DOUBLE RAMIFICATION CYCLES ON THE MODULI SPACES OF CURVES

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ABSTRACT

Curves of genus g which admit a map to $\mathbf{P}^1$ with specified ramification profile μ over $0 \in \mathbf{P}^1$ and ν over $\infty \in \mathbf{P}^1$ define a double ramification cycle $\mathrm{DR}_g(\mu, \nu)$ on the moduli space of curves. The study of the restrictions of these cycles to the moduli of nonsingular curves is a classical topic. In 2003, Hain calculated the cycles for curves of compact type. We study here double ramification cycles on the moduli space of Deligne-Mumford stable curves.

The cycle $\mathrm{DR}_g(\mu, \nu)$ for stable curves is defined via the virtual fundamental class of the moduli of stable maps to rubber. Our main result is the proof of an explicit formula for $\mathrm{DR}_g(\mu, \nu)$ in the tautological ring conjectured by Pixton in 2014. The formula expresses the double ramification cycle as a sum over stable graphs (corresponding to strata classes) with summand equal to a product over markings and edges. The result answers a question of Eliashberg from 2001 and specializes to Hain's formula in the compact type case.

When $\mu = \nu = \emptyset$, the formula for double ramification cycles expresses the top Chern class λ_g of the Hodge bundle of $\overline{\mathcal{M}}_g$ as a push-forward of tautological classes supported on the divisor of non-separating nodes. Applications to Hodge integral calculations are given.

0. Introduction

0.1. *Tautological rings*

Let $\overline{\mathcal{M}}_{g,n}$ be the moduli space of stable curves of genus g with n marked points. Since Mumford's article [28], there has been substantial progress in the study of the structure of the tautological rings[1]

$$\mathrm{R}^*(\overline{\mathcal{M}}_{g,n}) \subset \mathrm{A}^*(\overline{\mathcal{M}}_{g,n})$$

of the moduli spaces of curves. We refer the reader to [12] for a survey of the basic definitions and properties of $\mathrm{R}^*(\overline{\mathcal{M}}_{g,n})$.

The recent results [21, 31, 32] concerning relations in $\mathrm{R}^*(\overline{\mathcal{M}}_{g,n})$, conjectured to be *all* relations [33], may be viewed as parallel to the presentation

$$\mathrm{A}^*\big(\mathrm{Gr}(r, n)\big) = \mathbf{Q}\big[c_1(\mathrm{S}), \ldots, c_r(\mathrm{S})\big]/\big(s_{n-r+1}(\mathrm{S}), \ldots, s_n(\mathrm{S})\big)$$

of the Chow ring of the Grassmannian via the Chern and Segre classes of the universal subbundle

$$\mathrm{S} \to \mathrm{Gr}(r, n).$$

The Schubert calculus for the Grassmannian contains several explicit formulas for geometric loci. Our main result here is an explicit formula for the double ramification cycle in $\mathrm{R}^*(\overline{\mathcal{M}}_{g,n})$.

[1] All Chow groups will be taken with $\mathbf{Q}$-coefficients.

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0.2. *Double ramification cycles*

0.2.1. *Notation*

Double ramification data for maps will be specified by a vector

$$\mathrm{A} = (a_1, \ldots, a_n), \qquad a_i \in \mathbf{Z}$$

satisfying the balancing condition

$$\sum_{i=1}^{n} a_i = 0.$$

The integers a_i are the *parts* of A. We separate the positive, negative, and 0 parts of A as follows. The positive parts of A define a partition

$$\mu = \mu_1 + \cdots + \mu_{\ell(\mu)}.$$

The negatives of the negative parts of A define a second partition

$$\nu = \nu_1 + \cdots + \nu_{\ell(\nu)}.$$

Up to a reordering[2] of the parts, we have

$$\mathrm{A} = (\mu_1, \ldots, \mu_{\ell(\mu)}, -\nu_1, \ldots, -\nu_{\ell(\nu)}, \underbrace{0, \ldots, 0}_{n-\ell(\mu)-\ell(\nu)}). \tag{1}$$

Since the parts of A sum to 0, the partitions μ and ν must be of the same size. Let

$$\mathrm{D} = |\mu| = |\nu|$$

be the *degree* of A. Let I be the set of markings corresponding to the 0 parts of A.

