + \mathbf{w}_j),$$

where $\mathbf{v}_j \in \mathbf{E}_j(ug_{-t}x)$, $\mathbf{w}_j \in \mathbf{V}_{>j}(ug_{-t}x)$, and for all $j \in \Lambda'$,

$$(\mathbf{8.16}) \qquad \|\mathbf{v}_j\| \geq \beta'(\delta)\|\mathbf{w}_j\|.$$

Then,

$$\|(g_s)_* \mathbf{w}_j\| \leq e^{(\lambda_{j+1}+\epsilon)s}\|\mathbf{w}_j\|,$$

and,

$$(\mathbf{8.17}) \qquad \|(g_s)_* \mathbf{v}_j\| \geq e^{(\lambda_j-\epsilon)s}\|\mathbf{v}_j\| \geq e^{(\lambda_j-\epsilon)s}\beta'(\delta)\|\mathbf{w}_j\|.$$

Thus, for all $j \in \Lambda'$,

$$\|(g_s)_* \mathbf{w}_j\| \leq e^{-(\lambda_j-\lambda_{j+1}+2\epsilon)s}\beta'(\delta)^{-1}\|(g_s)_* \mathbf{v}_j\|.$$

Since $(g_s)_* \mathbf{v}_j \in \mathbf{E}$ and using part (a) of Proposition 4.15, we get (a) of Proposition 8.5.

To prove (b), suppose $\mathbf{v} \notin \mathbf{E}(x)$. We may write

$$\mathbf{v} = \sum_{i \in \Lambda'} \hat{\mathbf{v}}_i, \qquad \hat{\mathbf{v}}_i \in \mathbf{F}'_i(x)$$

Let j be minimal such that $\hat{\mathbf{v}}_j \notin \mathbf{E}_j(x)$. Let $k > j$ be such that $\mathbf{F}_{\geq k}(x) \subset \mathbf{F}_{\geq j}(x)$ is the subspace preceding $\mathbf{F}_{\geq j}(x)$ in (8.10). Then, $\mathbf{F}_{\geq i}(x) = \mathbf{F}_{\geq j}(x)$ for $k+1 \leq i \leq j$.

Since $\hat{\mathbf{v}}_j \notin \mathbf{E}_j(x)$, $\hat{\mathbf{v}}_j$ must have a component in $\mathcal{V}_i(\mathbf{H})(x)$ for some $i \geq j+1$. Therefore, by looking only at the component in $\mathcal{V}_i(\mathbf{H})$, we get

$$\|(g_{-t})_* \mathbf{v}\| \geq C(\mathbf{v})e^{-(\lambda_{j+1}+\epsilon)t}.$$

Also since $\mathbf{F}_{\geq k}$ is g_t-equivariant we have $\mathbf{F}_{\geq k}(x) = \bigoplus_m \mathbf{F}_{\geq k}(x) \cap \mathcal{V}_m(\mathbf{H})$. Note that by the multiplicative ergodic theorem, the restriction of g_{-t} to $\mathcal{V}_i(\mathbf{H})$ is of the form $e^{-\lambda_i t}h_t$, where $\|h_t\| = O(e^{\epsilon t})$. Therefore (again by looking only at the component in $\mathcal{V}_i(\mathbf{H})$ and using Proposition 4.15(a)), we get

$$d\big((g_{-t})_* \mathbf{v}, \mathbf{F}_{\geq k}(g_{-t}x)\big) \geq C(\mathbf{v})e^{-(\lambda_{j+1}+2\epsilon)t}.$$

(Here and below, $d(\cdot,\cdot)$ denotes the distance on $\mathbf{H}(x)$ given by the dynamical norm $\|\cdot\|_x$.) Therefore (since $(g_{-t})_*\mathbf{v} \in \mathbf{F}_{\geq j}(g_{-t}x)$), we see that if we decompose $(g_{-t})_*\mathbf{v}$ as in (8.14), we get

$$\|\mathbf{v}'_j\| \geq C(\mathbf{v})e^{-(\lambda_{j+1}+2\epsilon)t}.$$

We now decompose $(u)_*(g_{-t})_*\mathbf{v}$ as in (8.15). Then, from (8.16) and (8.17),

$$\|(g_s)_*\mathbf{v}_j\| \geq e^{(\lambda_j-\epsilon)s}\|\mathbf{v}_j\| \geq e^{(\lambda_j-\epsilon)s}\beta(\delta)\|\mathbf{v}'_j\| \geq e^{(\lambda_j-\epsilon)s}\beta(\delta)C(\mathbf{v})e^{-(\lambda_{j+1}+2\epsilon)t}. \tag{8.18}$$

If s satisfies (8.12), then $\|(g_s)_*\mathbf{v}_j\| = O(1)$. Therefore, in view of (8.18),

$$e^{(\lambda_j-\epsilon)s}e^{-(\lambda_{j+1}+2\epsilon)t} \leq c = c(\mathbf{v},\delta).$$

Therefore,

$$s \leq \frac{(\lambda_{j+1}+2\epsilon)t + \log c(\mathbf{v},\delta)}{(\lambda_j-\epsilon)}.$$

Since $\lambda_j > \lambda_{j+1}$, this contradicts (8.13) if ϵ is sufficiently small and t is sufficiently large. □

9. The action of the cocycle on E

In this section, we work on the finite cover X defined in Section 4.6. Recall that if $f(\cdot)$ is an object defined on X_0, then for $x \in X$ we write $f(x)$ instead of $f(\sigma_0(x))$ (where $\sigma_0 : X \to X_0$ is the covering map).

In this section and in Section 10, assertions will hold at best for a.e $x \in X$, and never for all $x \in X$. This will be sometimes suppressed from the statements of the lemmas.

9.1. *The Jordan canonical form of the cocycle on* $\mathbf{E}(x)$. — We consider the action of the cocycle on $\mathbf{E}$. The Lyapunov exponents are λ_i, $i \in \Lambda'$. We note that by Lemma 8.2, the bundle $\mathbf{E}$ admits the equivariant measurable flat U^+-connection given by the maps $(u)_* : \mathbf{E}(x) \to \mathbf{E}(y)$, where $(u)_*$ is as in Lemma 6.6. This connection satisfies the condition (4.5), since by Lemma 8.2, $(u)_*\mathbf{E}_j(x) = \mathbf{E}_j(y)$. For each $i \in \Lambda'$, we have the maximal flag as in Lemma 4.3,

$$\{0\} \subset \mathbf{E}_{i1}(x) \subset \cdots \subset \mathbf{E}_{i,n_i}(x) = \mathbf{E}_i(x). \tag{9.1}$$

Let Λ'' denote the set of pairs ij which appear in (9.1). By Proposition 4.12 and Remark 4.13, we have for a.e. $u \in \mathcal{B}(x)$,

$$(u)_*\mathbf{E}_{ij}(x) = \mathbf{E}_{ij}(ux).$$

Let $\|\cdot\|_x$ and $\langle\cdot,\cdot\rangle_x$ denote the restriction to $\mathbf{E}(x)$ of the norm and inner product on $\mathbf{H}(x)$ defined in Section 4.7 and Section 6. (We will often omit the subscript from $\langle\cdot,\cdot\rangle_x$

and $\|\cdot\|_x$.) Then, the distinct $\mathbf{E}_i(x)$ are orthogonal. For each $ij \in \Lambda''$ let $\mathbf{E}'_{ij}(x)$ be the orthogonal complement (relative to the inner product $\langle\cdot,\cdot\rangle_x$) to $\mathbf{E}_{i,j-1}(x)$ in $\mathbf{E}_{ij}(x)$.

Then, by Proposition 4.15, we can write, for $\mathbf{v} \in \mathbf{E}'_{ij}(x)$,

$$(g_t)_*\mathbf{v} = e^{\lambda_{ij}(x,t)}\mathbf{v}' + \mathbf{v}'', \tag{9.2}$$

where $\mathbf{v}' \in \mathbf{E}'_{ij}(g_t x)$, $\mathbf{v}'' \in \mathbf{E}_{i,j-1}(g_t x)$, and $\|\mathbf{v}'\| = \|\mathbf{v}\|$. Hence (since $\mathbf{v}'$ and $\mathbf{v}''$ are orthogonal),

$$\|(g_t)_*\mathbf{v}\| \geq e^{\lambda_{ij}(x,t)}\|\mathbf{v}\|.$$

In view of Proposition 4.15 there exists a constant $\kappa > 1$ such that for a.e $x \in \mathrm{X}$ and for all $\mathbf{v} \in \mathbf{E}(x)$ and all $t \geq 0$,

$$e^{\kappa^{-1}t}\|\mathbf{v}\| \leq \|(g_t)_*\mathbf{v}\| \leq e^{\kappa t}\|\mathbf{v}\|. \tag{9.3}$$

Lemma **9.1.** — *For a.e. $x \in \mathrm{X}$ and for a.e. $y = ux \in \mathcal{B}[x]$, the connection $(u)_* : \mathbf{E}(x) \to \mathbf{E}(y)$ agrees with the restriction to $\mathbf{E}$ of the connection $\mathbf{P}^+(x,y)$ induced from the map $\mathrm{P}^+(x,y)$ defined in Section 4.2.*

Proof. — Let $\mathcal{V}_{\leq i}(x) = \mathcal{V}_{\leq i}(\mathrm{H}^1)(x)$ and $\mathcal{V}_i(x) = \mathcal{V}_i(\mathrm{H}^1)(x)$, where $\mathcal{V}_{\leq i}(\mathrm{H}^1)(x)$ and $\mathcal{V}_i(\mathrm{H}^1)(x)$ are as in Section 4.1. Consider the definition (6.12) of u_* in Section 6. For a fixed $\mathrm{Y} = \log u \in \mathrm{Lie}(\mathrm{U}^+)(x)$ and $\mathrm{M} \in \mathcal{H}_{++}(x)$, let $h : \mathrm{W}^+(x) \to \mathrm{W}^+(ux)$ be given by

$$h(v) = \exp\big((\mathrm{I}+\mathrm{M})\mathrm{Y}\big)(x+v) - \exp(\mathrm{Y})x.$$

From the form of h, we see that $h(\mathcal{V}_{\leq i}(x)) = \mathcal{V}_{\leq i}(ux)$, and also, h induces the identity map on $\mathcal{V}_{\leq i}(x)/\mathcal{V}_{<i}(x) = \mathcal{V}_{\leq i}(ux)/\mathcal{V}_{<i}(ux)$. Thus, for $v \in \mathcal{V}_i(x)$,

$$h(v) \in \mathrm{P}^+(x,ux)v + \mathcal{V}_{<i}(ux).$$

Similarly, $\mathrm{M}'' \equiv tr(x,ux) \circ \mathrm{M} \circ tr(ux,x)$ agrees with M up to higher Lyapunov exponents. Then, in view of (6.12), (6.18) and Lemma 6.8, for $\mathbf{v} \in \mathbf{E}_i(x)$,

$$(u)_*\mathbf{v} \in \mathbf{P}^+(x,ux)\mathbf{v} + \mathbf{V}_{<i}(ux).$$

But, for $\mathbf{v} \in \mathbf{E}_i(x)$, $(u)_*\mathbf{v} \in \mathbf{E}_i(ux)$ (and thus has no component in $\mathbf{V}_{<i}(ux)$). Hence, for all $\mathbf{v} \in \mathbf{E}_i(x)$, we have $(u)_*\mathbf{v} = \mathbf{P}^+(x,ux)\mathbf{v}$. □

9.2. *Time changes.*

The flows g_t^{ij} and the time changes $\hat{\tau}_{ij}(x,t)$. — We define the time changed flow g_t^{ij} so that (after the time change) the cocycle $\lambda_{ij}(x,t)$ of (9.2) becomes $\lambda_i t$. We write $g_t^{ij}x = g_{\hat{\tau}_{ij}(x,t)}x$. Then, by construction, $\lambda_{ij}(x, \hat{\tau}_{ij}(x,t)) = \lambda_i t$. We note the following:

Lemma **9.2.** — *Suppose $y \in \mathfrak{B}_0[x]$. Then for any $ij \in \Lambda''$ and any $t > 0$,*

$$g_{-t}^{ij}y \in \mathfrak{B}_0\big[g_{-t}^{ij}x\big].$$

Proof. — This follows immediately from property (e) of Proposition 4.15, and the definition of the flow g_{-t}^{ij}. □

In view of Proposition 4.15, we have

$$\frac{1}{\kappa}|t-t'| \le \left|\hat{\tau}_{ij}(x,t) - \hat{\tau}_{ij}\big(x,t'\big)\right| \le \kappa|t-t'| \tag{9.4}$$

where κ depends only on the Lyapunov spectrum.

9.3. *The foliations $\mathcal{F}_{ij}$, $\mathcal{F}_{\mathbf{v}}$ and the parallel transport* $\mathrm{R}(x,y)$. — For $x \in \tilde{\mathrm{X}}$, let

$$\mathrm{G}[x] = \big\{g_s u g_{-t}x \;:\; t \ge 0,\, s \ge 0,\, u \in \mathcal{B}(g_{-t}x)\big\} \subset \tilde{\mathrm{X}}.$$

For $y = g_s u g_{-t}x \in \mathrm{G}[x]$, let

$$\mathrm{R}(x,y) = (g_s)_*(u)_*(g_{-t})_*.$$

Here $(g_s)_*$ is as in (6.16) and $(u)_* : \mathbf{H}(g_{-t}x) \to \mathbf{H}(ug_{-t}x)$ is as in Lemma 6.6. It is easy to see using Lemma 6.7 that $\mathrm{R}(x,y) : \mathbf{H}(x) \to \mathbf{H}(y)$ depends only on x, y and not on the choices of t, u, s. We will usually consider $\mathrm{R}(x,y)$ as a map from $\mathbf{E}(x) \to \mathbf{E}(y)$.

In view of (9.2), Lemma 9.1 and Proposition 4.15(e) and (f), we have, for $\mathbf{v} \in \mathbf{E}'_{ij}(x)$, and any $y = g_s u g_{-t}x \in \mathrm{G}[x]$,

$$\mathrm{R}(x,y)\mathbf{v} = e^{\lambda_{ij}(x,y)}\mathbf{v}' + \mathbf{v}'' \tag{9.5}$$

where $\mathbf{v}' \in \mathbf{E}'_{ij}(y)$, $\mathbf{v}'' \in \mathbf{E}_{i,j-1}(y)$, and $\|\mathbf{v}'\| = \|\mathbf{v}\|$. In (9.5), we have

$$\lambda_{ij}(x,y) = \lambda_{ij}(x,-t) + \lambda_{ij}(ug_{-t}x, s). \tag{9.6}$$

Notational convention. — We sometimes use the notation $R(x, y)$ when $x \in X$ (instead of $\tilde{X}$) and $y \in G[x]$.

For $x \in \tilde{X}$ and $ij \in \Lambda''$, let $\mathcal{F}_{ij}[x]$ denote the set of $y \in G[x]$ such that there exists $\ell \geq 0$ so that

$$(\mathbf{9.7}) \qquad g_{-\ell}^{ij} y \in \mathcal{B}\big[g_{-\ell}^{ij} x\big].$$

By Lemma 9.2, if (9.7) holds for some ℓ, it also holds for any bigger ℓ. Alternatively,

$$\mathcal{F}_{ij}[x] = \big\{g_{\ell}^{ij} u g_{-\ell}^{ij} x \;:\; \ell \geq 0,\; u \in \mathcal{B}\big(g_{-\ell}^{ij} x\big)\big\} \subset \tilde{X}.$$

As above, when $x \in X$, we can think of the leaf of the foliation $\mathcal{F}_{ij}[x]$ as a subset of X (not $\tilde{X}$).

In view of (9.6), it follows that

$$(\mathbf{9.8}) \qquad \lambda_{ij}(x, y) = 0 \quad \text{if } y \in \mathcal{F}_{ij}[x].$$

We refer to the sets $\mathcal{F}_{ij}[x]$ as *leaves*. Locally, the leaf $\mathcal{F}_{ij}[x]$ through x is a piece of $U^+[x]$. More precisely, for $y \in \mathcal{F}_{ij}[x]$,

$$\mathcal{F}_{ij}[x] \cap \mathfrak{B}_0[y] \subset U^+[y].$$

Then, for any compact subset $A \subset \mathcal{F}_{ij}[x]$ there exists ℓ large enough so that $g_{-\ell}^{ij}(A)$ is contained in a set of the form $\mathcal{B}[z] \subset U^+[z]$. Then the same holds for $g_{-t}^{ij}(A)$, for any $t > \ell$.

Recall (from the start of Section 6) that the sets $\mathcal{B}[x]$ support a "Lebesgue measure" $|\cdot|$, namely the pushforward of the Haar measure on $U^+(x)/(U^+(x) \cap Q_{++}(x))(x)$ to $\mathcal{B}[x]$ under the map $u \to ux$. (Recall that $Q_{++}(x)$ is the stabilizer of x in the affine group $\mathcal{G}_{++}(x)$.) As a consequence, the leaves $\mathcal{F}_{ij}[x]$ also support a Lebesgue measure (defined up to normalization), which we also denote by $|\cdot|$. More precisely, if $A \subset \mathcal{F}_{ij}[x]$ and $B \subset \mathcal{F}_{ij}[x]$ are compact subsets, we define

$$(\mathbf{9.9}) \qquad \frac{|A|}{|B|} \equiv \frac{|g_{-\ell}^{ij}(A)|}{|g_{-\ell}^{ij}(B)|},$$

where ℓ is chosen large enough so that both $g_{-\ell}^{ij}(A)$ and $g_{-\ell}^{ij}(B)$ are contained in a set of the form $\mathcal{B}[z]$, $z \in X$. It is clear that if we replace ℓ by a larger number, the right-hand-side of (9.9) remains the same.

We define the "balls" $\mathcal{F}_{ij}[x, \ell] \subset \mathcal{F}_{ij}[x]$ by

$$(\mathbf{9.10}) \qquad \mathcal{F}_{ij}[x, \ell] = \big\{ y \in \mathcal{F}_{ij}[x] \;:\; g_{-\ell}^{ij} y \in \mathcal{B}\big[g_{-\ell}^{ij} x\big]\big\}.$$

Lemma **9.3.** — *Suppose $x \in \tilde{X}$ and $y \in \mathcal{F}_{ij}[x]$. Then, for ℓ large enough,*

$$\mathcal{F}_{ij}[x, \ell] = \mathcal{F}_{ij}[y, \ell].$$

Proof. — Suppose $y \in \mathcal{F}_{ij}[x]$. Then, for ℓ large enough, $g_{-\ell}^{ij} y \in \mathcal{B}[g_{-\ell}^{ij} x]$, and then $\mathcal{B}[g_{-\ell}^{ij} y] = \mathcal{B}[g_{-\ell}^{ij} x]$. □

The "flows" $g_t^{\mathbf{v}}$. — Suppose $x \in \tilde{\mathrm{X}}$ and $\mathbf{v} \in \mathbf{E}(x)$. Let $g_t^{\mathbf{v}} x = g_{\hat{\tau}_{\mathbf{v}}(x,t)} x$, where the time change $\hat{\tau}_{\mathbf{v}}(x, t)$ is chosen so that

$$\left\| \left(g_t^{\mathbf{v}}\right)_* \mathbf{v} \right\|_{g_t^{\mathbf{v}} x} = e^t \|\mathbf{v}\|_x.$$

(Note that we are not defining $g_t^{\mathbf{v}} y$ for $y \neq x$.) We have, for $x \in \tilde{\mathrm{X}}$,

$$g_{t+s}^{\mathbf{v}} x = g_s^{(g_t)_* \mathbf{v}} g_t^{\mathbf{v}} x.$$

By (9.3), (9.4) holds for $\hat{\tau}_{\mathbf{v}}$ instead of $\hat{\tau}_{ij}$.

For $y \in \mathrm{G}[x]$ and $\ell \in \mathbb{R}$, let

$$\tilde{g}_{-\ell}^{\mathbf{v},x} = g_{-\ell}^{\mathbf{w}} y, \quad \text{where } \mathbf{w} = \mathrm{R}(x, y)\mathbf{v}. \tag{9.11}$$

(When there is no potential for confusion about the point x and the vector $\mathbf{v}$ used, we denote $\tilde{g}_{-\ell}^{\mathbf{v},x}$ by $\tilde{g}_{-\ell}$.) Note that Lemma 9.2 still holds if g_{-t}^{ij} is replaced by $\tilde{g}_{-t}^{\mathbf{v},x}$.

The foliations $\mathcal{F}_{\mathbf{v}}$. — For $\mathbf{v} \in \mathbf{E}(x)$ we can define the foliations $\mathcal{F}_{\mathbf{v}}[x]$ and the "balls" $\mathcal{F}_{\mathbf{v}}[x, \ell]$ as in (9.7) and (9.10), with $\tilde{g}_{-t}^{\mathbf{v},x}$ replacing the role of g_{-t}^{ij}.

For $y \in \mathcal{F}_{\mathbf{v}}[x]$, we have

$$\mathcal{F}_{\mathbf{v}}[x] = \mathcal{F}_{\mathbf{w}}[y], \quad \text{where } \mathbf{w} = \mathrm{R}(x, y)\mathbf{v}.$$

We can define the measure (up to normalization) $|\cdot|$ on $\mathcal{F}_{\mathbf{v}}[x, \ell]$ as in (9.9). Lemma 9.3 holds for $\mathcal{F}_{\mathbf{v}}[x]$ without modifications.

The following follows immediately from the construction:

Lemma **9.4.** — *For a.e. $x \in \tilde{\mathrm{X}}$, any $\mathbf{v} \in \mathbf{E}(x)$, and a.e. $y \in \mathcal{F}_{\mathbf{v}}[x]$, we have*

$$\|\mathrm{R}(x, y)\mathbf{v}\|_y = \|\mathbf{v}\|_x.$$

9.4. *A maximal inequality.*

Lemma **9.5.** — *Suppose $\mathrm{K} \subset \mathrm{X}$ with $\nu(\mathrm{K}) > 1 - \delta$. Then, for any $\theta' > 0$ there exists a subset $\mathrm{K}^* \subset \mathrm{X}$ with $\nu(\mathrm{K}^*) > 1 - 2\kappa^2\delta/\theta'$ such that for any $x \in \mathrm{K}^*$ and any $\ell > 0$,*

$$|\mathcal{F}_{ij}[x, \ell] \cap \mathrm{K}| > \left(1 - \theta'\right) |\mathcal{F}_{ij}[x, \ell]|. \tag{9.12}$$

Proof. — For $t > 0$ let

$$\mathcal{B}_t^{ij}[x] = g_{-t}^{ij}\big(\mathfrak{B}_0\big[g_t^{ij}x\big] \cap U^+\big[g_t^{ij}x\big]\big) = \mathcal{B}_\tau[x],$$

where τ is such that $g_\tau x = g_t^{ij}x$. Let $s > 0$ be arbitrary. Let $K_s = g_{-s}^{ij}K$. Then $\nu(K_s) > 1 - \kappa\delta$. Then, by Lemma 6.3, there exists a subset K_s' with $\nu(K_s') \geq (1 - 2\kappa\delta/\theta')$ such that for $x \in K_s'$ and all $t > 0$,

$$|K_s \cap \mathcal{B}_t^{ij}[x]| \geq \big(1 - \theta'/2\big)|K_s|.$$

Let $K_s^* = g_s^{ij}K_s'$, and note that $g_s^{ij}\mathcal{B}_t^{ij}[x] = \mathcal{F}_{ij}[g_s^{ij}x, s - t]$. Then, for all $x \in K_s^*$ and all $0 < s - t < s$,

$$|\mathcal{F}_{ij}[x, s - t] \cap K| \geq \big(1 - \theta'/2\big)|\mathcal{F}_{ij}[x, s - t]|.$$

We have $\nu(K_s^*) \geq (1 - 2\kappa^2\delta/\theta')$. Now take a sequence $s_n \to \infty$, and let K^* be the set of points which are in infinitely many $K_{s_n}^*$. □

10. Bounded subspaces and synchronized exponents

Recall that Λ'' indexes the "fine Lyapunov spectrum" on $\mathbf{E}$. In this section we define an equivalence relation called "synchronization" on Λ''; the equivalence class of $ij \in \Lambda''$ is denoted by $[ij]$ and the set of equivalence classes is denoted by $\tilde{\Lambda}$. For each $ij \in \Lambda''$ we define a g_t-equivariant and locally $(u)_*$-equivariant (in the sense of Lemma 6.6(b)) subbundle $\mathbf{E}_{ij,bdd}$ of the bundle $\mathbf{E}_i \equiv \mathcal{V}_i(\mathbf{E})$ and we define

$$\mathbf{E}_{[ij],bdd}(x) = \sum_{kr \in [ij]} \mathbf{E}_{kr,bdd}(x).$$

In fact we will show that there exists a subset $[ij]' \subset [ij]$ such that

$$\mathbf{E}_{[ij],bdd}(x) = \bigoplus_{kr \in [ij]'} \mathbf{E}_{kr,bdd}(x). \tag{10.1}$$

Then, we claim that the following three propositions hold:

Proposition **10.1.** — *There exists $\theta > 0$ depending only on ν and $n \in \mathbb{N}$ depending only on the dimension of* X *such that the following holds: for every $\delta > 0$ and every $\eta > 0$, there exists a subset $K = K(\delta, \eta)$ of measure at least $1 - \delta$ and $L_0 = L_0(\delta, \eta) > 0$ such that the following holds: Suppose $x \in X$, $\mathbf{v} \in \mathbf{E}(x)$, $L \geq L_0$, and*

$$|g_{[-1,1]}K \cap \mathcal{F}_{\mathbf{v}}[x, L]| \geq \big(1 - (\theta/2)^{n+1}\big)|\mathcal{F}_{\mathbf{v}}[x, L]|.$$

Then, for at least $(\theta/2)^n$*-fraction of* $y \in \mathcal{F}_{\mathbf{v}}[x, L]$,

$$d\left(\frac{R(x,y)\mathbf{v}}{\|R(x,y)\mathbf{v}\|}, \bigcup_{ij\in\tilde{\Lambda}} \mathbf{E}_{[ij],bdd}(y)\right) < \eta.$$

Proposition **10.2.** — *There exists a function* $C_3 : X \to \mathbb{R}^+$ *finite almost everywhere so that for all* $x \in \tilde{X}$*, for all* $y \in \mathcal{F}_{ij}[x]$*, for all* $\mathbf{v} \in \mathbf{E}_{[ij],bdd}(x)$,

$$C_3(x)^{-1}C_3(y)^{-1}\|\mathbf{v}\| \le \|R(x,y)\mathbf{v}\| \le C_3(x)C_3(y)\|\mathbf{v}\|.$$

(Recall from Section 2.2 that by $C_3(x)$ *we mean* $C_3(\pi(x))$*.)*

Proposition **10.3.** — *There exists* $\theta > 0$ *(depending only on* ν*) and a subset* $\Psi \subset X$ *with* $\nu(\Psi) = 1$ *such that the following holds:*

Suppose $x \in \Psi$*,* $\mathbf{v} \in \mathbf{H}(x)$*, and there exists* $C > 0$ *such that for all* $\ell > 0$*, and at least* $(1-\theta)$*-fraction of* $y \in \mathcal{F}_{ij}[x, \ell]$,

$$\|R(x,y)\mathbf{v}\| \le C\|\mathbf{v}\|.$$

Then, $\mathbf{v} \in \mathbf{E}_{[ij],bdd}(x)$.

Proposition 10.1 is what allows us to choose u so that there exists u' such that the vector in $\mathbf{H}$ associated to the difference between the generalized subspaces $U^+[g_t u' q_1']$ and $U^+[g_t u q_1]$ points close to a controlled direction, i.e. close to $\mathbf{E}_{[ij],bdd}(g_t u q_1)$. This allows us to address "Technical Problem #3" from Section 2.3. Then, Proposition 10.2 and Proposition 10.3 are used in Section 11 to define and control conditional measures f_{ij} associated to each $[ij] \in \tilde{\Lambda}$, so we can implement the outline in Section 2.3. We note that it is important for us to define a family of subspaces so that all three propositions hold.

The number $\theta > 0$, the synchronization relation and the subspaces $\mathbf{E}_{ij,bdd}$ are defined in Section 10.1*. Also Proposition 10.1 is proved in Section 10.1*. Proposition 10.2 and Proposition 10.3 are proved in Section 10.2*. Both subsections may be skipped on first reading.

Example. — To completely understand the example below, it necessary to read at least Section 10.1*. However, we include it here to give some flavor of the construction.

Suppose we have a basis $\{\mathbf{e}_1(x), \mathbf{e}_2(x), \mathbf{e}_3(x), \mathbf{e}_4(x)\}$ for $\mathbf{E}(x)$, relative to which the cocycle has the form (for $y \in G[x]$):

$$R(x,y) = \begin{pmatrix} e^{\lambda_{11}(x,y)} & u_{12}(x,y) & 0 & 0 \\ 0 & e^{\lambda_{12}(x,y)} & 0 & 0 \\ 0 & 0 & e^{\lambda_{31}(x,y)} & 0 \\ 0 & 0 & 0 & e^{\lambda_{41}(x,y)} \end{pmatrix}.$$

Suppose $\mathbf{E}_1(x) = \mathbb{R}\mathbf{e}_1(x) \oplus \mathbb{R}\mathbf{e}_2(x)$ (so $\mathbf{e}_1$ and $\mathbf{e}_2$ correspond to the Lyapunov exponent λ_1), $\mathbf{E}_3(x) = \mathbb{R}\mathbf{e}_3(x)$, $\mathbf{E}_4(x) = \mathbb{R}\mathbf{e}_4(x)$ (so that $\mathbf{e}_3$ and $\mathbf{e}_4$ correspond to the Lyapunov exponents λ_3 and λ_4 respectively). Therefore the Lyapunov exponents λ_3 and λ_4 have multiplicity 1, while λ_1 has multiplicity 2.

Then, we have

$$\mathbf{E}_{31,bdd}(x) = \mathbb{R}\mathbf{e}_3(x), \qquad \mathbf{E}_{41,bdd}(x) = \mathbb{R}\mathbf{e}_4(x), \qquad \mathbf{E}_{11,bdd}(x) = \mathbb{R}\mathbf{e}_1(x).$$

(For example, if $y \in \mathcal{F}_{31}[x]$ then $\lambda_{31}(x,y) = 0$, so that by (9.5), $\|\mathrm{R}(x,y)\mathbf{e}_3\| = \|\mathbf{e}_3\|$.)

Now suppose that 31 and 41 are synchronized, but all other pairs are not synchronized. (See Definition 10.8 for the exact definition of synchronization, but roughly this means that $|\lambda_{41}(x,y)|$ is bounded as y varies over $\mathcal{F}_{31}[x]$, but for all other distinct pairs ij and kl, $|\lambda_{ij}(x,y)|$ is essentially unbounded as y varies over $\mathcal{F}_{kl}[x]$.) Then,

$$\mathbf{E}_{[31],bdd}(x) = \mathbb{R}\mathbf{e}_3(x) \oplus \mathbb{R}\mathbf{e}_4(x).$$

Depending on the boundedness behavior of $u_{12}(x,y)$ as y varies over $\mathcal{F}_{12}[x]$ we would have either

$$\mathbf{E}_{12,bdd}(x) = \{0\} \quad \text{or} \quad \mathbf{E}_{12,bdd}(x) = \mathbb{R}\mathbf{e}_2(x).$$

Since $[11]' = \{11\}$ and $[12]' = \{12\}$, we have $\mathbf{E}_{[11],bdd}(x) = \mathbf{E}_{11,bdd}(x)$ and $\mathbf{E}_{[12],bdd}(x) = \mathbf{E}_{12,bdd}(x)$.

10.1*. *Bounded subspaces and synchronized exponents.* — For $x \in \tilde{\mathrm{X}}$, $y \in \tilde{\mathrm{X}}$, let

$$\rho(x,y) = \begin{cases} |t| & \text{if } y = g_t x, \\ \infty & \text{otherwise.} \end{cases}$$

If $x \in \tilde{\mathrm{X}}$ and $\mathrm{E} \subset \tilde{\mathrm{X}}$, we let $\rho(x,\mathrm{E}) = \inf_{y \in \mathrm{E}} \rho(x,y)$.

Lemma **10.4.** — *For every $\eta > 0$ and $\eta' > 0$ there exists $h = h(\eta',\eta)$ such that the following holds: Suppose $\mathbf{v} \in \mathbf{E}_{ij}(x)$ and*

$$d\left(\frac{\mathbf{v}}{\|\mathbf{v}\|}, \mathbf{E}_{i,j-1}(x)\right) > \eta'.$$

Then if $y \in \mathcal{F}_{\mathbf{v}}[x]$ and

$$\rho\big(y, \mathcal{F}_{ij}[x]\big) > h$$

then

$$d\big(\mathrm{R}(x,y)\mathbf{v}, \mathbf{E}_{i,j-1}(y)\big) \le \eta\|\mathbf{v}\|.$$

Proof. — There exists $t \in \mathbb{R}$ such that $y' = g_t y \in \mathcal{F}_{ij}[x]$. Then

$$\rho\big(y, \mathcal{F}_{ij}[x]\big) = \rho\big(y, y'\big) = |t| > h.$$

We have the orthogonal decomposition $\mathbf{v} = \hat{\mathbf{v}} + \mathbf{w}$, where $\hat{\mathbf{v}} \in \mathbf{E}'_{ij}(x)$ and $\mathbf{w} \in \mathbf{E}_{i,j-1}(x)$. Then by (9.5) we have the orthogonal decomposition.

$$\begin{aligned} &\mathrm{R}\big(x, y'\big)\hat{\mathbf{v}} = e^{\lambda_{ij}(x,y')}\mathbf{v}' + \mathbf{w}', \\ &\quad \text{where } \mathbf{v}' \in \mathbf{E}'_{ij}\big(y'\big),\ \mathbf{w}' \in \mathbf{E}_{i,j-1}\big(y'\big),\ \|\hat{\mathbf{v}}\| = \|\mathbf{v}'\|. \end{aligned}$$

Since $\mathrm{R}(x, y')\mathbf{w} \in \mathbf{E}_{i,j-1}(y')$, we have

$$\big\|\mathrm{R}\big(x, y'\big)\mathbf{v}\big\|^2 = e^{2\lambda_{ij}(x,y')}\|\hat{\mathbf{v}}\|^2 + \big\|\mathbf{w}' + \mathrm{R}\big(x, y'\big)\mathbf{w}\big\|^2 \geq e^{2\lambda_{ij}(x,y')}\|\hat{\mathbf{v}}\|^2.$$

By (9.8), we have $\lambda_{ij}(x, y') = 0$. Hence,

$$\big\|\mathrm{R}\big(x, y'\big)\mathbf{v}\big\| \geq \|\hat{\mathbf{v}}\| \geq \eta'\|\mathbf{v}\|.$$

Since $y \in \mathcal{F}_{\mathbf{v}}[x]$, $\|\mathrm{R}(x, y)\mathbf{v}\| = \|\mathbf{v}\|$. Since $|t| > h$, we have either $t > h$ or $t < -h$. If $t < -h$, then by (9.3) and Lemma 9.4,

$$\|\mathbf{v}\| = \|\mathrm{R}(x, y)\mathbf{v}\| = \big\|(g_{-t})_*\mathrm{R}\big(x, y'\big)\mathbf{v}\big\| \geq e^{\kappa^{-1}h}\big\|\mathrm{R}\big(x, y'\big)\mathbf{v}\big\| \geq e^{\kappa^{-1}h}\eta'\|\mathbf{v}\|,$$

which is a contradiction if $h > \kappa \log(1/\eta')$. Hence we may assume that $t > h$. We have,

$$\mathrm{R}(x, y)\mathbf{v} = e^{\lambda_{ij}(x,y)}\mathbf{v}'' + \mathbf{w}''$$

where $\mathbf{v}'' \in \mathbf{E}'_{ij}(y)$ with $\|\mathbf{v}''\| = \|\hat{\mathbf{v}}\|$, and $\mathbf{w}'' \in \mathbf{E}_{i,j-1}(y)$. Hence,

$$d\big(\mathrm{R}(x, y)\mathbf{v}, \mathbf{E}_{i,j-1}(y)\big) = e^{\lambda_{ij}(x,y)}\|\hat{\mathbf{v}}\| \leq e^{\lambda_{ij}(x,y)}\|\mathbf{v}\|.$$

But,

$$\lambda_{ij}(x, y) = \lambda_{ij}\big(x, y'\big) + \lambda_{ij}\big(y', -t\big) \leq -\kappa^{-1}t$$

by (9.8) and Proposition 4.15. Therefore,

$$d\big(\mathrm{R}(x, y)\mathbf{v}, \mathbf{E}_{i,j-1}(y)\big) \leq e^{-\kappa^{-1}t}\|\mathbf{v}\| \leq e^{-\kappa^{-1}h}\|\mathbf{v}\|. \qquad \square$$

The bounded subspace. — Fix $\theta > 0$. (We will eventually choose θ sufficiently small depending only on the dimension.)

Definition **10.5.** — *Suppose* $x \in \tilde{X}$. *A vector* $\mathbf{v} \in \mathbf{E}_{ij}(x)$ *is called* (θ, ij)-bounded *if there exists* $C < \infty$ *such that for all* $\ell > 0$ *and for* $(1-\theta)$*-fraction of* $y \in \mathcal{F}_{ij}[x, \ell]$,

$$\|R(x, y)\mathbf{v}\| \le C\|\mathbf{v}\|. \tag{10.2}$$

Remark. — From the definition and (9.5), it is clear that every vector in $\mathbf{E}_{i1}(x)$ is $(\theta, i1)$-bounded for every θ. Indeed, we have $\mathbf{E}'_{i1} = \mathbf{E}_{i1}$, and $\lambda_{i1}(x, y) = 0$ for $y \in \mathcal{F}_{i1}[x]$, thus for $y \in \mathcal{F}_{i1}[x]$ and $\mathbf{v} \in E_{i1}(x)$, $\|R(x, y)\mathbf{v}\| = \|\mathbf{v}\|$.

Lemma **10.6.** — *Let* $n = \dim \mathbf{E}_{ij}(x)$ *(for a.e* x*). If there exists no non-zero* θ/n*-bounded vector in* $\mathbf{E}_{ij}(x) \setminus \mathbf{E}_{i,j-1}(x)$, *we set* $\mathbf{E}_{ij,bdd} = \{0\}$. *Otherwise, we define* $\mathbf{E}_{ij,bdd}(x) \subset \mathbf{E}_{ij}(x)$ *to be the linear span of the* θ/n*-bounded vectors in* $\mathbf{E}_{ij}(x)$. *This is a subspace of* $\mathbf{E}_{ij}(x)$, *and any vector in this subspace is* θ*-bounded. Also,*

(a) $\mathbf{E}_{ij,bdd}(x)$ *is* g_t*-equivariant, i.e.* $(g_t)_*\mathbf{E}_{ij,bdd}(x) = \mathbf{E}_{ij,bdd}(g_t x)$.
(b) *For almost all* $u \in \mathcal{B}(x)$, $\mathbf{E}_{ij,bdd}(ux) = (u)_*\mathbf{E}_{ij,bdd}(x)$.

Proof. — Let $\mathbf{E}_{ij,bdd}(x) \subset \mathbf{E}_{ij}(x)$ denote the linear span of all $(\theta/n, ij)$-bounded vectors. If $\mathbf{v}_1, \dots, \mathbf{v}_n$ are any n $(\theta/n, ij)$-bounded vectors, then there exists $C > 1$ such that for $1-\theta$ fraction of y in $\mathcal{F}_{ij}[x, L]$, (10.2) holds. But then (10.2) holds (with a different C) for any linear combination of the $\mathbf{v}_i$. This shows that any vector in $\mathbf{E}_{ij,bdd}(x)$ is (θ, ij)-bounded. To show that (a) holds, suppose that $\mathbf{v} \in \mathbf{E}_{ij}(x)$ is $(\theta/n, ij)$-bounded, and $t < 0$. In view of Lemma 8.2, it is enough to show that $\mathbf{v}' \equiv (g_t^{ij})_*\mathbf{v} \in \mathbf{E}_{ij}(g_t^{ij}x)$ is $(\theta/n, ij)$-bounded. (This would show that for $t < 0$, $(g_t^{ij})_*\mathbf{E}_{ij,bdd}(x) \subset \mathbf{E}_{ij,bdd}(g_t^{ij}x)$ which, in view of the ergodicity of the action of g_t, would imply (a).)

Let $x' = g_t^{ij}x$. By (9.3), there exists $C_1 = C_1(t)$ such that for all $z \in X$ and all $\mathbf{w} \in \mathbf{E}(z)$,

$$C_1^{-1}\|\mathbf{w}\| \le \left\|\left(g_t^{ij}\right)_*\mathbf{w}\right\| \le C_1\|\mathbf{w}\|. \tag{10.3}$$

Suppose $y \in \mathcal{F}_{ij}[x, L]$ satisfies (10.2). Let $y' = g_t^{ij}y$. Then $y' \in \mathcal{F}_{ij}[x']$. Let $\mathbf{v}' = (g_t^{ij})_*\mathbf{v}$. (See Figure 3.) Note that

$$R\big(x', y'\big)\mathbf{v}' = R\big(y, y'\big)R(x, y)R\big(x', x\big)\mathbf{v}' = R\big(y, y'\big)R(x, y)\mathbf{v}$$

hence by (10.3), (10.2), and again (10.3),

$$\left\|R\big(x', y'\big)\mathbf{v}'\right\| \le C_1\|R(x, y)\mathbf{v}\| \le C_1 C\|\mathbf{v}\| \le C_1^2 C\|\mathbf{v}'\|.$$

Hence, for $y \in \mathcal{F}_{ij}[x, L]$ satisfying (10.2), $y' = g_t^{ij}y \in \mathcal{F}_{ij}[x']$ satisfies

$$\left\|R\big(x', y'\big)\mathbf{v}'\right\| < CC_1^2\|\mathbf{v}'\|. \tag{10.4}$$

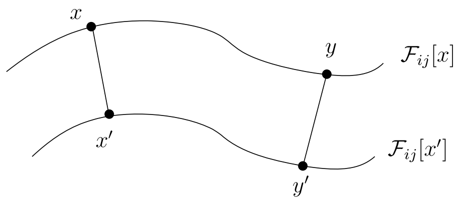

FIG. 3. — Proof of Lemma 10.6(a)

Therefore, since $\mathcal{F}_{ij}[g_t^{ij}x, L+t] = g_t^{ij}\mathcal{F}_{ij}[x, L]$, we have that for $1-\theta/n$ fraction of $y' \in \mathcal{F}_{ij}[x', L+t]$, (10.4) holds. Therefore, $\mathbf{v}'$ is $(\theta/n, ij)$-bounded. Thus, $\mathbf{E}_{ij,bdd}(x)$ is g_t-equivariant. This completes the proof of (a). Then (b) follows immediately from (a) since Lemma 9.3 implies that $\mathcal{F}_{ij}[ux, L] = \mathcal{F}_{ij}[x, L]$ for L large enough. □

Remark **10.7.** — Formally, from its definition, the subspace $\mathbf{E}_{ij,bdd}(x)$ depends on the choice of θ. It is clear that as we decrease θ, the subspace $\mathbf{E}_{ij,bdd}(x)$ decreases. In view of Lemma 10.6, there exists $\theta_0 > 0$ and $m \geq 0$ such that for all $\theta < \theta_0$ and almost all $x \in X$, the dimension of $\mathbf{E}_{ij,bdd}(x)$ is m. We will always choose $\theta \ll \theta_0$.

Synchronized exponents.

Definition **10.8.** — *Suppose* $\theta > 0$. *We say that* $ij \in \Lambda''$ *and* $kr \in \Lambda''$ *are* θ-synchronized *if there exists* $E \subset X$ *with* $\nu(E) > 0$, *and* $C < \infty$, *such that for all* $x \in \pi^{-1}(E)$, *for all* $\ell > 0$, *for at least* $(1-\theta)$*-fraction of* $y \in \mathcal{F}_{ij}[x, \ell]$, *we have*

$$\rho\big(y, \mathcal{F}_{kr}[x]\big) < C.$$

Remark **10.9.** — By the same argument as in the proof of Lemma 10.6(a), if ij and kr are θ-synchronized then we can replace the set E in Definition 10.8 by $\bigcup_{|s|<t} g_s E$. Therefore, we can take E in Definition 10.8 to have measure arbitrarily close to 1.

Remark **10.10.** — Clearly if ij and kr are not θ-synchronized, then they are also not θ'-synchronized for any $\theta' < \theta$. Therefore there exists $\theta_0' > 0$ such that if any pairs ij and kr are not θ-synchronized for some $\theta > 0$ then they are also not θ_0'-synchronized. We will always consider $\theta \ll \theta_0'$, and will sometimes use the term "synchronized" with no modifier to mean θ-synchronized for $\theta \ll \theta_0'$. Then in view of Remark 10.9, synchronization is an equivalence relation.

We now fix $\theta \ll \min(\theta_0, \theta_0')$.

If $\mathbf{v} \in \mathbf{E}(x)$, we can write

$$\mathbf{v} = \sum_{ij \in I_{\mathbf{v}}} \mathbf{v}_{ij}, \quad \text{where } \mathbf{v}_{ij} \in \mathbf{E}_{ij}(x), \text{ but } \mathbf{v}_{ij} \notin \mathbf{E}_{i,j-1}(x). \tag{10.5}$$

In the sum, $I_{\mathbf{v}}$ is a finite set of pairs ij where $i \in \Lambda'$ and $1 \leq j \leq n_i$. (Recall that Λ' denotes the Lyapunov spectrum of $\mathbf{E}$.) Since for a fixed i the $\mathbf{E}_{ij}(x)$ form a flag, without loss of generality we may (and always will) assume that $I_{\mathbf{v}}$ contains at most one pair ij for each $i \in \Lambda'$.

For $\mathbf{v} \in \mathbf{E}(x)$, and $y \in \mathcal{F}_{\mathbf{v}}[x]$, let

$$H_{\mathbf{v}}(x, y) = \sup_{ij \in I_{\mathbf{v}}} \rho\big(y, \mathcal{F}_{ij}[x]\big).$$

Lemma **10.11.** — *There exists a set $\Psi \subset X$ with $\nu(\Psi) = 1$ such that the following holds: Suppose $x \in \Psi$, $C < \infty$, and there exists $\mathbf{v} \in \mathbf{E}(x)$ so that for each $L > 0$, for at least $(1-\theta)$-fraction of $y \in \mathcal{F}_{\mathbf{v}}[x, L]$*

$$H_{\mathbf{v}}(x, y) < C.$$

Then, if we write $\mathbf{v} = \sum_{ij \in I_{\mathbf{v}}} \mathbf{v}_{ij}$ as in (10.5), then all $\{ij\}_{ij \in I_{\mathbf{v}}}$ are synchronized, and also for all $ij \in I_{\mathbf{v}}$, $\mathbf{v}_{ij} \in \mathbf{E}_{ij,bdd}(x)$.

Proof. — Let $\Psi = \bigcup_{t \in \mathbb{R}} g_t E$, where E is as in Definition 10.8. (In view of Remark 10.9, we may assume that the same E works for all synchronized pairs.) Suppose $ij \in I_{\mathbf{v}}$ and $kr \in I_{\mathbf{v}}$. We have for at least $(1-\theta)$-fraction of $y \in \mathcal{F}_{\mathbf{v}}[x, L]$,

$$\rho\big(y, \mathcal{F}_{ij}[x]\big) < C, \qquad \rho\big(y, \mathcal{F}_{kr}[x]\big) < C.$$

Let $y_{ij} \in \mathcal{F}_{ij}[x]$ be such that $\rho(y, \mathcal{F}_{ij}[x]) = \rho(y, y_{ij})$. Similarly, let $y_{kr} \in \mathcal{F}_{kr}[x]$ be such that $\rho(y, \mathcal{F}_{kr}[x]) = \rho(y, y_{kr})$. We have

$$\textbf{(10.6)} \qquad \rho(y_{ij}, y_{kr}) \leq \rho(y_{ij}, y) + \rho(y, y_{kr}) \leq 2C.$$

Note that $\tilde{g}^{\mathbf{v},x}_{-L}(\mathcal{F}_{\mathbf{v}}[x, L]) = g^{ij}_{-L'}(\mathcal{F}_{ij}[x, L'])$, where L' is chosen so that $g^{\mathbf{v}}_{-L}x = g^{ij}_{-L'}x$, where the notation $\tilde{g}$ is as in (9.11). Hence, in view of (10.6) and (9.9), for any $L' > 0$, for $(1-\theta)$-fraction of $y_{ij} \in \mathcal{F}_{ij}[x, L']$, $\rho(y_{ij}, \mathcal{F}_{kr}[x]) \leq 2C$. Then, for any $t \in \mathbb{R}$, for any $L'' > 0$, for $(1-\theta)$-fraction of $y_{ij} \in \mathcal{F}_{ij}[g_t x, L'']$, $\rho(y_{ij}, \mathcal{F}_{kr}[g_t x]) \leq C(t)$. Since $x \in \Psi$, we can choose t so that $g_t x \in E$ where E is as in Definition 10.8. This implies that ij and kr are synchronized.

Recall that $I_{\mathbf{v}}$ contains at most one j for each $i \in \Lambda'$. Since $R(x, y)$ preserves each $\mathbf{E}_i$, and the distinct $\mathbf{E}_i$ are orthogonal, for all $y'' \in G[x]$,

$$\big\|R\big(x, y''\big)\mathbf{v}\big\|^2 = \sum_{ij \in I_{\mathbf{v}}} \big\|R\big(x, y''\big)\mathbf{v}_{ij}\big\|^2.$$

Therefore, for each $ij \in I_{\mathbf{v}}$, and all $y'' \in G[x]$,

$$\big\|R\big(x, y''\big)\mathbf{v}_{ij}\big\| \leq \big\|R\big(x, y''\big)\mathbf{v}\big\|.$$

In particular,

$$\|R(x, y_{ij})\mathbf{v}_{ij}\| \leq \|R(x, y_{ij})\mathbf{v}\|.$$

We have for $(1-\theta)$-fraction of $y_{ij} \in \mathcal{F}_{ij}[x, L']$, $\rho(y_{ij}, y) < C$, where $y \in \mathcal{F}_{\mathbf{v}}(x)$. We have, by Lemma 9.4, $\|R(x, y)\mathbf{v}\| = \|\mathbf{v}\|$, and hence, by (9.3), for $(1-\theta)$-fraction of $y_{ij} \in \mathcal{F}_{ij}[x, L]$,

$$\|R(x, y_{ij})\mathbf{v}\| \leq C_2\|\mathbf{v}\|.$$

Hence, for $(1-\theta)$-fraction of $y_{ij} \in \mathcal{F}_{ij}[x, L']$,

$$\|R(x, y_{ij})\mathbf{v}_{ij}\| \leq C_2\|\mathbf{v}\|.$$

This implies that $\mathbf{v}_{ij} \in \mathbf{E}_{ij,bdd}(x)$. □

We write $ij \sim kr$ if ij and kr are synchronized. With our choice of $\theta > 0$, synchronization is an equivalence relation, see Remark 10.10. We write $[ij] = \{kr \ : \ kr \sim ij\}$. Let

$$\mathbf{E}_{[ij],bdd}(x) = \sum_{kr \in [ij]} \mathbf{E}_{kr,bdd}(x).$$

For $\mathbf{v} \in \mathbf{E}(x)$, write $\mathbf{v} = \sum_{ij \in I_{\mathbf{v}}} \mathbf{v}_{ij}$, as in (10.5). Define

$$\text{height}(\mathbf{v}) = \sum_{ij \in I_{\mathbf{v}}} (\dim \mathbf{E}) + j$$

The height is defined so it would have the following properties:

- If $\mathbf{v} \in \mathbf{E}_{ij}(x) \setminus \mathbf{E}_{i,j-1}(x)$ and $\mathbf{w} \in \mathbf{E}_{i,j-1}(x)$ then $\text{height}(\mathbf{w}) < \text{height}(\mathbf{v})$.
- If $\mathbf{v} = \sum_{i \in I_{\mathbf{v}}} \mathbf{v}_i$, $\mathbf{v}_i \in \mathbf{E}_i$, $\mathbf{v}_i \neq 0$, and $\mathbf{w} = \sum_{j \in J} \mathbf{w}_j$, $\mathbf{w}_j \in \mathbf{E}_j$, $\mathbf{w}_j \neq 0$, and also the cardinality of J is smaller then the cardinality of $I_{\mathbf{v}}$, then $\text{height}(\mathbf{w}) < \text{height}(\mathbf{v})$.

Let $\mathcal{P}_k(x) \subset \mathbf{E}(x)$ denote the set of vectors of height at most k. This is a closed subset of $\mathbf{E}(x)$.

Lemma **10.12.** — *For every $\delta > 0$ and every $\eta > 0$ there exists a subset $K \subset X$ of measure at least $1-\delta$ and $L'' > 0$ such that for any $x \in K$ and any unit vector $\mathbf{v} \in \mathcal{P}_k(x)$ with $d(\mathbf{v}, \bigcup_{ij} \mathbf{E}_{[ij],bdd}) > \eta$ and $d(\mathbf{v}, \mathcal{P}_{k-1}(x)) > \eta$, there exists $0 < L' < L''$ so that for at least θ-fraction of $y \in \mathcal{F}_{\mathbf{v}}[x, L']$,*

$$d\left(\frac{R(x, y)\mathbf{v}}{\|R(x, y)\mathbf{v}\|}, \mathcal{P}_{k-1}(y)\right) < \eta.$$

Proof. — Suppose $C > 1$ (we will later choose C depending on η). We first claim that we can choose K with $\nu(K) > 1 - \delta$ and $L'' > 0$ so that for every $x \in g_{[-1,1]}K$ and every $\mathbf{v} \in \mathcal{P}_k(x)$ such that $d(\mathbf{v}, \bigcup_{ij} \mathbf{E}_{[ij],bdd}) > \eta$ there exists $0 < L' < L''$ so that for θ-fraction of $y \in \mathcal{F}_{\mathbf{v}}[x, L']$,

$$H_{\mathbf{v}}(x, y) \geq C. \tag{10.7}$$

(Essentially, this follows from Lemma 10.11, but the argument given below is a bit more elaborate since we want to choose L'' uniformly over all $\mathbf{v} \in \mathcal{P}_k(x)$ satisfying $d(\mathbf{v}, \bigcup_{ij} \mathbf{E}_{[ij],bdd}) > \eta$.) Indeed, let $E_L \subset \mathcal{P}_k(x)$ denote the set of unit vectors $\mathbf{v} \in \mathcal{P}_k(x)$ such that for all $0 < L' < L$, for at least $(1-\theta)$-fraction of $y \in \mathcal{F}_{\mathbf{v}}[x, L']$, $H_{\mathbf{v}}(x, y) \leq C$. Then, the E_L are closed sets which are decreasing as L increases, and by Lemma 10.11,

$$\bigcap_{L=1}^{\infty} E_L \subset \left(\bigcup_{ij \in \tilde{\Lambda}} \mathbf{E}_{[ij],bdd}(x) \right) \cap \mathcal{P}_k(x).$$

Let F denote the subset of the unit sphere in $\mathcal{P}_k(x)$ which is the complement of the η-neighborhood of $\bigcup_{ij} \mathbf{E}_{[ij],bdd}(x)$. Then the E_L^c are an open cover of F, and since F is compact, there exists $L = L_x$ such that $F \subset E_L^c$. Now for any $\delta > 0$ we can choose L'' so that $L'' > L_x$ for all x in a set K of measure at least $(1 - \delta)$.

Now suppose $\mathbf{v} \in F$. Since $F \subset E_{L''}^c$, $\mathbf{v} \notin E_{L''}$, hence there exists $0 < L' < L''$ (possibly depending on $\mathbf{v}$) such that the fraction of $y \in \mathcal{F}_{\mathbf{v}}[x, L']$ which satisfies $H_{\mathbf{v}}(x, y) \geq C$ is greater than θ. Then, (10.7) holds.

Now suppose (10.7) holds (with a yet to be chosen $C = C(\eta)$). Write

$$\mathbf{v} = \sum_{ij \in I_{\mathbf{v}}} \mathbf{v}_{ij}$$

as in (10.5). Let

$$\mathbf{w} = R(x, y)\mathbf{v}, \qquad \mathbf{w}_{ij} = R(x, y)\mathbf{v}_{ij}.$$

Since $y \in \mathcal{F}_{\mathbf{v}}[x]$, by Lemma 9.4, $\|\mathbf{w}\| = \|\mathbf{v}\| = 1$. Let $ij \in I_{\mathbf{v}}$ be such that the supremum in the definition of $H_{\mathbf{v}}(x, y)$ is achieved for ij. If $\|\mathbf{w}_{ij}\| < \eta/2$ we are done, since $\mathbf{w}' = \sum_{kr \neq ij} \mathbf{w}_{kr}$ has smaller height than $\mathbf{v}$, and $d(\mathbf{w}, \frac{\mathbf{w}'}{\|\mathbf{w}'\|}) < \eta$. Hence we may assume that $1 \geq \|\mathbf{w}_{ij}\| \geq \eta/2$.

Since $d(\mathbf{v}, \mathcal{P}_{k-1}(x)) \geq \eta$, we have

$$d\big(\mathbf{v}_{ij}, \mathbf{E}_{i,j-1}(x)\big) \geq \eta \geq \eta \|\mathbf{v}_{ij}\|,$$

where the last inequality follows from the fact that $\|\mathbf{v}_{ij}\| \leq 1$. In particular, we have $1 \geq \|\mathbf{v}_{ij}\| \geq \eta$.

Let $y' = g_t y$ be such that $y' \in \mathcal{F}_{\mathbf{v}_{ij}}[x]$. Note that

$$1 = \big\| \mathrm{R}(x, y')\mathbf{v}_{ij} \big\| = \big\| \mathrm{R}(y, y')\mathbf{w}_{ij} \big\| = \|(g_t)_* \mathbf{w}_{ij}\| \quad \text{and} \quad 1 \geq \|\mathbf{w}_{ij}\| \geq \eta/2.$$

Then, in view of (9.3), $|t| \leq \mathrm{C}_0(\eta)$, and hence $\|\mathrm{R}(y', y)\| \leq \mathrm{C}_0'(\eta)$.

Let $\mathrm{C}_1 = \mathrm{C}_0(\eta) + h(\eta, \frac{1}{2}\eta/\mathrm{C}_0'(\eta))$, where $h(\cdot,\cdot)$ is as in Lemma 10.4. We now choose the constant C in (10.7) to be C_1. If $\mathrm{H}_{\mathbf{v}}(x, y) > \mathrm{C}_1$ then, by the choice of ij, $\rho(y, \mathcal{F}_{ij}[x]) > \mathrm{C}_1$. Since $y' = g_t y$ and $|t| \leq \mathrm{C}_0(\eta)$, we have

$$\rho\big(y', \mathcal{F}_{ij}[x]\big) > \mathrm{C}_1 - \mathrm{C}_0(\eta) = h\bigg(\eta, \frac{1}{2}\eta/\mathrm{C}_0'(\eta)\bigg).$$

Then, by Lemma 10.4 applied to $\mathbf{v}_{ij}$ and $y' \in \mathcal{F}_{\mathbf{v}_{ij}}[x]$,

$$d\big(\mathrm{R}(x, y')\mathbf{v}_{ij}, \mathbf{E}_{i,j-1}(y')\big) \leq \frac{1}{2}\big(\eta/\mathrm{C}_0'(\eta)\big)\|\mathbf{v}_{ij}\| \leq \frac{1}{2}\eta/\mathrm{C}_0'(\eta).$$

Then, since $\mathbf{w}_{ij} = \mathrm{R}(y', y)\mathrm{R}(x, y')\mathbf{v}_{ij}$,

$$\begin{aligned} \big\| d\big(\mathbf{w}_{ij}, \mathbf{E}_{i,j-1}(y)\big) \big\| &\leq \big\|\mathrm{R}(y', y)\big\| d\big(\mathrm{R}(x, y')\mathbf{v}_{ij}, \mathbf{E}_{i,j-1}(y')\big) \\ &\leq \big\|\mathrm{R}(y', y)\big\| \big(\eta/\mathrm{C}_0'(\eta)\big) \leq \frac{\eta}{2}. \end{aligned}$$

Let $\mathbf{w}_{ij}'$ be the closest vector to $\mathbf{w}_{ij}$ in $\mathbf{E}_{i,j-1}(y)$, and let $\mathbf{w}' = \mathbf{w}_{ij}' + \sum_{kr \neq ij} \mathbf{w}_{ij}$. Then $d(\mathbf{w}, \frac{\mathbf{w}'}{\|\mathbf{w}'\|}) < \eta$ and $\mathbf{w}' \in \mathcal{P}_{k-1}$. □

Proof of Proposition 10.1. — Let n denote the maximal possible height of a vector. Let $\delta' = \delta/n$. Let $\eta_n = \eta$. Let $\mathrm{L}_{n-1} = \mathrm{L}_{n-1}(\delta', \eta_n)$ and $\mathrm{K}_{n-1} = \mathrm{K}_{n-1}(\delta', \eta_n)$ be chosen so that Lemma 10.12 holds for $k = n-1$, $\mathrm{K} = \mathrm{K}_{n-1}$, $\mathrm{L}'' = \mathrm{L}_{n-1}$ and $\eta = \eta_n$. Let η_{n-1} be chosen so that $\exp(\mathrm{N}(\mathrm{L}_{n-1} + 1))\eta_{n-1} \leq \eta_n$, where N is as in Lemma 7.1. We repeat this process until we choose L_1, η_0. Let $\mathrm{L}_0 = \mathrm{L}_1 + 1$. Let $\mathrm{K} = \mathrm{K}_0 \cap \cdots \cap \mathrm{K}_{n-1}$. Then $\nu(\mathrm{K}) > 1 - \delta$.

Let

$$\mathrm{E}_k' = \left\{ y \in \mathcal{F}_{\mathbf{v}}[x, \mathrm{L}] \;:\; d\bigg(\frac{\mathrm{R}(x, y)\mathbf{v}}{\|\mathrm{R}(x, y)\mathbf{v}\|}, \mathcal{P}_k(y) \cup \bigcup_{ij \in \tilde{\Lambda}} \mathbf{E}_{[ij], bdd}(y)\bigg) < \eta_k \right\},$$

and let

$$\mathrm{E}_k = \tilde{g}_{-\mathrm{L}}\big(\mathrm{E}_k'\big),$$

so $\mathrm{E}_k \subset \mathcal{B}[z]$, where $z = \tilde{g}_{-\mathrm{L}} x$. Since $\mathrm{E}_n' = \mathcal{F}_{\mathbf{v}}[x, \mathrm{L}]$, we have $\mathrm{E}_n = \mathcal{B}[z]$. Let $\mathrm{Q} = \tilde{g}_{-\mathrm{L}}(g_{[-1,1]}\mathrm{K} \cap \mathcal{F}_{\mathbf{v}}[x, \mathrm{L}])$. Then, by assumption,

$$|\mathrm{Q}| \geq \big(1 - (\theta/2)^{n+1}\big)|\mathcal{B}[z]|. \tag{10.8}$$

By Lemma 10.12, for every point $uz \in (\mathrm{E}_k \cap \mathrm{Q}) \setminus \mathrm{E}_{k-1}$ there exists a "ball" $\mathcal{B}_t[uz]$ (where $t = \mathrm{L} - \mathrm{L}'$ and L' is as in Lemma 10.12) such that

$$|\mathrm{E}_{k-1} \cap \mathcal{B}_t[uz]| \geq \theta |\mathcal{B}_t[uz]|. \tag{10.9}$$

(When we are applying Lemma 10.12 we do not have $\mathbf{v} \in \mathcal{P}_k$ but rather $d(\mathbf{v}/\|\mathbf{v}\|, \mathcal{P}_k) < \eta_k$; however by the choice of the η's and the L's this does not matter.) The collection of balls $\{\mathcal{B}_t[uz]\}_{uz \in (\mathrm{E}_k \cap \mathrm{Q}) \setminus \mathrm{E}_{k-1}}$ as in (10.9) are a cover of $(\mathrm{E}_k \cap \mathrm{Q}) \setminus \mathrm{E}_{k-1}$. These balls satisfy the condition of Lemma 3.10(b); hence we may choose a pairwise disjoint subcollection which still covers $(\mathrm{E}_k \cap \mathrm{Q}) \setminus \mathrm{E}_{k-1}$. We get $|\mathrm{E}_{k-1}| \geq \theta |\mathrm{E}_k \cap \mathrm{Q}|$. Hence, by (10.8) and induction over k, we have

$$|\mathrm{E}_k| \geq (\theta/2)^{n-k} |\mathcal{B}[z]|.$$

Hence, $|\mathrm{E}_0| \geq (\theta/2)^n |\mathcal{B}[z]|$. Therefore $|\mathrm{E}'_0| \geq (\theta/2)^n |\mathcal{F}_{\mathbf{v}}[x, \mathrm{L}]|$. Since $\mathcal{P}_0 = \emptyset$, the Proposition follows from the definition of E'_0. □

10.2*. *Invariant measures on* $\mathrm{X} \times \mathbb{P}(\mathbf{L})$. — In this subsection we prove Proposition 10.2.

Recall that any bundle is measurably trivial.

Lemma **10.13.** — *Suppose* $\mathbf{L}(x)$ *is an invariant subbundle or quotient bundle of* $\mathbf{H}(x)$. *(In fact the arguments in this subsection apply to arbitrary vector bundles.) Let* $\tilde{\mu}_\ell$ *be the measure on* $\mathrm{X} \times \mathbb{P}(\mathbf{L})$ *defined by*

$$\tilde{\mu}_\ell(f) = \int_{\mathrm{X}} \int_{\mathbb{P}(\mathbf{L})} \frac{1}{|\mathcal{F}_{ij}[x, \ell]|} \int_{\mathcal{F}_{ij}[x,\ell]} f\big(x, \mathrm{R}(y, x)\mathbf{v}\big)\, dy\, d\rho_0(\mathbf{v})\, d\nu(x) \tag{10.10}$$

where ρ_0 *is the "round" measure on* $\mathbb{P}(\mathbf{L})$. *(In fact,* ρ_0 *can be any measure on* $\mathbb{P}(\mathbf{L})$ *in the measure class of Lebesgue measure, independent of x and fixed once and for all.) Let* $\hat{\mu}_\ell$ *be the measure on* $\mathrm{X} \times \mathbb{P}(\mathbf{L})$ *defined by*

$$\hat{\mu}_\ell(f) = \int_{\mathrm{X}} \int_{\mathbb{P}(\mathbf{L})} \frac{1}{|\mathcal{F}_{ij}[x, \ell]|} \int_{\mathcal{F}_{ij}[x,\ell]} f\big(y, \mathrm{R}(x, y)\mathbf{v}\big)\, dy\, d\rho_0(\mathbf{v})\, d\nu(x). \tag{10.11}$$

Then $\hat{\mu}_\ell$ *is in the same measure class as* $\tilde{\mu}_\ell$, *and*

$$\kappa^{-2} \leq \frac{d\hat{\mu}_\ell}{d\tilde{\mu}_\ell} \leq \kappa^2, \tag{10.12}$$

where κ *is as in Proposition 4.15.*

Proof. — Let

$$\mathrm{F}(x, y) = \int_{\mathbb{P}(\mathbf{L})} f\big(x, \mathrm{R}(y, x)\mathbf{v}\big)\, d\rho_0(\mathbf{v}).$$

Then,

$$\tilde{\mu}_\ell(f) = \int_X \frac{1}{|\mathcal{F}_{ij}[x,\ell]|} \int_{\mathcal{F}_{ij}[x,\ell]} F(x,y)\,dy\,d\nu(x) \tag{10.13}$$

$$\hat{\mu}_\ell(f) = \int_X \frac{1}{|\mathcal{F}_{ij}[x,\ell]|} \int_{\mathcal{F}_{ij}[x,\ell]} F(y,x)\,dy\,d\nu(x) \tag{10.14}$$

Let $x' = g_{-\ell}^{ij} x$. Then, in view of Proposition 4.15, $\kappa^{-1} d\nu(x) \le d\nu(x') \le \kappa\, d\nu(x)$. Then,

$$\frac{1}{\kappa}\tilde{\mu}_\ell(f) \le \int_X \frac{1}{|\mathcal{B}[x']|} \int_{\mathcal{B}[x']} F\left(g_\ell^{ij} x', g_\ell^{ij} z\right) dz\, d\nu\left(x'\right) \le \kappa\, \tilde{\mu}_\ell(f),$$

and

$$\frac{1}{\kappa}\hat{\mu}_\ell(f) \le \int_X \frac{1}{|\mathcal{B}[x']|} \int_{\mathcal{B}[x']} F\left(g_\ell^{ij} z, g_\ell^{ij} x'\right) dz\, d\nu\left(x'\right) \le \kappa\, \hat{\mu}_\ell(f)$$

Let X'' consist of one point from each $\mathcal{B}[x]$. In view of Definition 6.2(iii), we now disintegrate $d\nu(x') = d\beta(x'')dz'$ where $x'' \in X''$, $z' \in \mathcal{B}[x']$.

$$\begin{aligned}
&\int_X \frac{1}{|\mathcal{B}[x']|} \int_{\mathcal{B}[x']} F\left(g_\ell^{ij} x', g_\ell^{ij} z\right) dz\, d\nu\left(x'\right) \\
&\quad = \int_{X''} \int_{\mathcal{B}[x'']\times\mathcal{B}[x'']} F\left(g_\ell^{ij} z', g_\ell^{ij} z\right) dz'\, dz\, d\beta\left(x''\right) \\
&\quad = \int_{X''} \int_{\mathcal{B}[x'']\times\mathcal{B}[x'']} F\left(g_\ell^{ij} z, g_\ell^{ij} z'\right) dz'\, dz\, d\beta\left(x''\right) \\
&\quad = \int_X \frac{1}{|\mathcal{B}[x']|} \int_{\mathcal{B}[x']} F\left(g_\ell^{ij} z, g_\ell^{ij} x'\right) dz\, d\nu\left(x'\right).
\end{aligned}$$

Now (10.12) follows from (10.13) and (10.14). ☐

Lemma **10.14.** — *Let $\tilde{\mu}_\infty$ be any weak-star limit of the measures $\tilde{\mu}_\ell$. Then,*

(a) *We may disintegrate $d\tilde{\mu}_\infty(x,\mathbf{v}) = d\nu(x)d\lambda_x(\mathbf{v})$, where for each $x \in X$, λ_x is a measure on $\mathbb{P}(\mathbf{L})$.*

(b) *For $x \in \tilde{X}$ and $y \in \mathcal{F}_{ij}[x]$,*

$$\lambda_y = R(x,y)_* \lambda_x,$$

(where to simplify notation, we write λ_x and λ_y instead of $\lambda_{\pi(x)}$ and $\lambda_{\pi(y)}$).

(c) *Let* $\mathbf{w} \in \mathbb{P}(\mathbf{L})$ *be a point. For* $\eta > 0$ *let*

$$B(\mathbf{w}, \eta) = \{\mathbf{v} \in \mathbb{P}(\mathbf{L}) \;:\; d(\mathbf{v}, \mathbf{w}) \leq \eta\}.$$

Then, for any $t < 0$ *there exists* $c_1 = c_1(t, \mathbf{w}) > 0$ *and* $c_2 = c_2(t, \mathbf{w}) > 0$ *such that for* $x \in X$,

$$\lambda_{g_t x}\big(B(g_t \mathbf{w}, c_1 \eta)\big) \geq c_2 \lambda_x\big(B(\mathbf{w}, \eta)\big).$$

Consequently, for $t < 0$, *the support of* $\lambda_{g_t x}$ *contains the support of* $(g_t)_* \lambda_x$.

(d) *For almost all* $x \in X$ *there exist a measure* ψ_x *on* $\mathbb{P}(\mathbf{L})$ *such that*

$$\lambda_x = h(x) \psi_x$$

for some $h(x) \in SL(\mathbf{L})$, *and also for almost all* $y \in \mathcal{F}_{ij}[x]$, $\psi_y = \psi_x$ *(so that* ψ *is constant on the leaves* $\mathcal{F}_{ij}$*). The maps* $x \to \psi_x$ *and* $x \to h(x)$ *are both* ν*-measurable.*

Proof. — If $f(x, \mathbf{v})$ is independent of the second variable, then it is clear from the definition of $\tilde{\mu}_\ell$ that $\tilde{\mu}_\ell(f) = \int_X f d\nu$. This implies (a). To prove (b), note that $R(y', y) = R(x, y) R(y', x)$. Then,

$$\begin{aligned}
\lambda_y &= \lim_{k \to \infty} \frac{1}{|\mathcal{F}_{ij}[y, \ell_k]|} \int_{\mathcal{F}_{ij}[y, \ell_k]} \big(R(y', y)_* \rho_0\big) dy' \\
&= R(x, y)_* \lim_{k \to \infty} \frac{1}{|\mathcal{F}_{ij}[y, \ell_k]|} \int_{\mathcal{F}_{ij}[y, \ell_k]} \big(R(y', x)_* \rho_0\big) dy' \\
&= R(x, y)_* \lim_{k \to \infty} \frac{1}{|\mathcal{F}_{ij}[x, \ell_k]|} \int_{\mathcal{F}_{ij}[x, \ell_k]} \big(R(y', x)_* \rho_0\big) dy' \\
&= R(x, y)_* \lambda_x
\end{aligned}$$

where to pass from the second line to the third we used the fact that $\mathcal{F}_{ij}[x, \ell] = \mathcal{F}_{ij}[y, \ell]$ for ℓ large enough. This completes the proof of (b).

We now begin the proof of (c). Let $\mathbf{w}(x) = \mathbf{w}$. Working in the universal cover, we define for $y \in G[x]$, $\mathbf{w}(y) = R(x, y) \mathbf{w}(x)$. We define

$$\mathbf{w}_\eta(x) = \{\mathbf{v} \in \mathbb{P}\big(\mathbf{L}(x)\big) \;:\; d\big(\mathbf{v}, \mathbf{w}(x)\big) \leq \eta\}.$$

(Here we are thinking of the space as $X \times \mathbb{P}(\mathbf{L})$ and using the same metric on all the $\mathbb{P}(\mathbf{L})$ fibers.)

Let $x' = g_t^{ij} x$, $y' = g_t^{ij} y$. We have

$$R\big(y', x'\big) = R\big(x, x'\big) R(y, x) R\big(y', y\big).$$

Since $\|R(x, x')^{-1}\| \le c^{-1}$, where c depends on t, we have $R(x, x')^{-1}\mathbf{w}_{c\eta}(x') \subset \mathbf{w}_\eta(x)$. Then,

$$\begin{aligned}
&\rho_0\{\mathbf{v} : R(y', x')\mathbf{v} \in \mathbf{w}_{c\eta}(x')\} \\
&\quad = \rho_0\{\mathbf{v} : R(y, x)R(y', y)\mathbf{v} \in R(x, x')^{-1}\mathbf{w}_{c\eta}(x')\} \\
&\quad \ge \rho_0\{\mathbf{v} : R(y, x)R(y', y)\mathbf{v} \in \mathbf{w}_\eta(x)\} \\
&\quad = \rho_0\{R(y, y')^{-1}\mathbf{u} : R(y, x)\mathbf{u} \in \mathbf{w}_\eta(x)\} \\
&\quad = R(y, y')_*^{-1}\rho_0\{\mathbf{u} : R(y, x)\mathbf{u} \in \mathbf{w}_\eta(x)\} \\
&\quad \ge c'\rho_0\{\mathbf{u} : R(y, x)\mathbf{u} \in \mathbf{w}_\eta(x)\}.
\end{aligned}$$

Note that for $t < 0$, $g_t^{ij}\mathcal{F}_{ij}[x, \ell] \subset \mathcal{F}_{ij}[g_t^{ij}x, \ell]$ and $|g_t^{ij}\mathcal{F}_{ij}[x, \ell]| \ge c(t)|\mathcal{F}_{ij}[g_t^{ij}x, \ell]|$. Substituting into (10.10) completes the proof of (c).

To prove part (d), let $\mathcal{M}$ denote the space of measures on $\mathbb{P}(\mathbf{L})$. Recall that by [Zi2, Theorem 3.2.6] the orbits of the special linear group $SL(\mathbf{L})$ on $\mathcal{M}$ are locally closed. Then, by [Ef, Theorem 2.9 (13), Theorem 2.6(5)] [1] there exists a Borel cross section $\phi : \mathcal{M}/SL(\mathbf{L}) \to \mathcal{M}$. Then, let $\psi_x = \phi(\pi(\lambda_x))$ where $\pi : \mathcal{M} \to \mathcal{M}/SL(\mathbf{L})$ is the quotient map. □

We also recall the following well known Lemma of Furstenberg (see e.g. [Zi2, Lemma 3.2.1]):

Lemma **10.15.** — *Let* $\mathbf{L}$ *be a vector space, and suppose* μ *and* ν *are two probability measures on* $\mathbb{P}(\mathbf{L})$. *Suppose* $g_i \in SL(\mathbf{L})$ *are such that* $g_i \to \infty$ *and* $g_i\mu \to \nu$. *Then the support of* ν *is contained in a union of two proper subspaces of* $\mathbf{L}$.

In particular, if the support of a measure ν *on* $\mathbb{P}(\mathbf{L})$ *is not contained in a union of two proper subspaces, then the stabilizer of* ν *in* $SL(\mathbf{L})$ *is bounded.*

Lemma **10.16.** — *Suppose* $\mathbf{L}$ *is either a subbundle or a quotient bundle of* $\mathbf{H}$. *Suppose that* $\theta > 0$, *and suppose that for all* $\delta > 0$ *there exists a set* $K \subset X$ *with* $\nu(K) > 1 - \delta$ *and a constant* $C_1 < \infty$, *such that such that for all* $x \in K$, *all* $\ell > 0$ *and at least* $(1-\theta)$-*fraction of* $y \in \mathcal{F}_{ij}[x, \ell]$,

$$\|R(x, y)\mathbf{v}\| \le C_1\|\mathbf{v}\| \quad \textit{for all } \mathbf{v} \in \mathbf{L}. \tag{10.15}$$

Then for all $\delta > 0$ *and for all* $\ell > 0$ *there exists a subset* $K''(\ell) \subset X$ *with* $\nu(K''(\ell)) > 1 - c(\delta)$ *where* $c(\delta) \to 0$ *as* $\delta \to 0$, *and there exists* $\theta'' = \theta''(\theta, \delta)$ *with* $\theta'' \to 0$ *as* $\theta \to 0$ *and* $\delta \to 0$ *such that for all* $x \in K''(\ell)$, *for at least* $(1 - \theta'')$-*fraction of* $y \in \mathcal{F}_{ij}[x, \ell]$,

$$C_1^{-1}\|\mathbf{v}\| \le \|R(x, y)\mathbf{v}\| \le C_1\|\mathbf{v}\| \quad \textit{for all } \mathbf{v} \in \mathbf{L}. \tag{10.16}$$

[1] The "condition C" of [Ef] is satisfied since $SL(\mathbf{L})$ is locally compact and $\mathcal{M}$ is Hausdorff.

Proof. — Let f be the characteristic function of $\mathrm{K} \times \mathbb{P}(\mathbf{L})$. By (10.10), $\tilde{\mu}_\ell(f) \geq (1-\delta)$. By Lemma 10.13 we have $\hat{\mu}_\ell(f) \geq (1-\kappa^2\delta)$. Therefore, by (10.11) there exists a subset $\mathrm{K}'(\ell) \subset \mathrm{X}$ with $\nu(\mathrm{K}'(\ell)) \geq 1-(\kappa^2\delta)^{1/2}$ such that such that for all $x \in \mathrm{K}'(\ell)$,

$$|\mathcal{F}_{ij}[x,\ell] \cap \mathrm{K}| \geq \big(1-\big(\kappa^2\delta\big)^{1/2}\big)|\mathcal{F}_{ij}[x,\ell]|.$$

For $x_0 \in \mathrm{X}$, let

$$\mathrm{Z}_\ell[x_0] = \big\{(x,y) \in \mathcal{F}_{ij}[x_0,\ell] \times \mathcal{F}_{ij}[x_0,\ell] \ : \ x \in \mathrm{K},\ y \in \mathrm{K},\ \text{and (10.15) holds}\big\}.$$

Then, if $x_0 \in \mathrm{K}'(\ell)$ and $\theta' = \theta + (\kappa^2\delta)^{1/2}$ then, by Fubini's theorem,

$$|\mathrm{Z}_\ell[x_0]| \geq \big(1-\theta'\big)|\mathcal{F}_{ij}[x_0,\ell] \times \mathcal{F}_{ij}[x_0,\ell]|.$$

Let

$$\mathrm{Z}_\ell[x_0]^t = \big\{(x,y) \in \mathcal{F}_{ij}[x_0,\ell] \times \mathcal{F}_{ij}[x_0,\ell] \ : \ (y,x) \in \mathrm{Z}_\ell[x_0]\big\}.$$

Then, for $x_0 \in \mathrm{K}'(\ell)$,

$$|\mathrm{Z}_\ell[x_0] \cap \mathrm{Z}_\ell[x_0]^t| \geq \big(1-2\theta'\big)|\mathcal{F}_{ij}[x_0,\ell] \times \mathcal{F}_{ij}[x_0,\ell]|.$$

For $x \in \mathcal{F}_{ij}[x_0,\ell]$, let

$$\mathrm{Y}'_\ell(x) = \big\{\, y \in \mathcal{F}_{ij}[x,\ell] \ : \ (x,y) \in \mathrm{Z}_\ell[x] \cap \mathrm{Z}_\ell[x]^t\big\}.$$

Therefore, by Fubini's theorem, for all $x_0 \in \mathrm{K}'(\ell)$ and $\theta'' = (2\theta')^{1/2}$,

$$\Big|\big\{x \in \mathcal{F}_{ij}[x_0,\ell] \ : \ |\mathrm{Y}'_\ell(x)| \geq \big(1-\theta''\big)|\mathcal{F}_{ij}[x_0,\ell]|\big\}\Big| \geq \big(1-\theta''\big)|\mathcal{F}_{ij}[x_0,\ell]|. \tag{10.17}$$

(Note that $\mathcal{F}_{ij}[x_0,\ell] = \mathcal{F}_{ij}[x,\ell]$.) Let

$$\mathrm{K}''(\ell) = \big\{x \in \mathrm{X} \ : \ |\mathrm{Y}'_\ell(x)| \geq \big(1-\theta''\big)|\mathcal{F}_{ij}[x,\ell]|\big\}.$$

Therefore, by (10.17), for all $x_0 \in \mathrm{K}'(\ell)$,

$$|\mathcal{F}_{ij}[x_0,\ell] \cap \mathrm{K}''(\ell)| \geq \big(1-\theta''\big)|\mathcal{F}_{ij}[x_0,\ell]|.$$

Then, by the definition of $\hat{\mu}_\ell$,

$$\hat{\mu}_\ell\big(\mathrm{K}''(\ell) \times \mathbb{P}(\mathbf{L})\big) \geq \big(1-\theta''\big)\nu\big(\mathrm{K}'(\ell)\big) \geq \big(1-2\theta''\big),$$

and therefore, by Lemma 10.13,

$$\nu\big(\mathrm{K}''(\ell)\big) = \tilde{\mu}_\ell\big(\mathrm{K}''(\ell) \times \mathbb{P}(\mathbf{L})\big) \geq \big(1-2\kappa^2\theta''\big).$$

Now, for $x \in \mathrm{K}''(\ell)$, and $y \in \mathrm{Y}'_\ell(x)$, (10.16) holds. □

Lemma **10.17.** — *Suppose* $\mathbf{L}(x) = \mathbf{E}_{ij,bdd}(x)$. *Then there exists a* Γ*-invariant function* $C : \tilde{X} \to \mathbb{R}^+$ *finite almost everywhere such that for all* $x \in \tilde{X}$, *all* $\mathbf{v} \in \mathbf{L}(x)$, *and all* $y \in \mathcal{F}_{ij}[x]$,

$$C(x)^{-1}C(y)^{-1}\|\mathbf{v}\| \leq \|R(x,y)\mathbf{v}\| \leq C(x)C(y)\|\mathbf{v}\|,$$

Proof. — Let $\tilde{\mu}_\ell$ and $\hat{\mu}_\ell$ be as in Lemma 10.13. Take a sequence $\ell_k \to \infty$ such that $\tilde{\mu}_{\ell_k} \to \tilde{\mu}_\infty$, and $\hat{\mu}_{\ell_k} \to \hat{\mu}_\infty$. Then by Lemma 10.14(a), we have $d\tilde{\mu}_\infty(x,\mathbf{v}) = d\nu(x)d\lambda_x(\mathbf{v})$ where λ_x is a measure on $\mathbb{P}(\mathbf{L})$. Let $E \subset X$ be such that for $x \in E$, λ_x is supported on at most two subspaces. We will show that $\nu(E) = 0$.

Suppose not; then $\nu(E) > 0$, and for $x \in E$, λ_x is supported on $\mathbf{F}_1(x) \cup \mathbf{F}_2(x)$, where $\mathbf{F}_1(x)$ and $\mathbf{F}_2(x)$ are subspaces of $\mathbf{L}(x)$. We always choose $\mathbf{F}_1(x)$ and $\mathbf{F}_2(x)$ to be of minimal dimension, and if λ_x is supported on a single subspace $\mathbf{F}(x)$ (of minimal dimension), we let $\mathbf{F}_1(x) = \mathbf{F}_2(x) = \mathbf{F}(x)$. Then, for $x \in E$, $\mathbf{F}_1(x) \cup \mathbf{F}_2(x)$ is uniquely determined by x. After possibly replacing E by a smaller subset of positive measure, we may assume that $\dim \mathbf{F}_1(x)$ and $\dim \mathbf{F}_2(x)$ are independent of $x \in E$.

Let

$$\Psi = \{x \in X \,:\, g_t x \in E \text{ and } g_{-s}x \in E \text{ for some } t > 0 \text{ and } s > 0\}.$$

Then, $\nu(\Psi) = 1$. If $x \in \Psi$, then, by Lemma 10.14(c),

$$(g_s)_*\mathbf{F}_1(g_{-s}x) \cup (g_s)_*\mathbf{F}_2(g_{-s}x) \subset \operatorname{supp}\lambda_x \subset (g_{-t})_*\mathbf{F}_1(g_t x) \cup (g_{-t})_*\mathbf{F}_2(g_t x). \tag{10.18}$$

Since $\mathbf{F}_i(g_t x)$ and $\mathbf{F}_i(g_{-s}x)$ have the same dimension, the sets on the right and on the left of (10.18) coincide. Therefore, $E \supset \Psi$ (and so E has full measure) and the set $\mathbf{F}_1(x) \cup \mathbf{F}_2(x)$ is g_t-invariant. By Proposition 4.4 (see also the remark immediately following the Proposition) the set $\mathbf{F}_1(x) \cup \mathbf{F}_2(x)$ is also U^+-invariant.

Fix $\delta > 0$ (which will be chosen sufficiently small later). Suppose $\ell > 0$ is arbitrary. Since $\mathbf{L} = \mathbf{E}_{ij,bdd}$, there exists a constant C_1 independent of ℓ and a compact subset $K \subset X$ with $\nu(K) > 1 - \delta$ and for each $x \in K$ a subset $Y_\ell(x)$ of $\mathcal{F}_{ij}[x,\ell]$ with $|Y_\ell(x)| \geq (1-\theta)|\mathcal{F}_{ij}[x,\ell]|$, such that for $x \in K$ and $y \in Y_\ell(x) \cap K$ we have

$$\|R(x,y)\mathbf{v}\| \leq C_1\|\mathbf{v}\| \quad \text{for all } \mathbf{v} \in \mathbf{L}.$$

Therefore by Lemma 10.16, there exists $0 < \theta'' < 1/2$, $K''(\ell) \subset X$ and for each $x \in K''(\ell)$ a subset $Y'_\ell(x) \subset \mathcal{F}_{ij}[x,\ell]$ with $|Y'_\ell(x)| \geq (1-\theta'')|\mathcal{F}_{ij}[x,\ell]|$ such that for $x \in K''(\ell)$ and $y \in Y'_\ell(x)$, (10.16) holds.

Let

$$\mathbf{Z}(x,\eta) = \Big\{\mathbf{v} \in \mathbb{P}(\mathbf{L}) \,:\, d\big(\mathbf{v}, \mathbf{F}_1(x) \cup \mathbf{F}_2(x)\big) \geq \eta\Big\}.$$

We may choose $\eta > 0$ small enough so that there exists $K' \subset X$ with $\nu(K''(\ell) \cap K') > 0$ such that for all $x \in K'$,

$$\rho_0\big(\mathbf{Z}(x, C_1\eta)\big) > 1/2.$$

Let

$$S(\eta) = \left\{(x, \mathbf{v}) \;:\; x \in X,\; \mathbf{v} \in \mathbf{Z}(x, \eta)\right\}$$

Let f denote the characteristic function of the set

$$\left\{(x, \mathbf{v}) \;:\; x \in K''(\ell) \cap K',\; \mathbf{v} \in \mathbf{Z}(x, \eta)\right\} \subset S(\eta).$$

We now claim that for any ℓ,

$$\textbf{(10.19)} \qquad \hat{\mu}_\ell(f) \geq \nu\big(K''(\ell) \cap K'\big)\big(1 - \theta''\big)(1/2).$$

Indeed, if we restrict in (10.11) to $x \in K''(\ell) \cap K'$, $y \in Y'_\ell(x)$, and $\mathbf{v} \in \mathbf{Z}(x, C_1\eta)$, then by (10.16), $f(x, R(x, y)\mathbf{v}) = 1$. This implies (10.19). Thus, (provided $\delta > 0$ and $\theta > 0$ in Definition 10.5 are sufficiently small), there exists $c_0 > 0$ such that for all ℓ, $\hat{\mu}_\ell(S(\eta)) \geq c_0 > 0$. Therefore, by Lemma 10.13, $\tilde{\mu}_\ell(S(\eta)) \geq c_0/\kappa^2$.

There exists compact $K_0 \subset X$ with $\nu(K_0) > 1 - c_0/(2\kappa^2)$ such that the map $x \to \mathbf{F}_1(x) \cap \mathbf{F}_2(x)$ is continuous on K_0. Let $K'_0 = \{(x, \mathbf{v}) \;:\; x \in K_0\}$. Then $S(\eta) \cap K'_0$ is a closed set with $\tilde{\mu}_\ell(S(\eta) \cap K'_0) \geq c_0/(2\kappa^2)$. Therefore, $\tilde{\mu}_\infty(S(\eta) \cap K'_0) > c_0/(2\kappa^2) > 0$, which is a contradiction to the fact that λ_x is supported on $\mathbf{F}_1(x) \cup \mathbf{F}_2(x)$.

Thus, for almost all x, λ_x is not supported on a union of two subspaces. Thus the same holds for the measure ψ_x of Lemma 10.14(d). By combining (b) and (d) of Lemma 10.14 we see that for almost all x and almost all $y \in \mathcal{F}_{ij}[x]$,

$$R(x, y)h(x)\psi_x = h(y)\psi_x,$$

hence $h(y)^{-1}R(x, y)h(x)$ stabilizes ψ_x. Hence by Lemma 10.15,

$$h(y)^{-1}\bar{R}(x, y)h(x) \in K(x)$$

where $K(x)$ is a compact subset of $SL(\mathbf{L})$, and $\bar{R}(x, y)$ is the image of $R(x, y)$ under the natural map $GL(\mathbf{L}) \to SL(\mathbf{L})$. Thus, $\bar{R}(x, y) \in h(y)K(x)h(x)^{-1}$, and thus

$$\textbf{(10.20)} \qquad \|\bar{R}(x, y)\| \leq C(x)C(y).$$

Since $\bar{R}(x, y)^{-1} = \bar{R}(y, x)$, we get, by exchanging x and y,

$$\textbf{(10.21)} \qquad \|\bar{R}(x, y)^{-1}\| \leq C(x)C(y).$$

Note that by Lemma 10.6, there exists $\mathbf{v} \in \mathbf{L}(x) = \mathbf{E}_{ij,bdd}(x) \subset \mathbf{E}_{ij}(x)$ such that $\mathbf{v} \notin \mathbf{E}_{i,j-1}(x)$. Then, (9.5) and the fact that $\lambda_{ij}(x, y) = 0$ for $y \in \mathcal{F}_{ij}[x]$ shows that (10.20) and (10.21) must hold for $R(x, y)$ in place of $\bar{R}(x, y)$. This implies the statement of the lemma. □

Lemma **10.18.** — *Suppose that for all $\delta > 0$ there exists a constant $C > 0$ and a compact subset $K \subset X$ with $\nu(K) > 1 - \delta$ and for each $\ell > 0$ and $x \in K$ a subset $Y_\ell(x)$ of $\mathcal{F}_{ij}[x, \ell]$ with $|Y_\ell(x)| \geq (1-\theta)|\mathcal{F}_{ij}[x, \ell]|$, such that for $x \in K$ and $y \in Y_\ell(x)$ we have*

$$\lambda_{kr}(x, y) \leq C. \tag{10.22}$$

Then, ij and kr are synchronized, and there exists a function $C : X \to \mathbb{R}^+$ finite ν-almost everywhere such that for all $x \in X$, and all $y \in \mathcal{F}_{ij}[x]$,

$$\rho\big(y, \mathcal{F}_{kr}[x]\big) \leq C(x)C(y). \tag{10.23}$$

Proof. — The proof is a simplified version of the proof of Lemma 10.17. Let $\mathbf{L}_1 = \mathbf{E}_{ij}/\mathbf{E}_{i,j-1}$, $\mathbf{L}_2 = \mathbf{E}_{kr}/\mathbf{E}_{k,r-1}$, and $\mathbf{L} = \mathbf{L}_1 \times \mathbf{L}_2$.

We have, for $y \in G[x]$, and $(\bar{\mathbf{v}}, \bar{\mathbf{w}}) \in \mathbf{L}$,

$$\begin{aligned} R(x, y)(\bar{\mathbf{v}}, \bar{\mathbf{w}}) &= \big(e^{\lambda_{ij}(x,y)}\bar{\mathbf{v}}', e^{\lambda_{kr}(x,y)}\bar{\mathbf{w}}'\big), \\ &\text{where } \|\bar{\mathbf{v}}'\| = \|\bar{\mathbf{v}}\| \text{ and } \|\bar{\mathbf{w}}'\| = \|\bar{\mathbf{w}}\|. \end{aligned} \tag{10.24}$$

Recall that $\lambda_{ij}(x, y) = 0$ for all $y \in \mathcal{F}_{ij}[x]$. Therefore, (10.22) implies that for all $x \in K$, all $\ell > 0$ and all $y \in Y_\ell(x)$,

$$\|R(x, y)(\bar{\mathbf{v}}, \bar{\mathbf{w}})\| \leq C_1\|(\bar{\mathbf{v}}, \bar{\mathbf{w}})\|.$$

Therefore, by Lemma 10.16, there exists a subset $K''(\ell) \subset X$ with $\nu(K''(\ell)) > 1 - c(\delta)$ where $c(\delta) \to 0$ as $\delta \to 0$, and for each $x \in K''(\ell)$ a subset $Y'_\ell \subset \mathcal{F}_{ij}[x, \ell]$ with $|Y'_\ell| > (1-\theta'')|\mathcal{F}_{ij}[x, \ell]|$ such that for all $y \in Y'_\ell$,

$$C_1^{-1}\|(\bar{\mathbf{v}}, \bar{\mathbf{w}})\| \leq \|R(x, y)(\bar{\mathbf{v}}, \bar{\mathbf{w}})\| \leq C_1\|(\bar{\mathbf{v}}, \bar{\mathbf{w}})\|.$$

This implies that for $x \in K''(\ell)$, $y \in Y'_\ell(x)$,

$$|\lambda_{kr}(x, y)| = |\lambda_{ij}(x, y) - \lambda_{kr}(x, y)| \leq C_1. \tag{10.25}$$

Let $\tilde{\mu}_\ell$ and $\hat{\mu}_\ell$ be as in Lemma 10.13. Take a sequence $\ell_m \to \infty$ such that $\tilde{\mu}_{\ell_m} \to \tilde{\mu}_\infty$, and $\hat{\mu}_{\ell_m} \to \hat{\nu}_\infty$. Then by Lemma 10.14(a), we have $d\tilde{\mu}_\infty(x, \mathbf{v}) = d\nu(x)d\lambda_x(\mathbf{v})$ where λ_x is a measure on $\mathbb{P}(\mathbf{L})$. We will show that for almost all $x \in X$, λ_x is not supported on $\mathbf{L}_1 \times \{0\} \cup \{0\} \times \mathbf{L}_2$.

Suppose that for a set of positive measure λ_x is supported on $(\mathbf{L}_1 \times \{0\}) \cup (\{0\} \times \mathbf{L}_2)$. Then, in view of the ergodicity of g_t and Lemma 10.14(c), λ_x is supported on $(\mathbf{L}_1 \times \{0\}) \cup (\{0\} \times \mathbf{L}_2)$ for almost all $x \in X$. Let

$$\mathbf{Z}(x, \eta) = \big\{(\bar{\mathbf{v}}, \bar{\mathbf{w}}) \in \mathbf{L}(x),\ \|(\bar{\mathbf{v}}, \bar{\mathbf{w}})\| = 1,\ d(\bar{\mathbf{v}}, \mathbf{L}_1) \geq \eta,\ d(\bar{\mathbf{w}}, \mathbf{L}_2) \geq \eta\big\},$$

and let

$$S(\eta) = \big\{\big(x, (\bar{\mathbf{v}}, \bar{\mathbf{w}})\big)\ :\ x \in X, (\bar{\mathbf{v}}, \bar{\mathbf{w}}) \in \mathbf{Z}(x, \eta)\big\}.$$

Then we have $\tilde{\mu}_\infty(S(\eta)) = 0$. Therefore, by Lemma 10.13, $\hat{\mu}_\infty(S(\eta)) = 0$.

By (10.24) and (10.25), for $x \in K''(\ell_m)$ and $y \in Y'_{\ell_m}(x)$,

$$R(x, y)\mathbf{Z}(x, C_1\eta) \subset \mathbf{Z}(y, \eta). \tag{10.26}$$

Choose $\eta > 0$ so that there exists $K' = K'(\ell_m) \subset X$ with $\nu(K''(\ell_m) \cap K') > 0$ such that for $x \in K'$, $\rho_0(\mathbf{Z}(x, C_1\eta)) > (1/2)$. Let f be the characteristic function of $S(\eta)$. Then, if we restrict in (10.11) to $x \in K''(\ell_m) \cap K'$, $y \in Y'_{\ell_m}(x)$, and $\mathbf{v} \in \mathbf{Z}(x, C_1\eta)$, then by (10.26), $f(x, R(x, y)\mathbf{v}) = 1$. This implies that for all m,

$$\hat{\mu}_{\ell_m}\big(S(\eta)\big) \geq \nu\big(K''(\ell_m) \cap K'\big)\big(1 - \theta''\big)(1/2).$$

Hence $\hat{\mu}_\infty(S(\eta)) > 0$ which is a contradiction. Therefore, for almost all x, λ_x is not supported on $\mathbf{L}_1 \times \{0\} \cup \{0\} \times \mathbf{L}_2$. Thus the same holds for the measure ψ_x of Lemma 10.14(d). By combining (b) and (d) of Lemma 10.14 we see that for almost all $x \in X$ and almost all $y \in \mathcal{F}_{ij}[x]$,

$$R(x, y)h(x)\psi_x = h(y)\psi_x,$$

hence $h(y)^{-1}R(x, y)h(x)$ stabilizes ψ_x. Note that in view of (10.24),

$$h(y)^{-1}R(x, y)h(x)(\bar{\mathbf{v}}, \bar{\mathbf{w}}) = \big(e^{\alpha(x,y)}\bar{\mathbf{v}}', e^{\alpha'(x,y)}\bar{\mathbf{w}}'\big),$$
$$\text{where } \alpha(x, y) \in \mathbb{R},\ \alpha'(x, y) \in \mathbb{R},\ \|\bar{\mathbf{v}}'\| = \|\bar{\mathbf{v}}\| \text{ and } \|\bar{\mathbf{w}}'\| = \|\bar{\mathbf{w}}\|.$$

For $i = 1, 2$ let $\mathrm{Conf}_x(\mathbf{L}_i)$ denote the subgroup of $GL(\mathbf{L}_i)$ which preserves the inner product $\langle\cdot,\cdot\rangle_x$ up to a scaling factor. Let $\mathrm{Conf}_x(\mathbf{L}) = \mathrm{Conf}_x(\mathbf{L}_1) \times \mathrm{Conf}_x(\mathbf{L}_2)$. Then, by an elementary variant of Lemma 10.15, since ψ_x is not supported on $\mathbf{L}_1 \times \{0\} \cup \{0\} \times \mathbf{L}_2$, we get

$$h(y)^{-1}R(x, y)h(x) \in K(x)$$

where $K(x)$ is a compact subset of $\mathrm{Conf}_x(\mathbf{L})$. Thus, $R(x, y) \in h(y)K(x)h(x)^{-1}$, and thus

$$\|R(x, y)\| \leq C(x)C(y).$$

Note that by reversing x and y we get $\|R(x, y)^{-1}\| \leq C(x)C(y)$. Therefore, by (10.24),

$$|\lambda_{ij}(x, y) - \lambda_{kr}(x, y)| \leq C(x)C(y).$$

This completes the proof of (10.23).

For any $\delta > 0$ we can choose a compact $K \subset X$ with $\nu(K) > 1 - \delta$ and $N < \infty$ such that $C(x) < N$ for $x \in K$. Now, the fact that ij and kr are synchronized follows from applying Lemma 9.5 to K. □

Proof of Proposition 10.2. — This follows immediately from Lemmas 10.18 and 10.17. □

Proof of Proposition 10.3. — Choose $\epsilon < \epsilon'/10$, where ϵ' is as in Proposition 8.5(b). By the multiplicative ergodic theorem, there exists a set $K_1'' \subset X$ with $\nu(K_1'') > 1-\theta$ and $T > 0$, such that for $x \in K_1''$ and $t > T$,

$$|\lambda_{ij}(x,t) - \lambda_i t| < \epsilon t, \tag{10.27}$$

where $\lambda_{ij}(x,t)$ is as in (9.2). Then, by Fubini's theorem there exists a set $K_2'' \subset K_1''$ with $\nu(K_2'') > 1-3\theta$ such that for $x \in K_2''$, for $(1-\theta)$-fraction of $u \in \mathcal{B}(x)$, $ux \in K_1''$.

Let K'' be as in Proposition 8.5(b) with $\delta = \theta$. We may assume that the conull set Ψ in Proposition 10.3 is such so that for $x \in \Psi$, $g_{-t}x \in K'' \cap K_2''$ for arbitrarily large $t > 0$.

Suppose $g_{-t}x \in K'' \cap K_2''$ and $y \in \mathcal{F}_{ij}[x]$. We may write

$$y = g_{t'}^{ij} u g_{-t'}^{ij} x = g_{s'} u g_{-t} x.$$

By the definition of $\mathcal{F}_{ij}[x,t']$, and since $g_{-t}x \in K_2''$, we have $g_{-t}x \in K_1''$ and for at least $(1-\theta)$-fraction of $y \in \mathcal{F}_{ij}[x,t']$, we have $ug_{-t}x \in K_1''$, and thus, in view of (10.27),

$$|s' - \lambda_i t'| \le \epsilon t \quad \text{and} \quad |t - \lambda_i t'| \le \epsilon t.$$

Therefore for $(1-\theta)$-fraction of $y \in \mathcal{F}_{ij}[x,t']$ or equivalently for $(1-\theta)$-fraction of $u \in \mathcal{B}(g_{-t}x)$,

$$|s' - t| \le 2\epsilon t. \tag{10.28}$$

Now suppose $\mathbf{v} \in \mathbf{H}(x)$. Note that if $\|R(x,y)\mathbf{v}\| \le C\|\mathbf{v}\|$, and s is as in Proposition 8.5, then $s > s' - O(1)$ (where the implied constant depends on C). Therefore, in view of (10.28), for $(1-\theta)$-fraction of $u \in \mathcal{B}(g_{-t}x)$, (8.13) holds. Thus, by Proposition 8.5(b), we have $\mathbf{v} \in \mathbf{E}(x)$. Thus, we can write

$$\mathbf{v} = \sum_{kr \in I_{\mathbf{v}}} \mathbf{v}_{kr}$$

where the indexing set $I_{\mathbf{v}}$ contains at most one r for each $k \in \Lambda'$. Without loss of generality, Ψ is such that for $x \in \Psi$, $g_{-t}x$ satisfies the conclusions of Proposition 4.15 infinitely often. Note that for $y \in \mathcal{F}_{ij}[x]$,

$$\|R(x,y)\mathbf{v}\| \ge \|R(x,y)\mathbf{v}_{kr}\| \ge e^{\lambda_{kr}(x,y)}\|\mathbf{v}_{kr}\|.$$

By assumption, for all $\ell > 0$ and for at least $1-\theta$ fraction of $y \in \mathcal{F}_{ij}[x,\ell]$, $\|R(x,y)\mathbf{v}\| \le C$. Therefore, for all $\ell > 0$ and for at least $(1-\theta)$ fraction of $y \in \mathcal{F}_{ij}[x,\ell]$, (10.22) holds. Then, by Lemma 10.18, for all $kr \in I_{\mathbf{v}}$, kr and ij are synchronized, i.e. $kr \in [ij]$. Therefore, for at least $(1-2\theta)$-fraction of $y' \in \mathcal{F}_{kr}[x,\ell]$,

$$\left\|R(x,y')\mathbf{v}_{kr}\right\| \le \left\|R(x,y')\mathbf{v}\right\| \le C'.$$

Now, by Definition 10.5, $\mathbf{v}_{kr}(x) \in \mathbf{E}_{kr,bdd}(x)$. Therefore, $\mathbf{v} \in \mathbf{E}_{[ij],bdd}(x)$. □

It follows from the proof of Proposition 10.3 that (10.1) holds.

11. Equivalence relations on W^+

Let GSpc denote the space of generalized subspaces of W^+. Let $\bar{\mathcal{H}}_{++}(x)$ denote the set of $M \in \mathcal{H}_{++}(x)$ such that $(I + M)\,\mathrm{Lie}(U^+)(x)$ is a subalgebra of $\mathrm{Lie}(\mathcal{G}_{++})(x)$. We have a map $\mathcal{U}_x : \bar{\mathcal{H}}_{++}(x) \times W^+(x) \to \mathrm{GSpc}$ taking the pair (M, v) to the generalized subspace it parametrizes. Let $\mathcal{U}_x^{-1}$ denote the inverse of this map (given a Lyapunov-adapted transversal $Z(x)$).

For $ij \in \tilde{\Lambda}$, let

$$\mathcal{E}_{ij}[x] = \left\{ \mathcal{Q} \in \mathrm{GSpc} \;:\; \mathbf{j}\big(\mathcal{U}_x^{-1}(\mathcal{Q})\big) \in \mathbf{E}_{[ij],bdd}(x) \right\}.$$

Motivation. — In view of Proposition 10.2 and Lemma 6.9(b), for any sufficiently small $\epsilon > 0$, the conditions that $\mathcal{Q} \in \mathcal{E}_{ij}[x]$ and $hd_x^{X_0}(\mathcal{Q}, U^+[x]) = O(\epsilon)$ imply the following: for "most" $y \in \mathcal{F}_{ij}[x]$,

$$hd_y^{X_0}\big(R(x,y)\mathcal{Q}, U^+[y]\big) = O(\epsilon).$$

A partition of $W^+[x]$. — Let $\mathfrak{B}_0$ denote the measurable partition constructed in Section 3 (see also Section 4.6). We denote the atom containing x by $\mathfrak{B}_0[x]$, and let $\mathfrak{B}_0(x) = \{v \in W^+(x) \;:\; v + x \in \mathfrak{B}_0[x]\}$. In this section, the only properties of $\mathfrak{B}_0$ we will use is that it is subordinate to W^+, and that the atoms $\mathfrak{B}_0[x]$ are relatively open in $W^+[x]$.

Equivalence relations. — Fix $x_0 \in X$. For $x, x' \in W^+[x_0]$ we say that

$$x' \sim_{ij} x \text{ if } x' \in \mathfrak{B}_0[x] \text{ and } U^+\big[x'\big] \in \mathcal{E}_{ij}[x].$$

Proposition **11.1.** — *The relation $\sim_{ij}$ is a (measurable) equivalence relation.*

The main part of the proof of Proposition 11.1 is the following:

Lemma **11.2.** — *There exists a subset $\Psi \subset X$ with $\nu(\Psi) = 1$ such that for any $ij \in \tilde{\Lambda}$, if $x_0 \in \Psi$, $x_1 \in \Psi$, $x_1 \in \mathfrak{B}_0[x_0]$ (so in particular $d^{X_0}(x_0, x_1) < 1/100$), and $U^+[x_1] \in \mathcal{E}_{ij}[x_0]$, then $\mathcal{E}_{ij}[x_1] = \mathcal{E}_{ij}[x_0]$.*

Warning. — We will consider the condition $x' \sim_{ij} x$ to be undefined unless x and x' both belong to the set Ψ of Lemma 11.2.

Motivation. — In view of Proposition 10.1, we can ensure, in the notation of Section 2.3 that for some $ij \in \tilde{\Lambda}$, $U^+[q_2']$ is close to $\mathcal{E}_{ij}[q_2]$; then in the limit we would have $U^+[\tilde{q}_2'] \in \mathcal{E}_{ij}[\tilde{q}_2]$, and thus $\tilde{q}_2' \sim_{ij} \tilde{q}_2$.

Proof of Proposition 11.1, assuming Lemma 11.2. — We have $0 \in \mathbf{E}_{[ij],bdd}(x)$, therefore,

$$U^+[x] \in \mathcal{E}_{ij}[x]. \tag{11.1}$$

Thus $x \sim_{ij} x$.

Suppose $x' \sim_{ij} x$. Then, $x' \in \mathfrak{B}_0[x]$, and so $x \in \mathfrak{B}_0[x']$. By (11.1), $U^+[x] \in \mathcal{E}_{ij}[x]$, and by Lemma 11.2, $\mathcal{E}_{ij}[x'] = \mathcal{E}_{ij}[x]$. Therefore, $U^+[x] \in \mathcal{E}_{ij}[x']$, and thus $x \sim_{ij} x'$.

Now suppose $x' \sim_{ij} x$ and $x'' \sim_{ij} x'$. Then, $x'' \in \mathfrak{B}_0[x]$. Also, $U^+[x''] \in \mathcal{E}_{ij}[x'] = \mathcal{E}_{ij}[x]$, therefore $x'' \sim_{ij} x$. □

Remark. — By Lemma 11.2, for $x, x' \in \Psi$, $x' \sim_{ij} x$ if and only if $x' \in \mathfrak{B}_0[x]$ and $\mathcal{E}_{ij}[x'] = \mathcal{E}_{ij}[x]$.

Outline of the proof of Lemma 11.2. — Intuitively, the condition $U^+[x_1] \in \mathcal{E}_{ij}[x_0]$ is the same as "$\mathcal{F}_{ij}[x_1]$ and $\mathcal{F}_{ij}[x_0]$ stay close", and "$U^+[x_1]$ and $U^+[x_0]$ stay close as we travel along $\mathcal{F}_{ij}[x_0]$ or $\mathcal{F}_{ij}[x_1]$", which is clearly an equivalence relation. We give some more detail below. Throughout the proof we will be using Lemma 9.2, without mentioning it explicitly.

Fix $\epsilon \ll 1/100$. Suppose $x_1 \in \mathfrak{B}_0[x_0]$, so in particular $d^{X_0}(x_0, x_1) < 1/100$, and suppose

$$hd_{x_0}^{X_0}\big(U^+[x_1], U^+[x_0]\big) = \epsilon.$$

Then, by Lemma 6.9(b),

$$\mathbf{j}\big(\mathcal{U}_{x_0}^{-1}\big(U^+[x_1]\big)\big) = O(\epsilon).$$

We are given that $U^+[x_1] \in \mathcal{E}_{ij}[x_0]$, thus $\mathbf{j}(\mathcal{U}_{x_0}^{-1}(U^+[x_1])) \in \mathbf{E}_{[ij],bdd}(x_0)$. Then, by Proposition 10.2, for most $y_0 \in \mathcal{F}_{ij}[x_0]$,

$$\big\|R(x_0, y_0)\mathbf{j}\big(\mathcal{U}_{x_0}^{-1}\big(U^+[x_1]\big)\big)\big\| = O(\epsilon).$$

We have

$$R(x_0, y_0)\mathbf{j}\big(\mathcal{U}_{x_0}^{-1}\big(U^+[x_1]\big)\big) = \mathbf{j}\big(\mathcal{U}_{y_0}^{-1}\big(U^+\big[y_1'\big]\big)\big),$$

for some $y_1' \in G[x_1]$. Then, by Lemma 6.9(b), for most $y_0 \in \mathcal{F}_{ij}[x_0]$,

$$hd_{y_0}^{X_0}\big(U^+\big[y_1'\big], U^+[y_0]\big) = O(\epsilon) \quad \text{for some } y_1' \in G[x_1].$$

It is not difficult to show that y_1' is near a point $y_1 \in \mathcal{F}_{ij}[x_1]$. Thus, for most $y_0 \in \mathcal{F}_{ij}[x_0]$,

$$hd_{y_0}^{X_0}\big(U^+[y_1], U^+[y_0]\big) = O(\epsilon) \quad \text{for some } y_1 \in \mathcal{F}_{ij}[x_1]. \tag{11.2}$$

Thus, most of the time $\mathcal{F}_{ij}[x_0]$ and $\mathcal{F}_{ij}[x_1]$ remain close, and also that for most $y_0 \in \mathcal{F}_{ij}[x_0]$, $U^+[y_1]$ and $U^+[y_0]$ remain close, for some $y_1 \in \mathcal{F}_{ij}[x_1]$.

Now suppose $\mathcal{Q}_1 \in \mathcal{E}_{ij}[x_1]$, and

$$hd_{x_1}^{X_0}\big(\mathcal{Q}_1, U^+[x_1]\big) = O(\epsilon).$$

Then, $\mathbf{j}(\mathcal{U}_{x_1}^{-1}(\mathcal{Q}_1)) \in \mathbf{E}_{[ij],bdd}(x_1)$, and thus, for most $y_1 \in \mathcal{F}_{ij}[x_1]$, using Proposition 10.2 and Lemma 6.9(b) twice as above, we get that for most $y_1 \in \mathcal{F}_{ij}[x_1]$,

$$hd_{y_1}^{X_0}\big(R(x_1, y_1)\mathcal{Q}_1, U^+[y_1]\big) = O(\epsilon). \tag{11.3}$$

In our notation, $R(x_1, y_1)\mathcal{Q}_1$ is the same generalized subspace (i.e. the same subset of W^+) as $R(x_0, y_0)\mathcal{Q}_1$ for $y_0 \in \mathcal{F}_{ij}[x_0]$ close to y_1. Then, from (11.2) and (11.3), for most $y_0 \in \mathcal{F}_{ij}[x_0]$,

$$hd_{y_0}^{X_0}\big(R(x_0, y_0)\mathcal{Q}_1, U^+[y_0]\big) = O(\epsilon).$$

Thus, using Lemma 6.9(b) again, we get that for most $y_0 \in \mathcal{F}_{ij}[x_0]$,

$$\big\| R(x_0, y_0)\mathbf{j}\big(\mathcal{U}_{x_0}^{-1}(\mathcal{Q}_1)\big)\big\| = O(\epsilon).$$

By Proposition 10.3, this implies that $\mathbf{j}(\mathcal{U}_{x_0}^{-1}(\mathcal{Q}_1)) \in \mathbf{E}_{[ij],bdd}(x_0)$, and thus $\mathcal{Q}_1 \in \mathcal{E}_{ij}[x_0]$. Thus, $\mathcal{E}_{ij}[x_1] \subset \mathcal{E}_{ij}[x_0]$.

Conversely, if $\mathcal{Q}_0 \in \mathcal{E}_{ij}[x_0]$, then the same argument shows that $\mathcal{Q}_0 \in \mathcal{E}_{ij}[x_1]$. Therefore, $\mathcal{E}_{ij}[x_0] = \mathcal{E}_{ij}[x_1]$. □

The (tedious) formal verification of Lemma 11.2 is given in Section 11.1* below.

The equivalence classes $\mathcal{C}_{ij}[x]$. — For $x \in \Psi$ we define the equivalence class

$$\mathcal{C}_{ij}[x] = \big\{x' \in \mathfrak{B}_0[x] \;:\; x' \sim_{ij} x\big\}.$$

Let $\mathcal{C}_{ij}$ denote the σ-algebra of ν-measurable sets which are unions of the equivalence classes $\mathcal{C}_{ij}[x]$. We do not distinguish between σ-algebras which are equivalent mod sets of ν-measure 0, so we can assume that $\mathcal{C}_{ij}$ is countably generated (see [CK, §1.2]). We now want to show that (away from a set of measure 0), the atoms of the σ-algebra $\mathcal{C}_{ij}$ are the sets $\mathcal{C}_{ij}[x]$. More precisely, we want to show that the partition $\mathcal{C}_{ij}$ whose atoms are the sets $\mathcal{C}_{ij}[x]$ is a measurable partition in the sense of [CK, Definition 1.10].

To see this, note that each set $\mathcal{E}_{ij}[x]$ is an algebraic subset of GSpc, and is thus parametrized by a finite dimensional space Y. Let $\psi_{ij} : X \to Y$ be the map taking x to the parametrization of $\mathcal{E}_{ij}[x]$. We note that the functions ψ_{ij} are measurable. Also, in view of Lemma 11.2, we have

$$x \sim_{ij} y \quad \text{if and only if } y \in \mathfrak{B}_0[x] \text{ and } \psi_{ij}(y) = \psi_{ij}(x).$$

By Lusin's theorem, for each ij, there exists a Borel function $\tilde{\psi}_{ij}$ such that ν-almost everywhere, $\tilde{\psi}_{ij} = \psi_{ij}$. Now the measurability of $\mathcal{C}_{ij}$ follows from [CK, Theorem 1.14].

Lemma **11.3.** — *Suppose* $t \in \mathbb{R}$, $u \in U^+(x)$.

(a) $g_t\mathcal{C}_{ij}[x] \cap \mathfrak{B}_0[g_tx] \cap g_t\mathfrak{B}_0[x] = \mathcal{C}_{ij}[g_tx] \cap \mathfrak{B}_0[g_tx] \cap g_t\mathfrak{B}_0[x]$.
(b) $u\mathcal{C}_{ij}[x] \cap \mathfrak{B}_0[ux] \cap u\mathfrak{B}_0[x] = \mathcal{C}_{ij}[ux] \cap \mathfrak{B}_0[ux] \cap u\mathfrak{B}_0[x]$.

Proof. — Note that the sets $U^+[x]$ and $\mathbf{E}_{[ij],bdd}(x)$ are g_t-equivariant. Therefore, so are the $\mathcal{E}_{ij}[x]$, which implies (a). Part (b) is also clear, since locally, by Lemma 8.2, $(u)_*\mathbf{E}_{ij}(x) = \mathbf{E}_{ij}(ux)$. □

The measures $f_{ij}[x]$. — We now define $f_{ij}[x]$ to be the conditional measure of ν along the $\mathcal{C}_{ij}[x]$. In other words, $f_{ij}[x]$ is defined so that for any measurable $\phi : X \to \mathbb{R}$,

$$\mathbb{E}(\phi \mid \mathcal{C}_{ij})(x) = \int_X \phi \, df_{ij}[x].$$

We view $f_{ij}[x]$ as a measure on $W^+[x]$ which is supported on $\mathcal{C}_{ij}[x]$.

The measures $f_{ij}(x)$. — We can identify $W^+[x]$ with the vector space $W^+(x)$, where x corresponds to the origin. Let $f_{ij}(x)$ be the pullback to $W^+(x)$ of $f_{ij}[x]$ under this identification. We will also call the $f_{ij}(x)$ conditional measures. (The term "leaf-wise" measures is used in [EL] in a related context.) We abuse notation slightly and write formulas such as

$$\mathbb{E}(\phi \mid \mathcal{C}_{ij})(x) = \int_X \phi \, df_{ij}(x).$$

The "distance" $d_*(\cdot,\cdot)$. — Suppose E_1, E_2 are open subsets of a normed vector space V, with $E_1 \cap E_2 \neq \emptyset$. Suppose that for $i = 1, 2$, μ_i is a finite measure on E_i, with $\mu_i(E_1 \cap E_2) > 0$. Then, let $d_*(\mu_1, \mu_2)$ denote the Kontorovich-Rubinstein distance between (the normalized versions of) $\bar{\mu}_1$ and $\bar{\mu}_2$, i.e.

$$d_*(\mu_1, \mu_2) = \sup_f \left| \frac{1}{\mu_1(E_1 \cap E_2)} \int_{E_1 \cap E_2} f \, d\mu_1 - \frac{1}{\mu_2(E_1 \cap E_2)} \int_{E_1 \cap E_2} f \, d\mu_2 \right|,$$

where the sup is taken over all 1-Lipshitz functions $f : E_1 \cap E_2 \to \mathbb{R}$ with $\sup |f(x)| \le 1$.