The various constituents of A are permitted to be empty. The degree 0 case occurs when

$$\mu = \nu = \emptyset.$$

If $\mathrm{I} = \emptyset$, then $n = \ell(\mu) + \ell(\nu)$. The empty vector A is permitted, then

$$n = 0, \qquad \mathrm{D} = 0, \quad \text{and} \quad \mu = \nu = \mathrm{I} = \emptyset.$$

[2] There will be no disadvantage in assuming A is ordered as in (1).

0.2.2. *Nonsingular curves*

Let $\mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,n}$ be the moduli space of nonsingular pointed curves. Let

$$\mathrm{A} = (a_1, \ldots, a_n)$$

be a vector of double ramification data as defined in Section 0.2.1. Let

$$\mathcal{Z}_g(\mathrm{A}) \subset \mathcal{M}_{g,n}$$

be the locus[3] parameterizing curves $[\mathrm{C}, p_1, \ldots, p_n] \in \mathcal{M}_{g,n}$ satisfying

$$\mathcal{O}_{\mathrm{C}}\left(\sum_i a_i p_i\right) \overset{\sim}{=} \mathcal{O}_{\mathrm{C}}. \tag{2}$$

Condition (2) is algebraic and defines $\mathcal{Z}_g(\mathrm{A})$ canonically as a substack of $\mathcal{M}_{g,n}$.

If $[\mathrm{C}, p_1, \ldots, p_n] \in \mathcal{Z}_g(\mathrm{A})$, the defining condition (2) yields a rational function (up to $\mathbf{C}^*$-scaling),

$$f : \mathrm{C} \to \mathbf{P}^1, \tag{3}$$

of degree D with ramification profile μ over $0 \in \mathbf{P}^1$ and ν over $\infty \in \mathbf{P}^1$. The markings corresponding to 0 parts lie over $\mathbf{P}^1 \setminus \{0, \infty\}$. Conversely, every such morphism (3), up to $\mathbf{C}^*$-scaling, determines an element of $\mathcal{Z}_g(\mathrm{A})$.

We may therefore view $\mathcal{Z}_g(\mathrm{A}) \subset \mathcal{M}_{g,n}$ as the moduli space of degree D maps (up to $\mathbf{C}^*$-scaling),

$$f : \mathrm{C} \to \mathbf{P}^1,$$

with ramification profiles μ and ν over 0 and ∞ respectively. The term *double ramification* is motivated by the geometry of the map f.

The dimension of $\mathcal{Z}_g(\mathrm{A})$ is easily calculated via the theory of Hurwitz covers. Every map (3) can be deformed within $\mathcal{Z}_g(\mathrm{A})$ to a map with only *simple ramification* over $\mathbf{P}^1 \setminus \{0, \infty\}$. The number of simple branch points in $\mathbf{P}^1 \setminus \{0, \infty\}$ is determined by the Riemann-Hurwitz formula to equal

$$\ell(\mu) + \ell(\nu) + 2g - 2.$$

The dimension of the irreducible components of $\mathcal{Z}_g(\mathrm{A})$ can be calculated by varying the branch points [36] to equal

$$\ell(\mu) + \ell(\nu) + 2g - 2 + \ell(\mathrm{I}) - 1 = 2g - 3 + n.$$

The -1 on the left is the effect of the $\mathbf{C}^*$-scaling. Hence, $\mathcal{Z}_g(\mathrm{A})$ is of *pure codimension g* in $\mathcal{M}_{g,n}$.

[3] When considering $\mathcal{Z}_g(\mathrm{A})$, we *always* assume the stability condition $2g - 2 + n > 0$ holds.