The only property of $d_*(\cdot,\cdot)$ we will use is that it induces the topology of weak-* convergence on the domain of common definition of the measures, up to normalization.

The following proposition is the rigorous version of (2.5) in Section 2.3:

Proposition **11.4.** — *There exists* $0 < \alpha_0 < 1$ *depending only on the Lyapunov spectrum, and for every* $\delta > 0$ *there exists a compact set* $K_0 \subset X$ *with* $\nu(K_0) > 1 - \delta$ *such that the following holds: Suppose* $ij \in \tilde{\Lambda}$, $1 < C_1 < \infty$, $0 < \epsilon < C_1^{-1}/100$, $C < \infty$, $t > 0$, $t' > 0$, *and* $|t' - t| < C$. *Furthermore suppose* $q \in \pi^{-1}(K_0)$ *and* $q' \in W^-[q] \cap \pi^{-1}(K_0)$ *are such that* $d^X(q, q') < 1/100$. *Let* $q_1 = g_\ell q$, $q'_1 = g_\ell q'$. *Also let* $q_3 = g^{ij}_{t'} q_1$, $q'_3 = g^{ij}_{t'} q'_1$. *Suppose* q_1, q'_1, q_3, q'_3 *all belong to* $\pi^{-1}(K_0)$.

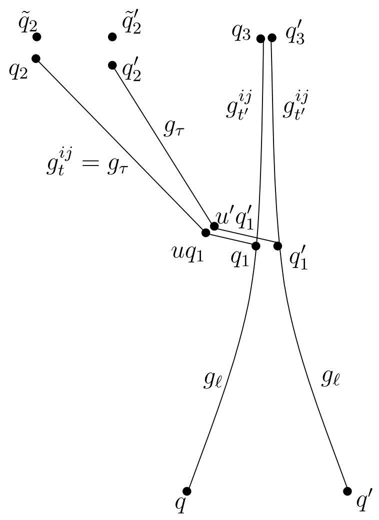

FIG. 4. — Proposition 11.4

Suppose $u \in \mathcal{B}(q_1, 1/100)$, $u' \in \mathcal{B}(q_1', 1/100)$. Let $q_2 = g_t^{ij} u q_1'$. We write $q_2 = g_\tau u q_1$ for some $\tau > 0$, and let $q_2' = g_\tau u' q_1'$ (see Figure 4). Also suppose $uq_1 \in \pi^{-1}(K_0)$, $u'q_1' \in \pi^{-1}(K_0)$, $q_2 \in \pi^{-1}(K_0)$, $q_2' \in \pi^{-1}(K_0)$ and

$$C_1^{-1}\epsilon \le hd_{q_2}^{X_0}\big(U^+[q_2], U^+[q_2']\big) \le C_1\epsilon \text{ and } \ell > \alpha_0\tau.$$

In addition, suppose there exist $\tilde{q}_2 \in \pi^{-1}(K_0)$ and $\tilde{q}_2' \in \pi^{-1}(K_0)$ such that $\sigma_0(\tilde{q}_2') \in W^+[\sigma_0(\tilde{q}_2)]$, and also $d^X(\tilde{q}_2, q_2) < \xi$ and $d^X(\tilde{q}_2', q_2') < \xi$. Then, provided ξ is small enough and t is large enough (depending on K_0),

$$\tilde{q}_2' \in W^+[\tilde{q}_2]. \tag{11.4}$$

Also, there exists $\xi''' > 0$ (depending on ξ, K_0 and C and t) with $\xi''' \to 0$ as $\xi \to 0$ and $t \to \infty$ such that

$$d_*\big(P^+(\tilde{q}_2, \tilde{q}_2') f_{ij}(\tilde{q}_2), f_{ij}(\tilde{q}_2')\big) \le \xi'''. \tag{11.5}$$

(In (11.5) we think of $f_{ij}(\tilde{q}_2')$ as a measure on $\mathfrak{B}_0[\tilde{q}_2']$, $P^+(\tilde{q}_2, \tilde{q}_2') f_{ij}(\tilde{q}_2)$ as a measure on $P^+(\tilde{q}_2, \tilde{q}_2')\mathfrak{B}_0[\tilde{q}_2]$, and we use the AGY norm $\|\cdot\|_Y$ on $W^+(\tilde{q}_2')$ for the norm in the definition of $d_*(\cdot,\cdot)$.)

Proposition 11.4 is proved in Section 11.2*. We give an outline of the argument below.

Outline of the proof of Proposition 11.4. — The initial intuition behind the proof of Proposition 11.4 is that "one goes from q_3' to q_2' by nearly the same linear map as from q_3 to q_2; since this map is bounded on the relevant subspaces, $f_{ij}(q_2)$ should be related to

$f_{ij}(q_3)$ and $f_{ij}(q_2')$ should be related to $f_{ij}(q_2)$; since $f_{ij}(q_3)$ and $f_{ij}(q_3')$ are close, $f_{ij}(q_2')$ should be related to $f_{ij}(q_2)$."

There are several problems with this argument. First, because of the need to change transversals, there is no linear map from the space $\mathrm{GSpc}(q_3)$ of generalized subspaces near q_3 to the space $\mathrm{GSpc}(q_2)$ of generalized subspaces near q_2. This difficulty is easily handled by working instead with the linear maps $\mathrm{R}(q_3, q_2) : \mathbf{H}(q_3) \to \mathbf{H}(q_2)$ and $\mathrm{R}(q_3', q_2') : \mathbf{H}(q_3') \to \mathbf{H}(q_2')$.

The second difficulty is connected to the first. We would like to say that the two maps $\mathrm{R}(q_3, q_2)$ and $\mathrm{R}(q_3', q_2')$ are close, but the domains and ranges of the maps are different. Thus we need "connecting" linear maps from $\mathbf{H}(q_3)$ to $\mathbf{H}(q_3')$, and also from $\mathbf{H}(q_2)$ to $\mathbf{H}(q_2')$. The first map is easy to construct: since q_3 and q_3' are in the same leaf of W^-, we can just use the linear map $\mathbf{P}^-(q_3, q_3')$ induced by the "W^--connection map" $\mathrm{P}^-(q_3, q_3')$ defined in Section 4.2.

Instead of constructing directly a map from $\mathbf{H}(q_2)$ to $\mathbf{H}(q_2')$ we construct, using the choice of transversal $\mathrm{Z}(\cdot)$, linear maps $\mathbf{P}^{\mathrm{Z}(q_2)}(q_2, \tilde{q}_2) : \mathbf{H}(q_2) \to \mathbf{H}(\tilde{q}_2)$ and $\mathbf{P}^{\mathrm{Z}(q_2')}(q_2', \tilde{q}_2') : \mathbf{H}(q_2') \to \mathbf{H}(\tilde{q}_2')$. Since q_2 and $\tilde{q}_2$ are close, and also since q_2' and $\tilde{q}_2'$ are close, these maps are in a suitable sense close to the identity. Then, since $\tilde{q}_2$ and $\tilde{q}_2'$ are on the same leaf of W^+, we have the map $\mathbf{P}^+(\tilde{q}_2, \tilde{q}_2')$ induced by the W^+-connection map $\mathrm{P}^+(\tilde{q}_2, \tilde{q}_2')$ of Section 4.2.

Thus, finally we have two maps from $\mathbf{H}(q_3)$ to $\mathbf{H}(\tilde{q}_2')$:

$$\mathbf{A} = \mathbf{P}^+\left(\tilde{q}_2, \tilde{q}_2'\right) \circ \mathbf{P}^{\mathrm{Z}(q_2)}(q_2, \tilde{q}_2) \circ \mathrm{R}(q_3, q_2)$$

and

$$\mathbf{A}' = \mathbf{P}^{\mathrm{Z}(q_2')}\left(q_2', \tilde{q}_2'\right) \circ \mathrm{R}\left(q_3', q_2'\right) \circ \mathbf{P}^-\left(q_3, q_3'\right).$$

Even though $\mathbf{A}$ and $\mathbf{A}'$ are defined on $\mathbf{H}(q_3)$, in what follows we only need to consider their restrictions to $\mathbf{E}_{[ij],bdd}(q_3) \subset \mathbf{H}(q_3)$; we will denote the restrictions by $\mathbf{B}$ and $\mathbf{B}'$ respectively.

We would like to show that $\mathbf{B}$ and $\mathbf{B}'$ are close. By linearity, it is enough to show that the restrictions of $\mathbf{B}$ and $\mathbf{B}'$ to each $\mathbf{E}_{ij,bdd}(q_3) \subset \mathbf{E}_{[ij],bdd}(q_3)$ are close. Note that by Proposition 4.12(a), $\mathbf{P}^-(q_3, q_3')\mathbf{E}_{ij,bdd}(q_3) = \mathbf{E}_{ij,bdd}(q_3')$. Continuing this argument, we see that the two subspaces $\mathbf{B}\mathbf{E}_{ij,bdd}(q_3)$ and $\mathbf{B}'\mathbf{E}_{ij,bdd}(q_3)$ are close to $\mathbf{E}_{ij,bdd}(\tilde{q}_2')$ (and thus are close to each other). Also, from the construction and Proposition 10.2, we see that both $\mathbf{B}$ and $\mathbf{B}'$ are uniformly bounded linear maps. However, this is still not enough to conclude that $\mathbf{B}$ and $\mathbf{B}'$ are close. In fact we also check that $\mathbf{B}$ and $\mathbf{B}'$ are close modulo $\mathbf{V}_{<i}(\tilde{q}_2)$. (This part of the argument uses the assumptions on q, q', q_1, q_1', etc.) Then we apply the elementary Lemma 11.5 below with $\mathrm{E} = \mathbf{E}_{ij,bdd}(q_3)$, $\mathrm{L} = \mathbf{H}(\tilde{q}_2')$, $\mathrm{F} = \mathbf{E}_{ij,bdd}(\tilde{q}_2')$, $\mathrm{V} = \mathbf{V}_{<i}(\tilde{q}_2')$ to get

$$\|\mathbf{B} - \mathbf{B}'\| \to 0 \text{ as } \xi \to 0. \tag{11.6}$$

The final part of the proof of Proposition 11.4 consists of deducing (11.5) from (11.6) and the fact that $\mathbf{B}$ and $\mathbf{B}'$ are uniformly bounded (Proposition 10.2).

Lemma **11.5.** — *Suppose* L *is a finite-dimensional normed vector space,* F *and* V *are subspaces of* L*, with* $F \cap V = \{0\}$*. Let* S *denote the unit sphere in* L*, and let* $hd(\cdot, \cdot)$ *denote the Hausdorff distance induced by the norm on* L*. Suppose* E *is another finite-dimensional normed vector space, and* $\mathbf{B} : E \to L$ *and* $\mathbf{B}' : E \to L$ *are two linear maps each of norm at most* C*. Let* π_V *denote the projection* $L \to L/V$*. Suppose* $\xi > 0$ *is such that*

(i) $\|\pi_V \circ \mathbf{B} - \pi_V \circ \mathbf{B}'\| \leq \xi$.
(ii) $hd(\mathbf{B}(E) \cap S, F \cap S) \leq \xi$.
(iii) $hd(\mathbf{B}'(E) \cap S, F \cap S) \leq \xi$.

Then, $\|\mathbf{B} - \mathbf{B}'\| \leq \xi'$*, where* ξ' *depends on* ξ*,* C *and the angle between* V *and* F*. Furthermore,* $\xi' \to 0$ *as* $\xi \to 0$ *(and the other parameters remain fixed).*

In the course of the proof, we will prove the following lemma, which will be used in Section 12:

Lemma **11.6.** — *For every* $\delta > 0$ *there exists a compact set* $K_0 \subset X$ *with* $\nu(K_0) > 1 - \delta$ *such that the following holds: Suppose* $x, x', y, y' \in \pi^{-1}(K_0)$*,* $y \in W^+[x]$*,* $y' \in W^+[x']$ *and* $x' \in W^-[x]$*. Suppose further that* $d^{X_0}(x, y) \leq 1/100$*,* $d^{X_0}(y, y') \leq 1/100$*, and that there exists* $s > 0$ *such that for all* $|\tau| \leq s$*,* $d^{X_0}(g_\tau x, g_\tau x') \leq 1/100$ *and* $d^{X_0}(g_\tau y, g_\tau y') \leq 1/100$*. Furthermore, suppose* $0 < \alpha_0 < 1$ *and that* $0 < t < \alpha_0^{-1} s$ *is such that* $d^{X_0}(g_t y, g_t y') < 1/100$*,* $g_t y \in K_0$ *and* $g_t y' \in K_0$*. Then, for all* $ij \in \Lambda''$*,*

$$\left|\hat{\tau}_{ij}(y, t) - \hat{\tau}_{ij}\big(y', t\big)\right| \leq C, \tag{11.7}$$

where C *depends only on* δ*,* α_0 *and the Lyapunov spectrum.*

11.1*. *Proof of Lemma 11.2.* — Let $\theta_1 > 0$ and $\delta > 0$ be small constants to be chosen later. Let $K \subset X$ and $C > 0$ be such that $\nu(K) > 1 - \delta$, for $x \in K$ the Lemma 6.9(b) holds with $c_1(x) > C^{-1}$, and for all $x \in K$, all $\mathbf{v} \in \mathbf{E}_{[ij],bdd}(x)$ and all $\ell > 0$, for at least $(1 - \theta_1)$ fraction of $y \in \mathcal{F}_{ij}[x, \ell]$,

$$\|R(x, y)\mathbf{v}\| < C\|\mathbf{v}\|. \tag{11.8}$$

By Lemma 9.5 there exists a subset $K^* \subset K$ with $\nu(K^*) \geq (1 - 2\kappa^2\delta^{1/2})$ such that for $x \in K^*$, (9.12) holds with $\theta' = \delta^{1/2}$. Furthermore, we may ensure that for $x \in K^*$, $K^* \cap \mathcal{F}_{ij}[x]$ is relatively open in $\mathcal{F}_{ij}[x]$. (Indeed, suppose $z \in \mathcal{F}_{ij}[x]$ is near $x \in K^*$. Then, there exists ℓ_0 such that for $\ell > \ell_0$, $\mathcal{F}_{ij}[x, \ell] = \mathcal{F}_{ij}[z, \ell]$ and thus (9.12) holds for z. For $\ell < \ell_0$, (9.12) holds for z sufficiently close to x by continuity.) Let

$$\Psi = \Big\{x \in X \,:\, \lim_{T\to\infty} \Big|\big\{t \in [0, T] \,:\, g_{-t}x \in K^*\big\}\Big| \geq \big(1 - 2\kappa^2\delta^{1/2}\big)\Big\}.$$

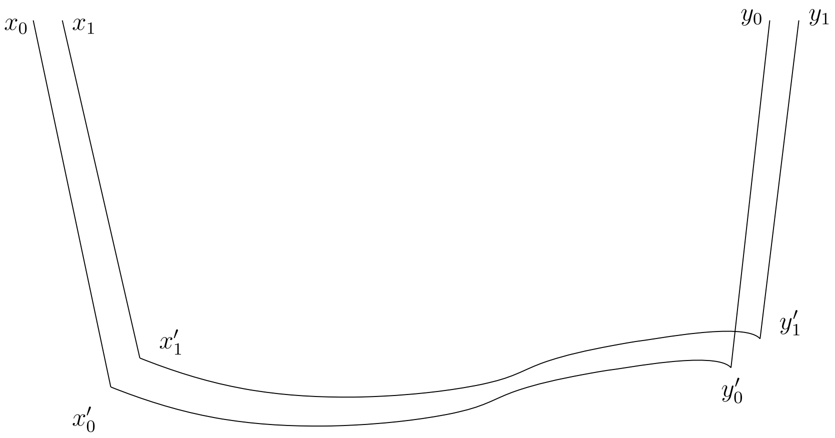

FIG. 5. — Proof of Lemma 11.2

Then $\nu(\Psi) = 1$. From its definition, Ψ is invariant under g_t. Since $K^* \cap \mathcal{F}_{ij}[x]$ is relatively open in $\mathcal{F}_{ij}[x]$, Ψ is saturated by the leaves of $\mathcal{F}_{ij}$. This implies that Ψ is (locally) invariant under U^+. Now, let

$$K_N = \Big\{x \in \Psi \ : \\ \text{for all } T > N, \left|\left\{t \in [0, T] \ : \ g_{-t}x \in K^*\right\}\right| \geq \left(1 - 4\kappa^2\delta^{1/2}\right)T\Big\}.$$

(We may assume that $4\kappa^2\delta^{1/2} \ll 1$.) We have $\bigcup_N K_N = \Psi$.

Suppose $x_0 \in K_N$, $x_1 \in \mathfrak{B}_0[x_0] \cap K_N$, so $d^{X_0}(x_0, x_1) < 1/100$. For $k = 0, 1$, let $\mathcal{Q}_k \subset \mathcal{E}_{ij}[x_k]$ be such that

$$hd^{X_0}_{x_k}\left(\mathcal{Q}_k, U^+[x_k]\right) \leq 1/100,$$

and the vector

$$\mathbf{v}_k = \mathbf{j}\left(\mathcal{U}_{x_k}^{-1}(\mathcal{Q}_{1-k})\right)$$

satisfies $\|\mathbf{v}_k\| \leq 1/100$.

We claim that $\mathbf{v}_k \in \mathbf{H}(x_k)$. Indeed, we may write $\mathcal{U}_{x_{1-k}}^{-1}(\mathcal{Q}_{1-k}) = (M_{1-k}, v_{1-k})$. Also we may write $\mathcal{U}_{x_k}^{-1}(U^+[x_{1-k}]) = (M_k', v_k')$. Then, $\mathcal{Q}_{1-k}$ is parametrized (from x_k) by a pair (M_k'', w_k) where $w_k \in W^+(x_k)$, and

$$M_k'' = (I + M_{1-k}) \circ \left(I + M_k'\right) - I$$

(This parametrization is not necessarily adapted to $Z(x_k)$.) Since M_{1-k} and M_k' are both in $\mathcal{H}_{++}$, $M_k'' \in \mathcal{H}_{++}(x_k)$. Thus, $\mathbf{v}_k = \mathbf{S}_{x_k}(\mathbf{j}(M_k'', w_k)) \in \mathbf{H}(x_k)$.

For $C_1(N)$ sufficiently large, we can find $C_1(N) < t < 2C_1(N)$ such that $x_0' \equiv g_{-t}^{ij}x_0 \in K^*$, $x_1' \equiv g_{-t}^{ij}x_1 \in K^*$, see Figure 5. By Lemma 9.2, $x_1' \in \mathfrak{B}_0[x_0']$. Let $\mathbf{v}_k' = g_{-t}^{ij}\mathbf{v}_k$,

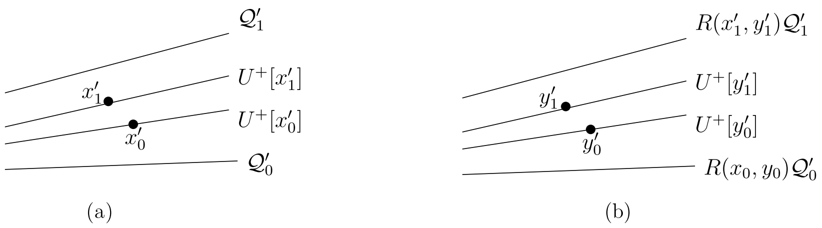

FIG. 6. — Proof of Lemma 11.2. In (b), the subspaces $\mathrm{U}^+[y'_0]$ and $\mathrm{U}^+[y'_1]$ stay close since $x'_1 \in \mathcal{E}_{ij}(x'_0)$, and also for $k \in \{0, 1\}$, the subspaces $\mathrm{R}(x'_k, y'_k)\mathcal{Q}'_k$ and $\mathrm{U}^+[y'_k]$ stay close since $\mathcal{Q}'_k \in \mathcal{E}_{ij,bdd}(x'_k)$

$\mathcal{Q}'_k = g^{ij}_{-t}\mathcal{Q}_k$, see Figure 6. By choosing $\mathrm{C}_1(\mathrm{N})$ sufficiently large (depending on N), we can ensure that

$$hd^{\mathrm{X}_0}_{x'_k}\big(\mathrm{U}^+[x'_k], \mathrm{U}^+[x'_{1-k}]\big) \le \mathrm{C}^{-3}, \qquad hd^{\mathrm{X}_0}_{x'_k}\big(\mathcal{Q}'_k, \mathrm{U}^+[x'_k]\big) \le \mathrm{C}^{-3}.$$

By Lemma 6.9, since $x'_k \in \mathrm{K}$,

$$\textbf{(11.9)} \qquad \big\|\mathbf{j}\big(\mathcal{U}^{-1}_{x'_k}\big(\mathrm{U}^+[x'_{1-k}]\big)\big)\big\| \le \mathrm{C}^{-2}, \qquad \big\|\mathbf{j}\big(\mathcal{U}^{-1}_{x'_k}(\mathcal{Q}'_k)\big)\big\| \le \mathrm{C}^{-2}.$$

Let $\ell > 0$ be arbitrary, and let ℓ' be such that $g^{ij}_t \mathcal{F}_{ij}[x, \ell'] = \mathcal{F}_{ij}[x, \ell]$. Then, for $k = 0, 1$, since $x'_k \in \mathrm{K}^*$,

$$\big|\big\{ y'_k \in \mathcal{F}_{ij}[x'_k, \ell'] \;:\; y'_k \in \mathrm{K}\big\}\big| \ge \big(1 - \delta^{1/2}\big)\big|\mathcal{F}_{ij}[x', \ell']\big|.$$

Since $\mathrm{U}^+[x_1] \in \mathcal{E}_{ij}[x_0]$, we have $\mathrm{U}^+[x'_1] \in \mathcal{E}_{ij}[x'_0]$, and thus $\mathbf{j}(\mathcal{U}^{-1}_{x'_0}(\mathrm{U}^+[x'_1])) \in \mathbf{E}_{[ij],bdd}(x'_0)$. Since $x'_0 \in \mathrm{K}$, we have by (11.8), for at least $(1 - \theta_1)$-fraction of $y'_0 \in \mathcal{F}_{ij}[x'_0, \ell']$,

$$\textbf{(11.10)} \qquad \big\|\mathrm{R}\big(x'_0, y'_0\big)\mathbf{j}\big(\mathcal{U}^{-1}_{x'_0}\big(\mathrm{U}^+[x'_1]\big)\big)\big\| \le \mathrm{C}\big\|\mathbf{j}\big(\mathcal{U}^{-1}_{x'_0}\big(\mathrm{U}^+[x'_1]\big)\big)\big\| \le \mathrm{C}^{-1},$$

where we have used (11.9) for the last estimate. Let $\theta'' = 2\theta_1 + 2\delta^{1/2}$. Then, for at least $1 - \theta''/2$ fraction of $y'_0 \in \mathcal{F}_{ij}[x'_0, \ell']$, $y'_0 \in \mathrm{K}$ and (11.10) holds. Therefore, by Lemma 6.9, for at least $(1 - \theta''/2)$-fraction of $y'_0 \in \mathcal{F}_{ij}[x'_0, \ell']$, for a suitable $y'_1 \in \mathcal{F}_{ij}[x'_1, \ell']$,

$$\textbf{(11.11)} \qquad hd^{\mathrm{X}_0}_{y'_0}\big(\mathrm{U}^+[y'_0], \mathrm{U}^+[y'_1]\big) \le 1/100.$$

Also, since $\mathcal{Q}_k \in \mathcal{E}_{ij}[x_k]$, $\mathcal{Q}'_k \in \mathcal{E}_{ij}[x'_k]$, and thus $\mathbf{j}(\mathcal{U}^{-1}_{x'_k}(\mathcal{Q}'_k)) \in \mathbf{E}_{[ij],bdd}(x'_k)$. Hence, by (11.8), for at least $(1 - \theta)$-fraction of $y'_k \in \mathcal{F}_{ij}[x'_k, \ell']$,

$$\textbf{(11.12)} \qquad \big\|\mathrm{R}\big(x'_k, y'_k\big)\mathbf{j}\big(\mathcal{U}^{-1}_{x'_k}(\mathcal{Q}'_k)\big)\big\| \le \mathrm{C}\big\|\mathbf{j}\big(\mathcal{U}^{-1}_{x'_k}(\mathcal{Q}'_k)\big)\big\| \le \mathrm{C}^{-1},$$

where we used (11.9) for the last estimate. Then, for at least $(1 - \theta''/2)$-fraction of $y'_k \in \mathcal{F}_{ij}[x'_k, \ell']$, $y'_k \in \mathrm{K}$ and (11.12) holds. Therefore, by Lemma 6.9, for at least $(1 - \theta''/2)$-fraction of $y'_k \in \mathcal{F}_{ij}[x'_k, \ell']$,

$$hd^{\mathrm{X}_0}_{y'_k}\big(\mathrm{U}^+[y'_k], \mathrm{R}\big(x'_k, y'_k\big)\mathcal{Q}'_k\big) \le 1/100.$$

Therefore, by (11.11), for at least $(1-\theta'')$-fraction of $y'_k \in \mathcal{F}_{ij}[x'_k, \ell']$, for a suitable $y'_{1-k} \in \mathcal{F}_{ij}[x'_{1-k}, \ell']$,

$$hd^{X_0}_{y'_k}\big(U^+[y'_k], R(x'_{1-k}, y'_{1-k})\mathcal{Q}'_{1-k}\big) \le 1/50. \tag{11.13}$$

Let

$$\mathbf{w}'_k = \mathbf{j}\big(\mathcal{U}^{-1}_{y'_k}\big(R(x'_{1-k}, y'_{1-k})\mathcal{Q}'_{1-k}\big)\big) = R(x'_k, y'_k)\mathbf{v}'_k.$$

Then, assuming $y'_0 \in K$ and (11.13) holds, by Lemma 6.9,

$$\|\mathbf{w}'_k\| \le C.$$

Let $y_k = g^{ij}_t y'_k$, and let

$$\mathbf{w}_k = R(y'_k, y_k)\mathbf{w}'_k = R(x_k, y_k)\mathbf{v}_k.$$

Then, for at least $(1-\theta'')$-fraction of $y_k \in \mathcal{F}_{ij}[x_k, \ell]$, $\|R(x_k, y_k)\mathbf{v}_k\| \le C_2(N)$. This implies, by Proposition 10.3, that $\mathbf{v}_k \in \mathbf{E}_{[ij],bdd}(x_k)$. (By making $\theta_1 > 0$ and $\delta > 0$ sufficiently small, we can make sure that $\theta'' < \theta$ where $\theta > 0$ is as in Proposition 10.3.)

Thus, for all $\mathcal{Q}_k \in \mathcal{E}_{ij}[x_k]$ such that $\mathbf{j}(\mathcal{U}^{-1}_{x_{1-k}}(\mathcal{Q}_k)) \le 1/100$, we have $\mathbf{j}(\mathcal{U}^{-1}_{x_{1-k}}(\mathcal{Q}_k)) \in \mathbf{E}_{[ij],bdd}(x_{1-k})$. Since both $\mathcal{U}^{-1}_{x_{1-k}}$ and $\mathbf{j}$ are analytic, this implies that $\mathbf{j}(\mathcal{U}^{-1}_{x_{1-k}}(\mathcal{Q}_k)) \in \mathbf{E}_{[ij],bdd}(x_{1-k})$ for all $\mathcal{Q}_k \in \mathcal{E}_{ij}[x_k]$. Thus, for $k = 0, 1$, $\mathcal{E}_{ij}[x_k] \subset \mathcal{E}_{ij}[x_{1-k}]$. This implies that $\mathcal{E}_{ij}[x_0] = \mathcal{E}_{ij}[x_1]$. □

11.2*. *Proof of Proposition 11.4.* — Let $\mathcal{O} \subset X$ be an open set contained in the fundamental domain, and let $x \to u_x \in U^+(x)$ be a function which is constant on each set of the form $U^+[x] \cap \mathcal{O}$. Let $T_u : \mathcal{O} \to X$ be the map which takes $x \to u_x x$.

Lemma **11.7.** — *Suppose* $E \subset \mathcal{O}$. *Then* $\nu(T_u(E)) = \nu(E)$.

Proof. — Without loss of generality, we may assume that $T_u(\mathcal{O}) \cap \mathcal{O} = \emptyset$. For each $x \in \mathcal{O}$, let $\tilde{U}[x]$ be a finite piece of $U^+[x]$ which contains both $U[x] \cap \mathcal{O}$ and $T_u(U[x] \cap \mathcal{O})$. We may assume that $\tilde{U}[x]$ is the same for all $x \in U[x] \cap \mathcal{O}$. Let $\tilde{\mathcal{U}}$ be the σ-algebra of functions which are constant along each $\tilde{U}[x]$. Then, for any measurable $\phi : X \to \mathbb{R}$,

$$\int_X \phi\, d\nu = \int_X \mathbb{E}(\phi \mid \tilde{\mathcal{U}})\, d\nu$$

Now suppose ϕ is supported on $\mathcal{O}$. We have $\mathbb{E}(\phi \circ T_u \mid \tilde{\mathcal{U}}) = \mathbb{E}(\phi \mid \tilde{\mathcal{U}})$ since the conditional measures along U^+ are Haar, and T_u restricted to $\mathcal{O} \cap U^+[x]$ is a translation. Thus

$$\int_X \phi \circ T_u\, d\nu = \int_X \mathbb{E}(\phi \circ T_u \mid \mathcal{U})\, d\nu = \int_X \mathbb{E}(\phi \mid \tilde{\mathcal{U}})\, d\nu = \int_X \phi\, d\nu. \qquad \square$$

We also recall the following standard fact:

Lemma **11.8.** — *Suppose* $\Psi : X \to X$ *preserves* ν, *and also for almost all* x, $\mathcal{C}_{ij}[\Psi(x)] \cap \mathfrak{B}_0[\Psi(x)] \cap \Psi(\mathfrak{B}_0[x]) = \Psi(\mathcal{C}_{ij}[x]) \cap \mathfrak{B}_0[\Psi(x)] \cap \Psi(\mathfrak{B}_0[x])$. *Then,*

$$f_{ij}\big(\Psi(x)\big) \propto \Psi_* f_{ij}(x),$$

in the sense that the restriction of both measures to the set $\mathfrak{B}_0[\Psi(x)] \cap \Psi(\mathfrak{B}_0[x])$ *where both make sense is the same up to normalization.*

Proof. — See [EL, Lemma 4.2(iv)]. □

Lemma **11.9.** — *We have (on the set where both are defined):*

$$f_{ij}(g_t T_u g_{-s} x) \propto (g_t T_u g_{-s})_* f_{ij}(x).$$

Proof. — This follows immediately from Lemma 11.7 and 11.8. □

The maps ϕ_x. — We have the map $\phi_x : W^+(x) \to \mathcal{H}_{++}(x) \times W^+(x)$ given by

$$\phi_x(z) = \mathcal{U}_x^{-1}\big(U^+[z]\big). \tag{11.14}$$

(Here $\mathcal{U}_x^{-1}$ is defined using the transversal $Z(x)$.)

Suppose $Z(x)$ is an admissible transversal to $U^+(x)$. Since $f_{ij}(x)$ is Haar along U^+, we can recover $f_{ij}(x)$ from its restriction to $Z(x)$. More precisely, the following holds:

Let $\pi_2 : \mathcal{H}_{++}(x) \times W^+(x) \to W^+(x)$ be projection onto the second factor. Then, for $z \in Z(x)$, $\pi_2(\phi(z)) = z$. Now, suppose Z' is another transversal to $U^+(x)$. Then,

$$(f_{ij}\,|_{Z'})(x) = \big(\pi_2 \circ S_x^{Z'} \circ \phi\big)_*(f_{ij}\,|_{Z(x)}).$$

The measures $\mathbf{f}_{ij}(x)$. — Let

$$\mathbf{f}_{ij}(x) = (\mathbf{j} \circ \phi_x)_* f_{ij}(x).$$

Then, $\mathbf{f}_{ij}(x)$ is a measure on $\mathbf{H}(x)$.

Lemma **11.10.** — *For* $y \in \mathcal{F}_{ij}[x]$, *we have (on the set where both are defined),*

$$\mathbf{f}_{ij}(y) \propto R(x,y)_* \mathbf{f}_{ij}(x).$$

Proof. — Suppose $t > 0$ is such that $x' = g_{-t}^{ij} x$ and $y' = g_{-t}^{ij} y$ satisfy $y' \in \mathcal{B}[x']$. Working in the universal cover, let $Z[x] = \{z \,:\, z - x \in Z(x)\}$. Let $Z[x'] = g_{-t}^{ij} Z[x]$, and let $Z[y'] = g_{-t}^{ij} Z[y]$. For $z \in Z[x']$ near x', let u_z be such that $u_z z \in Z[y']$. We extend the function $z \to u_z$ to be locally constant along U^+ in a neighborhood of $Z[x']$. Then, let

$$\Psi = g_t^{ij} \circ T_u \circ g_{-t}^{ij}.$$

Note that Ψ takes $Z[x]$ into $Z[y]$, and by Lemma 11.9,

$$\Psi_* f_{ij}(x) \propto f_{ij}(y). \tag{11.15}$$

By the definition of u_* in Section 6, for $z \in Z[x]$,

$$\big(R(x,y) \circ \mathbf{j} \circ \mathcal{U}_x^{-1}\big)U^+[z] = \big(\mathbf{j} \circ \mathcal{U}_y^{-1}\big)U^+\big[\Psi(z)\big].$$

Hence, by (11.14),

$$\big(R(x,y) \circ \mathbf{j} \circ \phi_x\big)(z) = (\mathbf{j} \circ \phi_y \circ \Psi)(z), \tag{11.16}$$

where ϕ_y is relative to the transversal $Z(y)$ and ϕ_x is relative to the transversal $Z(x)$. (Here we have used the fact that $\Psi(U^+[z]) = U^+[\Psi(z)]$ which follows from the equivariance of U^+. Also, in (11.16), $R(x,y)$ is as in Section 9.3.) Now the lemma follows from (11.15) and (11.16). □

Let $P^+(x,y)$ and $P^-(x,y)$ be as in Section 4.2. The maps $P^+(x,y)_* : \mathrm{Lie}(\mathcal{G}_{++})(x) \to \mathrm{Lie}(\mathcal{G}_{++})(y)$ (where we use the notation (6.11)) are an equivariant measurable flat W^+-connection on the bundle $\mathrm{Lie}(\mathcal{G}_{++})$ satisfying (4.5). Then, by Proposition 4.12(a),

$$P^+(x,y)_* \,\mathrm{Lie}\big(U^+\big)(x) = \mathrm{Lie}\big(U^+\big)(y). \tag{11.17}$$

The maps $\mathbf{P}^+(x,y)$ *and* $\mathbf{P}^-(x,y)$. — In view of (11.17), the maps $P^+(x,y)$ naturally induce a linear map (which we denote by $\tilde{\mathbf{P}}^+(x,y)$) from $\tilde{\mathbf{H}}(x)$ to $\tilde{\mathbf{H}}(y)$, so that for $(M, v) \in \mathcal{H}_{++}(x)$,

$$\tilde{\mathbf{P}}^+(x,y) \circ \mathbf{j}(M, v) = \mathbf{j}\big(P^+(x,y) \circ M \circ P^+(x,y)^{-1}, P^+(x,y)v\big).$$

Let $\mathbf{P}^+(x,y) = \mathbf{S}_y^{Z(y)} \circ \tilde{\mathbf{P}}^+(x,y)$. Then the maps $\mathbf{P}^+(x,y) : \mathbf{H}(x) \to \mathbf{H}(y)$ are an equivariant measurable flat W^+-connection on the bundle $\mathbf{H}$ satisfying (4.5). Then, by Proposition 4.12(a), we have

$$\mathbf{P}^+(x,y)\mathbf{E}_{ij,bdd}(x) = \mathbf{E}_{ij,bdd}(y). \tag{11.18}$$

For $y \in W^-[x]$, we have a map $\mathbf{P}^-(x,y)$ with analogous properties.

The maps $P^Z(x,y)$ *and* $\mathbf{P}^Z(x,y)$. — We also need to define a map between $\mathbf{H}(x)$ and $\mathbf{H}(y)$ even if x and y are not on the same leaf of W^+ or W^-. For every $v_i \in \mathcal{V}_i(x) \equiv \mathcal{V}_i(H^1)(x)$, and $i \in \Lambda$ (where Λ is the Lyapunov spectrum) we can write

$$v_i = v_i' + v_i'' \quad v_i' \in \mathcal{V}_i\big(H^1\big)(y), \quad v_i'' \in \bigoplus_{j \neq i} \mathcal{V}_i\big(H^1\big)(y).$$

Let $P^\sharp(x,y) : H^1(x) \to H^1(y)$ be the linear map whose restriction to $\mathcal{V}_i(H^1)(x)$ sends v_i to v_i'. By definition, $P^\sharp(x,y)$ sends $\mathcal{V}_i(H^1)(x)$ to $\mathcal{V}_i(H^1)(y)$, but it is not clear that

$P^\sharp(x,y)_* \operatorname{Lie}(U^+)(x) = \operatorname{Lie}(U^+)(y)$. To correct this, given a Lyapunov-adapted transversal $Z(x)$, note that (for y near x),

$$\operatorname{Lie}(\mathcal{G}_{++})(x) = P^\sharp(x,y)_*^{-1} \operatorname{Lie}\bigl(U^+\bigr)(y) \oplus Z(x).$$

Then, given $v \in \operatorname{Lie}(U^+)(x) \subset \operatorname{Lie}(\mathcal{G}_{++})(x)$, we can decompose

$$v = v' + v'', \quad v' \in P^\sharp(x,y)_*^{-1} \operatorname{Lie}\bigl(U^+\bigr)(y), \quad v'' \in Z(x). \tag{11.19}$$

Define $M(x;y) : \operatorname{Lie}(U^+)(x) \to \operatorname{Lie}(\mathcal{G}_{++})(x)$ by

$$Mv = -v''. \tag{11.20}$$

Then, since $Z(x)$ is Lyapunov adapted, $M(x;y) : \operatorname{Lie}(U^+)(x) \to \operatorname{Lie}(\mathcal{G}_{++})(x)$ is the linear map such that

$$\bigl(I + M(x;y)\bigr) \operatorname{Lie}\bigl(U^+\bigr)(x) = P^\sharp(x,y)_*^{-1} \operatorname{Lie}\bigl(U^+\bigr)(y), \tag{11.21}$$

and $M(x;y)\mathcal{V}_i(\operatorname{Lie}(U^+))(x) \subset Z_i(x)$, where $Z_i(x) = Z(x) \cap \mathcal{V}_i(\operatorname{Lie}(\mathcal{G}_{++}))(x)$ is as in Section 6. Then, let $P^{Z(x)}(x,y) : \mathcal{H}_{++}(x) \to \mathcal{H}_{++}(y)$ be the map taking $f \in \mathcal{H}_{++}(x)$ to

$$P^{Z(x)}(x,y)f \equiv P^\sharp(x,y)_* \circ f \circ \bigl(I + M(x;y)\bigr)^{-1} \circ P^\sharp(x,y)_*^{-1} \in \mathcal{H}_{++}(y).$$

Then, since $M(x;y)\mathcal{V}_i(\operatorname{Lie}(U^+))(x) \subset \mathcal{V}_i(\operatorname{Lie}(\mathcal{G}_{++}))(x)$ we have for a.e. x, y,

$$P^{Z(x)}(x,y)\mathcal{V}_i(\mathcal{H}_{++})(x) = \mathcal{V}_i(\mathcal{H}_{++})(y).$$

Then $P^{Z(x)}$ gives a map $\tilde{P}^{Z(x)}(x,y) : \mathcal{H}_{++}(x) \times W^+(x) \to \mathcal{H}_{++}(y) \times W^+(y)$ given by

$$\tilde{P}^{Z(x)}(x,y)(f, v) = \bigl(P^{Z(x)}(x,y)f, P^\sharp(x,y)v\bigr).$$

Therefore (after possibly composing with a change in transversal map $\mathbf{S}$) $\tilde{P}^{Z(x)}(x,y)$ induces a map we will call $\mathbf{P}^{Z(x)}(x,y)$ between $\mathbf{H}(x)$ and $\mathbf{H}(y)$. This map satisfies

$$\mathbf{P}^{Z(x)}(x,y)\mathcal{V}_i(\mathbf{H})(x) = \mathcal{V}_i(\mathbf{H})(y), \tag{11.22}$$

and has the equivariance property

$$\mathbf{P}^{g_{-t}Z(x)}(g_{-t}x, g_{-t}y) = g_{-t} \circ \mathbf{P}^{Z(x)}(x,y) \circ g_t.$$

Lemma **11.11.** — *For $y \in W^+[x]$, and any choice of $Z(x)$, we have*

$$\mathbf{P}^{Z(x)}(x,y) = \mathbf{P}^+(x,y). \tag{11.23}$$

Proof. — Suppose $y \in W^+[x]$. Then by Lemma 4.1, $P^\sharp(x,y) = P^+(x,y)$, thus

$$P^\sharp(x,y)_*^{-1} \operatorname{Lie}\big(U^+\big)(y) = P^+(x,y)_*^{-1} \operatorname{Lie}\big(U^+\big)(x) = \operatorname{Lie}\big(U^+\big)(x)$$

where for the last equality we used Proposition 4.12(a). Hence, $M(x;y) = 0$ and (11.23) follows. □

Lemma **11.12.** — *For any* $\delta > 0$ *there exists a compact subset* $K \subset X_0$ *with* $\nu(K) > 1 - \delta/2$ *such that the following holds: Suppose* x *and* $y \in \pi^{-1}(K)$, *and* $s > 0$ *are such that for all* $|t| < s$, $d^{X_0}(g_t x, g_t y) < 1/100$. *Then, there exists* $\alpha > 0$ *depending only on the Lyapunov spectrum, and* $C = C(\delta)$ *such that for all* i,

$$d_Y\big(P^{GM}(x,y)\mathcal{V}_i\big(H^1\big)(x), \mathcal{V}_i\big(H^1\big)(y)\big) \le C(\delta)e^{-\alpha s}.$$

Proof. — There exists a compact subset $K_1 \subset X_0$ such that the functions $x \to \mathcal{V}_i(H^1)(x)$ are uniformly continuous. (Here we are using the Gauss-Manin connection to identify $H_1(x)$ with $H^1(y)$ for y near x.) Then, there exists $\sigma > 0$ such that if $x \in \pi^{-1}(K_1)$, $y \in \pi^{-1}(K_1)$ and $d^{X_0}(x,y) < \sigma$ then $D^{(}x,y) < 1$ and $D^-(x,y) < 1$. (See Section 4.5 for the definition of $D^\pm(\cdot,\cdot)$.) We also may assume that there exists a constant $C_0(\delta)$ such that $C(x) < C_0(\delta)$ for all $x \in K_1$, where $C(\cdot)$ is as in Lemma 4.7. Then there exists a compact subset $K \subset X$ with $\nu(K) > 1 - \delta$, and $t_0 > 0$ such that for $x \in K$, for $t > t_0$, for $(1-\delta)$-fraction of $t \in [0,s]$, $g_t x \in K_1$, $g_{-t}x \in K_1$ also for at least half the fraction of $t \in [0,s]$, $g_t x$ and $g_{-t}x$ belong to K_{thick} where K_{thick} is as in Lemma 3.5.

Suppose $x \in \pi^{-1}(K)$, and $y \in \pi^{-1}(K)$. Then, by Lemma 3.5, there exists $\alpha_1 > 0$ depending only on the Lyapunov spectrum such that there exists $t \in [\alpha_1 s, s]$ with $g_t x \in K_1$, $g_t y \in K_1$ and $d^{X_0}(g_t x, g_t y) < \sigma$. Then, $D^-(g_t x, g_t y) < 1$. Then, by Lemma 4.7, applied to the points $g_t x$, $g_t y$, we get

$$d_Y\big(\mathcal{V}_{\ge i}\big(H^1\big)(x), \mathcal{V}_{\ge i}\big(H^1\big)(y)\big) \le C(\delta)e^{-\alpha t} = C(\delta)e^{-\alpha\alpha_1 s}.$$

Similarly, there exists $t \in [\alpha_1 s, s]$ with $g_{-t}x \in K_1$ and $g_{-t}y \in K_1$. Then, we get

$$d_Y\big(\mathcal{V}_{\le i}\big(H^1\big)(x), \mathcal{V}_{\le i}\big(H^1\big)(y)\big) \le C(\delta)e^{-\alpha t} = C(\delta)e^{-\alpha\alpha_1 s}.$$

The lemma follows. □

For every $\delta > 0$ and every $0 < \alpha < 1$ there exist compact sets $K_0 \subset K^\sharp \subset X$ with $\nu(K_0) > 1 - \delta$ such that the following hold:

($K^\sharp$1) The functions $U^+(x)$, $\mathcal{V}_i(H^1)(x)$ and more generally, $\mathcal{V}_i(H_{big})(x)$ for all i, are uniformly continuous on $K^\sharp$.
($K^\sharp$2) The functions $Z(x)$ are uniformly continuous on $K^\sharp$.
($K^\sharp$3) The functions $\mathbf{E}_{ij,bdd}(x)$ are uniformly continuous on $K^\sharp$.

($K^\sharp 4$) The functions $f_{ij}(x)$ and $\mathbf{f}_{ij}(x)$ are uniformly continuous on $K^\sharp$ (in the weak-* convergence topology).

($K^\sharp 5$) There exists $t_0 > 0$ and $\epsilon' < 0.25\alpha \min_{i \neq j} |\lambda_i - \lambda_j|$ such that for $t > t_0$, $x \in K^\sharp$, all i, and any $v \in \mathcal{V}_i(H^1)(x)$,

$$e^{(\lambda_i - \epsilon')t} \|v\| \leq \|(g_t)_* v\| \leq e^{(\lambda_i + \epsilon')t} \|v\|$$

($K^\sharp 6$) The function $C_3(\cdot)$ of Proposition 10.2 is uniformly bounded on $K^\sharp$.

($K^\sharp 7$) $\mathbf{E}_{ij,bdd}(x)$ and $\mathbf{V}_{<i}(x)$ are transverse for $x \in K^\sharp$.

($K^\sharp 8$) $K^\sharp \subset K''_{thick}$ where K''_{thick} is as in Lemma 3.5(c). Also $K^\sharp \subset K$ where K is as in Lemma 11.12.

($K^\sharp 9$) There exists $c_0(\delta) > 0$ with $c_0(\delta) \to 0$ as $\delta \to 0$ such that for all $x \in K^\sharp$, $d^{X_0}(x, \partial \mathfrak{B}_0[x]) > c_0(\delta)$ where $\mathfrak{B}_0[x]$ is as in Section 3.

($K^\sharp 10$) There exists a constant $C_4(\delta)$ such that for all $x \in K^\sharp$ and all $v \in H_{big}(x)$, $C_4(\delta)^{-1}\|v\| \leq \|v\|_Y \leq C_4(\delta)\|v\|$.

($K^\sharp 11$) There exists a constant $C_1 = C_1(\delta) < \infty$ such that for $x \in K_0$ and all $T > C_1(\delta)$ and all ij we have

$$\left|\left\{t \in [C_1, T] \ : \ g_{-t}^{ij} x \in K^\sharp\right\}\right| \geq 0.99(T - C_1).$$

Lemma **11.13.** — *Suppose $x, x', y, y' \in \pi^{-1}(K_0)$, $y \in W^+[x]$, $y' \in W^+[x']$ and $x' \in W^-[x]$. Suppose further that $d^{X_0}(x, y) < 1/100$, $d^{X_0}(y, y') < 1/100$, and that there exists $s > 0$ such that for all $|t| \leq s$, $d^{X_0}(g_t x, g_t x') < 1/100$ and $d^{X_0}(g_t y, g_t y') < 1/100$. Then,*

(a) *There exists α_2 depending only on the Lyapunov spectrum, such that*

$$\left\|P^\sharp\big(y, y'\big)P^{GM}\big(y', y\big) - I\right\|_Y = O\big(e^{-\alpha_2 s}\big). \tag{11.24}$$

(b) *There exists α_6 depending only on the Lyapunov spectrum such that*

$$\left\|P^+\big(x', y'\big) \circ P^-\big(x, x'\big) - P^{GM}\big(y, y'\big) \circ P^+(x, y)\right\|_Y = O\big(e^{-\alpha_6 s}\big). \tag{11.25}$$

Proof. — Note that part (a) follows immediately from Lemma 11.12, since we are assuming that $d^{X_0}(g_t y, g_t y') \leq 1/100$ for all t with $|t| \leq s$.

To prove (b) we abuse notation by identifying H^1_+ at all four points x, y, x', y' using the Gauss-Manin connection. We write $\mathcal{V}_i(x)$ for $\mathcal{V}_i(H^1_+)(x)$. Since

$$P^+\big(x', y'\big) \circ P^-\big(x, x'\big) \circ P^+(x, y)^{-1}\mathcal{V}_i(y) = \mathcal{V}_i\big(y'\big),$$

and by Lemma 11.12,

$$d_Y\big(\mathcal{V}_i(y), \mathcal{V}_i\big(y'\big)\big) = O\big(e^{-\alpha_2 s}\big),$$

it is enough to check that for $v \in \mathcal{V}_i(y)$,

$$\left\|\big(P^+\big(x', y'\big) \circ P^-\big(x, x'\big) \circ P^+(x, y)^{-1} - I\big)v + \mathcal{V}_{<i}(y)\right\|_Y = O\big(e^{-\alpha_6 s}\|v\|_Y\big). \tag{11.26}$$

But (11.26) follows from the following:

- $P^+(x,y)^{-1}$ is the identity map on $\mathcal{V}_{\le i}(y)/\mathcal{V}_{<i}(y) = \mathcal{V}_{\le i}(x)/\mathcal{V}_{<i}(x)$.
- $P^-(x,x')\mathcal{V}_{\le i}(x) = \mathcal{V}_{\le i}(x')$ and by Lemma 11.12, $\|P^-(x,x') - I\|_Y = O(e^{-\alpha_2 s})$.
- $P^+(x',y')$ is the identity on $\mathcal{V}_{\le i}(x')/\mathcal{V}_{<i}(x') = \mathcal{V}_{\le i}(y')/\mathcal{V}_{<i}(y')$.
- $d_Y(\mathcal{V}_{\le i}(y), \mathcal{V}_{\le i}(y')) = O(e^{-\alpha_2 s})$.

This completes the proof of (11.26) and thus (11.25). □

Lemma **11.14.**

(a) *Suppose* $x, \tilde{x}, y, \tilde{y}$ *all belong to* $\pi^{-1}(K^\sharp)$, $d^{X_0}(x,y) < 1/100$, $\tilde{y} \in W^+[\tilde{x}]$, $d^X(x,\tilde{x}) \le \xi$ *and* $d^X(y,\tilde{y}) \le \xi$. *Then*

$$\|\mathbf{P}^+(\tilde{x},\tilde{y}) \circ \mathbf{P}^{Z(x)}(x,\tilde{x}) - \mathbf{P}^{Z(y)}(y,\tilde{y}) \circ \mathbf{P}^{Z(x)}(x,y)\| \le \xi',$$

where $\xi' \to 0$ *as* $\xi \to 0$.

(b) *Suppose* $x, x', y, y' \in \pi^{-1}(K_0)$, $y \in W^+[x]$, $y' \in W^+[x']$ *and* $x' \in W^-[x]$. *Suppose further that* $d^{X_0}(x,y) \le 1/100$, $d^{X_0}(y,y') \le 1/100$, *and that there exists* $s > 0$ *such that for all* $|t| \le s$, $d^{X_0}(g_t x, g_t x') \le 1/100$ *and* $d^{X_0}(g_t y, g_t y') \le 1/100$. *Furthermore, suppose* $0 < \alpha_0 < 1$ *and that* $0 < \tau < \alpha_0^{-1} s$ *is such that* $d^{X_0}(g_\tau y, g_\tau y') < 1/100$ *and* $g_\tau y \in K^\sharp$. *Then,*

$$\left\|\mathbf{P}^+(x',y') \circ \mathbf{P}^-(x,x') - \mathbf{P}^{g_{-\tau}Z(g_\tau y)}(y,y') \circ \mathbf{P}^+(x,y)\right\| = O(e^{-\alpha s}),$$

where α *depends only on the Lyapunov spectrum and* α_0.

Proof of (a). — Since $y \in W^+[x]$, by Lemma 11.11 we have $\mathbf{P}^{Z(x)}(x,y) = \mathbf{P}^+(x,y)$. Since $\mathbf{P}^{Z(x)}(x,y)$ depends continuously on $x \in K^\sharp$ and $y \in K^\sharp$, part (a) follows from a compactness agreement.

Proof of (b). — We first claim that

$$\left\|P^{g_{-\tau}Z(g_\tau y)}(y,y')P^{GM}(y',y)_* - I\right\|_Y = O(e^{-\alpha' s}), \tag{11.27}$$

where α' depends only on α_0 and the Lyapunov spectrum.

By $(K^\sharp 1)$ there exists $\epsilon_0 > 0$ such that for $x_1 \in \pi^{-1}(K^\sharp)$, $y_1 \in \pi^{-1}(K^\sharp)$ with $d^{X_0}(x_1,y_1) < \epsilon_0$, $hd^{X_0}_{x_1}(U^+[x_1], U^+[y_1]) < 0.01$. By $(K^\sharp 10)$ there exists $t > s/2$ with $g_t y \in \pi^{-1}(K^\sharp)$, $g_t y' \in \pi^{-1}(K^\sharp)$ and $d^{X_0}(g_t y, g_t y') < 1/100$. Therefore, by Lemma 3.5(c) and Proposition 3.4 we have

$$hd^{X_0}_{x'}(U^+[y], U^+[y']) = O(e^{-\alpha_3 s}),$$

where α_3 depends only on the Lyapunov spectrum. Therefore, we get

$$d_Y(P^{GM}(y,y')_*^{-1}\,\mathrm{Lie}(U^+)(y), \mathrm{Lie}(U^+)(y')) = O(e^{-\alpha_3 s}).$$

Then, by (11.24),

$$\textbf{(11.28)}\qquad d_{\mathrm{Y}}\big(\mathrm{P}^{\sharp}(y,y')_*^{-1}\,\mathrm{Lie}(\mathrm{U}^+)(y'),\mathrm{Lie}(\mathrm{U}^+)(y)\big)=\mathrm{O}(e^{-\alpha_4 s})$$

where α_4 depends only on the Lyapunov spectrum.

Since $g_\tau y\in\pi^{-1}(\mathrm{K}^\sharp)$, by ($\mathrm{K}^\sharp 1$) and ($\mathrm{K}^\sharp 2$),

$$d_{\mathrm{Y}}\big(\mathrm{Z}(g_\tau y)\cap\mathcal{V}_i(\mathrm{Lie}(\mathcal{G}_{++}))(g_\tau y),\mathrm{Lie}(\mathrm{U}^+)(g_\tau y)\cap\mathcal{V}_i(\mathrm{Lie}(\mathcal{G}_{++}))(g_\tau y)\big)\geq c(\mathrm{K}^\sharp).$$

By ($\mathrm{K}^\sharp 5$) (i.e. the multiplicative ergodic theorem), the restriction of g_τ to $\mathcal{V}_i(\mathrm{Lie}(\mathcal{G}_{++}))$ is $e^{\lambda_i\tau}h_\tau$, where $\|h_\tau\|=\mathrm{O}(e^{\epsilon'\tau})$. Therefore,

$$\textbf{(11.29)}\qquad d_{\mathrm{Y}}\big(g_{-\tau}\mathrm{Z}(g_\tau y)\cap\mathcal{V}_i(\mathrm{Lie}(\mathcal{G}_{++}))(y),\mathrm{Lie}(\mathrm{U}^+)(y)\cap\mathcal{V}_i(\mathrm{Lie}(\mathcal{G}_{++}))(y)\big)\geq ce^{-\epsilon' s}$$

We may assume (since $\alpha>0$ in the choice of $\mathrm{K}^\sharp$ is arbitrary), that $\epsilon'<\alpha_4/2$. Then, it follows from (11.28), (11.29), (11.19) and (11.20) that

$$\textbf{(11.30)}\qquad \|\mathrm{M}(y;y')\|_{\mathrm{Y}}=\mathrm{O}(e^{-\alpha_5 s})$$

where $\mathrm{M}(\cdot;\cdot)$ is as in (11.21), and α_5 depends only on α_0 and the Lyapunov spectrum. Now, (11.27) follows from (11.24) and (11.30).

Combining (11.27), and (11.25) we get

$$\|\mathrm{P}^+(x',y')\circ\mathrm{P}^-(x,x')-\mathrm{P}^{g_{-\tau}\mathrm{Z}(g_\tau y)}(y,y')\circ\mathrm{P}^+(x,y)\|_{\mathrm{Y}}=\mathrm{O}(e^{-\alpha_6 s}).$$

Now (b) of Lemma 11.14 follows immediately, see also ($\mathrm{K}^\sharp 10$). □

Lemma **11.15.** — *Suppose $q_1\in\mathrm{K}^\sharp$ and $q_1'\in\mathrm{W}^-[q]\cap\mathrm{K}^\sharp$, are such that $d^{\mathrm{X}_0}(q_1,q_1')<1/100$. Suppose $u\in\mathcal{B}(q_1,1/100)$, $u'\in\mathcal{B}(q_1',1/100)$, with $uq_1\in\mathrm{K}^\sharp$, $u'q_1'\in\mathrm{K}^\sharp$. We write $q_2=g_\tau uq_1$ for some $\tau>0$, and let $q_2'=g_\tau u'q_1'$ (see Figure 4). Suppose $d^{\mathrm{X}_0}(q_2,q_2')<1/100$, and also there exists $\alpha_0>0$ depending only on the Lyapunov spectrum such that for $|t|<\alpha_0\tau$, $d^{\mathrm{X}_0}(g_tuq_1,g_tu'q_1')<1/100$.*

In addition, suppose there exist $\tilde{q}_2\in\mathrm{X}$ and $\tilde{q}_2'\in\mathrm{X}$ with $\sigma_0(\tilde{q}_2')\in\mathrm{W}^+[\sigma_0(\tilde{q}_2)]$ such that $d^{\mathrm{X}}(\tilde{q}_2,q_2)<\xi$ and $d^{\mathrm{X}}(\tilde{q}_2',q_2')<\xi$. Suppose further that q_2, q_2', $\tilde{q}_2$ and $\tilde{q}_2'$ all belong to $\mathrm{K}^\sharp$.

Then (assuming ϵ' in ($\mathrm{K}^\sharp 5$) is sufficiently small depending on α_0 and the Lyapunov spectrum), τ is sufficiently large and ξ is sufficiently small (both depending only on $\mathrm{K}^\sharp$), we have

$$\tilde{q}_2'\in\mathrm{W}^+[\tilde{q}_2].$$

Proof. — In this proof, α is a generic constant depending only on α_0 and the Lyapunov spectrum, with its value changing from line to line.

By Lemma 11.13(a),

$$\|\mathrm{P}^\sharp(uq_1,u'q_1')\circ\mathrm{P}^{\mathrm{GM}}(uq_1,u'q_1')^{-1}-\mathrm{I}\|_{\mathrm{Y}}=\mathrm{O}(e^{-\alpha\tau}).$$

By Lemma 11.13(b),

$$\left\| P^{GM}\big(uq_1, u'q_1'\big) \circ P^+(q_1, uq_1) - P^+\big(q_1', u'q_1'\big) \circ P^-\big(q_1, q_1'\big) \right\|_Y = O\big(e^{-\alpha\tau}\big).$$

Thus,

$$\left\| P^{\sharp}\big(uq_1, u'q_1'\big) \circ P^+(q_1, uq_1) - P^+\big(q_1', u'q_1'\big) \circ P^-\big(q_1, q_1'\big) \right\|_Y = O\big(e^{-\alpha\tau}\big). \tag{11.31}$$

Write $u'q_1' = (\sigma_0(u'q_1'), \mathfrak{F}')$, $uq_1 = (\sigma_0(uq_1), \mathfrak{F})$ where $\mathfrak{F}$ and $\mathfrak{F}'$ are as is in Section 4.6.