0.2.3. *The Abel-Jacobi map*

Let $\mathcal{M}^{\mathrm{ct}}_{g,n} \subset \overline{\mathcal{M}}_{g,n}$ be the moduli space of curves of compact type. The universal Jacobian

$$\mathrm{Jac}_{g,n} \to \mathcal{M}^{\mathrm{ct}}_{g,n}$$

is the moduli space of line bundles on the universal curve of degree 0 on *every* irreducible component of *every* fiber.

There is a natural compactification of $\mathcal{Z}_g(\mathrm{A})$ in the moduli space of compact type curves,

$$\mathcal{Z}_g(\mathrm{A}) \subset \mathcal{Z}^{\mathrm{ct}}_g(\mathrm{A}) \subset \mathcal{M}^{\mathrm{ct}}_{g,n},$$

obtained from the geometry of the universal Jacobian.

Consider the universal curve over the compact type moduli space

$$\pi^{\mathrm{ct}} : \mathcal{C}^{\mathrm{ct}} \to \mathcal{M}^{\mathrm{ct}}_{g,n}.$$

The double ramification vector $\mathrm{A} = (a_1, \ldots, a_n)$ determines a line bundle $\mathcal{L}$ on $\mathcal{C}^{\mathrm{ct}}$ of relative degree 0,

$$\mathcal{L} = \sum_i a_i[\mathrm{p}_i],$$

where $\mathrm{p}_i \subset \mathcal{C}^{\mathrm{ct}}$ is the section of π^{ct} corresponding to the marking p_i.

By twisting $\mathcal{L}$ by *components* of the π^{ct} inverse images of the boundary divisors of $\mathcal{M}^{\mathrm{ct}}_{g,n}$, we can easily construct a line bundle $\mathcal{L}'$ which has degree 0 on every component of every fiber of π^{ct}. The result $\mathcal{L}'$ is unique up to twisting by the π^{ct} inverse images of the boundary divisors of $\mathcal{M}^{\mathrm{ct}}_{g,n}$.

Via $\mathcal{L}'$, we obtain a section of the universal Jacobian,

$$\phi : \mathcal{M}^{\mathrm{ct}}_{g,n} \to \mathrm{Jac}_{g,n}. \tag{4}$$

Certainly the closure of $\mathcal{Z}_g(\mathrm{A})$ in $\mathcal{M}^{\mathrm{ct}}_{g,n}$ lies in the ϕ inverse image of the 0-section $\mathrm{S} \subset \mathrm{Jac}_{g,n}$ of the relative Jacobian. We define

$$\mathcal{Z}^{\mathrm{ct}}_g(\mathrm{A}) = \phi^{-1}(\mathrm{S}) \subset \mathcal{M}^{\mathrm{ct}}_{g,n}. \tag{5}$$

The substack $\mathcal{Z}^{\mathrm{ct}}_g(\mathrm{A})$ is independent of the choice of $\mathcal{L}'$. From the definitions, we see

$$\mathcal{Z}^{\mathrm{ct}}_g(\mathrm{A}) \cap \mathcal{M}_{g,n} = \mathcal{Z}_g(\mathrm{A}).$$

While $S \subset \mathrm{Jac}_{g,n}$ is of pure codimension g, $\mathcal{Z}_g(A)^{ct}$ typically has components of excess dimension (obtained from genus 0 components of the domain).[4] A cycle class of the expected dimension

$$\mathrm{DR}_g^{ct}(A) = \left[\mathcal{Z}_g^{ct}(A)\right]^{vir} \in A^g\left(\mathcal{M}_{g,n}^{ct}\right)$$

is defined by $\phi^*([S])$. A closed formula for $\mathrm{DR}_g^{ct}(A)$ was obtained by Hain [19]. A simpler approach was later provided by Grushevsky and Zakharov [18].

0.2.4. *Stable maps to rubber*

Let $A = (a_1, \ldots, a_n)$ be a vector of double ramification data. The moduli space

$$\overline{\mathcal{M}}_{g,I}\left(\mathbf{P}^1, \mu, \nu\right)^\sim$$

parameterizes *stable relative maps* of connected curves of genus g to rubber with ramification profiles μ, ν over the point $0, \infty \in \mathbf{P}^1$ respectively. The tilde indicates a rubber target. We refer the reader to [25, 26] for the foundations. There is a natural morphism

$$\epsilon : \overline{\mathcal{M}}_{g,I}\left(\mathbf{P}^1, \mu, \nu\right)^\sim \to \overline{\mathcal{M}}_{g,n}$$

forgetting everything except the marked domain curve.

Let $R/\{0, \infty\}$ be a rubber target,[5] and let

$$\left[f : C \to R/\{0, \infty\}\right] \in \overline{\mathcal{M}}_{g,I}\left(\mathbf{P}^1, \mu, \nu\right)^\sim$$

be a moduli point. If C is of compact type and $R \overset{\sim}{=} \mathbf{P}^1$, then

$$\epsilon\left([f]\right) \in \mathcal{Z}_g^{ct}(A) \subset \overline{\mathcal{M}}_{g,n}.$$

Hence, we have the inclusion

$$\mathcal{Z}_g^{ct}(A) \subset \mathrm{Im}(\epsilon) \subset \overline{\mathcal{M}}_{g,n}.$$

The virtual dimension of $\overline{\mathcal{M}}_{g,I}(\mathbf{P}^1, \mu, \nu)^\sim$ is $2g - 3 + n$. We define the *double ramification cycle* to be the push-forward

$$\mathrm{DR}_g(A) = \epsilon_*\left[\overline{\mathcal{M}}_{g,I}\left(\mathbf{P}^1, \mu, \nu\right)^\sim\right]^{vir} \in A^g(\overline{\mathcal{M}}_{g,n}).$$

[4] Hence, $\mathcal{Z}_g(A) \subset \mathcal{Z}_g^{ct}(A)$ is *not* always dense.

[5] R is a chain of $\mathbf{P}^1$s.

0.2.5. *Basic properties*

The first property of $\mathrm{DR}_g(A)$ is a compatibility with the Abel-Jacobi construction of Section 0.2.3. Let

$$\iota : \mathcal{M}_{g,n}^{\mathrm{ct}} \to \overline{\mathcal{M}}_{g,n}$$

be the open inclusion map.[6]

Proposition **1** *(Marcus-Wise [27]). — We have*

$$\iota^* \mathrm{DR}_g(A) = \mathrm{DR}_g^{\mathrm{ct}}(A) \in A^g\big(\mathcal{M}_{g,n}^{\mathrm{ct}}\big).$$

By definition, $\mathrm{DR}_g(A)$ is a Chow class on the moduli space of stable curves. In fact, the double ramification cycle lies in the tautological ring.

Proposition **2** *(Faber-Pandharipande [11]).* — $\mathrm{DR}_g(A) \in R^g(\overline{\mathcal{M}}_{g,n})$.

The proof of [11] provides an algorithm to calculate $\mathrm{DR}_g(A)$ in the tautological ring, but the complexity of the method is too great: there is no apparent way to obtain an explicit formula for $\mathrm{DR}_g(A)$ directly from [11].

0.3. *Stable graphs and strata*

0.3.1. *Summation over stable graphs*

The strata of $\overline{\mathcal{M}}_{g,n}$ are the quasi-projective subvarieties parameterizing pointed curves of a fixed *topological type*. The moduli space $\overline{\mathcal{M}}_{g,n}$ is a disjoint union of finitely many strata.