By Proposition 4.12 (see also (4.12) and (4.13)),

$$\mathfrak{F}' = P^+\big(q_1', u'q_1'\big) \circ P^-\big(q_1, q_1'\big) \circ P^+(uq_1, q_1)\mathfrak{F}.$$

Therefore, by (11.31),

$$d_Y\big(\mathfrak{F}', P^{\sharp}\big(uq_1, u'q_1'\big)\mathfrak{F}\big) = O\big(e^{-\alpha\tau}\big),$$

where the distance $d_Y(\cdot, \cdot)$ between flags is as in Section 4.6.

We now claim that

$$d_Y\big(g_\tau\mathfrak{F}', g_\tau P^{\sharp}\big(uq_1, u'q_1'\big)\mathfrak{F}\big) = O\big(e^{-\alpha\tau}\big). \tag{11.32}$$

Indeed to prove (11.32) it is enough to show that for each i,

$$d_Y\big(g_\tau\mathfrak{F}_i', g_\tau P^{\sharp}\big(uq_1, u'q_1'\big)\mathfrak{F}_i\big) = O\big(e^{-\alpha\tau}\big). \tag{11.33}$$

But $\mathfrak{F}_i' \subset \mathcal{V}_i(H_{big})(u'q_1')$, $\mathfrak{F}_i \subset \mathcal{V}_i(H_{big})(uq_1)$, and

$$P^{\sharp}\big(uq_1, u'q_1'\big)\mathcal{V}_i(H_{big})(uq_1) = \mathcal{V}_i(H_{big})\big(u'q_1'\big).$$

Thus, we have

$$\mathfrak{F}_i' \subset \mathcal{V}_i(H_{big})\big(u'q_1'\big), \qquad P^{\sharp}\big(uq_1, u'q_1'\big)\mathfrak{F}_i \subset \mathcal{V}_i(H_{big})\big(u'q_1'\big)$$

The geodesic flow g_τ restricted to $\mathcal{V}_i(H_{big})(u'q_1')$ is of the form $e^{\lambda_i\tau}h_\tau$, where $\|h_\tau\|_Y = O(e^{\epsilon'\tau})$. Thus, (11.33) and hence (11.32) follows. The equivariance property of $P^{\sharp}$ then implies that

$$d_Y\big(g_\tau\mathfrak{F}', P^{\sharp}\big(q_2, q_2'\big)g_\tau\mathfrak{F}\big) = O\big(e^{-\alpha\tau}\big). \tag{11.34}$$

We have since the $\mathcal{V}_i$ are continuous on $K^{\sharp}$ and Lemma 4.1,

$$\left\| P^{GM}\big(q_2', \tilde{q}_2'\big) \circ P^{\sharp}\big(q_2, q_2'\big) - P^+\big(\tilde{q}_2, \tilde{q}_2'\big) \circ P^{GM}(q_2, \tilde{q}_2) \right\|_Y \to 0,$$

as $\xi \to 0$. Combining this with (11.34), we get

$$d_Y\big(P^{GM}\big(q_2', \tilde{q}_2'\big)g_\tau\mathfrak{F}', P^+\big(\tilde{q}_2, \tilde{q}_2'\big) \circ P^{GM}(q_2, \tilde{q}_2)g_\tau\mathfrak{F}\big) \to 0, \tag{11.35}$$

as $\xi \to 0$ and $\tau \to \infty$.

Note that $q_2 = (\sigma_0(q_2), g_\tau \mathfrak{F})$, $q_2' = (\sigma_0(q_2'), g_\tau \mathfrak{F}')$. Write $\tilde{q}_2 = (\sigma_0(\tilde{q}_2), \tilde{\mathfrak{F}})$, $\tilde{q}_2' = (\sigma_0(\tilde{q}_2'), \tilde{\mathfrak{F}}')$. Then, since $d^X(q_2, \tilde{q}_2) \to 0$, in view of (4.14),

$$d_Y\big(P^{GM}(q_2, \tilde{q}_2) g_\tau \mathfrak{F}, \tilde{\mathfrak{F}}\big) \le \xi'$$
$$d_Y\big(P^{GM}\big(q_2', \tilde{q}_2'\big) g_\tau \mathfrak{F}', \tilde{\mathfrak{F}}'\big) \le \xi'$$

where $\xi' \to 0$ as $\xi \to 0$. Hence, by (11.35),

$$d_Y\big(\tilde{\mathfrak{F}}', P^+\big(\tilde{q}_2, \tilde{q}_2'\big)\tilde{\mathfrak{F}}\big) \to 0 \quad \text{as } \xi \to 0 \text{ and } \tau \to \infty.$$

This implies that $\tilde{q}_2' \in W^+[\tilde{q}_2]$ by (4.12). □

Proof of Lemma 11.6. — Note that, by the construction of $\mathbf{P}^{Z(\cdot)}(\cdot,\cdot)$, for all $i \in \Lambda'$,

$$\mathbf{P}^{Z(g_t y)}\big(g_t y, g_t y'\big)\mathcal{V}_i(\mathbf{H})(g_t y) = \mathcal{V}_i(\mathbf{H})\big(g_t y'\big). \tag{11.36}$$

However, even though for all $ij \in \Lambda''$, $\mathbf{E}_{ij}(x) \subset \mathcal{V}_i(\mathbf{H})(x)$, we may have

$$\mathbf{P}^{Z(g_t y)}\big(g_t y, g_t y'\big)\mathbf{E}_{ij}(g_t y) \neq \mathbf{E}_{ij}\big(g_t y'\big).$$

Suppose $\mathbf{v} \in \mathbf{E}_{ij}(y)$, and that $\mathbf{v}$ is orthogonal to $\mathbf{E}_{i,j-1}(y) \subset \mathbf{E}_{ij}(y)$. Let

$$\mathbf{v}' = \mathbf{P}^+\big(x', y'\big) \circ \mathbf{P}^-\big(x, x'\big) \circ \mathbf{P}^+(y, x)\mathbf{v}.$$

Then, by Proposition 4.12(a), $\mathbf{v}' \in \mathbf{E}_{ij}(y')$. By ($K^\sharp 1$), and the fact that

$$\mathbf{P}^+\big(x', y'\big) \circ \mathbf{P}^-\big(x, x'\big) \circ \mathbf{P}^+(y, x)\mathbf{E}_{i,j-1}(y) = \mathbf{E}_{i,j-1}\big(y'\big),$$

we have

$$C_1^{-1}\|\mathbf{v}\| \le \big\|\mathbf{v}' + \mathbf{E}_{i,j-1}\big(y'\big)\big\| \le C_1\|\mathbf{v}\|, \tag{11.37}$$

where C_1 depends only on K_0. By Lemma 11.14(b),

$$\big\|\mathbf{P}^{g_{-t}Z(g_t y)}\big(y, y'\big)\mathbf{v} - \mathbf{v}'\big\| = O\big(e^{-\alpha_1' t}\|\mathbf{v}\|\big),$$

where α_1' depends only on α_0 and the Lyapunov spectrum. By (11.36),

$$\mathbf{P}^{g_{-t}Z(g_t y)}\big(y, y'\big)\mathbf{v} \in \mathcal{V}_i(\mathbf{H})\big(y'\big).$$

Then by the multiplicative ergodic theorem (see also ($K^\sharp 5$)),

$$\big\|\mathbf{P}^{Z(g_t y)}\big(g_t y, g_t y'\big)(g_t \mathbf{v}) - g_t \mathbf{v}'\big\| = O\big(e^{-(\alpha_1' - \epsilon')t}\|g_t \mathbf{v}\|\big). \tag{11.38}$$

Since $\mathbf{v}$ is arbitrary, this implies that for all $ij \in \Lambda''$,

$$d\big(\mathbf{P}^{Z(g_t y)}\big(g_t y, g_t y'\big)\mathbf{E}_{ij}(g_t y), \mathbf{E}_{ij}\big(g_t y'\big)\big) = O\big(e^{-\alpha_1 t}\big), \tag{11.39}$$

where α_1 depends only on α_0 and the Lyapunov spectrum.

By ($K^\sharp 1$) and ($K^\sharp 2$),

$$\left\| \mathbf{P}^{Z(g_t y)}\big(g_t y, g_t y'\big) \right\| \le C_1'$$

where C_1' depends only on K_0. Therefore, by (11.38) and (11.39),

$$C_2^{-1} \|g_t \mathbf{v} + \mathbf{E}_{i,j-1}(g_t y)\| \le \left\| g_t \mathbf{v}' + \mathbf{E}_{i,j-1}\big(g_t y'\big) \right\| \le C_2 \|g_t \mathbf{v}' + \mathbf{E}_{i,j-1}(g_t y)\|, \tag{11.40}$$

where C_2 depends only on K_0, α_0 and the Lyapunov spectrum.

Note that

$$\hat{\tau}_{ij}(y, t) = \frac{\|g_t \mathbf{v} + \mathbf{E}_{i,j-1}(g_t y)\|}{\|\mathbf{v}\|}, \qquad \hat{\tau}_{ij}\big(y', t\big) = \frac{\|g_t \mathbf{v}' + \mathbf{E}_{i,j-1}(g_t y')\|}{\|\mathbf{v}' + \mathbf{E}_{i,j-1}(y')\|}.$$

Now (11.7) follows from (11.37) and (11.40). □

Proposition **11.16.** — *Suppose α, ϵ, s, ℓ, t, t', q, q', τ, q_1, q_1', q_3, q_3', u, u', q_2, q_2', $\tilde{q}_2$, $\tilde{q}_2'$, C, C_1, ξ are as in Proposition 11.4. Suppose also $\tilde{q}_2' \in W^+[\tilde{q}_2]$. Then (assuming ϵ' in ($K^\sharp 5$) is sufficiently small depending on α_0 and the Lyapunov spectrum),*

(a) *There exists $\xi' > 0$ (depending on ξ, K_0 and C and t) with $\xi' \to 0$ as $\xi \to 0$ and $t \to \infty$ such that for $\mathbf{v} \in \mathbf{E}_{[ij],bdd}(q_3)$,*

$$\begin{aligned} &\left\| \mathbf{P}^{Z(q_2')}\big(q_2', \tilde{q}_2'\big) \circ R\big(q_3', q_2'\big) \circ \mathbf{P}^-\big(q_3, q_3'\big)\mathbf{v} \right. \\ &\qquad \left. - \mathbf{P}^+\big(\tilde{q}_2, \tilde{q}_2'\big) \circ \mathbf{P}^{Z(q_2)}(q_2, \tilde{q}_2) \circ R(q_3, q_2)\mathbf{v} \right\| \le \xi' \|\mathbf{v}\|. \end{aligned} \tag{11.41}$$

(b) *There exists $\xi'' > 0$ (depending on ξ, K_0, C and t) with $\xi'' \to 0$ as $\xi \to 0$ and $t \to \infty$ such that*

$$d_*\big(\mathbf{P}^+\big(\tilde{q}_2, \tilde{q}_2'\big)\mathbf{f}_{ij}(\tilde{q}_2), \mathbf{f}_{ij}\big(\tilde{q}_2'\big)\big) \le \xi''.$$

Here $d_*(\cdot,\cdot)$ is any metric which induces the weak- convergence topology on the domain of common definition of the measures, up to normalization.*

Proof of (a). — Following the outline given after the statement of Proposition 11.4, the proof will consist of verifying conditions (i), (ii) and (iii) of Lemma 11.5, with $E = \mathbf{E}_{ij,bdd}(q_3)$, $L = \mathbf{H}(\tilde{q}_2')$, $F = \mathbf{E}_{ij,bdd}(\tilde{q}_2')$, $V = \mathbf{V}_{<i}(\tilde{q}_2')$, and $\mathbf{B}$ and $\mathbf{B}'$ as the linear maps on the first and second line of (11.41). (We note that $\mathbf{B}$ and $\mathbf{B}'$ are bounded by Proposition 10.2.) We start with (i).

Note that by (9.4), we have

$$\kappa^{-1}\tau \le t \le \kappa\tau, \tag{11.42}$$

where κ depends only on the Lyapunov spectrum. Also, by assumption we have

$$\ell > \alpha_0 \tau,$$

where α_0 depends only on the Lyapunov spectrum.

Suppose $\mathbf{w} \in \mathbf{E}_{ij,bdd}(q_1)$. We now apply Lemma 11.14(b), with $x = q_1$, $x' = q_1'$, $y = uq_1$, $y' = u'q_1'$ and $\tau = \tau$ to get

$$\begin{aligned}&\left\|\mathbf{P}^+\left(u'q_1', uq_1\right) \circ \mathbf{P}^-\left(q_1, q_1'\right)\mathbf{w} - \mathbf{P}^{g_{-\tau}Z(q_2)}\left(uq_1, u'q_1'\right) \circ \mathbf{P}^+(q_1, uq_1)\mathbf{w}\right\| \\ &\quad = O\left(e^{-\alpha\tau}\|\mathbf{w}\|\right).\end{aligned}$$

By Proposition 4.12(a), $\mathbf{P}^-(q_1, q_1')\mathbf{w} \in \mathbf{E}_{ij,bdd}(q_1') \subset \mathbf{E}(q_1')$. Therefore, by Lemma 9.1, this can be rewritten as

$$\left\|\left(u'\right)_* \circ \mathbf{P}^-\left(q_1, q_1'\right)\mathbf{w} - \mathbf{P}^{g_{-\tau}Z(q_2)}\left(uq_1, u'q_1'\right) \circ (u)_*\mathbf{w}\right\| = O\left(e^{-\alpha\tau}\|\mathbf{w}\|\right).$$

Hence,

$$\left(u'\right)_* \circ \mathbf{P}^-\left(q_1, q_1'\right)\mathbf{w} = \mathbf{P}^{g_{-\tau}Z(q_2)}\left(uq_1, u'q_1'\right) \circ (u)_*\mathbf{w} + \mathbf{w}' \tag{11.43}$$

where $\mathbf{w}' \in \mathbf{H}(u'q_1')$ satisfies

$$\|\mathbf{w}'\| = O\left(e^{-\alpha\tau}\|\mathbf{w}\|\right) = O_{\epsilon'}\left(e^{-(\lambda_i+\alpha-\epsilon')\tau}\|\mathbf{v}\|\right), \tag{11.44}$$

where we wrote $\mathbf{w} = g_{-t'}^{ij}\mathbf{v}$ for some $\mathbf{v} \in \mathbf{E}_{ij}(q_3)$, and we have used ($K^\sharp 5$), (11.42) and the assumption $|t - t'| < C$ for the last estimate. We now apply $g_\tau = g_t^{ij}$ to both sides of (11.43) and take the quotient mod $\mathbf{V}_{<i}(q_2')$. We get

$$\begin{aligned}&g_\tau \circ \left(u'\right)_* \circ \mathbf{P}^-\left(q_1, q_1'\right)\mathbf{w} + \mathbf{V}_{<i}\left(q_2'\right) \\ &\quad = \mathbf{P}^{Z(q_2)}\left(q_2, q_2'\right) \circ \left[g_\tau \circ (u)_*\mathbf{w} + g_\tau\mathbf{w}'\right] + \mathbf{V}_{<i}\left(q_2'\right).\end{aligned} \tag{11.45}$$

We may write

$$\mathbf{w}' = \sum_k \mathbf{w}_k, \qquad \mathbf{w}_k \in \mathcal{V}_k(\mathbf{H})\left(u'q_1'\right).$$

Then,

$$g_\tau\mathbf{w}' + \mathbf{V}_{<i}\left(q_2'\right) = \sum_k g_\tau\mathbf{w}_k' + \mathbf{V}_{<i}\left(q_2'\right) = \sum_{k\geq i} g_\tau\mathbf{w}_k' + \mathbf{V}_{<i}\left(q_2'\right),$$

since for $k < i$, $g_\tau\mathbf{w}_k' \in \mathbf{V}_{<i}(q_2')$. By ($K^\sharp 5$), for $k \geq i$,

$$\|g_\tau\mathbf{w}_k'\| = O\left(e^{(\lambda_k+\epsilon')\tau}\|\mathbf{w}_k'\|\right) = O\left(e^{(\lambda_i+\epsilon')\tau}\|\mathbf{w}_k'\|\right) = O\left(e^{-\alpha_5\tau}\|\mathbf{v}\|\right),$$

using (11.44) (and choosing ϵ' sufficiently small depending on α_0 and the Lyapunov spectrum). Therefore, substituting into (11.45), we get, for $\mathbf{v} \in \mathbf{E}_{ij,bdd}(q_3)$,

$$\begin{aligned}&R\left(q_3', q_2'\right) \circ \mathbf{P}^-\left(q_3, q_3'\right)\mathbf{v} + \mathbf{V}_{<i}\left(q_2'\right) \\ &\quad = \mathbf{P}^{Z(q_2)}\left(q_2, q_2'\right) \circ R(q_3, q_2)\mathbf{v} + O\left(e^{-\alpha_5\tau}\|\mathbf{v}\|\right) + \mathbf{V}_{<i}\left(q_2'\right).\end{aligned}$$

We now apply $\mathbf{P}^{Z(q_2')}(q_2', \tilde{q}_2')$ to both sides to get (using (11.22))

$$\mathbf{P}^{Z(q_2')}\big(q_2', \tilde{q}_2'\big) \circ R\big(q_3', q_2'\big) \circ \mathbf{P}^{-}\big(q_3, q_3'\big)\mathbf{v} + \mathbf{V}_{<i}\big(\tilde{q}_2'\big) = \mathbf{P}^{Z(q_2')}\big(q_2', \tilde{q}_2'\big) \circ \mathbf{P}^{Z(q_2)}\big(q_2, q_2'\big) \circ R(q_3, q_2)\mathbf{v} + O\big(e^{-\alpha_5\tau}\|\mathbf{v}\|\big) + \mathbf{V}_{<i}\big(\tilde{q}_2'\big). \tag{11.46}$$

Since q_2, $\tilde{q}_2$, q_2', $\tilde{q}_2'$ all belong to $K^\sharp$, we have by Lemma 11.14(a),

$$\|\mathbf{P}^{Z(q_2')}\big(q_2', \tilde{q}_2'\big) \circ \mathbf{P}^{Z(q_2)}\big(q_2, q_2'\big) - \mathbf{P}^{+}\big(\tilde{q}_2, \tilde{q}_2'\big) \circ \mathbf{P}^{Z(q_2)}(q_2, \tilde{q}_2)\| \le \xi_3,$$

where $\xi_3 \to 0$ as $\xi \to 0$. Therefore, substituting into (11.46), we get

$$\begin{aligned}&\mathbf{P}^{Z(q_2')}\big(q_2', \tilde{q}_2'\big) \circ R\big(q_3', q_2'\big) \circ \mathbf{P}^{-}\big(q_3, q_3'\big)\mathbf{v} + \mathbf{V}_{<i}\big(\tilde{q}_2'\big)\\ &\quad = \mathbf{P}^{+}\big(\tilde{q}_2, \tilde{q}_2'\big) \circ \mathbf{P}^{Z(q_2)}(q_2, \tilde{q}_2) \circ R(q_3, q_2)\mathbf{v} + O\big(e^{-\alpha_5\tau}\|\mathbf{v}\|\big) + O(\xi_3\|\mathbf{v}\|)\\ &\qquad + \mathbf{V}_{<i}\big(\tilde{q}_2'\big).\end{aligned}$$

This completes the verification of (i) of Lemma 11.5.

We now verify (ii) of Lemma 11.5. For $\mathbf{v} \in \mathbf{E}_{ij,bdd}(q_3)$, we have $R(q_3, q_2)\mathbf{v} \in \mathbf{E}_{ij,bdd}(q_2)$, and then

$$\begin{aligned}&\mathbf{P}^{+}\big(\tilde{q}_2, \tilde{q}_2'\big) \circ \mathbf{P}^{Z(q_2)}(q_2, \tilde{q}_2) \circ R(q_3, q_2)\mathbf{v}\\ &\quad \in \mathbf{P}^{+}\big(\tilde{q}_2, \tilde{q}_2'\big) \circ \mathbf{P}^{Z(q_2)}(q_2, \tilde{q}_2)\mathbf{E}_{ij,bdd}(q_2).\end{aligned}$$

By ($K^\sharp$2) and ($K^\sharp$3), since $d^X(q_2, \tilde{q}_2) < \xi$,

$$d_Y\big(\mathbf{P}^{Z(q_2)}(q_2, \tilde{q}_2)\mathbf{E}_{ij,bdd}(q_2), \mathbf{E}_{ij,bdd}(\tilde{q}_2)\big) < \xi_0,$$

where $\xi_0 \to 0$ as $\xi \to 0$. Then, using (11.18),

$$d_Y\big(\mathbf{P}^{+}\big(\tilde{q}_2, \tilde{q}_2'\big) \circ \mathbf{P}^{Z(q_2)}(q_2, \tilde{q}_2)\mathbf{E}_{ij,bdd}(q_2), \mathbf{E}_{ij,bdd}\big(\tilde{q}_2'\big)\big) < \xi_1,$$

where $\xi_1 \to 0$ as $\xi \to 0$. This completes the verification of condition (ii) of Lemma 11.5.

Also, by (11.18) (applied to $\mathbf{P}^{-}$), we have $\mathbf{P}^{-}(q_3, q_3')\mathbf{v} \in \mathbf{E}_{ij,bdd}(q_3')$. Then, $R(q_3', q_2') \circ \mathbf{P}^{-}(q_3, q_3')\mathbf{v} \in \mathbf{E}_{ij,bdd}(q_2')$, and

$$\mathbf{P}^{Z(q_2')}\big(q_2', \tilde{q}_2'\big) \circ R\big(q_3', q_2'\big) \circ \mathbf{P}^{-}\big(q_3, q_3'\big)\mathbf{v} \in \mathbf{P}^{Z(q_2')}\big(q_2', \tilde{q}_2'\big)\mathbf{E}_{ij,bdd}\big(q_2'\big).$$

By ($K^\sharp$2) and ($K^\sharp$3),

$$d_Y\big(\mathbf{P}^{Z(q_2')}\big(q_2', \tilde{q}_2'\big)\mathbf{E}_{ij,bdd}\big(q_2'\big), \mathbf{E}_{ij,bdd}\big(\tilde{q}_2'\big)\big) < \xi_2,$$

where $\xi_2 \to 0$ as $\xi \to 0$. This completes the verification of condition (iii) of Lemma 11.5.

Now (11.41) for arbitrary $\mathbf{v} \in \mathbf{E}_{ij,bdd}(q_3)$ follows from Lemma 11.5. The general case of (11.41) (i.e. for an arbitrary $\mathbf{v} \in \mathbf{E}_{[kr],bdd}(q_3)$) follows since $\mathbf{E}_{[kr],bdd}(q_3) = \bigoplus_{ij\in[kr]} \mathbf{E}_{ij,bdd}(q_3)$ and all the maps on the left-hand-side of (11.41) are linear.

Proof of (b). — By ($K^\sharp 4$),

$$d_*\big(\mathbf{P}^-\big(q_3, q_3'\big)_* \mathbf{f}_{ij}(q_3), \mathbf{f}_{ij}\big(q_3'\big)\big) \le \xi_1,$$

where $\xi_1 \to 0$ as and $t \to \infty$. In view of in view of condition ($K^\sharp 6$), the assumption $|t - t'| < C$ and Proposition 10.2, that $R(q_3, q_2)$ is a linear map with norm bounded depending only on $K^\sharp$ and C. It then follows from (a) that $R(q_3', q_2')$ is also a linear map whose norm is bounded depending only on $K^\sharp$ and C. Furthermore, by ($K^\sharp 9$) and Lemma 3.5 there exists a constant $C_2(\delta)$ such that if

$$C > t - t' > C_2(\delta), \tag{11.47}$$

then if we write $q_2 = g_t^{ij} u g_{-t'}^{ij} q_3$, then $g_t^{ij} u g_{-t'}^{ij} \mathfrak{B}_0[q_3] \cap \mathcal{C}_{ij}[q_3] \supset \mathfrak{B}_0[q_2] \cap \mathcal{C}_{ij}[q_2]$. Then, by Lemma 11.10,

$$\mathbf{f}_{ij}(q_2) \propto R(q_3, q_2)_* \mathbf{f}_{ij}(q_3) \quad \text{and} \quad \mathbf{f}_{ij}\big(q_2'\big) \propto R\big(q_3', q_2'\big)_* \mathbf{f}_{ij}\big(q_3'\big).$$

In view of ($K^\sharp 11$), we can assume that (11.47) holds: otherwise we can replace q_3 and q_3' by $g_{-s}^{ij} q_3 \in K^\sharp$ and $g_{-s}^{ij} q_3' \in K^\sharp$ where $C_2(\delta) < s < 2C_2(\delta)$. (Without loss of generality we may assume that $C > 2C_2(\delta)$.) Hence, we have

$$d_*\big(\big(R\big(q_3', q_2'\big) \circ \mathbf{P}^-\big(q_3, q_3'\big)\big)_* \mathbf{f}_{ij}(q_3), \mathbf{f}_{ij}\big(q_2'\big)\big) \le \xi_2, \tag{11.48}$$

where $\xi_2 \to 0$ as $t \to \infty$. Thus, by ($K^\sharp 1$), ($K^\sharp 2$), ($K^\sharp 3$),

$$d_*\big(\mathbf{P}^{Z(q_2')}\big(q_2', \tilde{q}_2'\big) \mathbf{f}_{ij}\big(q_2'\big), \mathbf{f}_{ij}\big(\tilde{q}_2'\big)\big) \le \xi_3,$$

where $\xi_3 \to 0$ as $\xi \to 0$ and $t \to \infty$. Hence,

$$d_*\big(\big(\mathbf{P}^{Z(q_2')}\big(q_2', \tilde{q}_2'\big) \circ R\big(q_3', q_2'\big) \circ \mathbf{P}^-\big(q_3, q_3'\big)\big)_* \mathbf{f}_{ij}(q_3), \mathbf{f}_{ij}\big(\tilde{q}_2'\big)\big) \le \xi_4, \tag{11.49}$$

where $\xi_4 \to 0$ as $\xi \to 0$ and $t \to \infty$. Also, in view of (11.48), and since $\mathbf{P}^+(\tilde{q}_2, \tilde{q}_2')$ is a linear map whose norm is bounded depending only on $K^\sharp$,

$$d_*\big(\big(\mathbf{P}^+\big(\tilde{q}_2, \tilde{q}_2'\big) \circ \mathbf{P}^{Z(q_2)}(q_2, \tilde{q}_2) \circ R(q_3, q_2)\big)_* \mathbf{f}_{ij}(q_3), \mathbf{P}^+\big(\tilde{q}_2, \tilde{q}_2'\big)_* \mathbf{f}_{ij}(\tilde{q}_2)\big) \le \xi_5, \tag{11.50}$$

where $\xi_5 \to 0$ as $\xi \to 0$ and $t \to \infty$. Now part (b) follows from (11.49), (11.50), and (11.41).

Proof of Proposition 11.4. — Note that (11.4) follows from Lemma 11.15. We assume this from now on.

Without loss of generality, and to simplify the notation, we may assume that $Z(\tilde{q}_2') = P^+(\tilde{q}_2, \tilde{q}_2') Z(\tilde{q}_2)$. (Otherwise, we can further compose with a reparametrization map at $\tilde{q}_2'$ which will not change the result.) We have

$$\mathbf{f}_{ij}(\tilde{q}_2) = (\mathbf{j} \circ \phi_{\tilde{q}_2})_* f_{ij}(\tilde{q}_2)$$

and

$$\mathbf{f}_{ij}(\tilde{q}_2') = (\mathbf{j} \circ \phi_{\tilde{q}_2'})_* f_{ij}(\tilde{q}_2')$$

As in Section 6, let $\mathrm{P}_*^+ : \mathcal{H}_{++}(\tilde{q}_2) \times \mathrm{W}^+(\tilde{q}_2) \to \mathcal{H}_{++}(\tilde{q}_2') \times \mathrm{W}^+(\tilde{q}_2')$ be given by

$$\mathrm{P}_*^+(\mathrm{M}, v) = \left(\mathrm{P}^+(\tilde{q}_2, \tilde{q}_2')^{-1} \circ \mathrm{M} \circ \mathrm{P}^+(\tilde{q}_2, \tilde{q}_2'), \mathrm{P}^+(\tilde{q}_2, \tilde{q}_2')v\right). \tag{11.51}$$

Then,

$$\mathbf{P}^+(\tilde{q}_2, \tilde{q}_2') \circ \mathbf{j}(\mathrm{M}, v) = \mathbf{j}\left(\mathrm{P}_*^+(\mathrm{M}, v)\right) \tag{11.52}$$

We write $\mathrm{A} \approx_{\xi,t} \mathrm{B}$ if $d(\mathrm{A}, \mathrm{B}) \to 0$ as $\xi \to 0$ and $t \to \infty$. Then, we have, by Proposition 11.16,

$$\begin{aligned}(\mathbf{j} \circ \phi_{\tilde{q}_2'})_* f_{ij}(\tilde{q}_2') = \mathbf{f}_{ij}(\tilde{q}_2') &\approx_{\xi,t} \mathbf{P}^+(\tilde{q}_2, \tilde{q}_2')_* \mathbf{f}_{ij}(\tilde{q}_2) \\ &= \left(\mathbf{P}^+(\tilde{q}_2, \tilde{q}_2') \circ \mathbf{j} \circ \phi_{\tilde{q}_2}\right)_* f_{ij}(\tilde{q}_2)\end{aligned}$$

By (11.52),

$$(\mathbf{j} \circ \phi_{\tilde{q}_2'})_* f_{ij}(\tilde{q}_2') \approx_{\xi,t} \left(\mathbf{j} \circ \mathrm{P}_*^+ \circ \phi_{\tilde{q}_2}\right)_* f_{ij}(\tilde{q}_2).$$

Therefore,

$$(\phi_{\tilde{q}_2'})_* f_{ij}(\tilde{q}_2') \approx_{\xi,t} \left(\mathrm{P}_*^+ \circ \phi_{\tilde{q}_2}\right)_* f_{ij}(\tilde{q}_2).$$

Let $\pi_2 : \mathcal{H}_{++}(x) \times \mathrm{W}^+(x) \to \mathrm{W}^+(x)$ be projection onto the second factor. Then, applying π_2 to both sides, we get

$$(\pi_2 \circ \phi_{\tilde{q}_2'})_* f_{ij}(\tilde{q}_2') \approx_{\xi,t} \left(\pi_2 \circ \mathrm{P}_*^+ \circ \phi_{\tilde{q}_2}\right)_* f_{ij}(\tilde{q}_2). \tag{11.53}$$

For $z \in \mathrm{Z}(\tilde{q}_2)$, $\pi_2(\phi_{\tilde{q}_2}(z)) = z$, and thus in view of (11.51),

$$\left(\pi_2 \circ \mathrm{P}_*^+ \circ \phi_{\tilde{q}_2}\right)(z) \approx_{\xi,t} \mathrm{P}^+(\tilde{q}_2, \tilde{q}_2')z. \tag{11.54}$$

By assumption, we have $\mathrm{Z}(\tilde{q}_2') = \mathrm{P}^+(\tilde{q}_2, \tilde{q}_2')\mathrm{Z}(\tilde{q}_2)$. Then, similarly, for $z \in \mathrm{Z}(\tilde{q}_2') = \mathrm{P}^+(\tilde{q}_2, \tilde{q}_2')\mathrm{Z}(\tilde{q}_2)$,

$$(\pi_2 \circ \phi_{\tilde{q}_2'})(z) = z. \tag{11.55}$$

Since $f_{ij}(x)$ is Haar along U^+, we can recover $f_{ij}(\tilde{q}_2)$ from its restrictions to $\mathrm{Z}(\tilde{q}_2)$ and $f_{ij}(\tilde{q}_2')$ from its restriction to $\mathrm{Z}(\tilde{q}_2')$. It now follows from (11.53), (11.54) and (11.55) that

$$f_{ij}(\tilde{q}_2') \approx_{\xi,t} \mathrm{P}^+(\tilde{q}_2, \tilde{q}_2')_* f_{ij}(\tilde{q}_2).$$

□

12. The inductive step

Proposition **12.1.** — *Suppose* ν *is a* P*-invariant measure on* X_0*. Suppose* $U^+(x)$ *is a family of subgroups of* $\mathcal{G}_{++}(x)$ *compatible with* ν *in the sense of Definition 6.2. Let* $L^-[x]$ *and* $L^+[x]$ *be as in Section 6.2, and suppose the equivalent conditions of Lemma 6.15 do not hold. Then, there exists a family of subgroups* $U^+_{new}(x)$ *of* $\mathcal{G}_{++}(x)$ *compatible with* ν *in the sense of Definition 6.2 such that for almost all* x, $U^+_{new}(x)$ *strictly contains* $U^+(x)$.

The rest of Section 12 will consist of the proof of Proposition 12.1. We assume that $L^-(x)$, $L^+(x)$ and $U^+(x)$ are as in Proposition 12.1, and the equivalent conditions of Lemma 6.15 do not hold. The argument has been outlined in Section 2.3, and we have kept the same notation (in particular, see Figure 1).

Let $f_{ij}(x)$ be the measures on $W^+(x)$ introduced in Section 11. We think of f_{ij} as a function from X to a space of measures (which is metrizable). Let $P^+(x, y)$ be the map introduced in Section 4.2. Proposition 12.1 will be derived from the following:

Proposition **12.2.** — *Suppose* U^+, L^+, L^- *are as in Proposition 12.1, and the equivalent conditions of Lemma 6.15 do not hold. Then there exists* $0 < \delta_0 < 0.1$*, a subset* $K_* \subset X$ *with* $\nu(K_*) > 1 - \delta_0$ *such that all the functions* f_{ij}, $ij \in \tilde{\Lambda}$ *are uniformly continuous on* K_**, and* $C > 1$ *(depending on* K_**) such that for every* $0 < \epsilon < C^{-1}/100$ *there exists a subset* $E \subset K_*$ *with* $\nu(E) > \delta_0$*, such that for every* $x \in \pi^{-1}(E)$ *there exists* $ij \in \tilde{\Lambda}$ *and* $y \in \mathcal{C}_{ij}[x] \cap \pi^{-1}(K_*)$ *with*

$$C^{-1}\epsilon \le hd_x^{X_0}\big(U^+[x], U^+[y]\big) \le C\epsilon \tag{12.1}$$

and (on the domain where both are defined)

$$f_{ij}(y) \propto P^+(x, y)_* f_{ij}(x). \tag{12.2}$$

We now begin the proof of Proposition 12.2.

Choice of parameters #1. — Fix $\theta > 0$ as in Proposition 10.1 and Proposition 10.2. We then choose $\delta > 0$ sufficiently small; the exact value of δ will we chosen at the end of this section. All subsequent constants will depend on δ. (In particular, $\delta \ll \theta$; we will make this more precise below.) Let $\epsilon > 0$ be arbitrary and $\eta > 0$ be arbitrary; however we will always assume that ϵ and η are sufficiently small depending on δ.

We will show that Proposition 12.2 holds with $\delta_0 = \delta/10$. Let $K_* \subset X$ be any subset with $\nu(K_*) > 1 - \delta_0$ on which all the functions f_{ij} are uniformly continuous. It is enough to show that there exists $C = C(\delta)$ such that for any $\epsilon > 0$ and for an arbitrary compact set $K_{00} \subset X$ with $\nu(K_{00}) \ge (1 - 2\delta_0)$, there exists $x \in K_{00} \cap K_*$, $ij \in \tilde{\Lambda}$ and $y \in \mathcal{C}_{ij}[x] \cap K_*$ satisfying (12.1) and (12.2). Thus, let $K_{00} \subset X$ be an arbitrary compact set with $\nu(K_{00}) > 1 - 2\delta_0$.

We can choose a compact set $K_0 \subset K_{00} \cap K_*$ with $\nu(K_0) > 1 - 5\delta_0 = 1 - \delta/2$ so that Proposition 11.4 holds. In addition, there exists $\epsilon_0'(\delta) > 0$ such that for all $x \in K_0$,

$$d^+\big(x, \partial\mathfrak{B}_0[x]\big) > \epsilon_0'(\delta). \tag{12.3}$$

(Here, $d^+(\cdot,\cdot)$ is as in Section 3 and by $\partial\mathfrak{B}_0[x]$ we mean the boundary of $\mathfrak{B}_0[x]$ as a subset of $W^+[x]$.)

Let $\kappa > 1$ be as in Proposition 7.4, and so that (9.4) holds. Without loss of generality, assume $\delta < 0.01$. We now choose a subset $K \subset K_0 \subset X$ with $\nu(K) > 1 - \delta$ such that the following hold:

- There exists a number $T_0(\delta)$ such that for any $x \in K$ and any $T > T_0(\delta)$,

  $$\big\{t \in [-T/2, T/2] \;:\; g_t x \in K_0\big\} \geq 0.9T.$$

  (This can be done by the Birkhoff ergodic theorem.)
- Proposition 8.5(a) holds.
- Proposition 10.1 holds.
- There exists a constant $C = C(\delta)$ such that for $x \in K$, $C_3(x)^2 < C(\delta)$ where C_3 is as in Proposition 10.2.
- There is a constant $C'' = C''(\delta)$ such that for $x \in K$, $C(x) < C''(\delta)$ where $C(x)$ is as in Lemma 6.10 or in Corollary 6.13. Also for $x \in K$, the function $c_1(x)$ of Lemma 6.9 is bounded from below by $C''(\delta)^{-1}$.
- Lemma 4.17 holds for $K = K(\delta)$ and $C_1 = C_1(\delta)$.
- There exists a constant $C' = C'(\delta)$ such that for $x \in K$, $C_1(x) < C'$, $C_2(x) < C'$ and $C(x) < C'$ where $C_1(x)$, $C_2(x)$ and $C(x)$ are as in Proposition 6.11. Also $K \subset K'$ and also $C_1'(\delta) < C'$, $C_2'(\delta) < C'$, $C_4'(\delta) < C'$ and $C_4(\delta) < C'$ where K', $C_1'(\delta)$, $C_2'(\delta)$ and $C_4'(\delta)$ are as in Lemma 6.12, and $C_4(\delta)$ is as in Corollary 6.13.
- Lemma 6.14 holds for K.
- Proposition 11.4 and Lemma 11.6 hold for K (in place of K_0).

Let

$$\tilde{\mathcal{D}}_{00}(q_1) = \tilde{\mathcal{D}}_{00}(q_1, K_{00}, \delta, \epsilon, \eta) = \{t > 0 \;:\; g_t q_1 \in K\}.$$

For $ij \in \tilde{\Lambda}$, let

$$\tilde{\mathcal{D}}_{ij}(q_1) = \tilde{\mathcal{D}}_{ij}(q_1, K_{00}, \delta, \epsilon, \eta) = \big\{\hat{\tau}_{ij}(q_1, t) \;:\; g_t q_1 \in \pi^{-1}(K),\; t > 0\big\}.$$

Then by the ergodic theorem and (9.4), there exists a set $K_{\mathcal{D}} = K_{\mathcal{D}}(K_{00}, \delta, \epsilon, \eta)$ with $\nu(K_{\mathcal{D}}) \geq 1 - \delta$ and $\ell_{\mathcal{D}} = \ell_{\mathcal{D}}(K_{00}, \delta, \epsilon, \eta) > 0$ such that for $q_1 \in \pi^{-1}(K_{\mathcal{D}})$ and all $ij \in \{00\} \cup \tilde{\Lambda}$, $\tilde{\mathcal{D}}_{ij}(q_1)$ has density at least $1 - 2\kappa\delta$ for $\ell > \ell_{\mathcal{D}}$. Let

$$E_2(q_1, u) = E_2(q_1, u, K_{00}, \delta, \epsilon, \eta) = \big\{\ell \;:\; g_{\hat{\tau}_{(\epsilon)}(q_1, u, \ell)} u q_1 \in \pi^{-1}(K)\big\},$$

$$\begin{aligned}E_3(q_1,u) &= E_3(q_1,u,K_{00},\delta,\epsilon,\eta)\\ &= \Big\{\ell\in E_2(q_1,u)\ :\ \forall ij\in\tilde\Lambda,\ \hat\tau_{ij}\big(uq_1,\hat\tau_{(\epsilon)}(q_1,u,\ell)\big)\in\tilde{\mathcal{D}}_{ij}(q_1)\Big\}.\end{aligned}$$

Note that $\hat\tau_{ij}(uq_1,\hat\tau_{(\epsilon)}(q_1,u,\ell))\in\tilde{\mathcal{D}}_{ij}(q_1)$ if and only if

$$\hat\tau_{ij}\big(uq_1,\hat\tau_{(\epsilon)}(q_1,u,\ell)\big)=\hat\tau_{ij}(q_1,s)\ \text{and}\ g_s q_1\in\pi^{-1}(K).$$

Claim **12.3.** — *There exists* $\ell_3=\ell_3(K_{00},\delta,\epsilon,\eta)>0$, *a set* $K_3=K_3(K_{00},\delta,\epsilon,\eta)$ *of measure at least* $1-c_3(\delta)$ *and for each* $q_1\in\pi^{-1}(K_3)$ *a subset* $Q_3=Q_3(q_1,K_{00},\ell,\delta,\epsilon,\eta)\subset\mathcal{B}(q_1,1/100)$ *of measure at least* $(1-c_3'(\delta))|\mathcal{B}(q_1,1/100)|$ *such that for all* $q_1\in\pi^{-1}(K_3)$ *and* $u\in Q_3$, $uq_1\in\pi^{-1}(K)$ *and the density of* $E_3(q_1,u)$ *(for* $\ell>\ell_3$*) is at least* $1-c_3''(\delta)$, *and we have* $c_3(\delta)$, $c_3'(\delta)$ *and* $c_3''(\delta)\to 0$ *as* $\delta\to 0$.

Proof of claim. — We choose $K_2=K\cap K_{\mathcal{D}}$, and

$$\begin{aligned}K_3=K_2\cap\Big\{x\in X\ :&\\ &\Big|\Big\{u\in\mathcal{B}(x,1/100)\ :\ ux\in K_2\Big\}\Big|>(1-\delta)|\mathcal{B}(x,1/100)|\Big\}.\end{aligned}$$

Suppose $q_1\in\pi^{-1}(K_3)$, and $uq_1\in\pi^{-1}(K_2)$. Let

$$E_{bad}=\Big\{t\ :\ g_t uq_1\in\pi^{-1}\big(K^c\big)\Big\}.$$

Then, since $uq_1\in\pi^{-1}(K_{\mathcal{D}})$, for $\ell>\ell_{\mathcal{D}}$, the density of E_{bad} is at most $2\kappa\delta$. We have

$$E_2(q_1,u)^c=\Big\{\ell\ :\ \hat\tau_{(\epsilon)}(q_1,u,\ell)\in E_{bad}\Big\}.$$

Then, by Proposition 7.4, for $\ell>\kappa\ell_{\mathcal{D}}$, the density of $E_2(q_1,u)$ is at least $1-4\kappa^2\delta$.

Let

$$\hat{\mathcal{D}}(q_1,u)=\hat{\mathcal{D}}(q_1,u,K_{00},\delta,\epsilon,\eta)=\Big\{t\ :\ \forall ij\in\tilde\Lambda,\ \hat\tau_{ij}(uq_1,t)\in\tilde{\mathcal{D}}_{ij}(q_1)\Big\}.$$

Since $q_1\in\pi^{-1}(K_{\mathcal{D}})$, for each j, for $\ell>\ell_{\mathcal{D}}$, the density of $\tilde{\mathcal{D}}_{ij}(q_1)$ is at least $1-2\kappa\delta$. Then, by (9.4), for $\ell>\kappa\ell_{\mathcal{D}}$, the density of $\hat{\mathcal{D}}(q_1,u)$ is at least $(1-4|\tilde\Lambda|\kappa^2\delta)$. Now

$$E_3(q_1,u)=E_2(q_1,u)\cap\Big\{\ell\ :\ \hat\tau_{(\epsilon)}(q_1,u,\ell)\in\hat{\mathcal{D}}(q_1,u)\Big\}.$$

Now the claim follows from Proposition 7.4. □

Claim **12.4.** — *There exists a set* $\mathcal{D}_4=\mathcal{D}_4(K_{00},\delta,\epsilon,\eta)\subset\mathbb{R}^+$ *and a number* $\ell_4=\ell_4(K_{00},\delta,\epsilon,\eta)>0$ *so that* $\mathcal{D}_4$ *has density at least* $1-c_4(\delta)$ *for* $\ell>\ell_4$, *and for* $\ell\in\mathcal{D}_4$ *a subset* $K_4(\ell)=K_4(\ell,K_{00},\delta,\epsilon,\eta)\subset X$ *with* $\nu(K_4(\ell))>1-c_4'(\delta)$, *such that for any* $q_1\in\pi^{-1}(K_4(\ell))$ *there exists a subset* $Q_4(q_1,\ell)\subset Q_3(q_1,\ell)\subset\mathcal{B}(q_1,1/100)$ *with density at least* $1-c_4''(\delta)$, *so that for all* $\ell\in\mathcal{D}_4$, *for all* $q_1\in\pi^{-1}(K_4(\ell))$ *and all* $u\in Q_4(q_1,\ell)$,

$$\ell\in E_3(q_1,u)\subset E_2(q_1,u). \qquad (\mathbf{12.4})$$

(We have $c_4(\delta)$, $c_4'(\delta)$ *and* $c_4''(\delta)\to 0$ *as* $\delta\to 0$.*)*

Proof of claim. — This follows from Claim 12.3 by applying Fubini's theorem to $X_{\mathcal{B}} \times \mathbb{R}$, where $X_{\mathcal{B}} = \{(x, u) \ : \ x \in X, \ u \in \mathcal{B}(x, 1/100)\}$. ☐

Suppose $\ell \in \mathcal{D}_4$. We now apply Proposition 5.3 with $K' = g_{-\ell} K_4(\ell)$. We denote the resulting set K by $K_5(\ell) = K_5(\ell, K_{00}, \delta, \epsilon, \eta)$. In view of the choice of ϵ_1, we have $\nu(K_5(\ell)) \geq 1 - c_5(\delta)$, where $c_5(\delta) \to 0$ as $\delta \to 0$.

Let $\mathcal{D}_5 = \mathcal{D}_4$ and let $K_6(\ell) = g_\ell K_5(\ell)$.

Choice of parameters #2: choice of q, q', q'_1 (depending on δ, ϵ, q_1, ℓ). — Suppose $\ell \in \mathcal{D}_5$ and $q_1 \in \pi^{-1}(K_6(\ell))$. Let $q = g_{-\ell} q_1$. Then, $q \in \pi^{-1}(K_5(\ell))$. Let $\mathcal{A}(q, u, \ell, t)$ be as in Section 6. (Note that following our conventions, we use the notation $\mathcal{A}(q_1, u, \ell, t)$ for $q_1 \in X$, even though $\mathcal{A}(q_1, u, \ell, t)$ was originally defined for $q_1 \in X_0$.) and for $u \in Q_4(q_1, \ell)$ let $\mathcal{M}_u$ be the subspace of Lemma 5.1 applied to the linear map $\mathcal{A}(q_1, u, \ell, \hat{\tau}_{(\epsilon)}(q_1, u, \ell))$. By Proposition 5.3 and the definition of $K_5(\ell)$, we can choose $q' \in \mathcal{L}^-[q] \cap \pi^{-1}(g_{-\ell} K_4(\ell))$ with $\rho'(\delta) \leq d^{X_0}(q, q') \leq 1/100$ and so that (5.4) and (5.5) hold with $\epsilon_1(\delta) \to 0$ as $\delta \to 0$. Let $q'_1 = g_\ell q'$. Then $q'_1 \in \pi^{-1}(K_4(\ell))$.

Standing assumption. — We assume $\ell \in \mathcal{D}_5$, $q_1 \in K_6(\ell)$ and q, q', q'_1 are as in Choice of parameters #2.

Notation. — For $u \in \mathcal{B}(q_1, 1/100)$, $u' \in \mathcal{B}(q'_1, 1/100)$, let

$$\tau(u) = \hat{\tau}_{(\epsilon)}(q_1, u, \ell), \qquad \tau'\big(u'\big) = \hat{\tau}_{(\epsilon)}\big(q'_1, u', \ell\big).$$

The maps ψ and ψ'. — For $u \in \mathcal{B}(q_1, 1/100)$, and $u' \in \mathcal{B}(q'_1, 1/100)$, let

$$\psi(u) = g_{\tau(u)} u q_1, \qquad \psi'\big(u'\big) = g_{\tau'(u')} u' q'_1.$$

Claim **12.5.** — *We have*

$$\psi\big(Q_4(q_1, \ell)\big) \subset \pi^{-1}(K), \quad \textit{and} \quad \psi'\big(Q_4(q'_1, \ell)\big) \subset \pi^{-1}(K). \tag{12.5}$$

Proof of claim. — Suppose $u \in Q_4(q_1, \ell)$. Since $q_1 \in K_4$ and $\ell \in \mathcal{D}_4$, it follows from (12.4) that $\ell \in E_2(q_1, u)$, and then from the definition of $E_2(q_1, u)$ is follows that $g_{\tau(u)} u q_1 \in \pi^{-1}(K)$. Hence $\psi(Q_4(q_1, \ell)) \subset \pi^{-1}(K)$. Similarly, since $q'_1 \in \pi^{-1}(K_4)$, $\psi'(Q_4(q'_1, \ell)) \subset \pi^{-1}(K)$, proving (12.5). ☐

The numbers t_{ij} and t'_{ij}. — Suppose $u \in Q_4(q_1, \ell)$, and suppose $ij \in \tilde{\Lambda}$. Let t_{ij} be defined by the equation

$$\hat{\tau}_{ij}\big(u q_1, \hat{\tau}_{(\epsilon)}(q_1, u, \ell)\big) = \hat{\tau}_{ij}(q_1, t_{ij}). \tag{12.6}$$

Then, since $\ell \in \mathcal{D}_4$ and in view of (12.4), we have $\ell \in E_3(q_1, u)$. In view of the definition of E_3, it follows that

$$g_{t_{ij}} q_1 \in \pi^{-1}(K). \tag{12.7}$$

Similarly, suppose $u' \in Q_4(q_1', \ell)$ and $ij \in \tilde{\Lambda}$. Let t_{ij}' be defined by the equation

$$\hat{\tau}_{ij}\big(u' q_1', \hat{\tau}_{(\epsilon)}\big(q_1', u', \ell\big)\big) = \hat{\tau}_{ij}\big(q_1', t_{ij}'\big). \tag{12.8}$$

Then, by the same argument,

$$g_{t_{ij}'} q_1' \in \pi^{-1}(K). \tag{12.9}$$

The map $\mathbf{v}(u)$ *and the generalized subspace* $\mathcal{U}(u)$. — For $u \in \mathcal{B}(q_1, 1/100)$, let

$$\mathbf{v}(u) = \mathbf{v}\big(q, q', u, \ell, t\big) = \mathcal{A}(q, u, \ell, t)\big(F(q) - F\big(q'\big)\big) \tag{12.10}$$

where $t = \hat{\tau}_{(\epsilon)}(q_1, u, \ell)$, F is as in Section 5 and $\mathcal{A}(\cdot, \cdot, \cdot, \cdot)$ is as in Section 6.1. By Proposition 6.11, we may write $\mathbf{v}(u) = \mathbf{j}(M'', v'')$, where $(M'', v'') \in \mathcal{H}_{++}(g_{\tau(u)} u q_1) \times W^+(g_{\tau(u)} u q_1)$. Let $\mathcal{U}(u) \equiv \mathcal{U}_{g_{\tau(u)} u q_1}(M'', v'')$ denote the generalized affine subspace corresponding to $\mathbf{v}(u)$. Thus, $\mathcal{U}(u)$ is the approximation to $U^+[g_{\tau(u)} q_1']$ near $g_{\tau(u)} u q_1$ defined in Proposition 6.11.

Standing assumption. — We have $C(\delta)\epsilon < 1/100$ for any constant $C(\delta)$ arising in the course of the proof. In particular, this applies to $C_2(\delta)$ and $C_2'(\delta)$ in the next claim.

Claim **12.6.** — *There exists a subset* $Q_5 = Q_5(q_1, \ell, K_{00}, \delta, \epsilon, \eta) \subset Q_4(q_1, \ell)$ *with* $|Q_5| \geq (1 - c_5''(\delta))|\mathcal{B}(q_1, 1/100)|$ *(with* $c_5''(\delta) \to 0$ *as* $\delta \to 0$*), and a number* $\ell_5 = \ell_5(\delta, \epsilon)$ *such that for all* $u \in Q_5$ *and* $\ell > \ell_5$,

$$\tau(u) < \frac{1}{2}\alpha_3 \ell, \tag{12.11}$$

where $\alpha_3 > 0$ *is as in Proposition 6.16 and Section 6.1. In addition,*

$$C_1(\delta)\epsilon \leq hd^{X_0}_{g_{\tau(u)} u q_1}\big(U^+[g_{\tau(u)} u q_1], U^+\big[g_{\tau(u)} q_1'\big]\big) \leq C_2(\delta)\epsilon, \tag{12.12}$$

$$hd^{X_0}_{g_{\tau(u)} u q_1}\big(U^+\big[g_{\tau(u)} q_1'\big], \mathcal{U}(u)\big) \leq C_7(\delta) e^{-\alpha \ell}, \tag{12.13}$$

where α *depends only on the Lyapunov spectrum. Also,*

$$C_1'(\delta)\epsilon \leq \|\mathbf{v}(u)\| \leq C_2'(\delta)\epsilon, \tag{12.14}$$

and if $u' \in U^+[q_1']$ *is such that*

$$d^{X_0}\big(g_{\tau(u)} u q_1, g_{\tau(u)} u' q_1'\big) < 1/100, \tag{12.15}$$

then $u' \in \mathcal{B}(q_1', 1/100)$.

Proof of claim. — Let $\mathcal{M}_u$ be the subspace of Lemma 5.1 applied to the linear map $\mathcal{A}(q_1, u, \ell, \hat{\tau}_{(\epsilon)}(q_1, u, \ell))$, where $\mathcal{A}(,,,)$ is as in Section 6. Let $Q(q_1)$ be as in Proposition 6.16, so $|Q(q_1)| \geq (1-\delta)|\mathcal{B}(q_1, 1/100)|$. Let $Q_5' \subset Q_4 \cap Q(q_1)$ be such that for all $u \in Q_5'$,

$$d_Y\big(F(q) - F(q'), \mathcal{M}_u\big) \geq \beta(\delta)$$

where F is as in Section 5. Then, (12.11) follows from Proposition 6.16 and the fact that $Q_5 \subset Q_1$. Also, by (5.5),

$$\begin{aligned}|Q_5'| &\geq |Q_4| - \big(\delta + \epsilon_1(\delta)\big)|\mathcal{B}(q_1, 1/100)| \\ &\geq \big(1 - \delta - \epsilon_1(\delta) - c_4''(\delta)\big)|\mathcal{B}(q_1, 1/100)|.\end{aligned}$$

Then, let $Q_5 = \{u \in Q_5' \,:\, d(u, \partial\mathcal{B}(q_1, 1/100)) > \delta\}$, hence

$$|Q_5| \geq \big(1 - c_5'(\delta) - c_4'(\delta) - c_n\delta\big)|\mathcal{B}(q_1, 1/100)|,$$

where c_n depends only on the dimension.

We have $C(\delta)^{-1}\epsilon \leq \|\mathcal{A}(q_1, u, \ell, t)\| \leq C(\delta)\epsilon$ by the definition of $t = \hat{\tau}_{(\epsilon)}(q_1, u, \ell)$. We now apply Lemma 5.1 to the linear map $\mathcal{A}(q_1, u, \ell, t)$. Then, for all $u \in Q_5$,

$$c(\delta)\|\mathcal{A}(q_1, u, \ell, t)\| \leq \big\|\mathcal{A}(q_1, u, \ell, t)\big(F(q) - F(q')\big)\big\| \leq \|\mathcal{A}(q_1, u, \ell, t)\|.$$

Therefore,

$$C'(\delta)^{-1}\epsilon \leq \big\|\mathcal{A}(q_1, u, \ell, t)\big(F(q) - F(q')\big)\big\| \leq C'(\delta)\epsilon$$

This immediately implies (12.14), in view of the definition of $\mathbf{v}(u)$. We now apply Proposition 6.11 and Lemma 6.12(a). (We assume ϵ is sufficiently small so that (6.29) holds. Also the condition (6.22) in Proposition 6.11 holds in view of Proposition 6.16.) Now (12.12) follows from (6.25). Also (12.13) follows from (6.27).

Finally, suppose $u \in Q_5$, and $u' \in U^+(q_1')$ is such that (12.15) holds. Then, by Lemma 6.14, we have $d^{X_0}(uq_1, u'q_1') = O_\delta(e^{-\alpha\ell})$. Then, assuming ℓ is sufficiently large (depending on δ) and using Proposition 3.4, we have $u' \in \mathcal{B}(q_1', 1/100)$. □

Standing assumption. — We assume $\ell > \ell_5$.

Claim **12.7.** — *Suppose* $u \in Q_5(q_1, \ell)$, $u' \in Q_4(q_1', \ell)$ *and* (12.15) *holds. Then, there exists* $C_0 = C_0(\delta)$ *such that*

$$\big|\hat{\tau}_{(\epsilon)}(q_1, u, \ell) - \hat{\tau}_{(\epsilon)}\big(q_1', u', \ell\big)\big| \leq C_0(\delta). \tag{12.16}$$

Proof of claim. — Let $t = \hat{\tau}_{(\epsilon)}(q_1, u, \ell)$, $t' = \hat{\tau}_{(\epsilon)}(q_1', u', \ell)$.
By Proposition 6.11(ii) (with q' and q reversed) and (5.4),

$$\begin{aligned}\epsilon &= \|\mathcal{A}(q_1', \ell, u', t')\| \geq \|\mathcal{A}(q_1', \ell, u', t')(\mathrm{F}(q') - \mathrm{F}(q))\| \\ &\geq c(\delta) hd^{\mathrm{X}_0}_{g_{t'}u'q_1'}(\mathrm{U}^+[g_{t'}u'q_1'], \mathrm{U}^+[g_{t'}uq_1])\end{aligned}$$

In view of Corollary 6.13(b), (12.11) and the fact that $g_{t'}u'q_1' \in \pi^{-1}(\mathrm{K})$, this contradicts (12.12), unless $t' < t + \mathrm{C}(\delta)$.

It remains give a lower bound on t'. Let $\mathcal{M}'$ denote the subspace as in Lemma 5.1 for $\mathcal{A}(q', u', \ell, t')$. Note that by Proposition 5.3 (with the function $u \to \mathcal{M}_u$ the constant function $\mathcal{M}'$) we can choose $q'' \in \mathrm{W}^-[q]$ with $d_{\mathrm{Y}}(\mathrm{F}(q'') - \mathrm{F}(q'), \mathcal{M}') > \rho(\delta)$, and also so the upper bounds in (5.3) and (5.4) hold with q'' in place of q'. Then,

$$\epsilon = \|\mathcal{A}(q', \ell, u', t')\| \leq c(\delta)\|\mathcal{A}(q', \ell, u', t')(\mathrm{F}(q'') - \mathrm{F}(q'))\|.$$

Write $q_1'' = g_\ell q''$. Then, by Proposition 6.11(ii), and Lemma 6.12(a),

$$hd^{\mathrm{X}_0}_{g_{t'}u'q_1'}(\mathrm{U}^+[g_{t'}u'q_1'], \mathrm{U}^+[g_{t'}q_1'']) \geq c_2(\delta)\epsilon. \tag{12.17}$$

By Corollary 6.13(a), (12.11) and (12.12), since $g_{t'}u'q_1' \in \pi^{-1}(\mathrm{K})$,

$$hd^{\mathrm{X}_0}_{g_{t'}u'q_1'}(\mathrm{U}^+[g_{t'}u'q_1'], \mathrm{U}^+[g_{t'}uq_1]) \leq \epsilon \mathrm{C}(\delta)e^{-\beta(t-t')} + \mathrm{C}_4(\delta)e^{-\alpha\ell}, \tag{12.18}$$

where α and β depend only on the Lyapunov spectrum. Then, by (12.17), (12.18), and the reverse triangle inequality,

$$hd^{\mathrm{X}_0}_{g_{t'}uq_1}(\mathrm{U}^+[g_{t'}uq_1], \mathrm{U}^+[g_{t'}q_1'']) \geq \epsilon(c_2(\delta) - \mathrm{C}(\delta)e^{-\beta(t-t')}) - \mathrm{C}_4(\delta)e^{-\alpha\ell}. \tag{12.19}$$

But,

$$\epsilon = \|\mathcal{A}(q, \ell, u, t)\| \geq c_3(\delta)\|\mathcal{A}(q, \ell, u, t)(\mathrm{F}(q'') - \mathrm{F}(q))\|,$$

and thus, by Proposition 6.11(ii) and Lemma 6.12(a),

$$hd^{\mathrm{X}_0}_{g_tuq_1}(\mathrm{U}^+[g_tuq_1], \mathrm{U}^+[g_tq_1'']) \leq c(\delta)\epsilon$$

In view of Corollary 6.13(b) (and the fact that $g_tuq_1 \in \pi^{-1}(\mathrm{K})$) this contradicts (12.19) unless $t' > t - \mathrm{C}_1(\delta)$. □

We note the following trivial lemma:

Lemma **12.8.** — *Suppose* P *and* P′ *are finite measure subsets of* $\mathbb{R}^n$ *with* $|\mathrm{P}| = |\mathrm{P}'|$, *and we have*

$$\mathrm{P} = \bigcup_{j=1}^{\mathrm{N}} \mathrm{P}_j, \qquad \mathrm{P}' = \bigcup_{j=1}^{\mathrm{N}} \mathrm{P}_j'.$$

Suppose there exists $k \in \mathbb{N}$ so that any point in P is contained in at most k sets P_j, and also any point in P' is contained in at most k sets P'_j. Also suppose $Q \subset P$ and $Q' \subset P'$ are subsets with $|Q| > (1-\delta)|P|$, $|Q'| > (1-\delta)|P'|$.

Suppose there exists $\kappa > 1$ such that for all $1 \le j \le N$ such that $P_j \cap Q \neq \emptyset$, $|P_j| \le \kappa |P'_j|$. Then there exists $\hat{Q} \subset Q$ with $|\hat{Q}| \ge (1 - 2\kappa k\delta)|P|$ such that if j is such that $\hat{Q} \cap P_j \neq \emptyset$, then $Q' \cap P'_j \neq \emptyset$.

Proof. — Let $J = \{j \,:\, P_j \cap Q \neq \emptyset\}$, and let $J' = \{j \,:\, Q' \cap P'_j \neq \emptyset\}$, and let

$$\hat{Q} = \left\{x \in Q : \text{ for all } j \text{ with } x \in P_j, \text{ we have } j \in J'\right\}.$$

Thus, if $x \in Q \setminus \hat{Q}$, then there exists $j \in J$ with $x \in Q \cap P_j$ but $j \notin J'$. Then,

$$|Q \setminus \hat{Q}| \le k \sum_{j \in J \setminus J'} |Q \cap P_j| \le k \sum_{j \in J \setminus J'} |P_j| \le \kappa k \sum_{j \notin J'} |P'_j| \le \kappa k \left|(Q')^c\right|,$$

since if $j \notin J'$ then $P'_j \subset (Q')^c$. Thus, $|Q \setminus \hat{Q}| \le \kappa k \delta |P|$, and so $|\hat{Q}| \ge (1 - 2\kappa k\delta)|P|$. □

The constant ϵ_0. — Let $\epsilon_0(\delta)$ be a constant to be chosen later (we will choose $\epsilon_0(\delta)$ following (12.33) of the form $\epsilon_0(\delta) = \epsilon'_0(\delta)/C(\delta)$), where $\epsilon'_0(\delta)$ is as in (12.3). We will always assume that $\epsilon < \epsilon_0(\delta) < \epsilon'(\delta)/10$.

Claim **12.9.** — *There exists a subset $Q_6(q_1, \ell) = Q_6(q_1, \ell, K_{00}, \delta, \epsilon, \eta) \subset Q_5(q_1, \ell)$ with $|Q_6(q_1, \ell)| > (1 - c'_6(\delta))|\mathcal{B}(q_1, 1/100)|$ and with $c'_6(\delta) \to 0$ as $\delta \to 0$ such that for all $u \in Q_6(q_1, \ell)$ there exists $u' \in Q_4(q'_1, \ell)$ such that*

$$d^{X_0}\left(g_{\tau(u)} u q_1, g_{\tau(u)} u' q'_1\right) < C(\delta)\epsilon_0(\delta). \tag{12.20}$$

Proof of claim. — Note that the sets $\{\mathcal{B}_{\tau(u)}[uq_1] \;:\; u \in Q_5(q_1, \ell)\}$ are a cover of $Q_5(q_1, \ell)q_1$. Then, since these sets satisfy the condition of Lemma 3.10(b), we can find a pairwise disjoint subcover, i.e. find $u_j \in Q_5(q_1, \ell)$, $1 \le j \le N$, with $Q_5(q_1, \ell)q_1 = \bigcup_{j=1}^N \mathcal{B}_{\tau(u_j)}[u_j q_1]$ and so that $\mathcal{B}_{\tau(u_j)}[u_j q_1]$ and $\mathcal{B}_{\tau(u_k)}[u_k q_1]$ are disjoint for $j \neq k$. Let

$$\mathcal{B}_j \equiv g_{\tau(u_j)} \mathcal{B}_{\tau(u_j)}[u_j q_1] = \mathcal{B}_0[g_{\tau(u_j)} u_j q_1] \subset \tilde{X}_0$$

In view of (12.3), Proposition 3.4, and the Besicovich covering lemma, there exists k, depending only on the dimension and points $x_{j,1}, \dots, x_{j,m(j)} \subset \mathcal{B}_j$ such that

$$\pi^{-1}(K) \cap \mathcal{B}_j \subset \bigcup_{m=1}^{m(j)} B^{X_0}\left(x_{j,m}, \epsilon_0(\delta)\right) \cap U^+[g_{\tau(u_j)} u q_1],$$

and also so that for a fixed j, each point is contained in at most k balls $B^{X_0}(x_{j,m}, \epsilon_0(\delta))$. Since $\epsilon_0(\delta) < \epsilon'_0(\delta)/10$, in view of (12.3) and (12.21), the same is true without fixing j.

For $1 \le j \le N$ and $1 \le m \le m(j)$, let

$$P_{j,m} = \{ u \in \mathcal{B}(q_1, 1/100) \; : \; g_{\tau(u_j)} u q_1 \in B^{X_0}\big(x_{j,m}, \epsilon_0(\delta)\big) \},$$

and let

$$P'_{j,m} = \{ u' \in \mathcal{B}\big(q'_1, 1/100\big) \; : \; g_{\tau(u_j)} u' q'_1 \in B^{X_0}\big(x_{j,m}, \epsilon_0(\delta)\big) \}.$$

By construction, each point is contained in at most k sets $P_{j,m}$, and at most k sets $P'_{j,m}$.

By (12.12) applied to u_j,

$$hd^{X_0}_{g_{\tau(u_j)}u_j q_1}\big(U^+[g_{\tau(u_j)}u_j q_1], U^+\big[g_{\tau(u_j)} q'_1\big]\big) \le C_2(\delta)\epsilon. \tag{12.21}$$

Suppose $\epsilon > 0$ is sufficiently small (depending on δ) so that Lemma 6.14 holds with $C_2(\delta)\epsilon$ in place of ϵ. Since for all $x \in X_0$, $\mathcal{B}_0[x] \subset B^{X_0}(x, 1/200)$ we have $d^{X_0}(x_{j,m}, g_t u_j q_1) < 1/200$, and

$$B^{X_0}\big(x_{j,m}, \epsilon_0(\delta)\big) \subset B^{X_0}(g_t u_j q_1, 1/100). \tag{12.22}$$

By Lemma 6.14, for $1 \le j \le N$, $1 \le m \le m(j)$, provided $\mathcal{B}_j \cap Q_5(q_1, \ell) \ne \emptyset$, we have $\kappa^{-1}|P_{j,m}| \le |P'_{j,m}| \le \kappa |P_j|$, where κ depends only on the Lyapunov spectrum, and we have normalized the measures $|\cdot|$ so that $|U^+[q_1] \cap B^+(q_1, 1/100)| = |U^+[q'_1] \cap B^+(q'_1, 1/100)| = 1$. Let $m(0) = 1$ and let

$$P_{0,1} = \mathcal{B}(q_1, 1/100) \setminus \bigcup_{j=1}^{N} \bigcup_{m=1}^{m(j)} P_{j,m}, \qquad P'_{0,1} = \mathcal{B}\big(q'_1, 1/100\big) \setminus \bigcup_{j=1}^{N} \bigcup_{m=1}^{m(j)} P'_{j,m}.$$

Then,

$$\mathcal{B}(q_1, 1/100) = \bigcup_{j=0}^{N} \bigcup_{m=1}^{m(j)} P_{j,m}, \qquad \mathcal{B}\big(q'_1, 1/100\big) = \bigcup_{j=0}^{N} \bigcup_{m=1}^{m(j)} P'_{j,m}.$$

Then, applying Lemma 12.8 with $P = \mathcal{B}(q_1, 1/100)$, $P' = \mathcal{B}(q'_1, 1/100)$, $Q = Q_5(q_1, \ell)$, $Q' = Q_4(q'_1, \ell)$, we get a set $\hat{Q} \equiv Q_6(q_1, \ell)$ with $|Q_6(q_1, \ell)| \ge (1 - c'_6(\delta))|\mathcal{B}(q_1, 1/100)|$ where $c'_6(\delta) \to 0$ as $\delta \to 0$, so that, in view of (12.22) and the definitions of $P_{j,m}$ and $P'_{j,m}$, for any $u \in Q_6(q_1, \ell)$ there exists $u_j \in Q_5(q_1, \ell)$ with $uq_1 \in \mathcal{B}_{\tau(u_j)}[u_j q_1]$ and $u' \in Q_4(q'_1, \ell)$ with

$$d^{X_0}\big(g_{\tau(u_j)} u q_1, g_{\tau(u_j)} u' q'_1\big) \le \epsilon_0(\delta). \tag{12.23}$$

It remains to replace $\tau(u_j)$ by $\tau(u)$ in (12.23). This can be done as follows: Since $uq_1 \in \mathcal{B}_{\tau(u_j)}[u_j q_1]$, we have, by (12.12) applied to u_j and Lemma 6.18,

$$C_2(\delta)^{-1}\epsilon \le hd_{g_{\tau(u_j)}uq_1}\big(U^+[g_{\tau(u_j)} u q_1], U^+\big[g_{\tau(u_j)} q'_1\big]\big) \le C_2(\delta)\epsilon$$

Then, since $g_{\tau(u)}uq_1 \in \pi^{-1}(K)$, by (12.12), (12.13), (12.11) and Corollary 6.13, we have

$$|\tau(u) - \tau(u_j)| \le C_1(\delta). \tag{12.24}$$

Then, provided ϵ is small enough depending on δ, (12.20) follows from (12.23), (12.24), and Lemma 3.6. □

Claim **12.10.** — *There exists a constants $c_7(\delta) > 0$ and $c_7'(\delta)$ with $c_7(\delta) \to 0$ and $c_7'(\delta) \to 0$ as $\delta \to 0$ and a subset $K_7(\ell) = K_7(\ell, K_{00}, \delta, \epsilon, \eta)$ with $K_7(\ell) \subset K_6(\ell)$ and $\nu(K_7(\ell)) > 1 - c_7(\delta)$ such that for $q_1 \in \pi^{-1}(K_7(\ell))$,*

$$|\mathcal{B}(q_1) \cap Q_6(q_1, \ell)| \ge \big(1 - c_7'(\delta)\big)|\mathcal{B}(q_1)|.$$

Proof of claim. — Recall that in view of Proposition 3.7, $\mathcal{B}(q_1) \subset \mathcal{B}(q_1, 1/100)$. Given $\delta > 0$, there exists $c_7''(\delta) > 0$ with $c_7''(\delta) \to 0$ as $\delta \to 0$ and a compact set $K_7' \subset X$ with $\nu(K_7') > 1 - c_7''(\delta)$, such that for $q_1 \in \pi^{-1}(K_7')$, $|\mathcal{B}(q_1) \cap \mathcal{B}(q_1, 1/100)| \ge c_6'(\delta)^{1/2}|\mathcal{B}(q_1, 1/100)|$. Then, for $q_1 \in \pi^{-1}(K_7' \cap K_6)$,

$$\begin{aligned} |\mathcal{B}(q_1) \cap Q_6(q_1, \ell)^c| &\le |Q_6(q_1, \ell)^c| \le c_6'(\delta)|\mathcal{B}(q_1, 1/100)| \\ &\le c_6'(\delta)^{1/2}|\mathcal{B}(q_1)|. \end{aligned}$$

Thus, the claim holds with $c_7(\delta) = c_6(\delta) + c_7''(\delta)$ and $c_7'(\delta) = c_6'(\delta)^{1/2}$. □

Standing assumption. — We assume that $q_1 \in \pi^{-1}(K_7(\ell))$.

The next few claims will help us choose u (once the other parameters have been chosen). Let

$$Q_7(q_1, \ell) = \mathcal{B}(q_1) \cap Q_6(q_1, \ell)$$

Claim **12.11.** — *There exists a subset $Q_7^*(q_1, \ell) = Q_7^*(q_1, \ell, K_{00}, \delta, \epsilon, \eta) \subset Q_7(q_1, \ell)$ with $|Q_7^*| \ge (1 - c_7^*(\delta))|\mathcal{B}(q_1)|$ such that for $u \in Q_7^*$ and any $\ell > \ell_7(\delta)$ we have*

$$|\mathcal{B}_\ell(uq_1) \cap Q_7(q_1, \ell)| \ge \big(1 - c_7^*(\delta)\big)|\mathcal{B}_\ell(uq_1)|,$$

where $c_7^*(\delta) \to 0$ as $\delta \to 0$.

Proof. — This follows immediately from Lemma 6.3. □

Claim **12.12.** — *There exist a number $\ell_8 = \ell_8(K_{00}, \delta, \epsilon, \eta)$ and a constant $c_8(\delta)$ with $c_8(\delta) \to 0$ as $\delta \to 0$ and for every $\ell > \ell_8$ a subset $Q_8(q_1, \ell) = Q_8(q_1, \ell, K_{00}, \delta, \epsilon, \eta) \subset \mathcal{B}(q_1)$ with $|Q_8(q_1, \ell)| \ge (1 - c_8(\delta))|\mathcal{B}(q_1)|$ so that for $u \in Q_8(q_1, \ell)$ we have*

$$d\left(\frac{\mathbf{v}(u)}{\|\mathbf{v}(u)\|}, \mathbf{E}(g_{\tau(u)}uq_1)\right) \le C_8(\delta)e^{-\alpha'\ell}, \tag{12.25}$$

where $\mathbf{v}(u)$ is defined in (12.10) and α' depends only on the Lyapunov spectrum.

Proof of claim. — Let $L' > L_2(\delta)$ be a constant to be chosen later, where $L_2(\delta)$ is as in Proposition 8.5(a). Also let $\ell_8 = \ell_8(\delta, \epsilon, K_{00}, \eta)$ be a constant to be chosen later. Suppose $\ell > \ell_8$, and suppose $u \in Q_7^*(q_1, \ell)$, so in particular $g_{\tau(u)}uq_1 \in \pi^{-1}(K)$. Let $t \in [L', 2L']$ be such that Proposition 8.5(a) holds for $\mathbf{v} = \mathbf{v}(u)$ and $x = g_{\tau(u)}uq_1$.

Let $B_u \subset \mathcal{B}(q_1)$ denote $\mathcal{B}_{\hat{\tau}_{(\epsilon)}(q_1,u,\ell)-t}(uq_1)u$, (where $\mathcal{B}_t(x)$ is defined in Section 6). Suppose $u_1 \in B_u \cap Q_7(q_1, \ell)$, and write

$$g_{\tau(u_1)}u_1q_1 = g_s u_2 g_t^{-1} g_{\tau(u)}uq_1.$$

Then, $u_2 \in \mathcal{B}(g_t^{-1}g_{\tau(u)}uq_1)$ and $t \le 2L'$.

We now claim that

$$s \le \frac{1}{2}\kappa t + C_0(\delta) \le \kappa L' + C_0(\delta) \tag{12.26}$$

where κ depends only on the Lyapunov spectrum. Let

$$\mathcal{U}_t = U^+[g_{-t}g_{\tau(u)}uq_1], \qquad \mathcal{U}'_t = U^+\big[g_{-t}g_{\tau(u)}q'_1\big].$$

By Corollary 6.13(b) applied at the point $g_{\tau(u)}uq_1 \in \pi^{-1}(K)$,

$$hd^{X_0}_{g_{-t}g_{\tau(u)}uq_1}\big(\mathcal{U}_t, \mathcal{U}'_t\big) \ge C(\delta)\epsilon e^{-\beta t} - c_0(\delta)e^{-\alpha\ell},$$

where β depends only on the Lyapunov spectrum, and by Corollary 6.13(a) applied at the point $g_{\tau(u_1)}u_1q_1 \in \pi^{-1}(K)$,

$$hd^{X_0}_{u_2g_{-t}g_{\tau(u)}uq_1}\big(\mathcal{U}_t, \mathcal{U}'_t\big) \le c(\delta)\epsilon e^{-2s} + c_0(\delta)e^{-\alpha\ell}$$

where β' also depends only on the Lyapunov spectrum. Also, by Lemma 6.18,

$$hd^{X_0}_{g_{-t}g_{\tau(u)}uq_1}\big(\mathcal{U}_t, \mathcal{U}'_t\big) \ge c_1 hd^{X_0}_{u_2g_{-t}g_{\tau(u)}uq_1}\big(\mathcal{U}_t, \mathcal{U}'_t\big) - c_0(\delta)e^{-\alpha\ell}$$

where c_1 is an absolute constant. Therefore,

$$\epsilon C(\delta)e^{-\beta t} - c_0(\delta)e^{-\alpha\ell} \le c_1\big(c(\delta)\epsilon e^{-2s} + c_0(\delta)e^{-\alpha\ell}\big).$$

This implies (12.26), assuming that ℓ is sufficiently large depending on ϵ.

Since $u \in Q_6(q_1, \ell)$, (12.12) and (12.13) hold. Therefore,

$$hd_{g_{\tau(u_1)}u_1q_1}\big((g_su_2g_t^{-1})\mathcal{U}(u), U^+\big[g_{\tau(u_1)}q'_1\big]\big) = O\big(e^{\kappa' L'}e^{-\alpha\ell}\big),$$

where κ' and α depend only on the Lyapunov spectrum. Thus, using (12.13) at the point $g_{\tau(u_1)}u_1q_1 \in \pi^{-1}(K)$,

$$hd_{g_{\tau(u_1)}u_1q_1}\big((g_su_2g_t^{-1})\mathcal{U}(u), \mathcal{U}(u_1)\big) = O\big(e^{\kappa' L'}e^{-\alpha\ell}\big).$$

Therefore,

$$\left\| \left(g_s u_2 g_t^{-1}\right)_* \mathbf{v}(u) - \mathbf{v}(u_1) \right\| = O\left(e^{\kappa' L'} e^{-\alpha \ell}\right). \tag{12.27}$$

In view of (12.14), $\|\mathbf{v}(u_1)\| \approx \epsilon$. Thus, $\|(g_s u_2 g_t^{-1})_* \mathbf{v}(u)\| \approx \epsilon$, and

$$\left\| \frac{(g_s u_2 g_t^{-1})_* \mathbf{v}(u)}{\|(g_s u_2 g_t^{-1})_* \mathbf{v}(u)\|} - \frac{\mathbf{v}(u_1)}{\|\mathbf{v}(u_1)\|} \right\| = O_\epsilon\left(e^{\kappa' L' - \alpha \ell}\right).$$

But, by Proposition 8.5(a), for $1-\delta$ fraction of $u_2 \in \mathcal{B}(g_t^{-1} g_{\tau(u)} u q_1)$,

$$d\left(\frac{(g_s u_2 g_{-t})_* \mathbf{v}(u)}{\|(g_s u_2 g_{-t})_* \mathbf{v}(u)\|}, \mathbf{E}(g_{\tau(u_1)} u_1 q_1) \right) \le C(\delta) e^{-\alpha L'}.$$

Note that

$$\mathcal{B}\left(g_t^{-1} g_{\tau(u)} u q_1\right) = g_{\hat{\tau}_{(\epsilon)}(q_1, u, \ell) - t} B_u.$$

Therefore, for $1-\delta$ fraction of $u_1 \in B_u$,

$$d\left(\frac{\mathbf{v}(u_1)}{\|\mathbf{v}(u_1)\|}, \mathbf{E}(g_{\tau(u_1)} u_1 q_1) \right) \le C(\epsilon, \delta)\left[e^{\kappa' L' - \alpha \ell} + e^{-\alpha L'} \right] \tag{12.28}$$

We can now choose $L' > 0$ to be $\alpha' \ell$ where $\alpha' > 0$ is a small constant depending only on the Lyapunov spectrum, and $\ell_8 > 0$ so that for $\ell > \ell_8$ the right-hand-side of the above equation is at most $e^{-\alpha' \ell}$.

The collection of balls $\{B_u\}_{u \in Q_7^*(q_1, \ell)}$ are a cover of $Q_7^*(q_1, \ell)$. These balls satisfy the condition of Lemma 3.10(b); hence we may choose a pairwise disjoint subcollection which still covers $Q_7^*(q_1, \ell)$. Then, by summing (12.28), we see that (12.25) holds for u in a subset $Q_8 \subset \mathcal{B}[q_1]$ of measure at least $(1 - c_8(\delta))|\mathcal{B}[q_1]| = (1-\delta)(1 - c_7^*(\delta))|\mathcal{B}[q_1]|$. □

Claim **12.13.** — *There exists a subset* $Q_8^*(q_1, \ell) = Q_8^*(q_1, \ell, K_{00}, \delta, \epsilon, \eta) \subset Q_8(q_1, \ell)$ *with* $|Q_8^*| \ge (1 - c_8^*(\delta))|\mathcal{B}(q_1)|$ *such that for* $u \in Q_8^*$ *and any* $t > \ell_8(\delta)$ *we have*

$$|\mathcal{B}_t(u q_1) \cap Q_8(q_1, \ell)| \ge \left(1 - c_8^*(\delta)\right)|\mathcal{B}_t(u q_1)|,$$

where $c_8^*(\delta) \to 0$ *as* $\delta \to 0$.

Proof. — This follows immediately from Lemma 6.3. □

Choice of parameters #3: choice of δ. — Let $\theta' = (\theta/2)^n$, where θ and n are as in Proposition 10.1. We can choose $\delta > 0$ so that

$$c_8^*(\delta) < \theta'/2. \tag{12.29}$$

Claim **12.14.** — *There exist sets* $Q_9(q_1,\ell) = Q_9(q_1,\ell,K_{00},\delta,\epsilon,\eta) \subset Q_8^*(q_1,\ell)$ *with* $|Q_9(q_1,\ell)| \geq (\theta'/2)(1-\theta'/2)|\mathcal{B}(q_1)|$ *and* $\ell_9 = \ell_9(K_{00},\delta,\epsilon,\eta)$*, such that for* $\ell > \ell_9$ *and* $u \in Q_9(q_1,\ell)$*,*

$$d\left(\frac{\mathbf{v}(u)}{\|\mathbf{v}(u)\|}, \bigcup_{ij\in\tilde{\Lambda}} \mathbf{E}_{[ij],bdd}(g_{\tau(u)}uq_1)\right) < 4\eta. \tag{12.30}$$

Proof of claim. — Suppose $u \in Q_8^*(q_1,\ell)$. Then, by (12.25) and (12.14), we may write

$$\mathbf{v}(u) = \mathbf{v}'(u) + \mathbf{v}''(u),$$

where $\mathbf{v}'(u) \in \mathbf{E}(g_{\tau(u)}uq_1)$ and $\|\mathbf{v}''(u)\| \leq C(\delta,\epsilon)e^{-\alpha'\ell}$. Arguing in the same way as in the proof of Claim 12.12, we see that for $(1-O(\delta))$-fraction of $y \in \mathcal{F}_{\mathbf{v}'(u)}[g_{\tau(u)}uq_1, L]$, we have $y \in g_{[-1,1]}K$. Then, by Proposition 10.1 applied with $L = L_0(\delta,\eta)$ and $\mathbf{v} = \mathbf{v}'(u)$, we get that for a at least θ'-fraction of $y \in \mathcal{F}_{\mathbf{v}'}[g_{\tau(u)}uq_1, L]$,

$$d\left(\frac{R(g_{\tau(u)}uq_1,y)\mathbf{v}'(u)}{\|R(g_{\tau(u)}uq_1,y)\mathbf{v}'(u)\|}, \bigcup_{ij\in\tilde{\Lambda}} \mathbf{E}_{[ij],bdd}(y)\right) < 2\eta.$$

Note that by Proposition 4.15(d), for $y \in \mathcal{F}_{\mathbf{v}'}[g_{\tau(u)}uq_1, L]$, $\|R(g_{\tau(u)}uq_1,y)\| \leq e^{\kappa^2 L}$, where κ is as in Proposition 4.15. Then, for at least θ'-fraction of $y \in \mathcal{F}_{\mathbf{v}'}[g_{\tau(u)}uq_1, L]$,

$$d\left(\frac{R(g_{\tau(u)}uq_1,y)\mathbf{v}(u)}{\|R(g_{\tau(u)}uq_1,y)\mathbf{v}(u)\|}, \bigcup_{ij\in\tilde{\Lambda}} \mathbf{E}_{[ij],bdd}(y)\right) < 3\eta + C(\epsilon,\delta)e^{2\kappa^2 L}e^{-\alpha'\ell}. \tag{12.31}$$

Let $B_u = \mathcal{B}_{\hat{\tau}_{(\epsilon)}(q_1,u,\ell)-L}(uq_1)u$. In view of (12.27) and (12.14) there exists $C = C(\epsilon,\delta)$ such that

$$\mathcal{F}_{\mathbf{v}'}[g_{\tau(u)}uq_1, L] \cap \pi^{-1}(K) \subset g_{[-C,C]}\psi(B_u) \quad \text{and}$$
$$\psi(B_u) \cap \pi^{-1}(K) \subset g_{[-C,C]}\mathcal{F}_{\mathbf{v}'}[g_{\tau(u)}uq_1, L].$$

Then, by (12.31) and (12.29), for $(\theta'/2)$-fraction of $u_1 \in B_u$, $g_{\tau(u_1)}u_1q_1 \in \pi^{-1}(K)$ and

$$d\left(\frac{R(g_{\tau(u)}uq_1, g_{\tau(u_1)}u_1q_1)\mathbf{v}(u)}{\|R(g_{\tau(u)}uq_1, g_{\tau(u_1)}u_1q_1)\mathbf{v}(u)\|}, \bigcup_{ij\in\tilde{\Lambda}} \mathbf{E}_{[ij],bdd}(g_{\tau(u_1)}u_1q_1)\right)$$
$$< C_1(\epsilon,\delta)\big(3\eta + e^{2\kappa^2 L}e^{-\alpha'\ell}\big).$$

Then, by (12.27), for $(\theta'/2)$-fraction of $u_1 \in B_u$,

$$d\left(\frac{\mathbf{v}(u_1)}{\|\mathbf{v}(u_1)\|}, \bigcup_{ij\in\tilde{\Lambda}} \mathbf{E}_{[ij],bdd}(g_{\tau(u_1)}u_1q_1)\right) < C_2(\epsilon,\delta)\big[3\eta + e^{2\kappa^2 L}e^{-\alpha'\ell} + e^{-\alpha'\ell}\big].$$

Hence, we may choose $\ell_9 = \ell_9(K_{00}, \epsilon, \delta, \eta)$ so that for $\ell > \ell_9$ the right-hand side of the above equation is at most 4η. Thus, (12.30) holds for $(\theta'/2)$-fraction of $u_1 \in B_u$.

The collection of balls $\{B_u\}_{u \in Q_8^*(q_1,\ell)}$ are a cover of $Q_8^*(q_1, \ell)$. These balls satisfy the condition of Lemma 3.10(b); hence we may choose a pairwise disjoint subcollection which still covers $Q_8^*(q_1, \ell)$. Then, by summing over the disjoint subcollection, we see that the claim holds on a set E of measure at least $(\theta'/2)|Q_8^*| \geq (\theta'/2)(1 - c_8^*(\delta)) \geq (\theta'/2)(1 - \theta'/2)$. ☐

Choice of parameters #4: choosing ℓ, q_1, q, q', q_1'. — Choose $\ell > \ell_9(K_{00}, \epsilon, \delta, \eta)$. Now choose $q_1 \in K_7(\ell)$, and let q, q', q_1' be as in Choice of Parameters #2.

Choice of parameters #5: choosing u, u', q_2, q_2', ij, $q_{3,ij}$, $q_{3,ij}'$ (depending on q_1, q_1', u, ℓ). — Choose $u \in Q_9(q_1, \ell)$, $u' \in Q_4(q_1', \ell)$ so that (12.12) and (12.13) hold. We have $\psi(u) = g_{\tau(u)} u q_1 \in \pi^{-1}(K)$ and $\psi'(u') \in \pi^{-1}(K)$. By (12.16),

$$\left|\hat{\tau}_{(\epsilon)}(q_1, u, \ell) - \hat{\tau}_{(\epsilon)}\left(q_1', u', \ell\right)\right| \leq C_0(\delta),$$

therefore,

$$g_{\tau(u)} u' q_1' \in \pi^{-1}(g_{[-C,C]} K),$$

where $C = C(\delta)$.

By the definition of K we can find $C_4(\delta)$ and $s \in [0, C_4(\delta)]$ such that

$$q_2 \equiv g_s g_{\tau(u)} u q_1 \in \pi^{-1}(K_0), \qquad q_2' \equiv g_s g_{\tau(u)} u' q_1' \in \pi^{-1}(K_0).$$

In view of (12.12), (12.13), the fact that $s \in [0, C_4(\delta)]$ and Corollary 6.13(a) we get

$$\frac{1}{C(\delta)} \epsilon \leq h d_{q_2}^{X_0}\left(U^+[q_2], U^+\left[q_2'\right]\right) \leq C(\delta)\epsilon. \tag{12.32}$$

By (12.20), the fact that $s \in [0, C_4(\delta)]$ and Lemma 3.6 we get

$$d^{X_0}\left(q_2, q_2'\right) = d^+\left(q_2, q_2'\right) \leq C(\delta)\epsilon_0(\delta). \tag{12.33}$$

We now choose $\epsilon_0(\delta)$ so that $C(\delta)\epsilon_0(\delta) < \epsilon_0'(\delta)$, where $C(\delta)$ is as in (12.33), and $\epsilon_0'(\delta)$ is as in (12.3).

Let $ij \in \tilde{\Lambda}$ be such that

$$d\left(\frac{\mathbf{v}(u)}{\|\mathbf{v}(u)\|}, \mathbf{E}_{[ij],bdd}(g_{\tau(u)} u q_1)\right) \leq 4\eta. \tag{12.34}$$

By Lemma 11.6,

$$\left|\hat{\tau}_{ij}\left(u q_1, \hat{\tau}_{(\epsilon)}(q_1, u, \ell)\right) - \hat{\tau}_{ij}\left(u' q_1', \hat{\tau}_{(\epsilon)}(q_1, u, \ell)\right)\right| \leq C_4'(\delta).$$

Then, by (12.16) and (9.4),

$$\left|\hat{\tau}_{ij}\big(uq_1, \hat{\tau}_{(\epsilon)}(q_1, u, \ell)\big) - \hat{\tau}_{ij}\big(u'q_1', \hat{\tau}_{(\epsilon)}\big(q_1', u', \ell\big)\big)\right| \leq C_4''(\delta).$$

Hence, by Proposition 4.15(e) (cf. Lemma 9.2), (12.6) and (12.8),

$$|t_{ij} - t_{ij}'| \leq C_5(\delta). \tag{12.35}$$

Therefore, by (12.7) and (12.9), we have

$$g_{t_{ij}} q_1 \in \pi^{-1}(K), \quad \text{and} \quad g_{t_{ij}} q_1' \in \pi^{-1}(g_{[-C_5(\delta), C_5(\delta)]}K).$$

By the definition of K, we can find $s'' \in [0, C_5''(\delta)]$ such that

$$q_{3,ij} \equiv g_{s''+t_{ij}} q_1 \in \pi^{-1}(K_0), \quad \text{and} \quad q_{3,ij}' \equiv g_{s''+t_{ij}} q_1' \in \pi^{-1}(K_0).$$

Let $\tau = s + \hat{\tau}_{(\epsilon)}(q_1, u, \ell)$, $\tau' = s'' + t_{ij}$. Then,

$$q_2 = g_\tau u q_1, \qquad q_2' = g_\tau u' q_1', \qquad q_{3,ij} = g_{\tau'} q_1, \qquad q_{3,ij}' = g_{\tau'} q_1'.$$

We may write $q_2 = g_t^{ij} u q_1$, $q_{3,ij} = g_{t'}^{ij} q_1$. Then, in view of (12.35) and (9.4),

$$|t - t'| \leq C_6(\delta).$$

We note that by Proposition 6.16, $\ell > \alpha_0 \tau$, where α_0 depends only on the Lyapunov spectrum.

Taking the limit as $\eta \to 0$. — For fixed δ and ϵ, we now take a sequence of $\eta_k \to 0$ (this forces $\ell_k \to \infty$) and pass to limits along a subsequence. Let $\tilde{q}_2 \in K_0$ be the limit of the q_2, and $\tilde{q}_2' \in K_0$ be the limit of the q_2'. We may also assume that along the subsequence $ij \in \tilde{\Lambda}$ is fixed, where ij is as in (12.34). By passing to the limit in (12.32), we get

$$\frac{1}{C(\delta)}\epsilon \leq hd_{\tilde{q}_2}^{X_0}\big(U^+[\tilde{q}_2], U^+[\tilde{q}_2']\big) \leq C(\delta)\epsilon. \tag{12.36}$$

We now apply Proposition 11.4 (with $\xi \to 0$ as $\eta_k \to 0$). By (11.4), $\tilde{q}_2' \in W^+[\tilde{q}_2]$. By applying g_s to (12.34) and then passing to the limit, we get $U^+[\tilde{q}_2'] \in \mathcal{E}_{ij}(\tilde{q}_2)$. Finally, it follows from passing to the limit in (12.33) that $d^+(\tilde{q}_2, \tilde{q}_2') \leq \epsilon_0'(\delta)$, and thus, since $\tilde{q}_2 \in K_0$ and $\tilde{q}_2' \in K_0$, it follows from (12.3) that $\tilde{q}_2' \in \mathfrak{B}_0[\tilde{q}_2]$. Hence,

$$\tilde{q}_2' \in \mathcal{C}_{ij}(\tilde{q}_2).$$

Now, by (11.5), we have

$$f_{ij}(\tilde{q}_2) \propto P^+\big(\tilde{q}_2, \tilde{q}_2'\big)_* f_{ij}\big(\tilde{q}_2'\big).$$

This concludes the proof of Proposition 12.2. We have $\tilde{q}_2 \in \pi^{-1}(K_0) \subset \pi^{-1}(K_{00} \cap K_*)$, and $\tilde{q}_2' \in \pi^{-1}(K_0 \subset K_*)$. □

Applying the argument for a sequence of ϵ's tending to 0. — Take a sequence $\epsilon_n \to 0$. We now apply Proposition 12.2 with $\epsilon = \epsilon_n$. After passing to a subsequence, we may assume ij is constant. We get, for each n a set $E_n \subset K_*$ with $\nu(E_n) > \delta_0$ and with the property that for every $x \in E_n$ there exists $y \in \mathcal{C}_{ij}(x) \cap K_*$ such that (12.1) and (12.2) hold for $\epsilon = \epsilon_n$. Let

$$F = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n \subset K_*$$

(so F consists of the points which are in infinitely many E_n). Suppose $x \in F$. Then there exists a sequence $y_n \to x$ such that $y_n \in \mathcal{C}_{ij}[x]$, $y_n \notin U^+[x]$, and so that $f_{ij}(y_n) \propto P^+(x, y_n)_* f_{ij}(x)$. Then (on the set where both are defined)

$$f_{ij}(x) \propto (\gamma_n)_* f_{ij}(x),$$

where $\gamma_n \in \mathcal{G}_{++}(x)$ is the affine map whose linear part is $P^+(x, y_n)$ and whose translational part is $y_n - x$. (Here we have used the fact that $y_n \in \mathcal{C}_{ij}[x]$, and thus by the definition of conditional measure, $f_{ij}(y_n) = (y_n - x)_* f_{ij}(x)$, where $(y_n - x)_* : W^+(x) \to W^+(x)$ is translation by $y_n - x$.)

Let $\tilde{f}_{ij}(x)$ denote the measure on $\mathcal{G}_{++}(x)$ given by

$$\tilde{f}_{ij}(x)(h) = \int_{W^+[x]} \bar{h} \, d f_{ij}(x),$$

where for a compactly supported real-valued continuous function h on $\mathcal{G}_{++}(x)$, $\bar{h} : W^+[x] \to \mathbb{R}$ is given by

$$\bar{h}(gx) = \int_{\mathcal{Q}_{++}(x)} h(gq)\, dm(q),$$

where m is the Haar measure on $\mathcal{Q}_{++}(x)$. (Thus, $\tilde{f}_{ij}(x)$ is the pullback of $f_{ij}(x)$ from $W^+[x] \cong \mathcal{G}_{++}(x)/\mathcal{Q}_{++}(x)$ to $\mathcal{G}_{++}(x)$.) Then,

$$(\gamma_n)_* \tilde{f}_{ij}(x) \propto \tilde{f}_{ij}(x) \tag{12.37}$$

on the set where both are defined.

For $x \in X$, let $U^+_{new}(x)$ denote the maximal connected subgroup of $\mathcal{G}_{++}(x)$ such that for $u \in U^+_{new}(x)$ (on the domain where both are defined),

$$(u)_* \tilde{f}_{ij}(x) \propto \tilde{f}_{ij}(x). \tag{12.38}$$

By (12.37) and Proposition D.3, for $x \in F$, $U^+_{new}(x)$ strictly contains $U^+(x)$.

Suppose $x \in F$, $y \in F$ and $y \in \mathcal{C}_{ij}[x]$. Then, since $\tilde{f}_{ij}(y) = Tr(x, y)_* \tilde{f}_{ij}(x)$, we have that (12.38) holds for $u \in Tr(y, x) U^+_{new}(y)$ (see Lemma 6.1). Therefore, by the maximality of $U^+_{new}(x)$, for $x \in F$, $y \in F \cap \mathcal{C}_{ij}[x]$,

$$Tr(y, x) U^+_{new}(y) = U^+_{new}(x). \tag{12.39}$$

Suppose $x \in F$, $t < 0$ and $g_t x \in F$. Then, since the measurable partition $\mathcal{C}_{ij}$ is g_t-equivariant (see Lemma 11.3) we have that (12.38) holds for $u \in g_{-t}U^+(g_t x)$. Therefore, by the maximality of $U^+_{new}(x)$, for $x \in F$, $t < 0$ with $g_t x \in F$ we have

$$\textbf{(12.40)} \qquad g_{-t}U^+_{new}(g_t x) = U^+_{new}(x),$$

and (12.38) and (12.39) still hold.

From (12.38), we get that for $x \in F$ and $u \in U^+_{new}(x)$,

$$\textbf{(12.41)} \qquad (u)_* \tilde{f}_{ij}(x) = e^{\beta_x(u)} \tilde{f}_{ij}(x),$$

where $\beta_x : U^+_{new}(x) \to \mathbb{R}$ is a homomorphism. Since $\nu(F) > \delta_0 > 0$ and g_t is ergodic, for almost all $x \in X$ there exist arbitrarily large $t > 0$ so that $g_{-t}x \in F$. Then, we define $U^+_{new}(x)$ to be $g_t U^+_{new}(g_{-t}x)$. (This is consistent in view of (12.40).) Then, (12.41) holds for a.e. $x \in X$. It follows from (12.41) that for a.e. $x \in X$, $u \in U^+_{new}(x)$ and $t > 0$,

$$\textbf{(12.42)} \qquad \beta_{g_{-t}x}(g_{-t}ug_t) = \beta_x(u).$$

We can write

$$\beta_x(u) = L_x(\log u),$$

where $L_x : \mathrm{Lie}(U^+)(x) \to \mathbb{R}$ is a Lie algebra homomorphism (which is in particular a linear map). Let $K \subset X$ be a positive measure set for which there exists a constant C with $\|L_x\| \leq C$ for all $x \in K$. Now for almost all $x \in X$ and $u \in U^+_{new}(x)$ there exists a sequence $t_j \to \infty$ so that $g_{-t_j}x \in K$ and $g_{-t_j}ug_{t_j} \to e$, where e is the identity element of U^+_{new}. Then, (12.42) applied to the sequence t_j implies that $\beta_x(u) = 0$ almost everywhere (cf. [BQ, Proposition 7.4(b)]). Therefore, for almost all $x \in X$, the conditional measure of ν along the orbit $U^+_{new}[x]$ is the push-forward of the Haar measure on $U^+_{new}(x)$.

The partition whose atoms are $U^+_{new}[x]$ is given by the refinement of the measurable partition $\mathcal{C}_{ij}$ into orbits of an algebraic group. (For the atom $\mathcal{C}_{ij}[x]$ this group is $U^+_{new}(y)$ for almost any $y \in \mathcal{C}_{ij}[x]$; in view of (12.39) and Lemma 6.1, this group, viewed as a group of affine maps of $W^+[x]$ is independent of the choice of y.) Therefore the partition whose atoms are sets of the form $U^+_{new}[x] \cap \mathfrak{B}_0[x]$ is a measurable partition.

In view of (12.39), and since for u near the identity, $U^+_{new}[x] \subset \mathcal{C}_{ij}[x]$ we have that (6.2) holds for U^+_{new}. Then, it also holds for any u in view of g_t-equivariance. Finally, since $U^+_{new}(x) \supset U^+(x)$ and $U^+(x) \supset \exp N(x)$, we have $U^+_{new}(x) \supset N(x)$.

Similarly, recall that the measure ν on X is the pullback of the measure on X_0 such that the conditionals on the fibers of the covering map $\sigma_0 : X \to X_0$ are the counting measure.

By (4.12) there exists a subset $\Omega_0 \subset X_0$ of full measure such that for any $x_0 \in \Omega_0$, for any $x \in \sigma^{-1}(x_0)$ we have an (almost-everywhere defined) identification σ_x between $W^+[x] \subset X$ and $W^+[x_0] \subset X_0$ and under this identification, the conditional measures coincide, i.e. $(\sigma_x)_* \nu_{W^+[x]} = \nu_{W^+[x_0]}$. Suppose $x_0 \in \Omega_0$ and $x \in \sigma_0^{-1}(x_0)$. After removing from Ω_0 a set of measure 0, we may assume that Definition 6.2(iii) holds for x and $U^+_{new}(x)$. Therefore it also holds for x_0 and $\sigma_x \circ U^+_{new}(x) \circ \sigma_x^{-1} \subset \mathcal{G}_{++}(x_0)$. Now for $x_0 \in \Omega_0$ define $U^+_{new}(x_0)$ to be the group generated by all the groups $\sigma_x \circ U^+_{new}(x) \circ \sigma_x^{-1}$ where x varies over $\sigma_0^{-1}(x_0)$. Then, Definition 6.2(iii) holds for x_0 and $U^+_{new}(x_0)$. In the same way, all of the other parts of Definition 6.2 hold for x_0 and $U^+_{new}(x_0)$ since they hold for x and $U^+_{new}(x)$ for any $x \in \sigma_0^{-1}(x_0)$.

This completes the proof of Proposition 12.1. □

13. Proof of Theorem 2.1

Let L^-, L^+, S^+ be as in Section 6.2. Apply Proposition 12.1 to get an equivariant system of subgroups $U^+_{new}(x) \subset \mathcal{G}_{++}(x)$ which is compatible with ν in the sense of Definition 6.2.

We have that $L^-[x]$ is smooth at x for almost all $x \in X$, see [AEM, §3]. Let $T_{\mathbb{R}}U^+(x) \subset W^+(x)$ denote the tangent subspace at x to the smooth manifold $U^+[x]$, and let $T_{\mathbb{R}}L^-(x) \subset W^-(x)$ denote the tangent subspace to $L^-[x]$ at x. (This exists for almost all x.)

If $L^+[x] \not\subset S^+[x]$ we can apply Proposition 12.1 again and repeat the process. When this process stops, the following hold:

(a) $L^+[x] \subset S^+[x] \subset U^+[x]$. In particular,

$$T_{\mathbb{R}}L^+(x) \equiv \hat{\pi}_x^+ \circ \left(\hat{\pi}_x^{-1}\right) T_{\mathbb{R}}L^-(x) \subset T_{\mathbb{R}}U^+(x).$$

(b) The conditional measures $\nu_{U^+[x]}$ are induced from the Haar measure on $U^+[x]$. These measures are g_t-equivariant.

(c) The subspaces $T_{\mathbb{R}}U^+(x) \subset W^+(x)$ is $P = AN$ equivariant. (This follows from the fact that the N direction is contained in $U^+(x)$, (6.2) and the fact that the N direction is in the center of $\mathcal{G}_{++}(x)$.) The subspaces $T_{\mathbb{R}}L^-(x)$ are g_t-equivariant.

(d) The conditional measures $\nu_{W^-[x]}$ are supported on $L^-[x]$.

Let $H^1_\perp$ denote the subspace of $H^1(M, \Sigma, \mathbb{R})$ which is orthogonal to the $SL(2,\mathbb{R})$ orbit, see (2.1). Let I denote the Lyapunov exponents (with multiplicity) of the cocycle in $T_{\mathbb{R}}U^+(x) \cap H^1_\perp$, J denote the Lyapunov exponents of the cocycle in $T_{\mathbb{R}}L^+(x) \cap H^1_\perp$. By (a), we have $J \subset I$.

Since $T_{\mathbb{R}}U^+(x) \cap H^1_\perp(x)$ is AN-invariant, by Theorem A.3 we have,

$$\sum_{i \in I} \lambda_i \geq 0. \tag{13.1}$$

We now compute the entropy of g_t. We have, by Theorem B.9(i) (applied to the flow in the reverse direction),

$$\frac{1}{t}h(g_t,\nu) \geq 2 + \sum_{i\in I}(1+\lambda_i) = 2 + |I| + \sum_{i\in I}\lambda_i \geq 2 + |I| \tag{13.2}$$

where the 2 comes from the direction of N, and for the last estimate we used (13.1). Also, by Theorem B.9(ii),

$$\begin{aligned}\frac{1}{t}h(g_{-t},\nu) &\leq 2 + \sum_{j\in J}(1-\lambda_j), && \text{where the 2 is the potential contribution of } \bar{N} \\ &\leq 2 + \sum_{i\in I}(1-\lambda_i) && \text{since } (1-\lambda_i) \geq 0 \text{ for all } i \\ &\leq 2 + |I| && \text{by (13.1)}\end{aligned} \tag{13.3}$$

However, $h(g_t,\nu) = h(g_{-t},\nu)$. Therefore, all the inequalities in (13.2) and (13.3) are in fact equalities. In particular, $I = J$, i.e.

$$T_{\mathbb{R}}L^+(x) = T_{\mathbb{R}}U^+(x). \tag{13.4}$$

Since $L^+[x] \subset S^+[x]$ and $S^+[x]$ is closed and star-shaped with respect to x, it follows that

$$T_{\mathbb{R}}L^+[x] \subset S^+[x]. \tag{13.5}$$

Since $S^+[x] \subset U^+[x]$, we get, in view of (13.4) and (13.5) that

$$T_{\mathbb{R}}U^+[x] \subset S^+[x] \subset U^+[x].$$

Thus $U^+[x]$ is an affine subspace of $W^+[x]$. Then, in view of (13.4), and the fact that $L^+[x] \subset U^+[x]$, we get that $L^+[x] = U^+[x]$. Thus, $L^+[x]$ is an affine subspace, hence $\mathcal{L}^-(x) = L^-(x)$.

We have

$$\frac{1}{t}h_\nu\left(g_{-t}, W^-\right) = 2 + \sum_{i\in I}(1-\lambda_i).$$

By applying Theorem B.9(iii) to the affine subspaces $\mathcal{L}^-(x)$, this implies that the conditional measures $\nu_{\mathcal{L}^-}(x)$ are Lebesgue, and that ν is $\bar{N}$-invariant (where $\bar{N}$ is as in Section 1.1). Hence ν is $SL(2,\mathbb{R})$-invariant.

By the definition of $\mathcal{L}^-$, the conditional measures $\nu_{W^-[x]}$ are supported on $\mathcal{L}^-[x]$. Thus, the conditional measures $\nu_{W^-[x]}$ are (up to null sets) precisely the Lebesgue measures on $\mathcal{L}^-[x]$.

Let $\mathcal{U}^+[x]$ denote the smallest linear subspace of $W^+[x]$ which contains the support of $\nu_{W^+[x]}$. Since ν is $SL(2,\mathbb{R})$-invariant, we can argue by symmetry that the conditional measures $\nu_{W^+[x]}$ are precisely the Lebesgue measures on $\mathcal{U}^+[x]$. Since $U^+[x]$ accounts for all the entropy of the flow, we must have $\mathcal{U}^+[x] = U^+[x]$. Since $\mathcal{U}^+[x] = \mathcal{L}^+[x]$, this completes the proof of Theorem 2.1. □

14. Random walks

In all of Sections 14–16, we work with the finite cover X_0 (which is a manifold), and do not use the measurable cover X.

We choose a compactly supported absolutely continuous measure μ on $SL(2, \mathbb{R})$. We also assume that μ is spherically symmetric. Let ν be any ergodic μ-stationary probability measure on X_0. By Furstenberg's theorem [NZ, Theorem 1.4],

$$\nu = \frac{1}{2\pi}\int_0^{2\pi} (r_\theta)_* \nu_0 d\theta$$

where r_θ is as in Section 1.1 and ν_0 is a measure invariant under $P = AN \subset SL(2, \mathbb{R})$. Then, by Theorem 2.1, ν_0 is $SL(2, \mathbb{R})$-invariant. Therefore the stationary measure ν is also in fact $SL(2, \mathbb{R})$-invariant.

We can think of $x \in X_0$ as a point in $H^1(M, \Sigma, \mathbb{C})$. For a subspace $U(x) \subset H^1(M, \Sigma, \mathbb{R})$ let $U_{\mathbb{C}} = \mathbb{C} \otimes U(x)$ denote its complexification, which is a subspace of $H^1(M, \Sigma, \mathbb{C})$. In all cases we will consider, $U(x)$ will either contain the space spanned by $\operatorname{Re} x$ and $\operatorname{Im} x$ or will be symplectically orthogonal to that space.

Let $\text{area}(x, 1) \subset H^1(M, \Sigma, \mathbb{C})$ denote the set of $y \in H^1(M, \Sigma, \mathbb{C})$ such that $x + y$ has area 1. We often abuse notation by referring to $U_{\mathbb{C}}(x) \cap \text{area}(1, x)$ also as $U_{\mathbb{C}}(x)$. We also write $U_{\mathbb{C}}[x]$ for the corresponding subset of X_0.

The map $p : H^1(M, \Sigma, \mathbb{R}) \to H^1(M, \mathbb{R})$ naturally extends to a map (also denoted by p) from $H^1(M, \Sigma, \mathbb{C}) \to H^1(M, \mathbb{C})$.

By Theorem 2.1, there is a $SL(2, \mathbb{R})$-equivariant family of subspaces $U(x) \subset H^1(M, \Sigma, \mathbb{R})$ containing $\operatorname{Re} x$ and $\operatorname{Im} x$ and such that the conditional measures of ν along $U_{\mathbb{C}}[x]$ are Lebesgue. Furthermore, for almost all x, the conditional measure of ν along $W^+[x]$ is supported on $W^+[x] \cap U_{\mathbb{C}}[x]$, and the conditional measure of ν along $W^-[x]$ is supported on $W^-[x] \cap U_{\mathbb{C}}[x]$.

Lemma **14.1.** — *There exists a volume form* $d\,\mathrm{Vol}(x)$ *on* $U(x)$ *which is invariant under the* $SL(2, \mathbb{R})$ *action. This form is non-degenerate on compact subsets of* X_0.

Proof. — The subspaces $p(U(x))$ form an invariant subbundle $p(U)$ of the Hodge bundle. By Theorem A.6(a) (after passing to a finite cover) we may assume that $p(U)$ is a direct sum of irreducible subbundles. Then, by Theorem A.6(b), we have a decomposition

$$p(U)(x) = U_{symp}(x) \oplus U_0(x)$$

where the symplectic form on U_{symp} is non-degenerate, the decomposition is orthogonal with respect to the Hodge inner product, and U_0 is isotropic. Then, by Theorems A.5 and A.4 the Hodge inner product on U_0 is equivariant under the $SL(2,\mathbb{R})$ action.

Then we can define the volume form on $p(U)$ to be the product of the appropriate power of the symplectic form on U_{symp} and the volume form induced by the Hodge inner product on U_0. The subbundle U_{symp} is clearly $SL(2,\mathbb{R})$ equivariant. By [Fi1, Corollary 5.4], applied to the section $c_1 \wedge \cdots \wedge c_k$ where $\{c_1, \dots, c_k\}$ is a symplectic basis for U_{symp}, it follows that the symplectic volume form on U_{symp} agrees with the volume form induced by the Hodge inner product on U_{symp} (which is non-degenerate on compact sets). This gives a volume form on $p(U)$ with the desired properties.

Since the Kontsevich-Zorich cocycle acts trivially on $\ker p$, the normalized Lebesgue measure on $\ker p$ is well defined. Thus, the volume form on $p(U)$ naturally induces a volume form on U. □

Remark. — In fact it follows from the results of [AEM] that U_0 is trivial.

Lemma **14.2.** — *There exists an* $SL(2,\mathbb{R})$*-equivariant subbundle* $p(U)^{\perp} \subset H^1(M,\mathbb{R})$ *such that*

$$p(U)(x) \oplus p(U)^{\perp}(x) = H^1(M,\mathbb{R}).$$

Proof. — This follows from the proof of Theorem A.6. □

The subbundles $\mathcal{L}_k$. — By Theorem A.6 we have

$$p(U)^{\perp}(x) = \bigoplus_{k\in\hat{\Lambda}} \mathcal{L}_k(x), \tag{14.1}$$

where $\hat{\Lambda}$ is an indexing set not containing 0, and for each $k \in \hat{\Lambda}$, $\mathcal{L}_k$ is an $SL(2,\mathbb{R})$-equivariant subbundle of the Hodge bundle. (In our notation, the action of the Kontsevich-Zorich cocycle may permute some of the $\mathcal{L}_k$.) Note that $\mathcal{L}_k(x)$ is symplectically orthogonal to the $SL(2,\mathbb{R})$ orbit of x. Without loss of generality, we may assume that the decomposition (14.1) is maximal, in the sense that on any (measurable) finite cover of X_0 each $\mathcal{L}_k$ does not contain a non-trivial proper $SL(2,\mathbb{R})$-equivariant subbundle. (If this was not true, we could without passing to a finite cover, write a version of (14.1) with a larger k.) If U does not contain the kernel of p, then we let $\hat{\lambda}_0 = 0$, and let $\tilde{\Lambda} = \hat{\Lambda} \cup \{0\}$.

The Forni subbundle. — Let $\tilde{\lambda}_k$ denote the top Lyapunov exponent of the geodesic flow g_t restricted to $\mathcal{L}_k$. Let

$$F(x) = \bigoplus_{\{k\,:\,\tilde{\lambda}_k=0\}} \mathcal{L}_k(x).$$

We call $F(x)$ the *Forni* subspace of ν. The subspaces $F(x)$ form a subbundle of the Hodge bundle which we call the Forni subbundle. It is an $SL(2,\mathbb{R})$-invariant subbundle, on which the Kontsevich-Zorich cocycle acts by Hodge isometries. In particular, all the Lyapunov exponents of $F(x)$ are 0. Let $F^{\perp}(x)$ denote the orthogonal complement to $F(x)$ in the Hodge norm. By Theorem A.9(b),

$$F^{\perp}(x) = \bigoplus_{\{k\,:\,\hat{\lambda}_k \neq 0\}} \mathcal{L}_k(x).$$

The following is proved in [AEM]:

Theorem **14.3.** — *There exists a subset* Φ *of the stratum with* $\nu(\Phi) = 1$ *such that for all* $x \in \Phi$ *there exists a neighborhood* $\mathcal{U}(x)$ *such that for all* $y \in \mathcal{U}(x) \cap \Phi$ *we have* $p(y - x) \in F^{\perp}_{\mathbb{C}}(x)$.

The backwards shift map. — Let B be the space of (one-sided) infinite sequences of elements of $SL(2,\mathbb{R})$. (We think of B as giving the "past" trajectory of the random walk.) Let $T : B \to B$ be the shift map. (In our interpretation, T takes us one step into the past.) We define the skew-product map $T : B \times X_0 \to B \times X_0$ by

$$T(b, x) = \big(Tb, b_0^{-1}x\big), \quad \text{where } b = (b_0, b_1, \dots)$$

(Thus the shift map and the skew-product map are denoted by the same letter.) We define the measure β on B to be $\mu \times \mu \cdots$. The skew product map T naturally acts on the bundle $H^1(M,\mathbb{R})$, and thus on each $\mathcal{L}_k$ for $k \in \hat{\Lambda}$.

For each $k \in \hat{\Lambda}$, by the multiplicative ergodic theorem we have the Lyapunov flag for this action (with respect to the invariant measure β):

$$\{0\} = \mathcal{V}^{(k)}_{\leq 0} \subset \mathcal{V}^{(k)}_{\leq 1}(b, x) \subset \cdots \mathcal{V}^{(k)}_{\leq n_k}(b, x) = \mathcal{L}_k(x).$$

By the multiplicative ergodic theorem applied to the action of $SL(2,\mathbb{R})$ on $\mathbb{R}^2$, for β-almost all $b \in B$,

$$\sigma_0 = \lim_{n \to \infty} \frac{1}{n} \log \|b_0 \dots b_n\|$$

where $\sigma_0 > 0$ is the Lyapunov exponent for the measure μ on $SL(2,\mathbb{R})$. Then, the Lyapunov exponents of the flow g_t and the Lyapunov exponents of the skew-product map T differ by a factor of σ_0. Let $\hat{\lambda}_k$ denote the top Lyapunov exponent of T restricted to $\mathcal{L}_k$.

The two-sided shift space. — Let $\tilde{B}$ denote the two-sided shift space. We denote the measure $\cdots \times \mu \times \mu \times \cdots$ on $\tilde{B}$ also by β.

Notation. — For $a, b \in \mathrm{B}$ let

$$a \vee b = (\ldots, a_2, a_1, b_0, b_1, \ldots) \in \tilde{\mathrm{B}}. \tag{14.2}$$

(Note that the indexing for $a \in \mathrm{B}$ starts at 1 not at 0.) If $\omega = a \vee b \in \tilde{\mathrm{B}}$, we think of the sequence

$$\ldots, \omega_{-2}, \omega_{-1} = \cdots a_2, a_1$$

as the "future" of the random walk trajectory. (In general, following [BQ], we use the symbols b, b' etc. to refer to the "past" and the symbols a, a' etc. to refer to the "future".)

The opposite Lyapunov flag. — Note that on the two-sided shift space $\tilde{\mathrm{B}} \times \mathrm{X}_0$, the map T is invertible. Thus, for each $a \vee b \in \mathrm{B}$, we have the Lyapunov flag for T^{-1}:

$$\{0\} = \mathcal{V}^{(k)}_{\geq n_k} \subset \mathcal{V}^{(k)}_{\geq n_k - 1}(a, x) \subset \cdots \mathcal{V}^{(k)}_{\geq 0}(a, x) = \mathcal{L}_k(x).$$

(As reflected in the above notation, this flag depends only on the "future" i.e. "a" part of $a \vee b$.)

The top Lyapunov exponent $\hat{\lambda}_k$. — Recall that $\hat{\lambda}_k \geq 0$ denotes the top Lyapunov exponent in $\mathcal{L}_k$. Then (since T steps into the past), for $v \in \mathcal{V}^{(k)}_{\leq 1}(b, x)$,

$$\lim_{n \to \infty} \frac{1}{n} \log \frac{\|\mathrm{T}^n(b, x)_* v\|}{\|v\|} = -\hat{\lambda}_k. \tag{14.3}$$

In the above equation we used the notation $\mathrm{T}^n(b, x)_*$ to denote the action of $\mathrm{T}^n(b, x)$ on $\mathrm{H}^1(\mathrm{M}, \mathbb{R})$.

Also, for $v \in \mathcal{V}^{(k)}_{>1}(a, x)$, for some $\alpha > 0$,

$$\lim_{n \to \infty} \frac{1}{n} \log \frac{\|\mathrm{T}^{-n}(a \vee b, x)_* v\|}{\|v\|} < \hat{\lambda}_k - \alpha.$$

Here, α is the minimum over k of the difference between the top Lyapunov exponent in $\mathcal{L}_k$ and the next Lyapunov exponent.

The following lemma is a consequence of the zero-one law Lemma C.10(i):

Lemma **14.4.** — *For every $\delta > 0$ and every $\delta' > 0$ there exists $\mathrm{E}_{good} \subset \mathrm{X}_0$ with $\nu(\mathrm{E}_{good}) > 1 - \delta$ and $\sigma = \sigma(\delta, \delta') > 0$, such that for any $x \in \mathrm{E}_{good}$, any k and any vector $w \in \mathbb{P}(\mathcal{L}_k(x))$,*

$$\beta\Big(\big\{a \in \mathrm{B} \,:\, d_{\mathrm{Y}}\big(w, \mathcal{V}^{(k)}_{>1}(a, x)\big) > \sigma\big\}\Big) > 1 - \delta' \tag{14.4}$$

(In (14.4), $d_Y(\cdot,\cdot)$ is the distance on the projective space $\mathbb{P}(H^1(M,\mathbb{R}))$ derived from the AGY norm.)

Proof. — It is enough to prove the lemma for a fixed k. For $F \subset Gr_{n_k-1}(\mathcal{L}_k(x))$ (the Grassmannian of $n_k - 1$ dimensional subspaces of $\mathcal{L}_k(x)$) let

$$\hat{\nu}_x^{(k)}(F) = \beta\Big(\Big\{a \in B \;:\; \mathcal{V}_{>1}^{(k)}(a,x) \in F\Big\}\Big),$$

and let $\hat{\nu}^{(k)}$ denote the measure on the bundle $X_0 \times Gr_{n_k-1}(\mathcal{L}_k)$ given by

$$d\hat{\nu}^{(k)}(x, L) = d\nu(x)\, d\hat{\nu}_x^{(k)}(L).$$

Then, $\hat{\nu}^{(k)}$ is a stationary measure for the (forward) random walk. For $w \in \mathbb{P}(\mathcal{L}_k(x))$ let $I(w) = \{L \in Gr_{n_k-1}(\mathcal{L}_k(x)) \;:\; w \in L\}$. Let

$$Z = \Big\{x \in X_0 \;:\; \hat{\nu}_x^{(k)}\big(I(w)\big) > 0 \text{ for some } w \in \mathbb{P}\big(\mathcal{L}_k(x)\big)\Big\}.$$

Suppose $\nu(Z) > 0$. Then, for each $x \in Z$ we can choose $w_x \in \mathbb{P}(\mathcal{L}_k(x))$ such that $\hat{\nu}_x^{(k)}(I(w_x)) > 0$. Then,

$$\hat{\nu}^{(k)}\left(\bigcup_{x\in Z} \{x\} \times I(w_x)\right) > 0. \tag{14.5}$$

Therefore, (14.5) holds for some ergodic component of $\hat{\nu}^{(k)}$. However, this contradicts Lemma C.10(i), since by the definition of $\mathcal{L}_k$, the action of the cocycle on $\mathcal{L}_k$ is strongly irreducible. Thus, $\nu(Z) = 0$ and $\nu(Z^c) = 1$. By definition, for all $x \in Z^c$ and all $w \in \mathcal{L}_k(x)$,

$$\beta\Big(\Big\{a \in B \;:\; w \in \mathcal{V}_{>1}^{(k)}(a,x)\Big\}\Big) = 0.$$

Fix $x \in Z^c$. Then, for every $w \in \mathbb{P}(\mathcal{L}_k(x))$ there exists $\sigma_0(x, w, \delta') > 0$ such that

$$\beta\Big(\Big\{a \in B \;:\; d_Y\big(\mathcal{V}_{>1}^{(k)}(a,x), w\big) > 2\sigma_0\big(x, w, \delta'\big)\Big\}\Big) > 1 - \delta'.$$

Let $\mathcal{U}(x,w) = \{z \in \mathbb{P}(\mathcal{L}_k(x)) \;:\; d_Y(z,w) < \sigma_0(x,w,\delta')\}$. Then the $\{\mathcal{U}(x,w)\}_{w\in\mathbb{P}(\mathcal{L}_k(x))}$ form an open cover of the compact space $\mathbb{P}(\mathcal{L}_k(x))$, and therefore there exist $w_1, \dots w_n$ with $\mathbb{P}(\mathcal{L}_k(x)) = \bigcup_{i=1}^n \mathcal{U}(x, w_i)$. Let $\sigma_1(x,\delta') = \min_i \sigma_0(x, w_i, \delta')$. Then, for all $x \in Z^c$ and all $w \in \mathbb{P}(\mathcal{L}_k(x))$,

$$\beta\Big(\Big\{a \in B \;:\; d_Y\big(\mathcal{V}_{>1}^{(k)}(a,x), w\big) > \sigma_1\big(x, \delta'\big)\Big\}\Big) > 1 - \delta'.$$

Let $E_N(\delta') = \{x \in Z^c \;:\; \sigma_1(x,\delta') > \frac{1}{N}\}$. Since $\bigcup_{N=1}^\infty E_N(\delta') = Z^c$ and $\nu(Z^c) = 1$, there exists $N = N(\delta, \delta')$ such that $\nu(E_N(\delta')) > 1 - \delta$. Let $\sigma = 1/N$ and let $E_{good} = E_N$. □

Lyapunov subspaces and relative homology. — The following lemma is well known:

Lemma **14.5.** — *The Lyapunov spectrum of the Kontsevich-Zorich cocycle acting on relative homology is the Lyapunov spectrum of the Kontsevich-Zorich cocycle acting on absolute homology, union n zeroes, where* $n = \dim \ker p$.

Let $\bar{\mathcal{L}}_k = p^{-1}(\mathcal{L}_k) \subset H^1(M, \Sigma, \mathbb{R})$. We have the Lyapunov flag

$$\{0\} = \bar{\mathcal{V}}^{(k)}_{\leq 0} \subset \bar{\mathcal{V}}^{(k)}_{\leq 1}(b, x) \subset \cdots \bar{\mathcal{V}}^{(k)}_{\leq \bar{n}_k}(b, x) = \bar{\mathcal{L}}_k(x),$$

corresponding to the action on the invariant subspace $\bar{\mathcal{L}}_k \subset H^1(M, \Sigma, \mathbb{R})$. Also for each $a \in B$, we have the opposite Lyapunov flag

$$\{0\} = \bar{\mathcal{V}}^{(k)}_{\geq \bar{n}_k} \subset \bar{\mathcal{V}}^{(k)}_{\geq \bar{n}_k - 1}(a, x) \subset \cdots \bar{\mathcal{V}}^{(k)}_{\geq 0}(a, x) = \bar{\mathcal{L}}_k(x).$$

Lemma **14.6.** — *Suppose* $\hat{\lambda}_k \neq 0$. *Then for almost all* (b, x),

$$p\big(\bar{\mathcal{V}}^{(k)}_{\leq 1}(b, x)\big) = \mathcal{V}^{(k)}_{\leq 1}(b, x),$$

and p is an isomorphism between these two subspaces. Similarly, for almost all (a, x),

$$\bar{\mathcal{V}}^{(k)}_{>1}(a, x) = p^{-1}\big(\mathcal{V}^{(k)}_{>1}(a, x)\big).$$

Proof. — In view of Lemma 14.5 and the assumption that $\hat{\lambda}_k \neq 0$, $\hat{\lambda}_k$ is the top Lyapunov exponent on both $\mathcal{L}_k$ and $\bar{\mathcal{L}}_k$. Note that

$$\bar{\mathcal{V}}^{(k)}_{\leq 1} = \left\{ \bar{v} \in \bar{\mathcal{L}}_k \;:\; \limsup_{t \to \infty} \frac{1}{t} \log \frac{\|T^n \bar{v}\|}{\|\bar{v}\|} \leq -\hat{\lambda}_k \right\}. \tag{14.6}$$

Also,

$$\mathcal{V}^{(k)}_{\leq 1} = \left\{ v \in \mathcal{L}_k \;:\; \limsup_{t \to \infty} \frac{1}{t} \log \frac{\|T^n v\|}{\|v\|} \leq -\hat{\lambda}_k \right\}. \tag{14.7}$$

It is clear from the definition of the Hodge norm on relative cohomology (A.1) that $\|p(v)\| \leq C\|v\|$ for some absolute constant C. Therefore, it follows from (14.7) and (14.6) that $p(\bar{\mathcal{V}}^{(k)}_{\leq 1}) \subset \mathcal{V}^{(k)}_{\leq 1}$. But by Lemma 14.5, $\dim(\bar{\mathcal{V}}^{(k)}_{\leq 1}) = \dim(\mathcal{V}^{(k)}_{\leq 1})$. Therefore, $p(\bar{\mathcal{V}}^{(k)}_{\leq 1}) = \mathcal{V}^{(k)}_{\leq 1}$. □

Remark. — Even though we will not use this, a version of Lemma 14.6 holds for all Lyapunov subspaces for non-zero exponents, and not just the subspace corresponding to the top Lyapunov exponent $\hat{\lambda}_k$.

The action on $H^1(M, \Sigma, \mathbb{C})$. — By the multiplicative ergodic theorem applied to the action of $SL(2, \mathbb{R})$ on $\mathbb{R}^2$, for β-almost all $b \in B$ there exists a one-dimensional subspace $W_+(b) \subset \mathbb{R}^2$ such that $v \in W_+(b)$,

$$\lim_{n\to\infty} \frac{1}{n} \log \| b_n^{-1} \dots b_0^{-1} v\| = -\sigma_0.$$

Let

$$W^+(b, x) = \big(W_+(b) \otimes H^1(M, \Sigma, \mathbb{R})\big) \cap \operatorname{area}(x, 1).$$

Since we identify $\mathbb{R}^2 \otimes H^1(M, \Sigma, \mathbb{R})$ with $H^1(M, \Sigma, \mathbb{C})$, we may consider $W^+(b, x)$ as a subspace of $H^1(M, \Sigma, \mathbb{C})$. This is the "stable" subspace for T. (Recall that T moves into the past.)

For a "future trajectory" $a \in B$, we can similarly define a 1-dimensional subspace $W_-(a) \subset \mathbb{R}^2$ such that

$$\lim_{n\to\infty} \frac{1}{n} \log \| a_n \dots a_1 v\| = -\sigma_0 \quad \text{for } v \in W_-(a).$$

Let $A : SL(2, \mathbb{R}) \times X_0 \to \operatorname{Hom}(H^1(M, \Sigma, \mathbb{R}), H^1(M, \Sigma, \mathbb{R}))$ denote the Kontsevich-Zorich cocycle. We then have the cocycle

$$\hat{A} : SL(2, \mathbb{R}) \times X_0 \to \operatorname{Hom}\big(H^1(M, \Sigma, \mathbb{C}), H^1(M, \Sigma, \mathbb{C})\big)$$

given by

$$\hat{A}(g, x)(v \otimes w) = gv \otimes A(g, x)w$$

and we have made the identification $H^1(M, \Sigma, \mathbb{C}) = \mathbb{R}^2 \otimes H^1(M, \Sigma, \mathbb{R})$. This cocycle can be thought of as the derivative cocycle for the action of $SL(2, \mathbb{R})$. From the definition we see that the Lyapunov exponents of $\hat{A}$ are of the form $\pm\sigma_0 + \lambda_i$, where the λ_i are the Lyapunov exponents of A.

15. Time changes and suspensions

There is a natural "forgetful" map $f : \tilde{B} \to B$. We extend functions on $B \times X_0$ to $\tilde{B} \times X_0$ by making them constant along the fibers of f. The measure $\beta \times \nu$ is a T-invariant measure on $\tilde{B} \times X_0$.

The cocycles θ_j. — By Theorem A.6, the restriction of the Kontsevich-Zorich cocycle to each $\mathcal{L}_j$ is semisimple. Then by Theorem C.5, the Lyapunov spectrum of T on each $\mathcal{L}_j$

is semisimple, and the restriction of T to the top Lyapunov subspace of each $\mathcal{L}_j$ consists of a single conformal block. This means that there is a inner product $\langle , \rangle_{j,b,x}$ defined on $W_+(b) \otimes \mathcal{V}^{(j)}_{\leq 1}(b,x)$ and a function $\theta_j : B \times X_0 \to \mathbb{R}$ such that for all $u, v \in W_+(b) \otimes \mathcal{V}^{(j)}_{\leq 1}(b,x)$,

$$(\mathbf{15.1}) \qquad \left\langle \hat{A}\left(b_0^{-1}, x\right)u, \hat{A}\left(b_0^{-1}, x\right)v \right\rangle_{j, Tb, b_0^{-1}x} = e^{-\theta_j(b,x)} \langle u, v\rangle_{j,b,x}.$$

To handle relative homology, we need to also consider the case in which the action of $A(\cdot,\cdot)$ on a subbundle is trivial. We thus define an inner product $\langle , \rangle_{0,b}$ on $\mathbb{R}^2$, and a cocycle $\theta_0 : B \to \mathbb{R}$ so that for $u, v \in W_+(b)$,

$$(\mathbf{15.2}) \qquad \left\langle b_0^{-1}u, b_0^{-1}v \right\rangle_{0,Tb} = e^{-\theta_0(b)} \langle u, v\rangle_{0,b}.$$

For notational simplicity, we let $\theta_0(b,x) = \theta_0(b)$.

Switch to positive cocycles. — The cocycle θ_j corresponds to the $\hat{A}(\cdot,\cdot)$-Lyapunov exponent $\sigma_0 + \hat{\lambda}_j$, where $\hat{\lambda}_j$ is the top Lyapunov exponent of $A(\cdot,\cdot)$ in $\mathcal{L}_j$. Since $\sigma_0 > 0$ and $\hat{\lambda}_j \geq 0$,

$$\sigma_0 + \hat{\lambda}_j = \int_{B \times X_0} \theta_j(b,x)\, d\beta(b)\, d\nu(x) > 0.$$

Thus, the cocycle θ_j has positive average on $B \times X_0$. However, we do not know that θ_j is positive, i.e. that for all $(b,x) \in B \times X_0$, $\theta_j(b,x) > 0$. This makes it awkward to use $\theta_j(b,x)$ to define a time change. Following [BQ] we use a positive cocycle τ_j equivalent to θ_j.

By [BQ, Lemma 2.1], we can find a *positive* cocycle $\tau_j : B \times X_0 \to \mathbb{R}$ and a measurable function $\phi_j : B \times X_0 \to \mathbb{R}$ such that

$$(\mathbf{15.3}) \qquad \theta_j - \phi_j \circ T + \phi_j = \tau_j$$

and

$$\int_{B \times X_0} \tau_j(b,x)\, d\beta(b)\, d\nu(x) < \infty.$$

For $v \in W_+(b) \otimes \mathcal{V}^{(j)}_{\leq 1}(b,x)$ we define

$$(\mathbf{15.4}) \qquad \|v\|'_{j,b,x} = e^{\phi_j(b,x)} \|v\|_{j,b,x},$$

where the norm $\langle \cdot, \cdot \rangle_j$ is as in (15.1) and (15.2). Then

$$(\mathbf{15.5}) \qquad \left\| \hat{A}\left(b_0^{-1}, x\right)v \right\|'_{j, T(b,x)} = e^{-\tau_j(b,x)} \|v\|'_{j,b,x}.$$

Suspension. — Let $B^X = B \times X_0 \times (0, 1]$. Recall that β denotes the measure on B which is given by $\mu \times \mu \cdots$. Let β^X denote the measure on B^X given by $\beta \times \nu \times dt$, where dt is the Lebesgue measure on $(0, 1]$. In B^X we identify $(b, x, 0)$ with $(T(b, x), 1)$, so that B^X is a suspension of T. We can then define a suspension flow $T_t : B^X \to B^X$ in the natural way. (Our suspensions are going downwards and not upwards, since we think of T as going into the past.) Then T_t preserves the measure β^X.

Let $\tilde{B}^X = \tilde{B} \times X_0 \times (0, 1]$. The suspension construction, the flow T_t, and the invariant measure β^X extend naturally from B^X to $\tilde{B}^X$.

Let $T_t(b, x, s)_*$ denote the action of $T_t(b, x, s)$ on $H^1(M, \Sigma, \mathbb{C})$ (i.e. the derivative cocycle on the tangent space). Then, for $t \in \mathbb{Z}$ and $v \in W_+(b) \otimes \mathcal{V}^{(j)}_{\le 1}(b, x)$ and $0 < s \le 1$ we have, in view of (15.5),

$$\|T_t(b, x, s)_* v\|'_{j, T_t(b,x)} = e^{-\tau_j(t,b,x)} \|v\|'_{j,b,x}, \tag{15.6}$$

where $\tau_j(t, b, x) = \sum_{n=0}^{t-1} \tau_j(T^n(b, x))$. We can extend the norm $\|\cdot\|'_j$ from $B \times X_0$ to B^X by

$$\|v\|'_{j,b,x,s} = \|v\|'_{j,b,x} e^{-(1-s)\tau_j(b,x)}.$$

Then (15.6) holds for all $t \in \mathbb{R}$ provided we set for $n \in \mathbb{Z}$ and $0 \le s < 1$,

$$\tau_j(n + s, b, x) = \tau_j(n, b, x) + s\tau_j\big(T^n(b, x)\big).$$

The time change. — Here we differ slightly from [BQ] since we would like to have several different time changes of the flow T_t on the same space. Hence, instead of changing the roof function, we keep the roof function constant, but change the speed in which one moves on the [0, 1] fibers.

Let $T^{\tau_j}_t : B^X \to B^X$ be the time change of T_t where on $(b, x) \times [0, 1]$ one moves at the speed $1/\tau_j(b, x)$. More precisely, we set

$$T^{\tau_j}_t(b, x, s) = \big(b, x, s - t/\tau_j(b, x)\big), \quad \text{if } 0 < s - t/\tau_j(b, x) \le 1, \tag{15.7}$$

and extend using the identification $((b, x), 0) = (T(b, x), 1)$.

Then $T^{\tau_k}_\ell$ is the operation of moving backwards in time far enough so that the cocycle multiplies the direction of the top Lyapunov exponent in $\mathcal{L}_k$ by $e^{-\ell}$. In fact, by (15.6) and (15.7), we have, for $v \in W_+(b) \otimes \mathcal{V}^{(k)}_{\le 1}(b, x)$,

$$\|T^{\tau_k}_\ell(b, x, s)_* v\|'_{j, T^{\tau_k}_\ell(b,x,s)} = e^{-\ell} \|v\|'_{j,b,x,s}. \tag{15.8}$$

The map T^{τ_k} and the two-sided shift space. — On the space $\tilde{B}^X$, T^{τ_k} is invertible, and we denote the inverse of $T^{\tau_k}_\ell$ by $T^{\tau_k}_{-\ell}$. We write

$$T^{\tau_k}_{-\ell}(a \vee b, x, s)_* \tag{15.9}$$

for the linear map on the tangent space $H^1(M, \Sigma, \mathbb{C})$ induced by $T^{\tau_k}_{-\ell}(a \vee b, x, s)$. In view of (15.4) and (15.8), we have for $v \in W_+(b) \otimes \mathcal{V}^{(k)}_{\leq 1}(b, x)$,

$$\|T^{\tau_k}_{-\ell}(a \vee b, x, s)_* v\| = \exp\big(\ell + \phi_k(b, x, s) - \phi_k\big(T^{\tau_k}_{-\ell}(a \vee b, x, s)\big)\big)\|v\|. \tag{15.10}$$

Here we have omitted the subscripts on the norm $\|\cdot\|_{k,b,x}$ and also extended the function $\phi_k(b, x, s)$ so that for all $(b, x, s) \in B^X$ and all $v \in W_+(b) \otimes \mathcal{V}^{(k)}_{\leq 1}(b, x)$,

$$\|v\|_{k,b,x} = e^{\phi_k(b,x,s)}\|v\|'_{k,b,x,s}.$$

Invariant measures for the time changed flows. — Let $\beta^{\tau_j,X}$ denote the measure on B^X given by

$$d\beta^{\tau_j,X}(b, x, t) = c_j \tau_j(b, x) d\beta(b) d\nu(x) dt,$$

where the $c_j \in \mathbb{R}$ is chosen so that $\beta^{\tau_j,X}(B^X) = 1$. Then the measures $\beta^{\tau_j,X}$ are invariant under the flows $T^{\tau_j}_t$. We note the following trivial:

Lemma **15.1.** — *The measures $\beta^{\tau_j,X}$ are all absolutely continuous with respect to β^X. For every $\delta > 0$ there exists a compact subset $\mathcal{K} = \mathcal{K}(\delta) \subset B^X$ and $L = L(\delta) < \infty$ such that for all j,*

$$\beta^{\tau_j,X}(\mathcal{K}) > 1 - \delta,$$

and also for $(b, x, t) \in \mathcal{K}$,

$$\frac{d\beta^{\tau_j,X}}{d\beta^X}(b, x, t) \leq L, \qquad \frac{d\beta^X}{d\beta^{\tau_j,X}}(b, x, t) \leq L.$$

16. The martingale convergence argument

Standing assumptions. — Let

$$W^+[b, x] = \big\{ y \,:\, y - x \in W^+(b, x).\big\}$$

Then, $W^+[b, x]$ is the stable subspace for T. From the definition, for almost all b, (locally) the sets $\{W^+[b, x] \,:\, x \in X\}$ form a measurable partition of X_0. Let

$$U^+(b, x) = W^+(b, x) \cap U_{\mathbb{C}}(x), \qquad U^+[b, x] = W^+[b, x] \cap U_{\mathbb{C}}[x].$$

We make the corresponding definitions for $W^-(b, x)$, $W^-[b, x]$, $U^+[b, x]$ and $U^-[b, x]$.

It follows from Theorem 2.1 applied to the flow $r_\theta g_t r_{-\theta}$, using the fact that $U_{\mathbb{C}}[r_\theta x] = U_{\mathbb{C}}[x]$, that for a.e. x, the conditional measures of ν along $W^\pm[b, x]$ are supported on $U^\pm[b, x]$, and also that the conditional measures of ν along $U^\pm[b, x]$ are Lebesgue.

Lemma **16.1.** — *There exists a subset* $\Psi \subset \mathrm{B}^{\mathrm{X}}$ *with* $\beta^{\mathrm{X}}(\Psi) = 1$ *such that for all* $(b, x) \in \Psi$,

$$\Psi \cap \mathrm{W}^+[b, x] \cap \textit{ball of radius } 1 \subset \Psi \cap \mathrm{U}^+[b, x].$$

Proof. — See [MaT] or [EL, 6.23]. □

The parameter δ. — Let $\delta > 0$ be a parameter which will eventually be chosen sufficiently small. We use the notation $c_i(\delta)$ and $c_i'(\delta)$ for functions which tend to 0 as $\delta \to 0$. In this section we use the notation $\mathrm{A} \approx \mathrm{B}$ to mean that the ratio A/B is bounded between two positive constants depending on δ.

We first choose a compact subset $\mathrm{K}_0 \subset \Psi \cap \Phi$ with $\beta^{\mathrm{X}}(\mathrm{K}_0) > 1 - \delta > 0.999$, the conull set Ψ is as in Lemma 16.1, and the conull set Φ is as in Theorem 14.3. By the multiplicative ergodic theorem and (14.3), we may also assume that there exists $\ell_1(\delta) > 0$ such that for all $(b, x, s) \in \mathrm{K}_0$ all k and all $v \in \mathcal{V}^{(k)}_{\le 1}(b, x)$ and all $\ell > \ell_1(\delta)$,

$$\|\mathrm{T}_\ell(b, x, s)_* v\| \le e^{-(\lambda_k/2)\ell} \|v\|. \tag{16.1}$$

(Here, as in (14.3) the notation $\mathrm{T}_\ell(b, x, s)_*$ denotes the action on $\mathrm{H}^1(\mathrm{M}, \Sigma, \mathbb{R})$.) By the norm $\|\cdot\|$ in this section, we mean the AGY norm (see Section A.1).

Lemma **16.2.** — *For every* $\delta > 0$ *there exists* $\mathrm{K} \subset \mathrm{B}^{\mathrm{X}}$ *and* $\mathrm{C} = \mathrm{C}(\delta) < \infty$, $\beta = \beta(\delta) > 0$ *and* $\mathrm{C}' = \mathrm{C}'(\delta) < \infty$ *such that*

(K1) *For all* $\mathrm{L} > \mathrm{C}'(\delta)$, *and all* $(b, x, s) \in \mathrm{K}$,

$$\frac{1}{\mathrm{L}} \int_0^{\mathrm{L}} \chi_{\mathrm{K}_0}\big(\mathrm{T}_t(b, x, s)\big)\, dt \ge 0.99.$$

(K2) $\beta^{\mathrm{X}}(\mathrm{K}) > 1 - c_1(\delta)$. *Also, for all* j, $\beta^{\tau_j, \mathrm{X}}(\mathrm{K}) > 1 - c_1(\delta)$.

(K3) *For all* j *and all* $(b, x, t) \in \mathrm{K}$, $|\phi_j(b, x, t)| < \mathrm{C}$, *where* ϕ_j *is as in* (15.3).

(K4) *For all* j, *all* $(b, x, t) \in \mathrm{K}$ *all* $k \neq 0$ *and all* $v \in \bar{\mathrm{V}}^{(k)}_{\le 1}(b, x)$,

$$\|p(v)\| \ge \beta(\delta) \|v\|. \tag{16.2}$$

(K5) *There exists* $\mathrm{C}_0 = \mathrm{C}_0(\delta)$ *such that for all* $(b, x, s) \in \mathrm{K}$ *all* j *and all* $v \in \mathrm{W}_+(b) \otimes \mathcal{V}^{(j)}_{\le 1}(b, x)$, *we have* $\mathrm{C}_0^{-1} \|v\| \le \|v\|_{j, b, x} \le \mathrm{C}_0 \|v\|$.

Proof. — By the Birkhoff ergodic theorem, there exists $\mathcal{K}'' \subset \mathrm{B}^{\mathrm{X}}$ such that $\beta^{\mathrm{X}}(\mathcal{K}'') > 1 - \delta/5$ and (K1) holds for $\mathcal{K}''$ instead of $\mathcal{K}$. We can choose $\mathcal{K}' \subset \mathrm{B}^{\mathrm{X}}$ and $\mathrm{C} = \mathrm{C}(\delta) < \infty$ such that $\beta^{\mathrm{X}}(\mathcal{K}') > 1 - \delta/5$ and (K3) holds for $\mathcal{K}'$ instead of K. Let $\mathcal{K} = \mathcal{K}(\delta/5)$ and $\mathrm{L} = \mathrm{L}(\delta/5)$ be as in Lemma 15.1 with $\delta/5$ instead of δ. Then choose $\mathrm{K}_j \subset \Psi$ with $\beta^{\tau_j, \mathrm{X}}(\mathrm{K}_j) > 1 - \delta/(5d\mathrm{L})$, where d is the number of Lyapunov exponents. In view of Lemma 14.6 there exists $\mathcal{K}''' \subset \mathrm{X}_0$ with $\beta^{\mathrm{X}}(\mathcal{K}''') > 1 - \delta/5$ so that (16.2) holds. Similarly, there exists a set $\mathcal{K}''''$ with $\mathcal{K}'''' > 1 - \delta/5$ where (K5) holds. Then, let $K = \mathcal{K}'''' \cap \mathcal{K}''' \cap \mathcal{K}'' \cap \mathcal{K}' \cap \mathcal{K} \cap \bigcap_j K_j$. The properties (K1), (K2), (K3) and (K4) are easily verified. □

Warning. — In the rest of this section, we will often identify K and K_0 with their pullbacks $f^{-1}(K) \subset \tilde{B}^X$ and $f^{-1}(K_0) \subset \tilde{B}^X$ where $f : \tilde{B}^X \to B^X$ is the forgetful map.

The martingale convergence theorem. — Let $\mathcal{B}^{\tau_j,X}$ denote the σ-algebra of $\beta^{\tau_j,X}$ measurable functions on B^X. As in [BQ], let

$$Q_\ell^{\tau_j,X} = \left(T_\ell^{\tau_j}\right)^{-1}\left(\mathcal{B}^{\tau_j,X}\right).$$

(Thus if a function F is measurable with respect to $Q_\ell^{\tau_j,X}$, then F depends only on what happened at least ℓ time units in the past, where ℓ is measured using the time change τ_j.)

Let

$$Q_\infty^{\tau_j,X} = \bigcap_{\ell>0} Q_\ell^{\tau_j,X}.$$

The $Q_\ell^{\tau_j,X}$ are a decreasing family of σ-algebras, and then, by the Martingale Convergence Theorem, for $\beta^{\tau_j,X}$-almost all $(b, x, s) \in B^X$,

$$\lim_{\ell\to\infty} \mathbb{E}_j\left(1_K \mid Q_\ell^{\tau_j,X}\right)(b, x, s) = \mathbb{E}_j\left(1_K \mid Q_\infty^{\tau_j,X}\right)(b, x, s) \tag{16.3}$$

where $\mathbb{E}_j$ denotes expectation with respect to the measure $\beta^{\tau_j,X}$.

The set S'. — In view of (16.3) and the condition (K2) we can choose $S' = S'(\delta) \subset B^X$ to be such that for all $\ell > \ell_0$, all j, and all $(b, x, s) \in S'$,

$$\mathbb{E}_j\left(1_K \mid Q_\ell^{\tau_j,X}\right)(b, x, s) > 1 - c_2(\delta). \tag{16.4}$$

By using Lemma 15.1 as in the proof of Lemma 16.2 we may assume that (by possibly making ℓ_0 larger) we have for all j,

$$\beta^{\tau_j,X}\left(S'\right) > 1 - c_2(\delta). \tag{16.5}$$

The set E_{good}. — By Lemma 14.4 we may choose a subset $E_{good} \subset \tilde{B}^X$ (which is actually of the form $\tilde{B} \times E'_{good}$ for some subset $E'_{good} \subset X \times [0, 1]$), with $\beta^X(E_{good}) > 1 - c_3(\delta)$, and a number $\sigma(\delta) > 0$ such that for any $(b, x, s) \in E_{good}$, any j and any unit vector $w \in \mathcal{L}_j(b, x)$,

$$\beta\left(\left\{a \in B \,:\, d_Y\left(w, \mathcal{V}_{>1}^{(j)}(a, x)\right) > \sigma(\delta)\right\}\right) > 1 - c_3'(\delta). \tag{16.6}$$

We may assume that $E_{good} \subset K$. By the Osceledets multiplicative ergodic theorem and Lemma 14.6, we may also assume that there exists $\alpha > 0$ (depending only on the Lyapunov spectrum), and $\ell_0 = \ell_0(\delta)$ such that for $(b, x, s) \in E_{good}$, $\ell > \ell_0$, at least $1 - c_3''(\delta)$ measure of $a \in B$, and all $\bar{v} \in \bar{\mathcal{V}}^{(j)}_{>1}(a, x)$,

$$\|T^{\tau_j}_{-\ell}(a \vee b, x, s)_* \bar{v}\| \leq e^{(1-\alpha)\ell}\|\bar{v}\|. \tag{16.7}$$

The sets Ω_ρ. — In view of (16.5) and the Birkhoff ergodic theorem, for every $\rho > 0$ there exists a set $\Omega_\rho = \Omega_\rho(\delta) \subset \tilde{B}^X$ such that

(Ω1) $\beta^X(\Omega_\rho) > 1 - \rho$.
(Ω2) There exists $\ell_0' = \ell_0'(\rho)$ such that for all $\ell > \ell_0'$, and all $(b, x, s) \in \Omega_\rho$,

$$\left|\left\{t \in [-\ell, \ell] \;:\; T_t(b, x, s) \in S' \cap E_{good}\right\}\right| \geq \left(1 - c_5(\delta)\right)2\ell.$$

Lemma **16.3.** — *Suppose the measure* ν *is not affine. Then there exists* $\rho > 0$ *so that for every* $\delta' > 0$ *there exist* $(b, x, s) \in \Omega_\rho$, $(b, y, s) \in \Omega_\rho$ *with* $\|y - x\| < \delta'$ *such that* $p(y - x) \in p(U)^{\perp}_{\mathbb{C}}(x)$,

$$d\left(y - x, U_{\mathbb{C}}(x)\right) > \frac{1}{10}\|y - x\| \tag{16.8}$$

and

$$d\left(y - x, W^+(b, x)\right) > \frac{1}{3}\|y - x\| \tag{16.9}$$

(so $y - x$ *is in general position with respect to* $W^+(b, x)$*).*

Remark. — In view of Theorem 14.3, it follows that for (b, x, s), (b, y, s) satisfying the conditions of Lemma 16.3, $p(y - x)$ is orthogonal to the complexification $F_{\mathbb{C}}(x)$ of the Forni subspace $F(x)$.

Proof. — By Fubini's theorem, there exists a subset $\Omega_\rho' \subset X$ with $\nu(\Omega_\rho') \geq 1 - \rho^{1/2}$ such that for $x \in \Omega_\rho'$,

$$(\beta \times dt)\left(\left\{(b, s) \;:\; (b, x, s) \in \Omega_\rho\right\}\right) \geq \left(1 - \rho^{1/2}\right). \tag{16.10}$$

Let $\mathcal{K}$ be an arbitrary compact subset of X_0 with $\nu(\mathcal{K}) > 1/2$, and let $\tilde{\mathcal{K}}$ denote its lift to $\tilde{X}_0$. Let $\pi : \tilde{X}_0 \to X_0$ denote the natural map. We have

$$\nu\left(\Omega_\rho'\right) \geq \left(1 - 2\rho^{1/2}\right)\nu(\mathcal{K}). \tag{16.11}$$

In view of Lemma 14.1 we can find finitely many sets $J_\alpha \subset K_\alpha \subset \tilde{X}_0$ and constants $N > 0$ and $\delta_0 > 0$ such that the following hold:

(i) For all α, K_α is diffeomorphic to an open ball, and the restriction of π to K_α is injective.
(ii) The sets J_α are disjoint, and up to a null set $\pi(\tilde{\mathcal{K}}) = \bigsqcup_\alpha \pi(J_\alpha)$.
(iii) Any point belongs to at most N of the sets $\pi(K_\alpha)$.
(iv) Recall that for $x \in \tilde{X}_0$, $U_{\mathbb{C}}[x]$ denotes the (infinite) affine space whose tangent space is $U_{\mathbb{C}}(x)$. We have, for ν-almost all $x \in J_\alpha$,

$$\mathrm{Vol}\big(U_{\mathbb{C}}[x] \cap K_\alpha\big) \geq \delta_0, \tag{16.12}$$

where Vol$(\cdot)$ is as in Lemma 14.1.

Let

$$\Omega''_\rho = \left\{x \in J_\alpha \ :\ \nu_{U_{\mathbb{C}}(x)}\big(\Omega'_\rho \cap K_\alpha\big) \geq \big(1-\rho^{1/4}\big)\nu_{U_{\mathbb{C}}(x)}(K_\alpha)\right\}. \tag{16.13}$$

In the above equation, $\nu_{U_{\mathbb{C}}(x)}$ is the conditional measure of ν along $U_{\mathbb{C}}[x]$ (which is in fact a multiple of the measure Vol of Lemma 14.1). By (16.11), properties (ii), (iii) and Fubini's theorem, $\nu(\Omega''_\rho) \geq (1-2N\rho^{1/4})\nu(\mathcal{K})$. In particular, $\bigcup_{\rho>0}\Omega''_\rho$ is conull in $\mathcal{K}$.

Note that by the definition of Ω''_ρ, if $x \in \Omega''_\rho \cap J_\alpha$ then $U_{\mathbb{C}}[x] \cap J_\alpha \subset \Omega''_\rho$. It follows that we may write, for some indexing set $I_\alpha(\rho)$,

$$\Omega''_\rho \cap J_\alpha = \bigsqcup_{x \in I_\alpha(\rho)} U_{\mathbb{C}}[x] \cap J_\alpha.$$

Suppose that for all α and all $\rho > 0$, $I_\alpha(\rho)$ is countable. Then, for a positive measure set of $x \in \tilde{X}_0$, x has an open neighborhood in $U_{\mathbb{C}}[x]$ whose ν-measure is positive. Then by ergodicity of the geodesic flow, this holds for ν-almost all $x \in \tilde{X}_0$ and without loss of generality, for all $x \in I_\alpha(\rho)$.

The restriction of ν to $U_{\mathbb{C}}[x]$ is a multiple of the measure Vol of Lemma 14.1, therefore there exists a constant $\psi(x) \neq 0$ such that for $E \subset U_{\mathbb{C}}[x]$, $\nu(E) = \psi(x)\,\mathrm{Vol}(E)$. Since both ν and Vol are invariant under the $SL(2,\mathbb{R})$ action, $\psi(x)$ is invariant, and thus by ergodicity ψ is constant almost everywhere.

Let $I'_\alpha = \bigcup_{\rho>0} I_\alpha(\rho)$. For $x, y \in I'_\alpha$ write $x \sim y$ if $U_{\mathbb{C}}[x] \cap J_\alpha = U_{\mathbb{C}}[y] \cap J_\alpha$, and let $I''_\alpha \subset I'_\alpha$ be the subset where we keep only one member of each $\sim$-equivalence class. Note that by properties (i) and (iv), for distinct $x, y \in I''_\alpha$, $U_{\mathbb{C}}[x] \cap K_\alpha$ and $U_{\mathbb{C}}[y] \cap K_\alpha$ are disjoint up to a set of measure 0. Then (16.12) implies that for each α,

$$\nu(K_\alpha) \geq \sum_{x \in I''_\alpha} \nu\big(U_{\mathbb{C}}[x] \cap K_\alpha\big) = \sum_{x \in I''_\alpha} \psi\,\mathrm{Vol}\big(U_{\mathbb{C}}[x] \cap K_\alpha\big) \geq \psi\delta_0|I''_\alpha|,$$

where $|\cdot|$ denotes the cardinality of a set. Since ν is a finite measure, we get that each I''_α is finite. Since for a fixed $\mathcal{K}$, there are only finitely many sets K_α, this implies that the support of restriction of ν to $\mathcal{K}$ is contained in a finite union of "affine pieces" each of the form $U_{\mathbb{C}}[x_j] \cap K_\alpha$ for some $x_j \in \mathcal{K}$, and the measure ν restricted to each affine piece coincides with ψ Vol. It follows from the ergodicity of g_t that the affine pieces fit together to form an (immersed) submanifold. Thus, ν is affine.

Thus, we may assume that there exist α and $\rho > 0$ such that $I_\alpha(\rho)$ is not countable. Then we can find $x_1 \in I_\alpha(\rho)$ and $y_n \in I_\alpha(\rho)$ such that

$$\lim_{n\to\infty} hd\big(U_{\mathbb{C}}[x_1] \cap K_\alpha, U_{\mathbb{C}}[y_n] \cap K_\alpha\big) = 0,$$

where hd denotes Hausdorff distance between sets (using the distance d^{X_0} defined in Section 3). Let $f_n : p(U)_{\mathbb{C}}[y_n] \to p(U)_{\mathbb{C}}[x_1]$ denote the function taking $z \in p(U)_{\mathbb{C}}[y_n]$ to the unique point in $p(U)_{\mathbb{C}}[x_1] \cap p(U)^{\perp}_{\mathbb{C}}[z]$. Then, for large n, the map f_n is almost measure preserving, in the sense that for $V \subset p(U)_{\mathbb{C}}(y_n)$,

$$(0.5)|V| \le |f_n(V)| \le 2|V|,$$

where $|\cdot|$ denotes Lebesgue measure. Then, in view of the definition (16.13) of Ω''_ρ, for sufficiently large n, there exist $x \in U_{\mathbb{C}}[x_1] \cap \Omega'_\rho$ and $y \in U_{\mathbb{C}}[y_n] \cap \Omega'_\rho$ such that $p(y-x) \in p(U)^{\perp}_{\mathbb{C}}(x)$, and $\|y-x\| < \delta'$. Then, by the definition (16.10) of Ω'_ρ, we can choose (b,s) so that $(b,x,s) \in \Omega_\rho$, $(b,y,s) \in \Omega_\rho$, and (16.8) and (16.9) holds. □

Standing assumption. — We fix $\rho = \rho(\delta)$ so that Lemma 16.3 holds.

The main part of the proof is the following:

Proposition **16.4.** — *There exists $C(\delta) > 1$ such that the following holds: Suppose for every $\delta' > 0$ there exist $(b,x,s), (b,y,s) \in \Omega_\rho$ with $\|x-y\| \le \delta'$, $p(x-y) \in p(U)^{\perp}_{\mathbb{C}}(x)$, and so that (16.8) and (16.9) hold. Then for every $\epsilon > 0$ there exist $(b'',x'',s'') \in K_0$, $(b'',y'',s'') \in K_0$, such that $y''-x'' \in U^{\perp}_{\mathbb{C}}(x'')$,*

$$\frac{\epsilon}{C(\delta)} \le \|y''-x''\| \le C(\delta)\epsilon,$$

$$d\big(y''-x'', U_{\mathbb{C}}(x'')\big) \ge \frac{1}{C(\delta)}\|y''-x''\|, \tag{16.14}$$

$$d\big(y''-x'', W^+(b'',x'')\big) < \delta'', \tag{16.15}$$

where δ'' depends only on δ', and $\delta'' \to 0$ as $\delta' \to 0$.

Proof. — Let $\tilde\Lambda \subset \hat\Lambda$ denote the subset $\{k : \hat\lambda_k \ne 0\}$. We may decompose

$$p(U)^{\perp}(x) = \bigoplus_{k\in\tilde\Lambda} \mathcal{L}_k(x) \bigoplus F(x) \tag{16.16}$$

as in Section 14. For $j \in \tilde\Lambda$, let π_j denote the projection to $\mathcal{L}_j$, using the decomposition (16.16). Note that by Theorem 14.3, the projection of $p(y-x)$ to $F(x)$ is always 0.

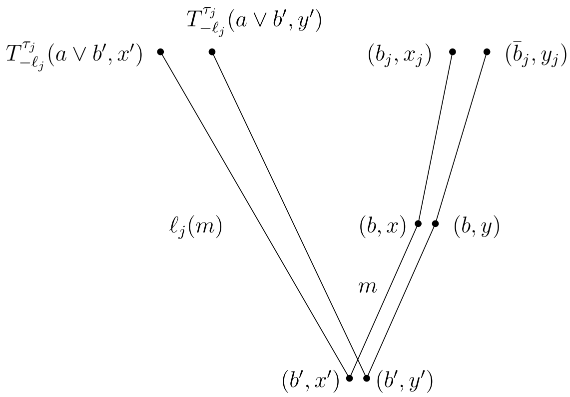

FIG. 7. — Proof of Proposition 16.4. In the figure, going "up" corresponds to the "future". The map T_m for $m > 0$ takes one m steps into the "past"

For $m \in \mathbb{R}^+$, write (see Figure 7)

$$\left(b', x', s'\right) = T_m(b, x, s), \quad \left(b', y', s'\right) = T_m(b, y, s),$$

and let

$$w_j(m) = \pi_j\left(x' - y'\right).$$

(We will always have m small enough so that the above equation makes sense.) Let $\ell_j(m)$ be such that

$$e^{\ell_j(m)} \|w_j(m)\| = \epsilon.$$

We also need to handle the relative homology part (where the action of the Kontsevich-Zorich cocycle is trivial). Set $\ell_0(m)$ to be the number such that

$$e^{\ell_0(m)} \|x' - y'\| = \epsilon.$$

Choose $0 < \sigma' \ll \lambda_{min}$ where $0 < \lambda_{min} = \min_{j \in \tilde{\Lambda}} \hat{\lambda}_j$. We will be choosing m so that

$$\frac{\sigma'}{2} |\log \|y - x\|| \leq m \leq \sigma' |\log \|y - x\||. \tag{16.17}$$

In view of (16.9) and Theorem A.1, (after some uniformly bounded time), $\|w_j(m)\|$ is an increasing function of m (since the factor of e^{-t} from the geodesic flow beats the contribution of the Kontsevich-Zorich cocycle). Therefore, $\ell_j(m)$ is a decreasing function of m.

For a bi-infinite sequence $b \in \tilde{B}$ and $x \in X_0$, let

$$G_j(b, x, s) = \left\{m \in \mathbb{R}_+ \ : \ T^{\tau_j}_{-\ell_j(m)} T_m(b, x, s) \in S'\right\}.$$

Let $G_{all}(b, x, s) = \bigcap_j G_j(b, x, s) \cap \{m \ : \ T_m(b, x, s) \in E_{good}\}$.

Lemma **16.5.** — *For* $(b, x, s) \in \Omega_\rho$ *and* N *sufficiently large,*

$$\frac{|G_{all}(b, x, s) \cap [0, N]|}{N} \geq 1 - c_6(\delta).$$

Proof. — We can write $T^{\tau_j}_{-\ell_j(m)} T_m = T_{-g_j(m)}$. By definition,

$$m \in G_j(b, x, s) \quad \text{if and only if} \quad T_{-g_j(m)}(b, x, s) \in S'.$$

Since $\ell_j(m)$ is a decreasing function of m, so is g_j, and in fact, for all $m_2 > m_1$

$$g_j(m_1) - g_j(m_2) > m_2 - m_1.$$

This implies that

$$g_j^{-1}(m_1) - g_j^{-1}(m_2) < m_1 - m_2. \tag{16.18}$$

Let $F = \{t \in [0, g_j(N)] \;:\; T_{-t}(b, x) \notin S'\}$. By condition ($\Omega 2$), for N large enough, $|F| \leq (1 - c_5(\delta)) g_j(N)$. Note that $G_j^c \cap [0, N] = g_j^{-1}(F)$. Then, by (16.18),

$$|G_j^c \cap [0, N]| = |g_j^{-1}(F)| \leq |F| \leq c_5(\delta) g_j(N) \leq c_6(\delta) N,$$

where as in our convention $c_6(\delta) \to 0$ as $\delta \to 0$. □

We now continue the proof of Proposition 16.4. We may assume that δ' is small enough so that the right-hand-side of (16.17) is smaller then the N of Lemma 16.5. Suppose $(b, x, s) \in \Omega_\rho$, $(b, y, s) \in \Omega_\rho$. By Lemma 16.5, we can fix $m \in G_{all}(x)$ such that (16.17) holds. Write $\ell_j = \ell_j(m)$. Let

$$\bigl(b', x', s'\bigr) = T_m(b, x, s), \qquad \bigl(b', y', s'\bigr) = T_m(b, y, s).$$

For $j \in \tilde{\Lambda}$, let

$$(b_j, x_j, s_j) = T^{\tau_j}_{-\ell_j(m)}\bigl(b', x', s'\bigr), \qquad (\bar{b}_j, y_j, \bar{s}_j) = T^{\tau_j}_{-\ell_j(m)}\bigl(b', y', s'\bigr).$$

Since $m \in G_{all}(b, x, s)$, we have $(b_j, x_j, s_j) \in S'$, $(\bar{b}_j, y_j, \bar{s}_j) \in S'$. Then, by (16.4), for all j,

$$\mathbb{E}_j\bigl(1_K \mid Q^{\tau_j, X}_{\ell_j}\bigr)(b_j, x_j, s_j) > \bigl(1 - c_2(\delta)\bigr),$$
$$\mathbb{E}_j\bigl(1_K \mid Q^{\tau_j, X}_{\ell_j}\bigr)(\bar{b}_j, y_j, \bar{s}_j) > \bigl(1 - c_2(\delta)\bigr).$$

Since $T^{\tau_j}_{\ell_j}(b_j, x_j, s_j) = (b', x', s')$, by [BQ, (7.5)] we have

$$\mathbb{E}_j\bigl(1_K \mid Q^{\tau_j, X}_{\ell_j}\bigr)(b_j, x_j, s_j) = \int_B 1_K\bigl(T^{\tau_j}_{-\ell_j}\bigl(a \vee b', x', s'\bigr)\bigr) d\beta(a),$$

where the notation $a \vee b'$ is as in (14.2). Thus, for all $j \in \tilde{\Lambda}$,

$$\beta\left(\left\{a \,:\, \mathrm{T}^{\tau_j}_{-\ell_j}\left(a \vee b', x', s'\right) \in \mathrm{K}\right\}\right) > 1 - c_2(\delta). \tag{16.19}$$

Similarly, for all $j \in \tilde{\Lambda}$,

$$\beta\left(\left\{a \,:\, \mathrm{T}^{\tau_j}_{-\ell_j}\left(a \vee b', y', s'\right) \in \mathrm{K}\right\}\right) > 1 - c_2(\delta).$$

Let $w = x' - y'$, and let $w_j = \pi_j(w)$. We can write

$$w = \bar{w}_0 + \sum_{j \in \hat{\Lambda}} \bar{w}_j \tag{16.20}$$

where $\bar{w}_0 \in \ker p$, and for $j > 0$, $\bar{w}_j$ are chosen so that $\pi_j(\bar{w}_j) = w_j$, and also $\|\bar{w}_j\| \approx \|w_j\|$.

For any $a \in \mathrm{B}$, we may write

$$w_j = \xi_j(a) + v_j(a),$$

where $\xi_j(a) \in \mathrm{W}_+(b') \otimes \mathcal{V}^{(j)}_{\leq 1}(b', x')$, and

$$v_j(a) \in \mathrm{W}_+(b) \otimes \mathcal{V}^{(j)}_{>1}\left(a, x'\right) + \mathrm{W}_-(a) \otimes \mathcal{L}_j\left(b', x'\right).$$

This decomposition is motivated as follows: if we consider the Lyapunov decomposition

$$\mathbb{C} \otimes \mathcal{L}_j(x) = \bigoplus_k \mathcal{V}_k(a \vee b, x)$$

then $\xi_j(a)$ belongs to the subspace $\mathcal{V}_{\leq 1}(a \vee b, x)$ corresponding to the top Lyapunov exponent $\sigma_0 + \hat{\lambda}_j$ for the action of T_{-t}, and $v_j \in \oplus_{k \geq 2} \mathcal{V}_k(a \vee b, x)$ will grow with a smaller Lyapunov exponent under T_{-t}. Then $v_j(a)$ will also grow with a smaller Lyapunov exponent then $\xi_j(a)$ under $\mathrm{T}^{\tau_j}_{-\ell}$.

Since $m \in \mathrm{G}_{all}(b, x, s)$, we have $(b', x', s') \in \mathrm{E}_{good}$. Then, by (16.6), for at least $1 - c_3'(\delta)$ fraction of $a \in \mathrm{B}$,

$$\|v_j(a)\| \approx \|\xi_j(a)\| \approx \|w_j\| \approx \epsilon e^{-\ell_j}, \tag{16.21}$$

where the notation $\mathrm{A} \approx \mathrm{B}$ means that A/B is bounded between two constants depending only on δ. Since $(b', x', s') \in \mathrm{E}_{good} \subset \mathrm{K}$, by condition (K3) we have $|\phi_j(b', x', s')| \leq \mathrm{C}(\delta)$. Also by (16.19), for at least $1 - c_2(\delta)$ fraction of $a \in \mathrm{B}$, we have $\mathrm{T}^{\tau_j}_{-\ell_j}(a \vee b', x', s') \in \mathrm{K}$, so again by condition (K3) we have

$$\left|\phi_j\left(\mathrm{T}^{\tau_j}_{-\ell_j}\left(a \vee b', x', s'\right)\right)\right| \leq \mathrm{C}(\delta).$$

Thus, by (16.21), (15.10) and (16.7), we have, for all $j \in \tilde{\Lambda}$, and at least $1 - c_4(\delta)$ fraction of $a \in \mathrm{B}$,

$$\left\|\mathrm{T}^{\tau_j}_{-\ell_j}\left(a \vee b', x', s'\right)_* \xi_j(a)\right\| \approx \epsilon, \quad \text{and} \quad \left\|\mathrm{T}^{\tau_j}_{-\ell_j}\left(a \vee b', x', s'\right)_* v_j(a)\right\| = \mathrm{O}\left(e^{-\alpha \ell_j}\right), \tag{16.22}$$

where $\alpha > 0$ depends only on the Lyapunov spectrum. (The notation in (16.22) is defined in (15.9).) Hence, for at least $1 - c_4(\delta)$ fraction of $a \in B$,

$$\| T^{\tau_j}_{-\ell_j}(a \vee b', x', s')_* w_j \| \approx \epsilon.$$

Since $\lambda_j \geq 0$ (and by Theorem 14.3, if $\lambda_j = 0$ then $j = 0$, and $\bar{w}_0 \in \ker p$ where the action of the Kontsevich-Zorich cocycle is trivial), we have for at least $1 - c_4(\delta)$ fraction of $a \in B$,

$$\| T^{\tau_j}_{-\ell_j}(a \vee b', x', s')_* \bar{w}_j \| \approx \epsilon. \tag{16.23}$$

Let

$$t_j(a) = \sup\{ t > 0 \ : \ \| T_{-t}(a \vee b', x', s')_* \bar{w}_j \| \leq \epsilon \},$$

and let $j(a)$ denote a $j \in \tilde{\Lambda}$ such that $t_j(a)$ is as small as possible as j varies over $\tilde{\Lambda}$. Then, if $j = j(a)$, then by (16.23),

$$\| T_{-t_j(a)}(a \vee b', x', s')_* \bar{w}_j \| \approx \| T^{\tau_j}_{-\ell_j}(a \vee b', x', s')_* \bar{w}_j \| \approx \epsilon. \tag{16.24}$$

Also, for at least $1 - c_4(\delta)$-fraction of $a \in B$, if $j = j(a)$ and $k \neq j$, then by (16.23),

$$\| T^{\tau_j}_{-\ell_j}(a \vee b', x', s')_* \bar{w}_k \| \leq C_1(\delta)\epsilon, \tag{16.25}$$

where $C_1(\delta)$ depends only on δ. Therefore, by (16.20), (16.24), and (16.25), for at least $1 - c_4(\delta)$-fraction of $a \in B$, if $j = j(a)$,

$$\| T^{\tau_j}_{-\ell_j}(a \vee b', x', s')_* (y' - x') \| \approx \epsilon. \tag{16.26}$$

We now choose $\delta > 0$ so that $c_4(\delta) + 2c_2(\delta) < 1/2$, and using (16.19) we choose $a \in B$ so that (16.26) holds, and also

$$T^{\tau_j}_{-\ell_j}(a \vee b', x', s') \in K, \qquad T^{\tau_j}_{-\ell_j}(a \vee b', y', s') \in K.$$

We may write

$$T^{\tau_j}_{-\ell_j}(a \vee b', x', s') = T_{-t}(a \vee b, x', s'),$$
$$T^{\tau_j}_{-\ell_j}(a \vee b', y', s') = T_{-t'}(a \vee b, y', s')$$

Then, $|t' - t| \leq C(\delta)$. Therefore by condition (K1), there exists t'' with $|t'' - t| \leq C(\delta)$ such that

$$(b'', x'', s'') = T_{-t''}(a \vee b', x', s') \in K_0,$$
$$(b'', y'', s'') = T_{-t''}(a \vee b', y', s') \in K_0.$$

Since $\|w\| \approx \epsilon e^{-\ell_j}$, and $\ell_j \to \infty$ as $\delta' \to 0$, we have $\|w\| = \|x' - y'\| \to 0$ as $\delta' \to 0$. Since $T_{-t''}$ does not expand the W^- components, the W^- component of $x'' - y''$ is bounded by the W^- component of $x' - y'$. Thus, the size of the W^- component of $x'' - y''$ tends to 0 as $\delta' \to 0$. Thus (16.15) holds.

It remains to prove (16.14). If

$$\|p(y'' - x'')\| \geq \frac{1}{C(\delta)} \|y'' - x''\| \tag{16.27}$$

then (16.14) holds since $p(y'' - x'') \in p(U)^{\perp}(x'')$. This automatically holds for the case where $|\Sigma| = 1$ (and thus, in particular, there are no marked points). If not, we may write

$$y'' - x'' = w''_+ + \bar{w}''_0$$

where $\|w''_+\| \leq c(\delta)\|\bar{w}''_0\|$ and $\bar{w}''_0 \in \ker p$. We will need to rule out the case where $\bar{w}''_0$ is very close to $U^+(x'') \cap \ker p$. We will show that this contradicts the assumption (16.8).

Let w'_+, $\bar{w}'_0$ be such that

$$w''_+ = T_{-t''}\big(a \vee b, x', s'\big)_* w'_+, \qquad \bar{w}''_0 = T_{-t''}\big(a \vee b, x', s'\big)_* \bar{w}'_0.$$

Then $y' - x' = w'_+ + \bar{w}'_0$ and in view of (16.1) and (16.21),

$$\|w'_+\| \leq e^{-\lambda_{min} t''/2} \|\bar{w}'_0\| \approx e^{-\lambda_{min} t''/2} \|y' - x'\|.$$

Applying $T_{-m}(b, x', s')$ to both sides we get

$$y - x = w_+ + \bar{w}_0,$$

where $\bar{w}_0 \in \ker p$, and

$$\|w_+\| \leq e^{2m} \|w'_+\| \leq e^{2m - \frac{\lambda_{min} t''}{2}} \|x - y\|.$$

By (16.17), $2m - \frac{\lambda_{min} t''}{2} \leq -\frac{\lambda_{min} t''}{4}$. Thus, $\|w_+\| \leq (1/100)\|y - x\|$. Therefore, by (16.8), we have

$$d\big(\bar{w}_0, \ker p \cap U_{\mathbb{C}}(x)\big) > \frac{1}{20}\|w_0\|.$$

Since the action of the cocycle on $\ker p$ is trivial (and we have shown that in our situation the component in $\ker p$ dominates throughout the process), this implies

$$d\big(\bar{w}''_0, \ker p \cap U_{\mathbb{C}}(x'')\big) > \frac{1}{20}\|w''_0\| \geq \frac{1}{40}\|y'' - x''\|.$$

This, together with the assumption that (16.27) does not hold, implies (16.14). □

Proof of Theorem 1.4. — It was already proved in Theorem 2.1 that ν is $SL(2, \mathbb{R})$-invariant. Now suppose ν is not affine. We can apply Lemma 16.3, and then iterate Proposition 16.4 with $\delta' \to 0$ and fixed ϵ and δ. Taking a limit along a subsequence we get points $(b_\infty, x_\infty, s_\infty) \in K_0$ and $(b_\infty, y_\infty, s_\infty) \in K_0$ such that $\|x_\infty - y_\infty\| \approx \epsilon$, $y_\infty \in W^+(b_\infty, x_\infty)$ and $y_\infty \in (U^{\perp})^+(b_\infty, x_\infty)$. This contradicts Lemma 16.1 since $K_0 \subset \Psi$. Hence ν is affine. □

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Appendix A: Forni's results on the $SL(2,\mathbb{R})$ action

In this appendix, we summarize the results we use from the fundamental work of Forni [Fo]. The recent preprint [FoMZ] contains an excellent presentation of these ideas and also some additional results which we will use as well.

A.1 The Hodge norm and the geodesic flow

Let $\mathcal{M}_g$ denote the moduli space of genus g curves. Fix a point S in $\mathcal{H}(\alpha)$; then S is a pair (M,ω) where $M \in \mathcal{M}_g$ and ω is a holomorphic 1-form on M. Let $\|\cdot\|_{H,t}$ denote the Hodge norm (see e.g. [ABEM]) at the surface $M_t = \pi(g_t S)$. Here $\pi : \mathcal{H}(\alpha) \to \mathcal{M}_g$ is the natural map taking (M,ω) to M. We recall that the Hodge norm is a norm on $H^1(M,\mathbb{R})$.

The following fundamental result is due to Forni [Fo, §2]:

Theorem **A.1.** — *For any* $\lambda \in H^1(M,\mathbb{R})$ *and any* $t \geq 0$,

$$\|\lambda\|_{H,t} \leq e^t \|\lambda\|_{H,0}.$$

If in addition λ *is orthogonal to* ω, *and for some compact subset* $\mathcal{K}$ *of* $\mathcal{M}_g$, *the geodesic segment* $[S, g_t S]$ *spends at least half the time in* $\pi^{-1}(\mathcal{K})$, *then we have*

$$\|\lambda\|_{H,t} \leq e^{(1-\alpha)t} \|\lambda\|_{H,0},$$

where $\alpha > 0$ *depends only on* $\mathcal{K}$.

The Hodge norm on relative cohomology. — Let Σ denote the set of zeroes of ω. Let $p : H^1(M,\Sigma,\mathbb{R}) \to H^1(M,\mathbb{R})$ denote the natural map. We define a norm $\|\cdot\|'$ on the relative cohomology group $H^1(M,\Sigma,\mathbb{R})$ as follows:

$$\|\lambda\|' = \|p(\lambda)\|_H + \sum_{(z,w)\in\Sigma\times\Sigma} \left| \int_{\gamma_{z,w}} (\lambda - h) \right|, \qquad (\mathbf{A.1})$$

where $\| \cdot \|_H$ denotes the Hodge norm on $H^1(M, \mathbb{R})$, h is the harmonic representative of the cohomology class $p(\lambda)$ and $\gamma_{z,w}$ is any path connecting the zeroes z and w. Since $p(\lambda)$ and h represent the same class in $H^1(M, \mathbb{R})$, Equation (A.1) does not depend on the choice of $\gamma_{z,w}$.

Let $\| \cdot \|'_t$ denote the norm (A.1) on the surface M_t. Then, up to a fixed multiplicative constant, the analogue of Theorem A.1 holds, for $\| \cdot \|'_t$, as long as $S \equiv (M, \omega)$ and $g_t S$ belong to a fixed compact set. This assertion is essentially Lemma 4.4 from [AthF]. For a self-contained proof in this notation see [EMR, §8].

The Avila-Gouëzel-Yoccoz (AGY) norm. — The Hodge norm on relative cohomology behaves badly in the thin part of Teichmüller space. Therefore, we will use instead the Avila-Gouëzel-Yoccoz norm $\| \cdot \|_Y$ defined in [AGY], some properties of which were further developed in [AG]. The norms $\| \cdot \|_Y$ and $\| \cdot \|'$ are equivalent on compact subsets of the strata $\mathcal{H}_1(\alpha)$, and therefore the decay estimates on $\| \cdot \|'$ in the style of Theorem A.1 also apply to the Avila-Gouëzel-Yoccoz norm. Furthermore, we have the following:

Theorem **A.2.** — *Suppose* $S = (M, \omega) \in \mathcal{H}(\alpha)$. *Let* $\| \cdot \|_t$ *denote the Avila-Gouëzel-Yoccoz (AGY) norm on the surface* $g_t S$. *Then,*

(a) *For all* $\lambda \in H^1(M, \Sigma, \mathbb{R})$ *and all* $t > 0$,

$$\|\lambda\|_t \leq e^t \|\lambda\|_0.$$

(b) *Suppose for some compact subset* $\mathcal{K}$ *of* $\mathcal{M}_g$, *the geodesic segment* $[S, g_t S]$ *spends at least half the time in* $\pi^{-1}(\mathcal{K})$. *Suppose* $\lambda \in H^1(M, \Sigma, \mathbb{R})$ *with* $p(\lambda)$ *orthogonal to* ω. *Then we have*

$$\|\lambda\|_t \leq C e^{(1-\alpha)t} \|\lambda\|_0,$$

where $\alpha > 0$ *depends only on* $\mathcal{K}$.

A.2 The Kontsevich-Zorich cocycle

We recall that X_0 denotes a finite cover of a stratum which is a manifold (see Section 3). In the sequel, a subbundle L of the Hodge bundle is called *isometric* if the action of the Kontsevich-Zorich cocycle restricted to L is by isometries in the Hodge metric. We say that a subbundle is *isotropic* if the symplectic form vanishes identically on the sections, and *symplectic* if the symplectic form is non-degenerate on the sections. A subbundle is *irreducible* if it cannot be decomposed as a direct sum, and *strongly irreducible* if it cannot be decomposed as a direct sum on any (measurable) finite cover of X_0.

Theorem **A.3.** — *Let* ν *be a* P*-invariant measure on* X_0, *and suppose* L *is a* P*-invariant* ν*-measurable subbundle of the Hodge bundle. Let* $\lambda_1, \ldots, \lambda_n$ *be the Lyapunov exponents of the restriction* *of the Kontsevich-Zorich cocycle to* L. *Then,*

$$\sum_{i=1}^{n} \lambda_i \geq 0.$$

Proof. — Let the symplectic complement $L^\dagger$ of L be defined by

$$L^\dagger(x) = \{v \,:\, v \wedge u = 0 \text{ for all } u \in L(x)\}. \tag{A.2}$$

Then, $L^\dagger$ is a P-invariant subbundle, and we have the short exact sequence

$$0 \to L \cap L^\dagger \to L \to L/(L \cap L^\dagger) \to 0.$$

The bundle $L/(L \cap L^\dagger)$ admits an invariant non-degenerate symplectic form, and therefore, the sum of the Lyapunov exponents on $L/(L \cap L^\dagger)$ is ≥ 0. Therefore, it is enough to show that the sum of the Lyapunov exponents on the isotropic subspace $L \cap L^\dagger$ is 0. Thus, without loss of generality, we may assume that L is isotropic.

Let $\{c_1, \ldots, c_n\}$ be a Hodge-orthonormal basis for the bundle L at the point $S = (M, \omega)$, where M is a Riemann surface and ω is a holomorphic 1-form on M. For $g \in SL(2, \mathbb{R})$, let $V_S(g)$ denote the Hodge norm of the polyvector $c_1 \wedge \cdots \wedge c_n$ at the point gS, where the vectors c_i are transported following a path from the identity to g using the Gauss-Manin connection. (The result does not depend on the path since the Gauss-Manin connection is flat, and X_0 has no orbifold points). Since $V_S(kg) = V_S(g)$ for $k \in SO(2)$, we can think of V_S as a function on the upper half plane $\mathbb{H}$. From the definition of V_S and the multiplicative ergodic theorem, we see that for ν-almost all $S \in X_0$,

$$\lim_{t\to\infty} \frac{\log V_S(g_t)}{t} = \sum_{i=1}^{n} \lambda_i, \tag{A.3}$$

where the λ_i are as in the statement of Theorem A.3.

Let Δ_{hyp} denote the hyperbolic Laplacian operator (along the Teichmüller disk). By [FoMZ, Lemma 2.8] (see also [Fo, Lemma 5.2 and Lemma 5.2′]) there exists a non-negative function $\Phi : X_0 \to \mathbb{R}$ such that for all $S \in X_0$ and all $g \in SL(2, \mathbb{R})$,

$$(\Delta_{hyp} \log V_S)(g) = \Phi(gS).$$

We now claim that the Kontsevich-Forni type formula

$$\sum_{i=1}^{n} \lambda_i = \int_{X_0} \Phi(S)\, d\nu(S) \tag{A.4}$$

holds, which clearly implies the theorem. The formula (A.4) is proved in [FoMZ] (and for the case of the entire stratum in [Fo]) under the assumption that the measure ν

is invariant under $SL(2,\mathbb{R})$. However, in the proofs, only averages over "large circles" in $\mathbb{H} = SO(2)\backslash SL(2,\mathbb{R})$ are used. Below we show that a slightly modified version of the proof works under the a-priori weaker assumption that ν is invariant under $P = AN \subset SL(2,\mathbb{R})$. This is not at all surprising, since large circles in $\mathbb{H}$ are approximately horocircles (i.e. orbits of N).

We now begin the proof of (A.4), following the proof of [FoMZ, Theorem 1].

Since (A.3) holds for ν-almost all S and ν is N-invariant, (A.3) also holds for almost all $S_0 \in X_0$ and almost all $S \in \Omega_N S_0$, where

$$\Omega_N = \left\{ \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} : |s| \le 1 \right\} \subset N.$$

We identify $SO(2)\backslash SL(2,\mathbb{R})S_0$ with $\mathbb{H}$ so that $SO(2)gS_0$ corresponds to $g^{-1} \cdot i$. Then $\Omega_N S_0$ corresponds to the horizontal line segment connecting $-1+i$ to $1+i$. Let $\epsilon = e^{-4t}$. Then, $g_t\Omega_N S_0$ corresponds to the line segment connecting $-1+i\epsilon$ to $1+i\epsilon$.

Let $f(z) = \log V_{S_0}(SO(2)z)$. Note that $\nabla_{hyp} f$ is bounded (where ∇_{hyp} is the gradient with respect to the hyperbolic metric on $\mathbb{H}$). Then, (A.3) implies that for almost all $x \in [-1,1]$,

$$\sum_{i=1}^{n} \lambda_i = \lim_{T\to\infty} \frac{f(x+ie^{-2T}) - f(x+i)}{T} = \lim_{T\to\infty} \frac{1}{T}\int_0^T \frac{\partial}{\partial t}\left[f\left(x+ie^{-2t}\right)\right]dt$$

Integrating the above formula from $x=-1$ to $x=1$, we get (using the bounded convergence theorem),

$$\sum_{i=1}^{n} \lambda_i = \lim_{T\to\infty} \frac{1}{T}\int_0^T \left(\int_{-1}^{1} \frac{\partial}{\partial t}\left[f\left(x+ie^{-2t}\right)\right]dx\right)dt$$

Let R_t denote the rectangle with corners at $-1+ie^{-2t}$, $1+ie^{-2t}$, $1+i$ and $-1+i$, see Figure 8. We now claim that

$$\textbf{(A.5)} \qquad \int_{-1}^{1} \frac{\partial}{\partial t}\left[f\left(x+ie^{-2t}\right)\right]dx = e^{-4t}\int_{\partial R_t} \frac{\partial f}{\partial n} + O\left(te^{-4t}\right),$$

where $\frac{\partial f}{\partial n}$ denotes the (outgoing) normal derivative of f with respect to the hyperbolic metric. Indeed, the integral over the bottom edge of the rectangle R_t on the left hand side of (A.5) coincides with the right hand side of (A.5) (the factor of e^{-4t} appears because the hyperbolic length element is $dx/y^2 = e^{-4t}dx$.) The partial derivative $\frac{\partial f}{\partial n}$ is uniformly bounded, and the hyperbolic lengths of the other three sides of ∂R_t are $O(t)$. Therefore (A.5) follows.

Now, by Green's formula (in the hyperbolic metric),

$$\int_{\partial R_t} \frac{\partial f}{\partial n} = \int_{R_t} \Delta_{hyp} f = \int_{R_t} \Phi.$$

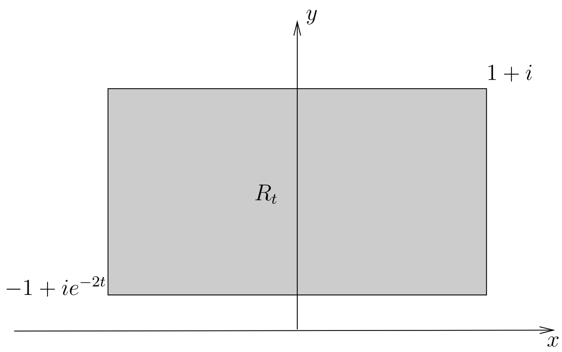

FIG. 8. — Proof of Theorem A.3

We get, for almost all S_0,

$$\sum_{i=1}^{n} \lambda_i = \lim_{T\to\infty} \frac{1}{T}\int_0^T \left(e^{-4t}\int_{R_t} \Phi\right) dt \geq 0.$$

This completes the proof of the Theorem. It is also easy to conclude (by integrating over S_0) that (A.4) holds. □

Theorem **A.4.** — *Let* ν *be an ergodic* $SL(2,\mathbb{R})$*-invariant measure, and suppose* L *is an* $SL(2,\mathbb{R})$*-invariant* ν*-measurable subbundle of the Hodge bundle. Suppose all the Lyapunov exponents of the restriction of the Kontsevich-Zorich cocycle to* L *vanish. Then, the action of the Kontsevich-Zorich cocycle on* L *is isometric with respect to the Hodge inner product, and the orthogonal complement* $L^\perp$ *of* L *with respect to the Hodge inner product is also an* $SL(2,\mathbb{R})$*-invariant subbundle.*

Proof. — The first assertion is the content of [FoMZ, Theorem 3]. The second assertion then follows from [FoMZ, Lemma 4.3]. □

Theorem **A.5.** — *Let* ν *be an ergodic* $SL(2,\mathbb{R})$*-invariant measure, and suppose* L *is an* $SL(2,\mathbb{R})$*-invariant* ν*-measurable subbundle of the Hodge bundle. Suppose* L *is isotropic. Then all the Lyapunov exponents of the restriction of the Kontsevich-Zorich cocycle to* L *vanish (and thus Theorem A.4 applies to* L*).*

Proof. — For a point $x \in X_0$ and an isotropic k-dimensional subspace I_k, let $\Phi_k(x, I_k)$ be as in [FoMZ, (2.46)] (or [Fo, Lemma 5.2']). We have from [FoMZ, Lemma 2.8] that

$$\Phi_k(x, I_k) \leq \Phi_j(x, I_j) \quad \text{if } i < j \text{ and } I_k \subset I_j.$$

Let $\lambda_1 \geq \cdots \geq \lambda_n$ be the Lyapunov exponents of the restriction of the Kontsevich-Zorich cocycle to L. Let $\mathcal{V}_{\leq j}(x)$ denote the direct sum of all the Lyapunov subspaces corresponding to exponents $\lambda_i \geq \lambda_j$. By definition, $V_n(x) = L(x)$. Suppose $j = n$ or $\lambda_j \neq \lambda_{j+1}$. Then, by [FoMZ, Corollary 3.1] the following formula holds:

$$\lambda_1 + \cdots + \lambda_j = \int_{X_0} \Phi_j\big(x, \mathcal{V}_{\leq j}(x)\big)\, d\nu(x)$$

(This formula is proved in [Fo] for the case where ν is Lebesgue measure and L is the entire Hodge bundle.)

We will first show that all the λ_j have the same sign. Suppose not, then we must have $\lambda_n < 0$ but not all $\lambda_j < 0$. Let k be maximal such that $\lambda_k \neq \lambda_n$. Then

$$\lambda_1 + \cdots + \lambda_k = \int_{X_0} \Phi_k\big(x, V_k(x)\big)\, d\nu(x)$$

and

$$\lambda_1 + \cdots + \lambda_n = \int_{X_0} \Phi_n\big(x, L(x)\big)\, d\nu(x)$$

But $\Phi_k(x, V_k(x)) \leq \Phi_n(x, L(x))$ since $V_k(x) \subset L(x)$. Thus,

$$\lambda_{k+1} + \cdots + \lambda_n \geq 0. \tag{A.6}$$

But by the choice of k, all the terms in (A.6) are equal to each other. This implies that $\lambda_n \geq 0$, contradicting our assumption that $\lambda_n < 0$. Thus all the λ_j, $1 \leq j \leq n$ have the same sign. Since ν is assumed to be $SL(2, \mathbb{R})$-invariant, and any diagonalizable $g \in SL(2, \mathbb{R})$ is conjugate to its inverse, we see that e.g. the λ_j cannot all be positive. Hence, all the Lyapunov exponents λ_j are 0. □

Algebraic hulls. — The algebraic hull of a cocycle is defined in [Zi2]. We quickly recap the definition: Let G be a group acting on a space X, preserving an ergodic measure ν. Suppose H is an $\mathbb{R}$-algebraic group, and let $A : G \times X \to H$ be a measurable cocycle. We say that the $\mathbb{R}$-algebraic subgroup H' of H is the *algebraic hull* of A if H' is the smallest $\mathbb{R}$-algebraic subgroup of H such that there exists a ν-measurable map $C : X \to H$ such that

$$C(gx)^{-1}A(g, x)C(x) \in H' \quad \text{for almost all } g \in G \text{ and } \nu\text{-almost all } x \in X.$$

It is shown in [Zi2] (see also [MZ, Theorem 3.8]) that the algebraic hull exists and is unique up to conjugation.

Theorem **A.6.** — *Let ν be an ergodic $SL(2, \mathbb{R})$-invariant measure. Then,*

(a) *The ν-algebraic hull H' of the Kontsevich-Zorich cocycle is semisimple.*
(b) *Every ν-measurable $SL(2, \mathbb{R})$-invariant irreducible subbundle of the Hodge bundle is either symplectic or isotropic.*

Remark. — The fact that the algebraic hull is semisimple for $SL(2,\mathbb{R})$-invariant measures is key to our approach.

Proof. — Suppose L is an invariant subbundle. It is enough to show that there exists an invariant complement to L. Let the symplectic complement $L^\dagger$ of L be defined as in (A.2). Then, $L^\dagger$ is also an $SL(2,\mathbb{R})$-invariant subbundle, and $K = L \cap L^\dagger$ is isotropic. By Theorem A.5, K is isometric, and $K^\perp$ is also $SL(2,\mathbb{R})$-invariant. Then,

$$L = K \oplus (L \cap K^\perp), \qquad L^\dagger = K \oplus (L^\dagger \cap K^\perp),$$

and

$$H^1(M,\mathbb{R}) = K \oplus (L \cap K^\perp) \oplus (L^\dagger \cap K^\perp)$$

Thus, $L^\dagger \cap K^\perp$ is an $SL(2,\mathbb{R})$-invariant complement to L. This proves (a). To prove (b), let L be any irreducible $SL(2,\mathbb{R})$-invariant ν-measurable irreducible subbundle of the Hodge bundle, and let $K = L \cap L^\dagger$. Since $K \subset L$ and L is irreducible, we have either $K = 0$ (so L is symplectic), or $K = L$ and so L is isotropic. The same could be done on any finite cover. □

The Forni subspace.

Definition **A.7** *(Forni subspace). — Let*

$$F(x) = \bigcap_{g \in SL(2,\mathbb{R})} g^{-1}\left(\operatorname{Ann} B^{\mathbb{R}}_{gx}\right), \tag{A.7}$$

where for $\omega \in X_0$ the quadratic form $B^{\mathbb{R}}_\omega(\cdot,\cdot)$ is as defined in [FoMZ, (2.33)].

Remark. — It is clear from the definition, that as long as its dimension remains constant, $F(x)$ varies real-analytically with x.

Theorem **A.8.** — *Suppose ν is an ergodic* $SL(2,\mathbb{R})$*-invariant measure. Then the subspaces* $F(x)$ *where x varies over the support of ν form the maximal ν-measurable* $SL(2,\mathbb{R})$*-invariant isometric subbundle of the Hodge bundle.*

Proof. — Let $F(x)$ be as defined in (A.7). Then, F is an $SL(2,\mathbb{R})$-invariant subbundle of the Hodge bundle, and the restriction of $B^{\mathbb{R}}_x$ to $F(x)$ is identically 0. Then, by [FoMZ, Lemma 1.9], F is isometric.

Now suppose M is any other ν-measurable isometric $SL(2,\mathbb{R})$-invariant subbundle of the Hodge bundle. Then by [FoMZ, Theorem 2], $M(x) \subset \operatorname{Ann} B^{\mathbb{R}}_x$. Since M is $SL(2,\mathbb{R})$-invariant, we have $M \subset F$. Thus F is maximal. □

Theorem **A.9.** — *Let* ν *be an ergodic* $SL(2,\mathbb{R})$*-invariant measure on any finite cover of* X_0.

(a) *For* ν*-almost all* $x \in X_0$*, the Forni subspace* $F(x)$ *is symplectic, and its symplectic complement* $F^{\dagger}(x)$ *coincides with its Hodge complement* $F^{\perp}(x)$.

(b) *Any* ν*-measurable* $SL(2,\mathbb{R})$*-invariant subbundle of* $F^{\perp}$ *is symplectic, and the restriction of the Kontsevich-Zorich cocycle to any invariant subbundle of* $F^{\perp}$ *has at least one non-zero Lyapunov exponent.*

Proof. — Suppose the subspace $F^{\perp}$ is not symplectic. Let $L = F^{\perp} \cap (F^{\perp})^{\dagger}$. Then L is isotropic, and therefore by Theorems A.5 and A.4, L is an $SL(2,\mathbb{R})$-invariant isometric subspace. Hence $L \subset F$ by Theorem A.8. As $L \subset F^{\perp}$ we get $L = 0$. Therefore $F^{\perp}$ is symplectic.

Let M be an irreducible subbundle of $F^{\perp}$. Then, in view of Theorem A.4 and the maximality of F, M must have at least one non-zero Lyapunov exponent. In particular, in view of Theorem A.5, M cannot be isotropic, so it must be symplectic in view of Theorem A.6(b). This proves the statement (b).

Since $F^{\perp}$ is symplectic, $(F^{\perp})^{\dagger}$ is $SL(2,\mathbb{R})$-invariant and complementary to $F^{\perp}$. Note that F is also $SL(2,\mathbb{R})$-invariant and complementary to $F^{\perp}$. In order to conclude that $(F^{\perp})^{\dagger} = F$, it is enough to show that there is a unique $SL(2,\mathbb{R})$-invariant complement to $F^{\perp}$.

Note that another complement to $F^{\perp}$ would be the graph of an equivariant linear map $A : F \to F^{\perp}$. If A is nonzero, then an invariant complement of its kernel in F exists by Theorem A.6, and it even contains an irreducible subbundle M_2. Then A induces an equivariant isomorphism between M_2 and its image, an irreducible subbundle M_1 of $F^{\perp}$. Now, to get a contradiction, it is enough to show that for any irreducible subbundles $M_1 \subset F^{\perp}$ and $M_2 \subset F$, the algebraic hulls $H'(M_i)$ of the restriction of the Kontsevich-Zorich cocycle to M_i are not isomorphic to each other. But the later statement is clear, since $H'(M_2)$ is compact and $H'(M_1)$ is not (since it has at least one non-zero Lyapunov exponent by (b)). Thus, $(F^{\perp})^{\dagger} = F$. Since we already showed that $F^{\perp}$ is symplectic, this implies that so is F, which completes the proof of (a). □

Appendix B: Entropy and the Teichmüller geodesic flow

The contents of this section are well-known, see e.g. [LY], [MaT] and also [BG]. However, for technical reasons, the statements we need do not formally follow from the results of any of the above papers. Our setting is intermediate between the homogeneous dynamics setting of [MaT] and the general C^2-diffeomorphism on a compact manifold setup of [LY], but it is closer to the former than the latter. What follows is a lightly edited but almost verbatim reproduction of [MaT, §9], adapted to the setting of Teichmüller space. It is included here primarily for the convenience of the reader. The (minor) differences between our presentation and that of [MaT] are related to the lack of uniform hyperbolicity outside of compact subsets of the space, and some notational changes due to the fact that our space is not homogeneous.

Notation. — We recall some notation from Section 2.2. Let X_0 denote the finite cover of $\mathcal{H}_1(\alpha)$ defined in Section 3 (which has no orbifold points). Let g_t denote the Teichmüller geodesic flow. In this section, ν is an ergodic g_t-invariant probability measure on X_0. Let $V(x)$ denote a subset of $H^1(M, \Sigma, \mathbb{R}^2)$. Then we denote

$$V[x] = \{ y \in X_0 \; : \; y - x \in V(x) \}.$$

This makes sense in a neighborhood of x.

Let $d^{X_0}(\cdot, \cdot)$ denote the AGY distance on X_0, defined in Section 3. Fix a point $p \in X_0$ (so p is not an orbifold point), and such that every neighborhood of p in X_0 has positive ν-measure. Fix relatively compact neighborhoods $C'(p)$ and $Q(p)$ of 0 in $W^+(p)$ and $\mathbb{R}$ respectively. Let

$$C = \bigcup_{t \in Q(p)} g_t C'[p].$$

For each $c \in \overline{C}$ choose a relatively compact neighborhood $B'(c)$ of 0 in $W^-(c)$ with diameter in the AGY distance at most $1/200$ so that the $B'(c)$ vary continuously with c. For $c \in C$, let

$$B'[c] = \{ c + v \; : \; v \in B'(c) \}, \qquad D = \bigsqcup_{c \in C} B'[c].$$

We assume that $C'(p)$, $Q(p)$ and the $B'(c)$ are sufficiently small so that D is open and contractible.

Lemma **B.1** *(Cf. [MaT, Lemma 9.1]).* — *There exists $s > 0$, $C_1 \subset C$ and for each $c \in C_1$ there exists a subset $E[c] \subset W^-[c]$ such that*

(1) $E[c] \subset B'[c]$.
(2) $E[c]$ *is open in* $W^-[c]$, *and the subset* $E \equiv \bigcup_{c \in C_1} E[c]$ *satisfies* $\nu(E) > 0$.
(3) *Let $T = g_s$ denote the time s map of the geodesic flow. Then whenever*

$$T^n E[c] \cap E \neq 0, \quad c \in C_1, \; n > 0,$$

we have $T^n E[c] \subset E$.

Proof. — Fix a compact subset $K_1 \subset X_0$, with $\nu(K_1^c) < 0.01$. Then by the Birkhoff ergodic theorem, for every $\delta > 0$ there exists $R > 0$ and a subset E_1 with $\nu(E_1) > 1 - \delta$ such that for all $x \in E_1$ and all $N > R$,

$$\left| \{ n \in [1, N] \; : \; g_n x \in K_1 \} \right| \geq (1/2)N.$$

By choosing $\delta > 0$ small enough, we may assume that $\nu(D \cap E_1) > 0$. Let

$$C_1 = \left\{ c \in C \;:\; c + v \in D \cap E_1 \text{ for some } v \in B'(c) \right\}.$$

Then there exists a compact $K \supset K_1$ such that for all $c \in C_1$ and all $x \in B'[c]$,

$$\left| \left\{ n \in [1, N] \;:\; g_n x \in K \right\} \right| \geq (1/2)N.$$

By Lemma 3.5 there exists $\alpha > 0$ such that for all $c \in C_1$ and all $x \in B'[c]$,

$$d^{X_0}(g_n x, g_n c) \leq \begin{cases} d^{X_0}(x, c) & \text{if } n \leq R \\ d^{X_0}(x, c) e^{-\alpha(n-R)} & \text{if } n > R \end{cases}$$

Therefore we may choose $s > 0$ such that if we let $T = g_s$ denote the time s map of the geodesic flow, then for all $c \in C_1$ and all $x \in B'[c]$,

$$d^{X_0}(Tx, Tc) \leq \frac{1}{10} d^{X_0}(x, c).$$

There exists $a > 0$ so that for all $c \in C_1$, $B'[c]$ contains the intersection with $W^-[c]$ of a ball in the AGY metric of radius a and centered at c. Let

$$a_0 = \frac{a}{10}$$

Let $B'_0[c] \subset W^-[c]$ denote the ball in the AGY metric of radius a_0 and centered at c. Let $E^{(0)}[c] = B'_0[c]$, and for $j > 0$ let

$$E^{(j)}[c] = E^{(j-1)}[c] \\ \cup \left\{ T^n B'_0\left[c'\right] \;:\; c' \in C_1, n > 0 \text{ and } T^n B'_0\left[c'\right] \cap E^{(j-1)}[c] \neq 0 \right\}.$$

Let

$$E[c] = \bigcup_{j \geq 0} E^{(j)}[c], \quad \text{and} \quad E = \bigcup_{c \in C_1} E[c].$$

It easily follows from the above definition that $E[c]$ has the properties (2) and (3). To show (1), it is enough to show that for each j,

$$d^{X_0}(x, c) < a/2, \quad \text{for all } x \in E^{(j)}[c]. \tag{B.1}$$

This is done by induction on j. The case $j = 0$ holds since $a_0 = a/10 < a/2$. Suppose (B.1) holds for $j - 1$, and suppose $x \in E^{(j)}[c] \setminus E^{(j-1)}[c]$. Then there exist $c_0 = c, c_1, \ldots, c_j = x$ in C_1 and non-negative integers $n_0 = 0, \ldots, n_j$ such that for all $1 \leq k \leq j$,

$$T^{n_k}\left(B'_0[c_k]\right) \cap T^{n_{k-1}}\left(B'_0[c_{k-1}]\right) \neq \emptyset. \tag{B.2}$$

Let $1 \leq k \leq j$ be such that n_k is minimal. Recall that $B'[y] \cap B'[z] = \emptyset$ if $y \neq z$, $y \in C_1$, $z \in C_1$. Therefore, in view of the inductive assumption, $n_k \geq 1$. Applying T^{-n_k} to (B.2) we get

$$\left(\bigcup_{i=1}^{k-1} T^{n_i - n_k} B'_0[c_i]\right) \cap B'_0[c_k] \neq \emptyset, \quad \text{and} \quad \left(\bigcup_{i=k+1}^{j} T^{n_i - n_k} B'_0[c_i]\right) \cap B'_0[c_k] \neq \emptyset.$$

Therefore, in view of (B.2), and the definition of the sets $E^{(j)}[c]$,

$$\left(\bigcup_{i=1}^{k} T^{n_i - n_k} B'_0[c_i]\right) \subset E^{(k-1)}[c_k], \quad \text{and} \quad \left(\bigcup_{i=k}^{j} T^{n_i - n_k} B'_0[c_i]\right) \subset E^{(j-k)}[c_k]$$

By the induction hypothesis, $\operatorname{diam}(E^{(k-1)}[c_k]) < a/2$, and $\operatorname{diam}(E^{(j-k)}[c_k]) < a/2$. Therefore,

$$\operatorname{diam}\left(\bigcup_{i=1}^{j} T^{n_i - n_k} B'_0[c_i]\right) \leq a.$$

Then, applying T^{n_k} we get,

$$\operatorname{diam}\left(\bigcup_{i=1}^{j} T^{n_i} B'_0[c_i]\right) \leq \frac{a}{10}$$

Since $\operatorname{diam}(B'_0[c]) \leq a/10$, we get

$$\operatorname{diam}\left(\bigcup_{i=0}^{j} T^{n_i} B'_0[c_i]\right) \leq \operatorname{diam}\big(B'_0[c_0]\big) + \operatorname{diam}\left(\bigcup_{i=1}^{j} T^{n_i} B'_0[c_i]\right)$$
$$\leq \frac{a}{10} + \frac{a}{10} < \frac{a}{2}.$$

But the set on the left-hand-side of the above equation contains both $c = c_0$ and $x = c_j$. Therefore $d^{X_0}(c, x) < a/2$, proving (B.1). Thus (1) holds. □

Lemma **B.2** *(Mañé). — Let* E *be a measurable subset of* X_0*, with* $\nu(E) > 0$*. If* ν *is a compactly supported measure on* E *and* $q : E \to (0, 1)$ *is such that* $\log q$ *is* ν*-integrable, then there exists a countable partition* $\mathcal{P}$ *of* E *with entropy* $H(\mathcal{P}) < \infty$ *such that, if* $\mathcal{P}(x)$ *denotes the atom of* $\mathcal{P}$ *containing* x*, then* $\operatorname{diam} \mathcal{P}(x) < q(x)$.

Proof. — See [M1] or [M2, Lemma 13.3]. □

Let $V(x)$ be a system of real-algebraic subsets of $W^-(x)$.

Definition **B.3.** — *The system* $V(x)$ *is admissible if it is* T*-equivariant and also for almost all* $x \in X_0$*,* x *is a smooth point of* $V[x]$.

Definition **B.4.** — *We say that a measurable partition* ξ *of the measure space* (X_0, ν) *is* subordinate *to an admissible system of real-algebraic subsets* $V(x) \subset W^-(x)$ *if for almost all (with respect to* ν*)* $x \in X_0$*, we have*

(a) $\xi[x] \subset V[x]$ *where* $\xi[x]$ *denotes, as usual, the element of* ξ *containing* x.
(b) $\xi[x]$ *is relatively compact in* $V[x]$.
(c) $\xi[x]$ *contains a neighborhood of* x *in* $V[x]$.

Let η and η' be measurable partitions of (X_0, ν). We write $\eta \leq \eta'$ if $\eta[x] \supset \eta'[x]$ for almost all (with respect to ν) $x \in X_0$. We define a partition $T\eta$ by $(T\eta)[x] = T(\eta[T^{-1}(x)])$.

Proposition **B.5.** — *Assume that* ν *is* T*-ergodic (where* T *is as in Lemma B.1(3)). Then there exists a measurable partition* η *of the measure space* (X_0, ν) *with the following properties:*

(i) η *is subordinate to* W^-.
(ii) η *is* T*-invariant, i.e.* $\eta \leq T\eta$.
(iii) *The mean conditional entropy* $H(T\eta \mid \eta)$ *is equal to the entropy* $h(T, \nu)$ *of the automorphism* $x \to Tx$ *of the measure space* (X_0, ν).

Proof. — Let $E[c]$ and E be as in Lemma B.1. Denote by $\pi : E \to C_1$ the natural projection ($\pi(x) = c$ if $x \in E[c]$). We set $\eta[x] = E(\pi(x))$ for every $x \in E$.

We claim that it is enough to find a countable measurable partition ξ of (X_0, ν) such that $H(\xi) < \infty$ and $\eta[x] = \xi^-[x]$ for almost all $x \in E$ where $\xi^- = \bigvee_{n=0}^{\infty} T^{-n}\xi$ is the product of the partitions $T^{-n}\xi$, $0 \leq n < \infty$.

Indeed, suppose the claim holds. Then it is clear that η is T-invariant. The set of $x \in X_0$ for which properties (a) and (b) (resp. (c)) in the definition of a subordinate partition are satisfied is T^{-1}-invariant (resp. T-invariant) and contains E. But $\nu(E) > 0$ and ν is T-ergodic. Therefore, η is subordinate to W^-. To check the property (iii) it is enough to show that the partition $\xi_s = \bigvee_{k=-\infty}^{\infty} T^k\xi$ is the partition into points, see [R, §9], or [KH, §4.3]. By [Fo] or [ABEM, Theorem 8.12] $\xi_s(x) = \{x\}$ if $T^{-n}x \in E$ for infinitely many n. (Recall that by the construction of E, any such geodesic will spend at least half the time in the compact set K.) But $\nu(E) > 0$ and ν is T-ergodic. Hence $\xi_s[x] = \{x\}$ for almost all x, which completes the proof of the claim.

Let us construct the desired partition ξ. For $x \in E$, let $n(x)$ be the smallest positive integer n such that $T^n x \in E$. We have the classical Kac formula [Ka]

$$\int_E n(x)\,d\nu(x) = 1. \tag{B.3}$$

Define a probability measure ν' on C_1 by

$$\nu'(F) = \frac{\nu(\pi^{-1}(F))}{\nu(E)}, \quad F \subset C_1. \tag{B.4}$$

Property (3) of the family $\{E[c] \ : \ c \in C_1\}$ implies that $n(x)$ is constant on every $E[c]$, $c \in C_1$. Therefore, in view of (B.3) and (B.4),

$$\int_{C_1} n(c) d\nu'(c) < \infty.$$

By Lemma 3.6, there exists $\kappa > 1$ such that for all $x, y \in X_0$,

$$d^{X_0}(Tx, Ty) \leq \kappa d^{X_0}(x, y).$$

Since the function $n(c)$ is ν'-integrable, one can find a positive function $q(c) < \kappa^{-2n(c)}$, $c \in C_1$ such that $\log q$ is ν'-integrable, and the ν'-essential infimum $\operatorname{ess\,inf}_{c \in C_1} q(c)$ is 0.

After replacing, if necessary, $C'(p)$, $Q(p)$ and the $B'(c)$ for $c \in \overline{C}$ by smaller subsets we can find $\epsilon > 0$ such that the minimum distance between lifts of E is at most $\epsilon/10$ and also

(a) $d^{X_0}(x, y) < 2d(\pi(x), \pi(y))$ whenever $x, y \in E$ and $d^{X_0}(x, y) < \epsilon$, and
(b) if $x, y \in C_1$ then $d^{X_0}(x, y) < \epsilon$.

Since the function $\log q(c)$ is ν'-integrable, there exists a countable measurable partition $\mathcal{P}$ of C_1 such that $H(\mathcal{P}) < \infty$ and $\operatorname{diam} \mathcal{P}(x) < \frac{\epsilon}{2} q(x)$ for almost all $x \in C_1$ (see Lemma B.2). After possibly replacing $\mathcal{P}$ by a countable refinement, we may assume that the function $x \to n(x)$ is constant on the atoms of $\mathcal{P}$. Now we define a countable measurable partition ξ of X_0 by

$$\xi(x) = \begin{cases} \pi^{-1}(\mathcal{P}(\pi(x))) & \text{if } x \in E \\ X_0 \setminus E & \text{if } x \notin E. \end{cases}$$

Since $H(\mathcal{P}) < \infty$ we get using (B.4) that $H(\xi) < \infty$. It remains to show that $\xi^-[x] = \eta[x]$ for almost all $x \in E$. It follows from the property (3) of the family $\{E[c]\}$ that $\eta[z] \subset \xi^-[z]$. Let x and y be elements in E with $\xi^-[x] = \xi^-[y]$. Since $\eta[z] \subset \xi[z]$, we can assume that $x, y \in C_1$. Then $d^{X_0}(x, y) < \epsilon$. Set $x_1 = x, y_1 = y$ and define by induction

$$x_{k+1} = T^{n(x_k)} x_k, \qquad y_{k+1} = T^{n(y_k)} y_k.$$

Then, the sequence $\{x_k\}_{k \in \mathbb{N}}$ (resp. $\{y_k\}_{k \in \mathbb{N}}$) is the part of the T-orbit of x (resp. T-orbit of y) which lies in E.

Let $\tilde{x}_1, \tilde{y}_1$ be the lifts of $x_1 = x$ and $y_1 = y$ to Teichmüller space, and let $\tilde{x}_k, \tilde{y}_k$ be defined inductively by

$$\tilde{x}_{k+1} = T^{n(x_k)} \tilde{x}_k, \qquad \tilde{y}_{k+1} = T^{n(y_k)} \tilde{y}_k.$$

Then $\tilde{x}_k$ and $\tilde{y}_k$ are lifts of x_k and y_k respectively. We now claim that for all $k \geq 0$,

$$d^{X_0}(\tilde{x}_k, \tilde{y}_k) < \epsilon q\big(\pi(x_k)\big). \tag{B.5}$$

If $k = 1$, the inequality (B.5) is true because $\operatorname{diam} \mathcal{P}(x) < \frac{\epsilon}{2} q(\pi(x))$ and $\mathcal{P}(x) = \mathcal{P}(y)$. Assume that (B.5) is proved for k. Then

$$\begin{aligned} d^{X_0}(\tilde{x}_{k+1}, \tilde{y}_{k+1}) &= d^{X_0}\big(T^{n(x_k)}\tilde{x}_k, T^{n(x_k)}\tilde{y}_k\big) \leq \kappa^{n(x_k)} d^{X_0}(\tilde{x}_k, \tilde{y}_k) \\ &\leq \kappa^{n(x_k)} \epsilon q\big(\pi(x_k)\big) \leq \epsilon. \end{aligned}$$

Then since x_{k+1} and y_{k+1} belong to the same element of the partition ξ (because $\xi^-[x] = \xi^-[y]$) and $\operatorname{diam}(\mathcal{P}(x_{k+1})) \leq \frac{\epsilon}{2} q\pi(x_{k+1})$, we get from condition (b) in the definition of $\epsilon > 0$ that (B.5) is true for $k+1$.

Since the measure ν is T-ergodic and $\operatorname{ess\,inf} q(c) = 0$ we may assume that $\liminf_{k\to\infty} q(\pi(x_k)) = 0$ (since this holds for almost all $x \in E$). Then (B.5) implies that

$$\liminf_{k\to\infty} d^{X_0}(\tilde{x}_k, \tilde{y}_k) = 0.$$

By the definition of $\tilde{x}_k, \tilde{y}_k$, there exists a sequence $m_k \to +\infty$ such that $\tilde{x}_k = T^{m_k}\tilde{x}$, $\tilde{y}_k = T^{m_k}\tilde{y}$. Thus,

$$d^{X_0}\big(T^{m_k}\tilde{x}, T^{m_k}\tilde{y}\big) = 0.$$

But, by construction $\tilde{x}$ and $\tilde{y}$ are on the same leaf of W^{0+}. This contradicts the non-contraction property of the Hodge distance [ABEM, Theorem 8.2], unless $\tilde{x} = \tilde{y}$. Thus we must have $x = y$. ☐

Lemma **B.6** *(See [LS, Proposition 2.2]). — Let* T *be an automorphism of a measure space* (X_0, ν), $\nu(X_0) < \infty$, *and let* f *be a positive finite measurable function defined on* X_0 *such that*

$$\log^- \frac{f \circ T}{f} \in L^1(X, \nu), \quad \textit{where } \log^-(a) = \min(\log a, 0).$$

Then

$$\int_{X_0} \log \frac{f \circ T}{f} \, d\nu = 0.$$

Suppose $V^-(x) \subset W^-(x)$ is an admissible T-equivariant family of real-algebraic subsets. Let $(T_{\mathbb{R}}V^-)(x) \subset W^-(x)$ denote the tangent space to smooth manifold $V^-[x]$ at x. (Recall that since V^- is admissible, for almost every x, $V^-[x]$ is smooth at x.)

Definition **B.7** *(Margulis property). — Suppose* $V^-(x) \subset W^-(x)$ *is an admissible* T*-equivariant family of real-algebraic subsets. Let* $\tau = \tau(x)$ *be a measure on each* $V^-[x]$. *We say that* τ *has the* Margulis Property *if for almost all* x, $\tau(x)$ *is in the Lebesgue measure class on* $V^-[x]$, *and also* $T_*\tau(x)$ *agrees with* $\tau(Tx)$ *up to normalization. (In other words the Radon-Nykodym derivative* $\frac{dT_*\tau(x)}{d\tau(Tx)}$ *is locally constant along* $V^-[x]$.*)*

Proposition **B.8.** — *Let* $T = g_s$ *as in Lemma B.1(iii). Let* $V^-(x) \subset W^-(x)$ *be a* T*-equivariant family of real-algebraic subsets. Suppose there exists a* T*-invariant measurable partition* η *of* (X_0, ν) *subordinate to* V^-*. Then the following hold:*

(a) *We have*

$$H(T\eta \mid \eta) \le s\Delta\big(V^-\big),$$

where $H(T\eta \mid \eta)$ *is the mean conditional entropy, and*

$$\Delta\big(V^-\big) = \sum_{i \in I(V)} (1 - \lambda_i),$$

where $I(V)$ *are the Lyapunov subspaces in* $T_{\mathbb{R}}V$ *(counted with multiplicity), and* λ_i *are the corresponding Lyapunov exponents of the Kontsevich-Zorich cocycle.*

(b) *Suppose that for almost all* x *there exists a measure* $\tau = \tau(x)$ *on each* $V^-[x]$ *with the Margulis property. Then*

(b1) *If the conditional measures of* ν *along* $V^-[x]$ *agree with* $\tau(x)$ *(up to normalization), then*

$$H(T\eta \mid \eta) = s\Delta\big(V^-\big)$$

(b2) *If* $H(T\eta \mid \eta) = s\Delta(V^-)$ *then the conditional measures of* ν *along* $V^-[x]$ *agree with* $\tau(x)$ *(up to normalization).*

Proof. — Since $\eta \le T\eta$ for almost all $x \in X_0$ we have a partition η_x of $\eta[x]$ such that $\eta_x[y] = (T\eta)[y]$ for almost all $y \in \eta[x]$. Let $\tau(x)$ be a measure on $V^-(x)$ in the Lebesgue measure class. (To simplify notation, we will sometimes denote $\tau(x)$ simply by τ.) (Here we pick some normalization of the Lebesgue measure on the connected components of the intersections of the leaves of V^- with a fixed fundamental domain.) Since $\eta[x] \subset V^-[x]$, τ induces a measure on $\eta[x]$ which we will denote also by τ. Let $J(x)$ denote the Jacobian of the restriction of the map T to $V^-[x]$ at x (with respect to the Lebesgue measure class measures τ on $V^-[x]$ and $V^-[Tx]$). Then, by the Osceledets multiplicative ergodic theorem, for almost all $x \in X_0$,

$$-s\Delta\big(V^-\big) = \lim_{N\to\infty} \frac{1}{N} \log \frac{d(T^{-N}\tau)(x)}{d\tau(x)} = -\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \log J\big(T^{-n}x\big).$$

Integrating both sides over X_0, we get

$$-\int_{X_0} \log J(x)\, d\nu(x) = s\Delta\big(V^-\big). \qquad \textbf{(B.6)}$$

Put $L(x) = \tau(\eta[x])$ and $\tau_x = \tau/L(x)$, $x \in X_0$. Note that on $\eta[x]$ we have a conditional probability measure ν_x induced by ν. Put $p(x) = \tau_x(\eta_x[x])$ and $r(x) = \nu_x(\eta_x[x])$.

Let

$$\eta' = \eta \vee T\eta \vee \cdots \vee T^k\eta. \tag{B.7}$$

Then, η' is also T-invariant, and $H(T\eta' \mid \eta') = H(T\eta \mid \eta)$. Thus, we can replace η by η'.

Suppose $\epsilon > 0$ is given. Then, we can choose k large enough in (B.7) so that (after replacing η by η'), on a set of measure at least $(1-\epsilon)$, we have

$$(1-\epsilon) \le \frac{p(x)L(x)}{J(T^{-1}x)L(T^{-1}x)} \le (1+\epsilon) \tag{B.8}$$

From its definition, $p(x) \le 1$. Also

$$-\int_{X_0} \log r(x) d\nu(x) = H(T\eta \mid \eta). \tag{B.9}$$

Let $Y_i(x)$, $1 \le i < \infty$ denote the elements of the countable partition η_x of $\eta[x]$. Then we have

$$\int_{\eta(x)} \log p(y) d\nu_x(y) - \int_{\eta(x)} \log r(y) d\nu_x(y) = \sum_{i=1}^{\infty} \log \frac{\tau_x(Y_i(x))}{\nu_x(Y_i(x))} \nu_x\big(Y_i(x)\big). \tag{B.10}$$

We have that

$$\sum_{i=1}^{\infty} \tau_x\big(Y_i(x)\big) \le 1, \tag{B.11}$$

and

$$\sum_{i=1}^{\infty} \nu_x\big(Y_i(x)\big) = 1. \tag{B.12}$$

(In (B.11), we can have strict inequality because apriori it is possible that the measure τ_x of $\eta[x] \setminus \bigcup_{i=1}^{\infty} Y_i(x)$ is positive.) From (B.10), (B.11) and (B.12), using the convexity of log we get that

$$\int_{\eta(x)} \log p(y) d\nu_x(y) \le \int_{\eta(x)} \log r(y) d\nu_x(y),$$

and the equality holds if and only if $p(y) = r(y)$ i.e. $\tau_x(\eta_x[y]) = \nu_x(\eta_x[y])$ for all $y \in \eta[x]$. Now using integration over the quotient space $(X_0, \nu)/\eta$ of the measure space (X_0, ν) by η, we get from (B.9) that

$$H(T\eta \mid \eta) \le -\int_{X_0} \log p(x) d\nu(x), \tag{B.13}$$

and the equality holds if and only if $\tau_x((T\eta)[x]) = \nu_x((T\eta)[x])$ for almost all $x \in X_0$.

In view of (B.8) and the fact that $p(x) \le 1$,

$$-\int_{X_0} \log p(x) d\nu(x) \le 2\epsilon - \int_{X_0} \log J(x) d\nu(x) + \int_{X_0} \log_{-}\big(L(T^{-1}x)/L(x)\big) d\nu(x).$$

The last term vanishes by Lemma B.6. Since $\epsilon > 0$ is arbitrary, we have, by (B.13) and (B.6) that (a) holds.

Now suppose that τ is as in (b). Then since $\eta_x[x] = T(\eta[T^{-1}x])$ one easily sees that $p(x) = J(T^{-1}x)L(T^{-1}x)/L(x)$. Therefore, by (B.6) and Lemma B.6,

$$-\int_{X_0} \log p(x) d\nu(x) = s\Delta\big(V^{-}\big).$$

If the conditional measures of ν along V^- coincide with τ, then $p(x) = r(x)$ and therefore equality in (B.13) holds. This proves (b1). Conversely, assume that $H(T\eta \mid \eta) = s\Delta(V^-)$. Then $H(T^k\eta \mid \eta) = ks\Delta(V^-)$ for every $k > 0$. Using the same argument as above and replacing T by T^k, we get that $\tau_x((T^k\eta)[x]) = \nu_x((T^k\eta)[x])$ for any $k > 0$ and almost all $x \in X_0$. On the other hand since η is subordinate to V^- and T is contracting on V^-, we have that $\bigvee_{k=1}^{\infty} T^k\eta$ is the partition into points. Hence the conditional measures of ν along V agree with τ. This proves (b2). □

Theorem **B.9.** — *Let* $T = g_s$ *denote the time s map of the geodesic flow. Assume that* T *acts ergodically on* (X_0, ν). *Let* $V^-(x)$ *be an admissible* T*-equivariant system of real-algebraic subsets of* $W^-(x)$, *and let* $\Delta(V^-)$ *be as in Proposition B.8.*

(i) *Suppose* V^- *has a system of measures* τ *with the Margulis property, and suppose that for almost all x, the conditional measures of* ν *along* $V^-[x]$ *agree with* $\tau(x)$ *up to normalization. Then,* $h(T, \nu) \ge s\Delta(V^-)$.

(ii) *Assume that there exists a subset* $\Psi \subset X_0$ *with* ν*-measure* 1 *such that* $\Psi \cap W^-[x] \subset V^-[x]$ *for every* $x \in \Psi$. *Then* $h(T, \nu) \le s\Delta(V^-)$.

(iii) *Assume that there exists a subset* $\Psi \subset X_0$ *with* ν*-measure* 1 *such that* $\Psi \cap W^-[x] \subset V^-[x]$ *for every* $x \in \Psi$. *Also assume that* V^- *has a system of measures* τ *with the Margulis property, and that* $h(T, \nu) = s\Delta(V^-)$. *Then, for almost all x, the conditional measures of* ν *along* $V^-[x]$ *agree with* $\tau(x)$ *up to normalization.*

Proof. — According to Proposition B.5, there exists a measurable T-invariant partition η of (X_0, ν), subordinate to W^-, such that $H(T\eta \mid \eta) = h(T, \nu)$. By Lemma 3.2, we may assume that the affine exponential map $W^-(x) \to W^-[x]$ is one-to-one and onto, and thus $W^-[x]$ has an affine structure. Set $\eta'(x) = V^-[x] \cap \eta[x]$.

Suppose the assumptions of (i) hold. Then,

$$h(T, \nu) \ge H\big(T\eta' \mid \eta'\big). \tag{B.14}$$

By Proposition B.8(b1), $H(T\eta' \mid \eta') = s\Delta(V^-)$. This, together with (B.14) implies the conclusion of (i).

Now suppose the assumptions of (ii) or (iii) hold. Then η and η' coincide on Ψ, i.e. $\eta[x] \cap \Psi = \eta'[x] \cap \Psi$. Hence $H(T\eta \mid \eta) = H(T\eta' \mid \eta')$. By Proposition B.5(iii), $h(T, \nu) = H(T\eta \mid \eta)$. Using Proposition B.8(a) we obtain (ii), and using Proposition B.8(b2) we obtain (iii). □

Appendix C: Semisimplicity of the Lyapunov spectrum

In this section we work with a bit more generality than we need. Let X be a space on which $SL(2, \mathbb{R})$ acts. Let μ be a compactly supported probability measure on $SL(2, \mathbb{R})$ and let ν be an ergodic μ-stationary probability measure on X. Let L be a finite dimensional real vector space, and suppose $A : SL(2, \mathbb{R}) \times X \to SL(L)$ is a cocycle, such that for any $g \in SL(2, \mathbb{R})$, the map $x \to \log^+ \|A(g, x)\|$ is in $L^1(X, \nu)$. Let H' be the algebraic hull of the cocycle A (see Section A.2 for the definition). We may assume that a basis at every point is chosen so that for all $g \in SL(2, \mathbb{R})$ and all $x \in X$, $A(g, x) \in H'$.

Definition **C.1.** — *We say that a measurable map* $W : X \to L$ *is an* invariant system of subspaces *for* $A(\cdot, \cdot)$ *if for* μ*-a.e.* $g \in SL(2, \mathbb{R})$ *and* ν*-a.e.* $x \in X$, $A(g, x)W(x) = W(gx)$.

Definition **C.2** *(Strongly irreducible).* — *We say that* A *is* strongly irreducible *if on any measurable finite cover of* X *there is no nontrivial proper invariant system of subspaces for* $A(\cdot, \cdot)$.

Remark. — If a cocycle is strongly irreducible, then its algebraic hull is a simple Lie group.

Let B be the space of (one-sided) infinite sequences of elements of $SL(2, \mathbb{R})$. We define the measure β on B to be $\mu \times \mu \cdots$. Let $\hat{T} : B \times X \to B \times X$ be the forward shift, with $\beta \times \nu$ as the invariant measure. We denote elements of B by the letter a (following the convention that these refer to "future" trajectories). If we write $a = (a_1, a_2, \dots)$ then

$$\hat{T}(a, x) = (Ta, a_1 x)$$

(and we use the letter T to denote the shift $T(a_1, a_2, \dots) = (a_2, a_3, \dots)$). By the Osceledets multiplicative ergodic theorem, for $\beta \times \nu$ almost every $(a, x) \in B \times X$ there exists a Lyapunov flag

$$\{0\} = \mathcal{V}_{\geq k}(a, x) \subset \mathcal{V}_{\geq k-1}(a, x) \subset \cdots \subset \mathcal{V}_{\geq 0}(a, x) = L. \tag{C.1}$$

Definition **C.3.** — *The map* $\hat{T} : B \times X \to B \times X$ *has* semisimple Lyapunov spectrum *if (after passing to a measurable finite cover), the algebraic hull of the cocycle* $\mathbb{Z} \times (B \times X) \to SL(L)$ *given by*

$$(n, a, x) \to A(a_n \dots a_1, x)$$

is block-conformal, see Section 4.3. In other words, $\hat{T}$ *has semisimple Lyapunov spectrum if all the off-diagonal blocks labelled* $*$ *in* (4.4) *are* 0.

In Appendix C our aim is to prove the following general fact:

Theorem **C.4.** — *Suppose* A *is strongly irreducible and* ν *is* μ*-invariant. Then* $\hat{T}$ *has semisimple Lyapunov spectrum. Furthermore, the restriction of* $\hat{T}$ *to the top Lyapunov subspace* $\mathcal{V}_{\geq 1}/\mathcal{V}_{>1}$ *consists of a single conformal block, i.e. for* $\beta \times \nu$ *almost every* (a, x) *there exists an inner product* $\langle \cdot, \cdot \rangle_{a,x}$ *on* $\mathcal{V}_{\geq 1}(a, x)/\mathcal{V}_{>1}(a, x)$ *and a function* $\lambda : B \times X \to \mathbb{R}$ *such that for all* $u, v \in \mathcal{V}_{\geq 1}(a, x)/\mathcal{V}_{>1}(a, x)$,

$$\langle a_1 u, a_1 v \rangle_{(Ta, ax)} = \lambda(a_1, x) \langle u, v \rangle_{a,x}. \tag{C.2}$$

If the algebraic hull H' *is all of* $SL(L)$*, then all the Lyapunov subspaces consist of a single conformal block, i.e. for all* $1 \leq i \leq k - 1$ *one can define an inner product* $\langle \cdot, \cdot \rangle_{a,x}$ *on* $\mathcal{V}_{\geq i}(a, x)/\mathcal{V}_{>i}(a, x)$ *so that* (C.2) *holds for some function* $\lambda = \lambda_i$.

The backwards shift. — We will actually use the analogue of Theorem C.4 for the backwards shift. Let $T : B \times X \to B \times X$ be the (backward) shift as in Section 14, with β^X as defined in [BQ, Lemma 3.1] as the invariant measure. By the Osceledets multiplicative ergodic theorem, for β^X almost every $(b, x) \in B \times X$ there exists a Lyapunov flag

$$\{0\} = \mathcal{V}_{\leq 0}(b, x) \subset \mathcal{V}_{\leq 1}(b, x) \subset \mathcal{V}_{\leq 2}(b, x) \subset \mathcal{V}_{\leq k}(b, x) = L. \tag{C.3}$$

We need the following:

Theorem **C.5.** — *Suppose* A *is strongly irreducible and* ν *is* μ*-invariant. Then* T *has semisimple Lyapunov spectrum. Furthermore, the restriction of* T *to the top Lyapunov subspace* $\mathcal{V}_{\leq 1}$ *consists of a single conformal block, i.e. for* β^X *almost every* (b, x) *there exists an inner product* $\langle \cdot, \cdot \rangle_{b,x}$ *on* $\mathcal{V}_{\leq 1}(b, x)$ *and a function* $\lambda : B \times X \to \mathbb{R}$ *such that for all* $u, v \in \mathcal{V}_{\leq 1}(b, x)$,

$$\left\langle b_0^{-1} u, b_0^{-1} v \right\rangle_{(Tb, b_0^{-1} x)} = \lambda(b_0, x) \langle u, v \rangle_{b,x}. \tag{C.4}$$

If the algebraic hull H' *is all of* $SL(L)$*, then all the Lyapunov subspaces consist of a single conformal block, i.e. for all* $1 \leq i \leq k - 1$ *one can define an inner product* $\langle \cdot, \cdot \rangle_{b,x}$ *on* $\mathcal{V}_{\leq i}(b, x)/\mathcal{V}_{<i}(b, x)$ *so that* (C.4) *holds for some function* $\lambda = \lambda_i$.

The two-sided shift. — As in Section 14, let $\tilde{B}$ be the space of bi-infinite sequences of elements of $SL(2, \mathbb{R})$, and we consider the two-sided random walk as a shift map on $\tilde{B} \times X$. We abuse notation by using the same letter T both for the backwards shift and the bi-infinite shift. We denote a point in $\tilde{B}$ by $a \vee b$ where a denotes the "future" of the trajectory and b denotes the "past". Let $\tilde{\beta}^X$ denote the T-invariant measure on $\tilde{B} \times X$ which projects to the measure $\beta \times \nu$ on the future trajectories, and to the measure β^X on the past trajectories. Then, at $\tilde{\beta}^X$ almost all points $(a \vee b, x)$ we have both the flags

(C.1) and (C.3). The two flags are generically in general position (see e.g. [GM, Lemma 1.5]) and thus we can intersect the flags to define the (shift-invariant) Lyapunov subspaces $\mathcal{V}_i(a \vee b, x)$ so that

$$\mathcal{V}_{\leq i}(b, x) = \bigoplus_{j=1}^{i} \mathcal{V}_j(a \vee b, x), \qquad \mathcal{V}_{\geq i}(a, x) = \bigoplus_{j=i}^{m} \mathcal{V}_j(a \vee b, x).$$

Then

$$\mathcal{V}_{\leq i}(b, x)/\mathcal{V}_{<i}(b, x) \cong \mathcal{V}_i(a \vee b, x) \cong \mathcal{V}_{\geq i}(a, x)/\mathcal{V}_{>i}(a, x). \tag{C.5}$$

We will prove the following:

Theorem **C.6.** — *Suppose* A *is strongly irreducible and* ν *is* μ*-invariant. Then* T *has semisimple Lyapunov spectrum. Furthermore, the restriction of* T *to the top Lyapunov subspace* $\mathcal{V}_{\leq 1}$ *consists of a single conformal block, i.e. for* $\tilde{\beta}^{X}$ *almost every* $(a \vee b, x)$ *there exists an inner product* $\langle \cdot, \cdot \rangle_{a \vee b, x}$ *on* $\mathcal{V}_1(a \vee b, x)$ *and a function* $\lambda : \tilde{B} \times X \to \mathbb{R}$ *such that for all* $u, v \in \mathcal{V}_1(a \vee b, x)$,

$$\langle a_1 u, a_1 v \rangle_{(T(a \vee b), a_1 x)} = \lambda(a \vee b, x) \langle u, v \rangle_{a \vee b, x}. \tag{C.6}$$

If the algebraic hull H' *is all of* SL(L)*, then all the Lyapunov subspaces consist of a single conformal block, i.e. for all* $1 \leq i \leq k-1$ *one can define an inner product* $\langle \cdot, \cdot \rangle_{a \vee b, x}$ *on* $\mathcal{V}_i(b, x)$ *so that* (C.6) *holds for some function* $\lambda = \lambda_i$.

Remark **1.** — The proof of Theorems C.4–C.6 we give is essentially taken from [GM], and is originally from [GR1] and [GR2].

For most of the proof, we assume only that ν is μ-stationary (and not necessarily μ-invariant). The exceptions are Lemma C.10 and Claim C.14.

We follow [GM] and present the proof of Theorems C.4–C.6 for the easier to read case where the algebraic hull H' of the cocycle A is all of SL(L). The general case of semisimple H' is treated in [EMat].

Remark **2.** — It is possible to define semisimplicity of the Lyapunov spectrum in the context of the action of $g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \subset SL(2, \mathbb{R})$ (instead of the random walk). Then the analogue of Theorems C.4–C.6 remains true; the proof would use an argument similar to the proof of Proposition 4.12. Since we will not use this statement we will omit the details.

C.3 An ergodic lemma

We recall the following well-known lemma:

Lemma **C.7.** — *Let* $T : \Omega \to \Omega$ *be a transformation preserving a probability measure* β. *Let* $F : \Omega \to \mathbb{R}$ *be an* L^1 *function. Suppose that for* β*-a.e.* $x \in \Omega$,

$$\liminf \sum_{i=1}^{n} F\big(T^i x\big) = +\infty.$$

Then $\int_\Omega F d\beta > 0$.

Proof. — This lemma is due to Atkinson [At] and Kesten [Ke]. See also [GM, Lemma 5.3], and the references quoted there. □

We will need the following variant:

Lemma **C.8.** — *Let* $T : \Omega \to \Omega$ *be a transformation preserving an ergodic probability measure* β. *Let* $F : \Omega \to \mathbb{R}$ *be an* L^1 *function. Suppose there exists* $K' \subset \Omega$ *with* $\beta(K') > 0$ *such that for* β*-a.e.* $x \in \Omega$,

$$\liminf \left\{ \sum_{i=1}^{n} F\big(T^i x\big) \;:\; T^n x \in K' \right\} = +\infty. \tag{C.7}$$

Then $\int_\Omega F d\beta > 0$.

Proof. — After passing to the natural extension, we may assume that T is invertible. We can choose a subset $K \subset K'$ with $\beta(K) > 0$, and $C > 0$ such that for all $x \in K$, we have

$$|F(x)| < C.$$

Since $K \subset K'$, (C.7) holds with K' replaced by K.

Let $A_{-1} = \{x \;:\; x \notin K\}$, $A_0 = \{x \;:\; x \in K, Tx \in K\}$, and for $n \geq 0$,

$$A_{n+1} = \big\{x \;:\; x \in K,\; Tx \notin K, \dots,\; T^n x \notin K,\; T^{n+1} x \in K\big\}.$$

Also let $A = \bigsqcup_{n=-1}^{\infty} A_n$. Note that by the ergodicity of T, for almost every $x \in \Omega$,

$$\Big|\big\{i \;:\; i \geq 0, T^i(x) \in K\big\}\Big| = \infty \quad (*).$$

Define $G : \Omega \to \mathbb{R}$ defined on A (which has full measure) by

- $G(x) = 0$ if $x \in A_{-1}$.
- $G(x) = F(x)$ if $x \in A_0$.
- $G(x) = F(x) + F(Tx) + \cdots + F(T^n x)$ if $x \in A_{n+1}$.

We now claim the following hold:

(1) For almost every $x \in \Omega$ we have

$$\lim_{n\to\infty} \mathrm{G}(x) + \mathrm{G}(\mathrm{T}x) + \cdots + \mathrm{G}\big(\mathrm{T}^n x\big) = \infty. \tag{C.8}$$

(2) $\int_\Omega |\mathrm{G}| d\beta \leq \int_\Omega |\mathrm{F}| d\beta < \infty$.

(3) $\int_\Omega \mathrm{G}(x) d\beta(x) = \int_\Omega \mathrm{F}(x) d\beta(x)$.

Proof of (1). — Note that almost every $x \in \Omega$ satisfies (C.7) (with K′ replaced by K). Also, we have,

$$\mathrm{G}(x) + \mathrm{G}(\mathrm{T}x) + \cdots + \mathrm{G}\big(\mathrm{T}^n x\big) = \sum_{i=m_0}^{m-1} \mathrm{F}\big(\mathrm{T}^i x\big),$$

where $m_0 = \inf\{k \,:\, \mathrm{T}^k x \in \mathrm{K}\}$, and $m = \inf\{k \,:\, k \geq n, \mathrm{T}^k x \in \mathrm{K}\}$. Thus,

$$\sum_{j=0}^{n} \mathrm{G}\big(\mathrm{T}^j x\big) = \sum_{i=1}^{m} \mathrm{F}\big(\mathrm{T}^i x\big) - \sum_{i=0}^{m_0-1} \mathrm{F}\big(\mathrm{T}^i x\big) - \mathrm{F}\big(\mathrm{T}^m x\big).$$

Since m_0 is independent of n, $\mathrm{T}^m x \in \mathrm{K}$ and for every $x \in \mathrm{K}$, we have $|\mathrm{F}(x)| < \mathrm{C}$, Equation (C.7) implies (C.8). □

Proof of (3) assuming (2). — By the definition of G we can use the dominated convergence theorem, and get that

$$\int_\Omega \mathrm{G} d\beta = \int_\mathrm{K} \mathrm{F} d\beta + \sum_{i=1}^{\infty} \int_{\mathrm{A}^i} \mathrm{F}\big(\mathrm{T}^i x\big) d\beta(x)$$

where $\mathrm{A}^i = \bigcup_{j \geq i} \mathrm{A}_i$. Then

$$\mathrm{T}^i \mathrm{A}^i = \mathrm{T}^i \mathrm{K} - \big(\mathrm{K} \cup \cdots \mathrm{T}^{i-1} \mathrm{K}\big).$$

Also $\mathrm{K} \cup \bigcup_{i=1}^{\infty} \mathrm{T}^i \mathrm{A}^i$ has full measure in Ω, and for $i \neq j$, $\mathrm{T}^i \mathrm{A}^i \cap \mathrm{T}^j \mathrm{A}^j$ and $\mathrm{K} \cap \mathrm{T}^i \mathrm{A}^i$ have measure zero. Note that $\mathrm{A}^i = \mathrm{T}^{-i}(\mathrm{T}^i \mathrm{A}^i)$. Since β is T invariant, we have

$$\int_{\mathrm{A}^i} \mathrm{F}\big(\mathrm{T}^i x\big) d\beta(x) = \int_{\mathrm{T}^i \mathrm{A}^i} \mathrm{F}(x) d\beta(x),$$

and hence

$$\int_\Omega \mathrm{G} d\beta = \int_\mathrm{K} \mathrm{F} d\beta + \sum_{i=1}^{\infty} \int_{\mathrm{T}^i \mathrm{A}^i} \mathrm{F}(x) d\beta(x) = \int_\Omega \mathrm{F} d\beta.$$

□

Proof of (2). — This follows by applying (3) to $|\mathrm{F}|$ instead of F, and then using the triangle inequality. □

Proof of Lemma C.8. — Now by (1), and (2), the function G satisfies the assumptions of Lemma C.7. Hence we have $\int_\Omega F d\beta = \int_\Omega G d\beta > 0$. □

C.4 A zero-one law

Lemma **C.9.** — *Suppose h is a bounded non-negative μ-subharmonic function, i.e. for ν-almost all $x \in X$,*

$$h(x) \leq \int_G h(gx)\, d\mu(g). \tag{C.9}$$

Then h is constant ν-almost everywhere.

Proof. — By the random ergodic theorem [Fu, Theorem 3.1], for ν-almost all $x \in X$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \int_G h(gx)\, d\mu^n(g) = \int_X h\, d\nu$$

Therefore, by (C.9), for ν-almost all $x \in X$,

$$h(x) \leq \int_X h\, d\nu. \tag{C.10}$$

Let $s_0 \geq 0$ denote the essential supremum of h, i.e.

$$s_0 = \inf\big\{s \in \mathbb{R} \,:\, \nu\big(\{h > s\}\big) = 0\big\}.$$

Suppose $\epsilon > 0$ is arbitrary. We can pick $x \in X$ such that (C.10) holds and $h(x) > s_0 - \epsilon$. Then,

$$s_0 - \epsilon \leq h(x) \leq \int_X h\, d\nu \leq s_0.$$

Since $\epsilon > 0$ is arbitrary, $\int_X h\, d\nu = s_0$. Thus $h(x) = s_0$ for ν-almost all x. □

Let ν be an ergodic stationary measure on X. Fix $1 \leq s < \dim(L)$, and let Gr_s denote the Grassmannian of s-dimensional subspaces in L. Let $\hat{X} = X \times Gr_s$. We then have an action of $SL(2,\mathbb{R})$ on $\hat{X}$, by

$$g \cdot (x, W) = \big(gx, A(g,x)W\big).$$

Let $\hat{\nu}$ be a μ-stationary measure on $\hat{X}$ which projects to ν under the natural map $\hat{X} \to X$. We may write

$$d\hat{\nu}(x, U) = d\nu(x)\, d\eta_x(U),$$

where η_x is a measure on Gr_s.

Let $m = \dim(\mathrm{L})$. For a subspace W of L, let

$$\mathrm{I}(\mathrm{W}) = \{\mathrm{U} \in \mathrm{G}r_s \,:\, \dim(\mathrm{U} \cap \mathrm{W}) > \max(0, m - \dim(\mathrm{U}) - \dim(\mathrm{W}))\}$$

Then $\mathrm{U} \in \mathrm{I}(\mathrm{W})$ if and only if U and W intersect more than general position subspaces of dimension dim(U) and dim(W).

Lemma **C.10** *(Cf. [GM, Lemma 4.2], [GR1, Theorem 2.6]).*

(i) *Suppose the cocycle is strongly irreducible on* L. *Then for almost all* $x \in \mathrm{X}$, *and any* 1*-dimensional subspace* $\mathrm{W}_x \subset \mathrm{L}$, $\eta_x(\mathrm{I}(\mathrm{W}_x)) = 0$.

(ii) *Suppose the algebraic hull* H′ *of the cocycle is* SL(L). *Then for almost all* $x \in \mathrm{X}$, *for any nontrivial proper subspace* $\mathrm{W}_x \subset \mathrm{L}$, $\eta_x(\mathrm{I}(\mathrm{W}_x)) = 0$.

Proof of Lemma C.10. — We give the proof under the extra assumption that ν is μ-invariant (and not just μ-stationary). The general case is proved in [EMat].

Suppose there exists a subset $\mathrm{E} \subset \mathrm{X}$ with $\nu(\mathrm{E}) > 0$ and for all $x \in \mathrm{E}$, a nontrivial subspace $\mathrm{W}_x \subset \mathrm{L}$ such that $\eta_x(\mathrm{I}(\mathrm{W}_x)) > 0$. Let $\vec{\mathrm{W}} = (\mathrm{W}_1, \ldots, \mathrm{W}_k)$ denote a finite collection of subspaces of L. If the assumptions of (i) hold, we are requiring the W_i to be one-dimensional; if the assumptions of (ii) hold, the W_i are allowed to be any dimension. Write

$$\mathrm{I}(\vec{\mathrm{W}}) = \mathrm{I}(\mathrm{W}_1) \cap \cdots \cap \mathrm{I}(\mathrm{W}_k).$$

For $x \in \mathrm{E}$, let $\mathcal{S}_x$ denote the set of $\mathrm{I}(\vec{\mathrm{W}}_x)$ such that for any $\vec{\mathrm{W}}'_x$ so that $\mathrm{I}(\vec{\mathrm{W}}'_x)$ is a proper subset of $\mathrm{I}(\vec{\mathrm{W}}_x)$, we have $\nu_x(\mathrm{I}(\vec{\mathrm{W}}'_x)) = 0$. For $x \in \mathrm{E}$, $\mathcal{S}_x$ is non-trivial since the subsets $\mathrm{I}(\vec{\mathrm{W}})$ are algebraic and thus there cannot be an infinite descending chain of them. For $\vec{\mathrm{W}} \in \mathcal{S}_x$, let

$$f_{\mathrm{I}(\vec{\mathrm{W}})}(x) = \eta_x(\mathrm{I}(\vec{\mathrm{W}})).$$

Since $\hat{\nu}$ is μ-stationary and ν is assumed to be μ-invariant, we have

$$f_{\mathrm{I}(\vec{\mathrm{W}})}(x) = \int_{\mathrm{G}} f_{\mathrm{I}(\mathrm{A}(g,x)\vec{\mathrm{W}})}(gx)\, d\mu(g) \qquad (\mathbf{C.11})$$

Let $\mathcal{S}(x) = \{\mathrm{I}(\vec{\mathrm{W}}) \in \mathcal{S}_x \,:\, f_{\mathrm{I}(\vec{\mathrm{W}})}(x) > 0\}$. Then, for $\mathrm{I}(\vec{\mathrm{W}}_1) \in \mathcal{S}(x)$, $\mathrm{I}(\vec{\mathrm{W}}_2) \in \mathcal{S}(x)$,

$$\eta_x(\mathrm{I}(\vec{\mathrm{W}}_1) \cap \mathrm{I}(\vec{\mathrm{W}}_2)) = 0.$$

Thus

$$\sum_{\mathrm{I}(\vec{\mathrm{W}}) \in \mathcal{S}(x)} f_{\mathrm{I}(\vec{\mathrm{W}})}(x) \le 1.$$

Therefore $\mathcal{S}(x)$ is at most countable. Let

$$f(x) = \max_{\mathrm{I}(\vec{\mathrm{W}}) \in \mathcal{S}(x)} f_{\mathrm{I}(\vec{\mathrm{W}})}(x). \tag{C.12}$$

Applying (C.11) to some $\mathrm{I}(\vec{\mathrm{W}})$ for which the max is achieved, we get

$$f(x) \leq \int_{\mathrm{G}} f(gx)\, d\mu(g)$$

i.e. f is a subharmonic function on X. By Lemma C.9, f is constant almost everywhere. Now substituting again into (C.11) we get that the cocycle A permutes the finite set of $\mathrm{I}(\vec{\mathrm{W}})$ where the maximum (C.12) is achieved. Therefore the same is true for the algebraic hull H'. If the assumptions of (ii) hold, this is a contradiction since H' acts transitively on subspaces of L. If the assumptions of (i) hold then, for $\vec{\mathrm{W}} = (\mathrm{W}_1, \dots, \mathrm{W}_k)$, since the W_i are 1-dimensional, we have

$$\begin{aligned} \mathrm{I}(\vec{\mathrm{W}}) &\equiv \mathrm{I}(\mathrm{W}_1) \cap \dots \mathrm{I}(\mathrm{W}_k) \\ &= \{\text{subspaces } \mathrm{M} \subset \mathrm{L} \text{ such that } \mathrm{W}_1 + \dots + \mathrm{W}_k \subset \mathrm{M}\}. \end{aligned}$$

Since H' must permute some finite set of $\mathrm{I}(\vec{\mathrm{W}})$ it must thus permute a finite set of subspaces of L which contradicts the strong irreducibility assumption. □

C.5 Proof of Theorem C.6

Recall that we are assuming that the algebraic hull of the cocycle is SL(L) for some vector space L. Let $m = \dim \mathrm{L}$.

Definition **C.11** *((ϵ, δ)-regular). — Suppose $\epsilon > 0$ and $\delta > 0$ are fixed. A measure η on* $\mathrm{G}r_k(\mathrm{L})$ *is* (ϵ, δ)-regular *if for any subspace* U *of* L,

$$\eta\big(\mathrm{N}bhd_\epsilon(\mathrm{I}(\mathrm{U}))\big) < \delta.$$

Lemma **C.12.** — *Suppose $g_n \in \mathrm{GL}(\mathrm{L})$ is a sequence of linear transformations, and η_n is a sequence of uniformly (ϵ, δ)-regular measures on $\mathrm{G}r_k(\mathrm{L})$ for some k. Suppose $\delta \ll 1$. Write*

$$g_n = \mathrm{K}(n)\mathrm{D}(n)\mathrm{K}'(n),$$

where $\mathrm{K}(n)$ and $\mathrm{K}'(n)$ are orthogonal relative to the standard basis $\{e_1, \dots e_m\}$, and $\mathrm{D}(n) =$ $\mathrm{diag}(d_1(n), \dots, d_m(n)\}$ with $d_1(n) \geq \dots \geq d_m(n)$.

(a) *Suppose*

$$\frac{d_k(n)}{d_{k+1}(n)} \to \infty \tag{C.13}$$

Then, for any subsequential limit λ *of* $g_n\eta_n$ *there exists a subspace* $W \in Gr_k(L)$ *such that*

$$K(n)\,\mathrm{span}\{e_1, \dots, e_k\} \to W, \tag{C.14}$$

and $\lambda(\{W\}) \geq 1 - \delta$.

(b) *Suppose* $g_n\eta_n \to \lambda$ *where* λ *is some measure on* $Gr_k(L)$. *Suppose also that there exists a subspace* $W \in Gr_k(L)$ *such that* $\lambda(\{W\}) > 5\delta$. *Then, as* $n \to \infty$, (C.13) *holds. As a consequence, by part (a),* (C.14) *holds and* $\lambda(\{W\}) \geq 1 - \delta$.

Proof of (a). — This statement is standard. Suppose $g_n\eta_n \to \lambda$. Without loss of generality, $K'(n)$ is the identity (or else we replace η_n by $K'(n)\eta_n$). By our assumptions, for $j_1 < \cdots < j_k$,

$$\frac{\|g_n(e_{j_1} \wedge \cdots \wedge e_{j_k})\|}{\|g_n(e_1 \wedge \cdots \wedge e_k)\|} \to 0 \quad \text{unless } j_i = i \text{ for } 1 \leq i \leq k.$$

Therefore, if $U \notin I(\mathrm{span}\{e_{k+1}, \dots, e_m\})$,

$$d\big(g_nU, K(n)\,\mathrm{span}\{e_1, \dots, e_k\}\big) \to 0,$$

where $d(\cdot,\cdot)$ denotes some distance in $Gr_k(L)$. After passing to a further subsequence, we may assume that for some $W \in Gr_k(L)$, (C.14) holds. It follows from the (ϵ, δ)-regularity of η_n that $\lambda(W) \geq 1 - \delta$. Since $\delta < 1/2$, W is uniquely determined by λ, and therefore (C.14) holds without passing to a further subsequence (but only assuming $g_n\eta_n \to \lambda$).

Proof of (b). — This is similar to [GM, Lemma 3.9]. Suppose $d_k(n)/d_{k+1}(n)$ does not go to ∞. Then, there is a subsequence of the g_n (which we again denote by g_n) that $K(n) \to K_*$ and that for every j, either $d_j(n)/d_{j+1}(n)$ converges as $n \to \infty$ or $d_j(n)/d_{j+1}(n) \to \infty$ as $n \to \infty$. Also without loss of generality we may assume that $K'(n)$ is the identity (or else we replace η_n by $K'(n)\eta_n$).

Let $1 \leq s \leq k < r \leq m$ be such that s is as small as possible, r is as large as possible, and $d_j(n)/d_{j+1}(n)$ is bounded for $s \leq j \leq r-1$. Then, for $j_1 < \cdots < j_k$,

$$\frac{\|g_n(e_{j_1} \wedge \cdots \wedge e_{j_k})\|}{\|g_n(e_1 \wedge \cdots \wedge e_k)\|} \to 0 \quad \text{unless } j_i = i \text{ for } 1 \leq i \leq s-1 \text{ and } s \leq j_i \leq r \text{ for } s \leq i \leq k. \tag{C.15}$$

Let

$$V_- = \mathrm{span}\{e_1, \dots, e_{s-1}\}, \qquad V_+ = \mathrm{span}\{e_1, \dots, e_r\}.$$

Let $D_* = \mathrm{diag}(d_*(1), \dots, d_*(m))$ be any diagonal matrix such that for $s \leq j \leq r-1$,

$$d_*(j)/d_*(j+1) = \lim_{n\to\infty} d_j(n)/d_{j+1}(n).$$

Then, in view of (C.15), for U such that $U \notin I(V_+^\perp) \cup I(V_-^\perp)$, if along some subsequence $g_n U \to U'$, we have

$$K_* V_- \subset U' \subset K_* V_+.$$

Therefore, we must have $V_- \subset K_*^{-1} W \subset V_+$. Furthermore, for $U \notin I(V_+^\perp) \cup I(V_-^\perp)$,

$$\text{if } g_n U \to W \text{ then } U \in I\big(D_*^{-1} K_*^{-1} W \cap V_-^\perp + V_+^\perp\big).$$

But, since η_n is (ϵ, δ)-regular,

$$\eta_n\big(Nbhd_\epsilon\big(I\big(V_+^\perp\big) \cup I\big(V_-^\perp\big) \cup I\big(D_*^{-1} K_*^{-1} W \cap V_-^\perp + V_+^\perp\big)\big)\big) < 3\delta.$$

Therefore $\lambda(\{W\}) < 3\delta$ which is a contradiction. Thus $d_k(n)/d_{k+1}(n) \to \infty$. Now by part (a) (C.14) holds, and $\lambda(\{W\}) \geq 1 - \delta$. □

Let $\mathcal{F} = \mathcal{F}(L)$ denote the space of full flags on L. Let $\hat{X} = X \times \mathcal{F}$. The cocycle A satisfies the cocycle relation

$$A(g_1 g_2, x) = A(g_1, g_2 x) A(g_2, x).$$

The group $SL(2, \mathbb{R})$ acts on the space $\hat{X}$ by

$$g \cdot (x, f) = \big(gx, A(g, x) f\big). \tag{C.16}$$

Let $\hat{\nu}$ be an ergodic μ-stationary measure on $\hat{X}$ which projects to ν under the natural map $\hat{X} \to X$. (Note there is always at least one such: one chooses $\hat{\nu}$ to be an extreme point among the measures which project to ν. If $\hat{\nu} = \hat{\nu}_1 + \hat{\nu}_2$ where the $\hat{\nu}_i$ are μ-stationary measures then $\nu = \pi_*(\hat{\nu}) = \pi_*(\hat{\nu}_1) + \pi_*(\hat{\nu}_2)$. Since ν is μ-ergodic, this implies that $\pi_*(\hat{\nu}_1) = \pi_*(\hat{\nu}_2) = \nu$, hence the $\hat{\nu}_i$ also project to ν. Since $\hat{\nu}$ is an extreme point among such measures, we must have $\hat{\nu}_1 = \hat{\nu}_2 = \hat{\nu}$. Thus $\hat{\nu}$ is μ-ergodic.)

Lemma **C.13** *(Furstenberg). — For $1 \leq s \leq \dim L$, let $\bar{\sigma}_s : SL(2, \mathbb{R}) \times \hat{X} \to \mathbb{R}$ be given by*

$$\bar{\sigma}_s(g, x, f) = \log \frac{\|A(g, x)\xi_s(f)\|}{\|\xi_s(f)\|}$$

where $\xi_s(f)$ is the s-dimensional component of the flag f. (The norms in the above equation are on $\bigwedge^s(V)$, and here and in the following we make sense of such expressions by picking the same basis for the $\xi_s(f)$ in the numerator and denominator.) Then, we have

$$\lambda_1 + \cdots + \lambda_s = \int_{SL(2,\mathbb{R})} \int_{\hat{X}} \bar{\sigma}_s(g, x, f)\, d\hat{\nu}(x, f)\, d\mu(g),$$

where λ_i is the i'th Lyapunov exponent of the cocycle A.

Proof. — See the proof of [GM, Lemma 5.2]. □

We may disintegrate

$$d\hat{\nu}(x,f) = d\nu(x)d\eta_x(f).$$

Note that Lemma C.10 applies to the projections of the measures η_x to the various Grassmannians which are components of $\mathcal{F}$.

For $a \in \tilde{\mathrm{B}}$, let the measures ν_a, $\hat{\nu}_a$ be as defined in [BQ, Lemma 3.2], i.e.

$$\nu_a = \lim_{n\to\infty} (a_n \dots a_1)_*^{-1}\nu$$

$$\hat{\nu}_a = \lim_{n\to\infty} (a_n \dots a_1)_*^{-1}\hat{\nu}.$$

The limits exist by the martingale convergence theorem. We disintegrate

$$d\hat{\nu}_a(x,f) = d\nu_a(x)d\eta_{a,x}(f).$$

For $1 \le k \le m$, let $\eta_x^k = (\xi_k)_*\eta_x$ and $\eta_{a,x}^k = (\xi_k)_*\eta_{a,x}$, where $\xi_k : \mathcal{F}(\mathrm{L}) \to \mathrm{G}r_k(\mathrm{L})$ is the natural projection. Then, η_x^k and $\eta_{a,x}^k$ are measures on $\mathrm{G}r_k(\mathrm{L})$.

Claim **C.14.** — *On a set of $\beta \times \nu$ full measure,*

$$\lim_{n\to\infty} (a_n \dots a_1)_*^{-1}\eta_{a_n\dots a_1 x} = \eta_{a,x}.$$

Equivalently, using (C.16),

$$\lim_{n\to\infty} \mathrm{A}\big((a_n \dots a_1)^{-1}, a_n \dots a_1 x\big)\eta_{a_n\dots a_1 x} = \eta_{a,x}.$$

Proof of claim. — In this claim, we use the invariance of ν. Let $\mathrm{C} \subset \mathrm{X}$ and $\mathrm{D} \subset \mathcal{F}$ be measurable, and let χ_{C} denote the characteristic functions of C. Recall that $d\hat{\nu}(x,z) = d\nu(x)d\eta_x(z)$ is μ-stationary, so that

$$\begin{aligned}\int_{\mathrm{C}} \eta_x(\mathrm{D})d\nu(x) &= \hat{\nu}(\mathrm{C}\times\mathrm{D}) = (\mu * \hat{\nu})(\mathrm{C}\times\mathrm{D})\\
&= \int \chi_{\mathrm{C}}(gy)\mathrm{A}(g,y)\eta_y(\mathrm{D})d\nu(y)d\mu(g)\\
&= \int \chi_{\mathrm{C}}(x)\mathrm{A}\big(g,g^{-1}x\big)\eta_{g^{-1}x}(\mathrm{D})d\nu(x)d\mu(g)\\
&= \int_{\mathrm{C}}\left(\int_{\mathrm{G}} \mathrm{A}\big(g,g^{-1}x\big)\eta_{g^{-1}x}(\mathrm{D})d\mu(g)\right)d\nu(x)\end{aligned}$$

Since C and D are arbitrary, we see that

$$\eta_x = \int_G A\big(g, g^{-1}x\big)\eta_{g^{-1}x}\,d\mu(g)$$

Therefore (replacing x by $a_{n-1}\dots a_1x$ and g by a_n^{-1}), we have

$$\eta_{a_{n-1}\dots a_1x} = \int_G A\big(a_n^{-1}, a_n\dots a_1x\big)\eta_{a_n\dots a_1x}\,d\mu(a_n).$$

Multiplying both sides on the left by $A((a_{n-1}\dots a_1)^{-1}, a_{n-1}\dots a_1x)$ and using the cocycle identity

$$\begin{aligned} &A\big((a_n\dots a_1)^{-1}, a_n\dots a_1x\big)\\ &\quad = A\big((a_{n-1}\dots a_1)^{-1}, a_{n-1}\dots a_1x\big)A\big(a_n^{-1}, a_n\dots a_1x\big), \end{aligned}$$

we get

$$\begin{aligned} &A\big((a_{n-1}\dots a_1)^{-1}, a_{n-1}\dots a_1x\big)\eta_{a_{n-1}\dots a_1x}\\ &\quad = \int_G A\big((a_n\dots a_1)^{-1}, a_n\dots a_1x\big)\eta_{a_n\dots a_1x}\,d\mu(a_n). \end{aligned} \tag{C.17}$$

In view of (C.17), the expression

$$A\big((a_n\dots a_1)^{-1}, a_n\dots a_1x\big)\eta_{a_n\dots a_1x}$$

is a (measure-valued) martingale. Therefore, the claim follows from the martingale convergence theorem. □

If the Lyapunov spectrum is simple, we expect the measures $\eta_{a,x}$ to be supported at one point. In the general case, let

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$$

denote the Lyapunov exponents, and let

$$I = \{1 \leq r \leq m-1 \;:\; \lambda_r = \lambda_{r+1}\}.$$

Then, by the multiplicative ergodic theorem, Lemmas C.10 and C.12(a), for $r \notin I$, we have $\eta_{a,x}^{m-r}$ is supported at one point. (This point is the part of the flag (C.1) corresponding to the Lyapunov exponents $\lambda_{r+1}, \dots, \lambda_m$.)

Claim **C.15.** — *For any* $r \in \mathrm{I}$ *and* $\beta \times \nu$*-almost all* (a, x)*, for any subspace* $\mathrm{W}(a, x) \in \mathrm{Gr}_{m-r}(\mathrm{L})$*, we have* $\eta^{m-r}_{a,x}(\{\mathrm{W}(a, x)\}) = 0$.

Proof of claim. — Suppose there exists $\delta > 0$ so that for some $r \in \mathrm{I}$ for a set (a, x) of positive measure, there exists $\mathrm{W}(a, x) \in \mathrm{Gr}_{m-r}(\mathrm{L})$ with $\eta^{r}_{a,x}(\{\mathrm{W}(a, x)\}) > \delta$. Then this happens for a subset of full measure by ergodicity.

Note that by the cocycle relation,

$$\mathrm{A}\big(g^{-1}, gx\big) = \mathrm{A}(g, x)^{-1}.$$

Therefore,

$$\mathrm{A}\big((a_n \dots a_1)^{-1}, a_n \dots a_1 x\big) = \mathrm{A}(a_n \dots a_1, x)^{-1}.$$

Hence, on a set of $\beta \times \nu$-full measure,

$$\lim_{n\to\infty} \mathrm{A}(a_n \dots a_1, x)^{-1} \eta_{a_n \dots a_1 x} = \eta_{a,x}.$$

In view of Lemma C.10 (cf. the proof of Lemma 14.4), there exists $\epsilon > 0$ and a compact $\mathcal{K}_\delta \subset \mathrm{X}$ with $\nu(\mathcal{K}_\delta) > 1 - \delta$ such that the family of measures $\{\eta_x\}_{x \in \mathcal{K}_\delta}$ is uniformly $(\epsilon, \delta/5)$-regular. Let

$$\mathcal{N}_\delta(a, x) = \{n \in \mathbb{N} \; : \; a_n \dots a_1 x \in \mathcal{K}_\delta\}.$$

Write

$$\mathrm{A}(a_n \dots a_1, x)^{-1} = \mathrm{K}_n(a, x)\mathrm{D}_n(a, x)\mathrm{K}'_n(a, x) \tag{C.18}$$

where K_n and K'_n are orthogonal, and D_n is diagonal with non-increasing entries. We also write

$$\mathrm{A}(a_n \dots a_1, x) = \bar{\mathrm{K}}_n(a, x)\bar{\mathrm{D}}_n(a, x)\bar{\mathrm{K}}'_n(a, x), \tag{C.19}$$

where $\bar{\mathrm{K}}_n$ and $\bar{\mathrm{K}}'_n$ are orthogonal, and $\bar{\mathrm{D}}_n$ is diagonal with non-increasing entries. Let $d_1(n, a, x) \geq \dots \geq d_m(n, a, x)$ be the entries of $\mathrm{D}_n(a, x)$, and let $\bar{d}_1(n, a, x) \geq \bar{d}_2(n, a, x) \geq \bar{d}_m(n, a, x)$ be the entries of $\bar{\mathrm{D}}_n(a, x)$. Then,

$$\begin{gathered}\bar{d}_j(n, a, x) = d^{-1}_{m+1-j}(n, a, x), \\ \bar{\mathrm{K}}'_n(a, x) = w_0 \mathrm{K}_n(a, x)^{-1} w_0^{-1}, \quad \bar{\mathrm{K}}_n(a, x) = w_0 \mathrm{K}'_n(a, x)^{-1} w_0,\end{gathered} \tag{C.20}$$

where $w_0 = w_0^{-1}$ is the permutation matrix mapping e_j to e_{m+1-j}. Then, by Lemma C.12(b), for $\beta \times \nu$ almost all (a, x), $\eta^{m-r}_{a,x}(\{\mathrm{W}(a, x)\}) \geq 1 - \delta$ (and thus $\mathrm{W}(a, x)$ is unique) and as $n \to \infty$ along $\mathcal{N}_\delta(a, x)$ we have:

$$d_{m-r}(n, a, x)/d_{m+1-r}(n, a, x) \to \infty,$$

and

$$\text{(C.21)} \qquad K_n(a,x)\,\mathrm{span}\{e_1,\dots,e_{m-r}\} \to W(a,x),$$

where the e_i are the standard basis for L. Then, by (C.20),

$$\text{(C.22)} \qquad \bar{d}_r(n,a,x)/\bar{d}_{r+1}(n,a,x) \to \infty,$$

and

$$\bar{K}'_n(a,x)^{-1}\,\mathrm{span}\{e_{r+1},\dots,e_m\} \to w_0 W(a,x)$$

Therefore for any $\epsilon_1 > 0$ there exists a subset $H_{\epsilon_1} \subset B \times X$ of $\beta \times \nu$-measure at least $1 - \epsilon_1$ such that the convergence in (C.22) and (C.21) is uniform over $(a,x) \in H_{\epsilon_1}$. Hence there exists $M > 0$ such that for any $(a,x) \in H_{\epsilon_1}$, and any $n \in \mathcal{N}_\delta(a,x)$ with $n > M$,

$$\text{(C.23)} \qquad \bar{K}'_n(a,x)^{-1}\,\mathrm{span}\{e_{r+1},\dots,e_m\} \in Nbhd_{\epsilon_1}\big(w_0 W(a,x)\big).$$

By Lemma C.10 (cf. the proof of Lemma 14.4) there exists a subset $H''_{\epsilon_1} \subset X$ with $\nu(H''_{\epsilon_1}) > 1 - c_2(\epsilon_1)$ with $c_2(\epsilon_1) \to 0$ as $\epsilon_1 \to 0$ such that for all $x \in H''_{\epsilon_1}$, and any $U \in Gr_{m-r}(L)$,

$$\eta^r_x\big(Nbhd_{2\epsilon_1}\big(I(U)\big)\big) < c_3(\epsilon_1),$$

where $c_3(\epsilon_1) \to 0$ as $\epsilon_1 \to 0$. Let

$$\text{(C.24)} \qquad H'_{\epsilon_1} = \Big\{(a,x,f) \;:\; (a,x) \in H_{\epsilon_1},\; x \in H''_{\epsilon_1} \text{ and } d\big(\xi_r(f), I\big(w_0 W(a,x)\big)\big) > 2\epsilon_1\Big\}.$$

Then, $(\beta \times \hat{\nu})(H'_{\epsilon_1}) > 1 - \epsilon_1 - c_2(\epsilon_1) - c_3(\epsilon_1)$, hence $(\beta \times \hat{\nu})(H'_{\epsilon_1}) \to 1$ as $\epsilon_1 \to 0$. Furthermore, by (C.23) and the definition of H'_{ϵ_1}, for $(a,x,f) \in H'_{\epsilon_1}$ and $n \in N_\delta(a,x)$ with $n > M$, we have

$$d\big(\xi_r(f), I\big(\bar{K}'_n(a,x)^{-1}\,\mathrm{span}\{e_{r+1},\dots,e_m\}\big)\big) > \epsilon_1.$$

Therefore, in view of (C.19) there exists $C = C(\epsilon_1)$, such that for any $(a,x,f) \in H'_{\epsilon_1}$, any $n \in N_\delta(a,x)$ with $n > M$,

$$\text{(C.25)} \qquad C > \frac{\|A(a_n \dots a_1, x)\xi_r(f)\|}{\|\xi_r(f)\|} \prod_{i=1}^{r} \bar{d}_i(n,a,x)^{-1} > \frac{1}{C}$$

(cf. [GM, Lemma 5.1]). Note that for all $(a,x,f) \in B \times \hat{X}$, all $n \in \mathbb{N}$ and $j = r-1$ or $j = r+1$ we have

$$\text{(C.26)} \qquad \frac{\|A(a_n \dots a_1, x)\xi_j(f)\|}{\|\xi_r(f)\|} \le \|A(a_n \dots a_1, x)\|_{\bigwedge^j(L)} \le \prod_{i=1}^{j} \bar{d}_i(n,a,x).$$

Then, in view of (C.25) and (C.26), for all $(a,x,f)\in H'_{\epsilon_1}$, as $n\to\infty$ in $\mathcal{N}_\delta(a,x)$,

$$\log \frac{\|(A(a_n\dots a_1,x))\xi_r(f)\|^2}{\|\xi_r(f)\|^2}\frac{\|\xi_{r-1}(f)\|}{\|(A(a_n\dots a_1,x))\xi_{r-1}(f)\|} \times \frac{\|\xi_{r+1}(f)\|}{\|(A(a_n\dots a_1,x))\xi_{r+1}(f)\|} \geq \log\frac{\bar{d}_r(n,a,x)}{\bar{d}_{r+1}(n,a,x)} \to \infty \tag{C.27}$$

Since $(\beta\times\hat{\nu})(H'_{\epsilon_1})\to 1$ as $\epsilon_1\to 0$, (C.27) holds as $n\to\infty$ along $\mathcal{N}_\delta(a,x)$ for $\beta\times\hat{\nu}$ almost all $(a,x,f)\in B\times\hat{X}$.

For $1\leq s\leq m$, let $\sigma_s : B\times\hat{X}\to\mathbb{R}$ be defined by $\sigma_s(a,x,f)=\bar{\sigma}_s(a_1,x,f)$, where $\bar{\sigma}$ is as in Lemma C.13. Then, the left hand side of (C.27) is exactly

$$\sum_{j=0}^{n-1}(2\sigma_r-\sigma_{r-1}-\sigma_{r+1})\big(\hat{T}^j(a,x,f)\big).$$

Also, we have $n\in\mathcal{N}_\delta(a,x)$ if and only if $\hat{T}^n(a,x)\in\mathcal{K}_\delta$. Then, by Lemma C.8,

$$\int_{B\times\hat{X}}(2\sigma_r-\sigma_{r-1}-\sigma_{r+1})(q)\,d(\beta\times\hat{\nu})(q)>0.$$

By Furstenberg's formula Lemma C.13, the left hand side of the above equation is $\lambda_r-\lambda_{r+1}$. Thus $\lambda_r>\lambda_{r+1}$, contradicting our assumption that $r\in I$. This completes the proof of the claim. □

Proof of Theorem C.6. — Pick an orthonormal basis at each point of X, and let $C(a\vee b,x): L\to L$ be a map which makes the subspaces $\mathcal{V}_i(a\vee b,x)$ orthonormal. Let $\tilde{A}$ denote the cocycle obtained by

$$\tilde{A}(n,a\vee b,x)=C\big(T^n(a\vee b,x)\big)^{-1}A(a_n\dots a_1,x)C(a\vee b,x).$$

Then $\tilde{A}$ is cohomologous to A. Let

$$\hat{\eta}(a\vee b,x)=C(a\vee b,x)_*\eta_x,\qquad \tilde{\eta}_{a\vee b,x}=C(a\vee b,x)_*\eta_{a,x}.$$

We have, on a set of $\tilde{\beta}^X$ full measure,

$$\tilde{\eta}_{a\vee b,x}=\lim_{n\to\infty}\tilde{A}(n,a\vee b,x)_*^{-1}\hat{\eta}\big(T^n(a\vee b,x)\big).$$

In view of Lemma C.10 there exists $\epsilon>0$ and a compact $\mathcal{K}_\delta\subset\tilde{B}\times X$ with $\tilde{\beta}^X(\mathcal{K}_\delta)>1-\delta$ such that the family of measures $\{\hat{\eta}(a\vee b,x)\}_{(a\vee b,x)\in\mathcal{K}_\delta}$ is uniformly $(\epsilon,\delta/5)$-regular. Write

$$\tilde{A}(n,a\vee b,x)^{-1}=K_n(a\vee b,x)D_n(a\vee b,x)K'_n(a\vee b,x)$$

where K_n and K'_n are orthogonal, and D_n is diagonal with non-increasing entries. Let $d_1(n, a \vee b, x) \geq \cdots \geq d_m(n, a, x)$ be the entries of $D_n(a \vee b, x)$.

By Claim C.15, for $r \in I$ and almost all $(a \vee b, x)$ $\tilde{\eta}^{m-r}_{a \vee b, x}$ has no atoms. It follows that for every $\delta > 0$ there exists $\mathcal{K}_1 = \mathcal{K}_1(\delta) \subset \tilde{B} \times X$ and $\epsilon_1 = \epsilon_1(\delta) > 0$, such that for $(a \vee b, x) \in \mathcal{K}_1$, $\eta^{m-r} a \vee b, x$ gives measure at most δ to the ϵ_1-neighborhood of any point. Then, by Lemma C.12(a), there exists $C_1 = C_1(\delta)$ such that if $(a \vee b, x) \in \mathcal{K}_1(\delta)$ and $T^n(a \vee b, x) \in \mathcal{K}_\delta$ then for $r \in I$

$$d_{m-r}(n, a \vee b, x)/d_{m+1-r}(n, a \vee b, x) \leq C_1. \qquad \textbf{(C.28)}$$

Note that the matrix of $\tilde{A}(n, a \vee b, x)$ is block diagonal. We can write each block as a scaling factor times a determinant one matrix which we denote by $\tilde{A}_i(n, a \vee b, x)$. (Thus $\tilde{A}_i(n, a \vee b, x)$ is, up to a scaling factor, a conjugate of the restriction of $A(n, a \vee b, x)$ to $\mathcal{V}_i(a \vee b, x)$.) Since the subspaces defining the blocks are by construction orthogonal, the KAK decomposition of $\tilde{A}(n, a \vee b, x)^{-1}$ is compatible with the KAK decompositions of each $\tilde{A}_i(n, a \vee b, x)^{-1}$. Then, (C.28) for all $r \in I$ implies that for all $(a \vee b, x) \in \mathcal{K}_1(\delta)$ such that $T^n(a \vee b, x) \in \mathcal{K}_\delta$ we have

$$\|\tilde{A}_i(n, a \vee b, x)\| \leq C'_1(\delta) \quad \text{for all } i.$$

It follows that for all $n \in \mathbb{Z}$

$$\tilde{\beta}^X\Big(\big\{(a \vee b, x) \in B \times X \,:\, \|\tilde{A}_i(n, a \vee b, x)\| > C'_1(\delta)\big\}\Big) \leq 2\delta.$$

Since $\delta > 0$ is arbitrary, this means (by definition) that the cocycle $\tilde{A}_i$ is bounded in the sense of Schmidt, see [Sch]. It is proved in [Sch] that any bounded cocycle is conjugate to a cocycle taking values in an orthogonal group. Therefore the same holds for the determinant one part of the cocycle $A|_{\mathcal{V}_i}$. □

Proof of Theorems C.4 and C.5. — To prove Theorem C.4, for the case where the algebraic hull is all of SL(L), it is enough to show that for almost all (a, x), the inner product $\langle\cdot,\cdot\rangle_{a \vee b, x}$ does not depend on b. The proof is similar to the proof of (4.16).

For any $\epsilon > 0$ exists a compact set $K \subset \tilde{B} \times X$ of measure $1 - \epsilon$ such that the map $(a \vee b, x) \to \langle\cdot,\cdot\rangle_{a \vee b, x}$ is uniformly continuous on K. Then there exists $\Omega \subset \tilde{B} \times X$ such that $\tilde{\beta}^X(\Omega) = 1$ and $T^n(a \vee b, x) \in K$ for set of n of asymptotic density at least 1/2.

For $(a \vee b, x) \in \tilde{B} \times X$ and $v, w \in \mathcal{V}_{\geq i}(a, x)/\mathcal{V}_{>i}(a, x)$, let

$$[v, w]_{i,(a \vee b, x)} = \frac{\langle v, w\rangle_{i,(a \vee b, x)}}{\langle v, v\rangle^{1/2}_{i,(a \vee b, x)} \langle w, w\rangle^{1/2}_{i,(a \vee b, x)}}$$

Now suppose $(a \vee b, x) \in \Omega$, and $(a \vee b', x) \in \Omega$. Consider the points $T^n(a \vee b, x)$ and $T^n(a \vee b', x)$, as $n \to \infty$. Then $d(T^n(a \vee b, x), T^n(a \vee b', x)) \to 0$. Let

$$v_n = A(a_n \ldots a_1)v, \qquad w_n = A(a_n \ldots a_1)w.$$

Then, by Theorem C.6, we have

$$(\mathbf{C.29})\qquad [v_n, w_n]_{i,T^n(a\vee b,x)} = [v, w]_{i,x}, \quad [v_n, w_n]_{i,T^n(a\vee b',x)} = [v, w]_{i,(a\vee b',x)}.$$

Now take a sequence $n_k \to \infty$ with $T^n(a \vee b, x) \in K$, $T^n(a \vee b', x) \in K$ (such a sequence exists by the definition of Ω). Then,

$$[v_{n_k}, w_{n_k}]_{i,T^{n_k}(a\vee b,x)} - [v_{n_k}, w_{n_k}]_{i,T^n(a\vee b',x)} \to 0.$$

Now from (C.29), we get

$$[v, w]_{i,(a\vee b,x)} = [v, w]_{i,(a\vee b',x)}.$$

Therefore, for all $v, w \in \mathcal{V}_{\geq i}(a, x)/\mathcal{V}_{>i}(a, x)$

$$\langle v, w\rangle_{i,(a\vee b,x)} = c\big(a, b, b', x\big)\langle v, w\rangle_{i,(a\vee b',x)},$$

where $c(a, b, b', x) \in \mathbb{R}^+$. We can (measurably) choose, for almost all (a, x) some $b_0 \in B$ so that $(a \vee b_0, x) \in \Omega$, and then replace $\langle\cdot,\cdot\rangle_{i,(a\vee b,x)}$ by

$$\langle v, w\rangle'_{i,(a,x)} = \langle v, w\rangle_{i,a\vee b_0,x}.$$

Then $\langle\cdot,\cdot\rangle'_{i,(a,x)}$ satisfies all the conditions of Theorem C.4. This concludes the proof of Theorem C.4 for the case where the algebraic hull is all of SL(L).

The proof of Theorem C.5 is identical. □

Appendix D: Dense subgroups of nilpotent groups

The aim of this appendix is to prove Proposition D.3 which is used in Section 12.

Let N be a nilpotent Lie group. For a subgroup $\Gamma \subset N$, let $\bar{\Gamma}$ denote the topological closure of Γ, and let $\bar{\Gamma}^0$ denote the connected component of $\bar{\Gamma}$ containing the identity e of N. Let $B(x, \epsilon)$ denote the ball of radius ϵ centered at x in some left-invariant metric on N.

Lemma **D.1.** — *Suppose* N *is a Lie group, and* $S \subset N$ *is a subset. For* $\epsilon > 0$, *let* Γ_ϵ *denote the subgroup generated by* $S \cap B(e, \epsilon)$. *Then there exists* $\epsilon_1 > 0$ *and a connected closed Lie subgroup* N_1 *of* N *such that for* $\epsilon < \epsilon_1$, $\overline{\Gamma}_\epsilon = N_1$.

Proof. — By Cartan's theorem (see e.g. [Kn, §0.4]), any closed subgroup of a Lie group is a closed Lie subgroup. Let $\epsilon > 0$ be arbitrary. Since we have $\bar{\Gamma}^0_{\epsilon'} \subset \bar{\Gamma}^0_\epsilon$ for $\epsilon' < \epsilon$, there exists $\epsilon_0 > 0$ such that for $\epsilon \leq \epsilon_0$, the dimension of the Lie algebra of $\bar{\Gamma}^0_\epsilon$ (and thus $\bar{\Gamma}^0_\epsilon$ itself) is independent of ϵ. Thus there exists a connected closed subgroup $N_1 \subset N$ such that for $\epsilon \leq \epsilon_0$, $\bar{\Gamma}^0_\epsilon = N_1$. In particular,

$$(\mathbf{D.1})\qquad \bar{\Gamma}_\epsilon \supset N_1.$$

From the definition it is immediate that $\bar{\Gamma}_{\epsilon_0}$ is a closed subgroup of N. Thus, by Cartan's theorem, $\bar{\Gamma}_{\epsilon_0}$ and $N_1 = \bar{\Gamma}^0_{\epsilon_0}$ are closed submanifolds of N. Therefore, there exists $\epsilon_1 < \epsilon_0$ such that

$$B(e, \epsilon_1) \cap \bar{\Gamma}_{\epsilon_0} = B(e, \epsilon_1) \cap \bar{\Gamma}^0_{\epsilon_0} = B(e, \epsilon_1) \cap N_1.$$

Then, for $\epsilon < \epsilon_1 < \epsilon_0$,

$$\Gamma_\epsilon \cap B(e, \epsilon_1) \subset \bar{\Gamma}_{\epsilon_0} \cap B(e, \epsilon_1) \subset N_1.$$

Therefore, $\Gamma_\epsilon \subset N_1$, and hence $\bar{\Gamma}_\epsilon \subset N_1$. In view of (D.1), the lemma follows. □

Lemma **D.2.** — *Suppose* N *is a simply connected nilpotent Lie group, and let* $S \subset N$ *be an (infinite) subset. For each* $\epsilon > 0$ *let* $\Gamma_\epsilon \subset N$ *denote the subgroup of* N *generated by the elements* $\gamma \in S \cap B(e, \epsilon)$. *Suppose that for all* $\epsilon > 0$, Γ_ϵ *is dense in* N.

Then, for every $\epsilon > 0$ *there exist* $0 < \theta < \epsilon$ *(depending on* ϵ *and* S*) such that for every* $\gamma \in \Gamma_\epsilon$ *with* $d(\gamma, e) < \theta$ *there exists* $n \in \mathbb{N}$ *and for* $1 \le i \le n$ *elements* $\gamma_i \in S$ *with*

$$\gamma = \gamma_n \dots \gamma_1 \tag{D.2}$$

and for each $1 \le j \le n$,

$$d(\gamma_j \dots \gamma_1, e) < \epsilon. \tag{D.3}$$

Proof. — We will proceed by induction on dim N. Let $N' = [N, N]$

For $k \in \mathbb{N}$, let S^k_ϵ be the product of at most k elements in $(S \cup S^{-1}) \cap B(e, \epsilon)$. Let $T^k_\epsilon = [S^k_\epsilon, S^k_\epsilon]$. This decreases with ϵ, so a variant of Lemma D.1 shows that, for small enough ϵ, the closure of the group generated by T^k_ϵ is a closed connected group N_k (and N_k is independent of ϵ for ϵ small enough). Since N_k increases with k, it is constant for large k. Fix k so that $N_k = N_{k+2}$. We will show that $N_k = N'$.

First, we show that N_k is normal. For $a, b \in S^k_\epsilon$ and $s \in S_\epsilon$, we have $s[a, b]s^{-1} = [sas^{-1}, sbs^{-1}] \in T^{k+2}_\epsilon$. So, $sT^k_\epsilon s^{-1} \subset T^{k+2}_\epsilon$. Taking the closure of the generated groups, we get $sN_k s^{-1} \subset N_{k+2} = N_k$. Hence, N_k is normalized by S_ϵ. Since S_ϵ generates a dense subset of N, N_k is normal.

We have $[ab, c] = a[b, c]a^{-1}[a, c]$. This shows that, if $[a, c]$ and $[b, c]$ both belong to N_k, then $[ab, c]$ also belongs to N_k, by normality. For $x, y \in S^k_\epsilon$, we have $[x, y] \in N_k$. Taking products, and since S^k_ϵ generates a dense subgroup of N, we get $[z, y] \in N_k$ for all $z \in N$. Doing the same argument with the other variable, we finally have $[z, z'] \in N_k$ for all $z, z' \in N$, and therefore $N_k = N'$ as desired.

Let $S' = T^k_{\epsilon/4k} \subset N'$. For $\delta > 0$ let Γ'_δ denote the subgroup of N' generated by $S' \cap B(e, \delta)$. Since (for sufficiently small δ) $[B(e, \delta), B(e, \delta)] \subset B(e, \delta)$, we have, for $\delta < \epsilon/4k$,

$$\overline{\Gamma'_\delta} \supset \overline{\{\text{the subgroup generated by } T^k_{\delta/4k}\}} = N'.$$

Therefore, $S' \subset N'$ satisfies the conditions of the Lemma. Let $\epsilon' > 0$ be such that

$$\textbf{(D.4)} \qquad B\big(e,\epsilon'\big)^k \subset B(e,\epsilon/100).$$

Since $\dim N' < \dim N$, by the inductive assumption there exist $0 < \theta' < \epsilon'$ such that for any $\gamma' \in \Gamma'_{\theta'}$ with $d(\gamma', e) < \theta'$, there exist $\gamma'_i \in S'$ such that (D.2) holds, and (D.3) holds with ϵ' in place of ϵ.

Suppose $\epsilon > \eta > 0$. By construction, N/N' is abelian. Note that N is connected and simply connected. Then, since $\bar{\Gamma}_\eta = N$, there exists a finite set

$$S_0 \equiv \{\lambda_1, \dots, \lambda_k\} \subset \Gamma_\eta \cap S$$

with $d(\lambda_i, e) < \eta$ for $1 \le i \le k$ so that $\lambda_1 N', \dots, \lambda_k N'$ form a basis over $\mathbb{R}$ for the vector space N/N'. Let Λ denote the subgroup generated by the λ_i, and let $F' \subset N/N'$ denote the parallelogram centered at the origin whose sides are parallel to the vectors $\lambda_i N'$. Then F' is a fundamental domain for the action of Λ on N/N', and

$$\operatorname{diam} F' = O(\eta).$$

Let N_0 be a local complement to N' in N near the identity e. We can choose N_0 to be a smooth manifold transversal to N' (N_0 need not be a subgroup). Let $\pi : N \to N/N'$ be the natural map, and let $\pi^{-1} : N/N' \to N_0$ be the inverse. Let $F = \pi^{-1}(F')$. We can now choose η sufficiently small so that $F \subset B(e, \rho)$, where $\theta' > \rho > \eta > 0$ is such that

$$\begin{aligned} B(e,\rho)^5 \cap N' &= \big[B(e,\rho)B(e,\rho)B(e,\rho)B(e,\rho)B(e,\rho)\big] \cap N' \\ &\subset B\big(e,\theta'\big) \cap N'. \end{aligned}$$

We now choose $\theta > 0$ so that $B(e,\theta) \subset F\mathcal{O}$ where $\mathcal{O} \subset N' \cap B(e,\rho)$ is some neighborhood of the origin. We now claim that for any $x \in F\mathcal{O}$ and any $s \in B(e,\theta)$, there exist $\lambda' \in S_0 \cup S_0^{-1}$ and $\gamma' \in \Gamma'_{\theta'}$ such that $\gamma'\lambda' s x \in F\mathcal{O}$. Indeed, since $B(e,\theta)N' \subset FN'$, for any $x \in FN'$,

$$B(x,\theta)N' \subset \bigcup_{\lambda \in S_0 \cup S_0^{-1}} \lambda B(x,\theta)N'.$$

Thus, we can find $\lambda' \in S_0 \cup S_0^{-1}$ such that $\lambda' s x \in FN'$. Since $\Gamma'_{\theta'}$ is dense in N', there exists $\gamma' \in \Gamma'_{\theta'}$ such that $\gamma'\lambda' s x \in F\mathcal{O}$, completing the proof of the claim.

Now suppose $\gamma \in \Gamma_\theta$ and $\gamma \in B(e,\theta) \subset F\mathcal{O}$. Then, we have

$$\gamma = s_n \dots s_1, \qquad \text{where } s_i \in S \cap B(e,\theta).$$

Note that $s_1 \in F\mathcal{O}$. We now define elements $\lambda'_j \in S_0 \cup S_0^{-1}$ and $\gamma'_j \in \Gamma'_{\theta'}$ inductively as follows. At every stage of the induction, we will have $x_j \equiv \gamma'_j \lambda'_j s_j \dots \gamma'_1 \lambda'_1 s_1 \in F\mathcal{O}$. Suppose

$\gamma'_1, \dots, \gamma'_{j-1}$ and $\lambda'_1, \dots \lambda'_{j-1}$ have already been chosen. Now choose $\lambda'_j \in S_0 \cup S_0^{-1}$ and $\gamma'_j \in \Gamma'_{\theta'}$ so that $x_j = \gamma'_j \lambda'_j s_j x_{j-1} \in F\mathcal{O}$. Such λ'_j and γ'_j exist by the claim.

Note that

$$\gamma'_j = x_j x_{j-1}^{-1} s_j^{-1} (\lambda'_j)^{-1}$$
$$\in (F\mathcal{O})(F\mathcal{O})^{-1} B(e,\theta)^{-1} (S_0 \cup S_0^{-1}) \subset B(e,\rho)^5 \subset B(e,\theta').$$

Since $x_n = \lambda'_n \gamma'_n s_n \dots \lambda'_1 \gamma'_1 s_1 \in FN'$, we have $\lambda'_n s_n \dots \lambda'_1 s_1 \in FN'$. Also $\gamma = s_n \dots s_1 \in B(x,\theta) \subset FN'$. Since FN' is a fundamental domain for the action of Λ on N/N', $\lambda'_n \dots \lambda'_1 \in N'$. Thus,

$$\gamma = \gamma' \gamma'_n \lambda'_n s_n \dots \gamma'_1 \lambda'_1 s_1, \tag{D.5}$$

where $\gamma' \in N'$. We have

$$\gamma' = \gamma x_n^{-1} \in B(e,\theta)(F\mathcal{O})^{-1} \subset B(e,\theta').$$

For notational convenience, denote γ' by γ'_{n+1}. By the inductive assumption, for $1 \leq i \leq n+1$, we can express $\gamma'_i = s'_{i1} \dots s'_{im_i}$ such that $s'_{ij} \in S' \cap B(e,\theta')$ and so that for all i, j,

$$d(s'_{ij} \dots s'_{i1}, e) \leq \epsilon'.$$

We now substitute this into (D.5). Finally, we express each s'_{ij} as a commutator of a product of at most k elements of $S \cap B(e, \epsilon/4k)$. Then, in view of (D.4), the resulting word satisfies (D.3). □

Proposition **D.3.** — *Suppose* N *is a simply connected nilpotent Lie group,* $\mathcal{O}$ *a neighborhood of the identity in* N*, and* μ *a measure on* N *supported on* $\mathcal{O}$*. Suppose* $S \subset N$ *is a subset containing elements arbitrarily close to (and distinct from) e, and suppose for each* $\gamma \in S$*,*

$$\gamma_* \mu \propto \mu \tag{D.6}$$

on $\mathcal{O} \cap \gamma^{-1}\mathcal{O}$ *where both sides make sense. Then, there exists a nontrivial connected subgroup* H *of* N *and a neighborhood* $\mathcal{O}'$ *of the identity in* H *such that for all* $h \in \mathcal{O}'$, $h_*\mu \propto \mu$ *on* $\mathcal{O} \cap h^{-1}\mathcal{O}$*. Furthermore, if* U *is a connected subgroup of* N *and* S *contains arbitrarily small elements not contained in* U*, then* H *is not contained in* U*.*

Proof. — Let N_1 and ϵ_1 be as in Lemma D.1. By our assumptions on S, N_1 is nontrivial (and also N_1 is not contained in U). Now suppose $\epsilon > 0$ is such that $B(e,\epsilon) \subset \mathcal{O}$, and let $\theta > 0$ be as in Lemma D.2, with N replaced by N_1. Without loss of generality, we may assume that $\theta < \epsilon_1$. Let Γ_θ be the subgroup of N_1 generated by $S \cap B(e,\theta)$. Since $\theta < \epsilon_1$, Γ_θ is dense in N_1. Now suppose $\bar{\gamma} \in N_1$, and $d(\bar{\gamma}, e) < \theta$. Then, there exists $\gamma_k \in \Gamma_\theta$ such that $\gamma_k \to \gamma$, and $d(\gamma_k, e) < \theta$. We can write each $\gamma_k = \gamma_{k,n} \dots \gamma_{k,1}$ as in Lemma D.2. Then, by applying (D.6) repeatedly, we get that $(\gamma_k)_*\mu \propto \mu$. Then, taking the limit as $k \to \infty$ we see that $(\bar{\gamma})_*\mu \propto \mu$. Thus, μ is invariant (up to normalization) under a neighborhood of the origin in N_1. □

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Index of Notation §1-§16

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