The main result of the paper is a proof of an explicit formula conjectured by Pixton [34] in 2014 for $\mathrm{DR}_g(A)$ in the tautological ring $R^*(\overline{\mathcal{M}}_{g,n})$. The formula is written in terms of a summation over stable graphs Γ indexing the strata of $\overline{\mathcal{M}}_{g,n}$. We review here the standard indexing of the strata of $\overline{\mathcal{M}}_{g,n}$ by stable graphs.

0.3.2. *Stable graphs*

The strata of the moduli space of curves correspond to *stable graphs*

$$\Gamma = (V, H, L, g : V \to \mathbf{Z}_{\geq 0}, v : H \to V, \iota : H \to H)$$

satisfying the following properties:

[6] Compatibility on the smaller open set of curves with rational tails was proven earlier in [4].

(i) V is a vertex set with a genus function $\mathrm{g} : \mathrm{V} \to \mathbf{Z}_{\geq 0}$,
(ii) H is a half-edge set equipped with a vertex assignment $v : \mathrm{H} \to \mathrm{V}$ and an involution ι,
(iii) E, the edge set, is defined by the 2-cycles of ι in H (self-edges at vertices are permitted),
(iv) L, the set of legs, is defined by the fixed points of ι and is placed in bijective correspondence with a set of markings,
(v) the pair (V, E) defines a *connected* graph,
(vi) for each vertex v, the stability condition holds:

$$2\mathrm{g}(v) - 2 + \mathrm{n}(v) > 0,$$

where $\mathrm{n}(v)$ is the valence of Γ at v including both half-edges and legs.

An automorphism of Γ consists of automorphisms of the sets V and H which leave invariant the structures L, g, v, and ι. Let $\mathrm{Aut}(\Gamma)$ denote the automorphism group of Γ.

The genus of a stable graph Γ is defined by:

$$\mathrm{g}(\Gamma) = \sum_{v \in \mathrm{V}} \mathrm{g}(v) + h^1(\Gamma).$$

A quasi-projective stratum of $\overline{\mathcal{M}}_{g,n}$ corresponding to Deligne-Mumford stable curves of fixed topological type naturally determines a stable graph of genus g with n legs by considering the dual graph of a generic pointed curve parameterized by the stratum.

Let $\mathrm{G}_{g,n}$ be the set of isomorphism classes of stable graphs of genus g with n legs. The set $\mathrm{G}_{g,n}$ is finite.

0.3.3. *Strata classes*

To each stable graph $\Gamma \in \mathrm{G}_{g,n}$, we associate the moduli space

$$\overline{\mathcal{M}}_\Gamma = \prod_{v \in \mathrm{V}} \overline{\mathcal{M}}_{\mathrm{g}(v),\mathrm{n}(v)}.$$

There is a canonical morphism

$$\xi_\Gamma : \overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}_{g,n} \qquad (\mathbf{6})$$

with image[7] equal to the closure of the stratum associated to the graph Γ. To construct ξ_Γ, a family of stable pointed curves over $\overline{\mathcal{M}}_\Gamma$ is required. Such a family is easily defined by attaching the pull-backs of the universal families over each of the $\overline{\mathcal{M}}_{\mathrm{g}(v),\mathrm{n}(v)}$ along the sections corresponding to the two halves of each edge in $\mathrm{E}(\Gamma)$.

[7] The degree of ξ_Γ is $|\mathrm{Aut}(\Gamma)|$.

Each half-edge $h \in \mathrm{H}(\Gamma)$ determines a cotangent line

$$\mathcal{L}_h \to \overline{\mathcal{M}}_\Gamma.$$

If $h \in \mathrm{L}(\Gamma)$, then $\mathcal{L}_h$ is the pull-back via ξ_Γ of the corresponding cotangent line of $\overline{\mathcal{M}}_{g,n}$. If h is a side of an edge $e \in \mathrm{E}(\Gamma)$, then $\mathcal{L}_h$ is the cotangent line of the corresponding side of a node. Let

$$\psi_h = c_1(\mathcal{L}_h) \in \mathrm{A}^1(\overline{\mathcal{M}}_\Gamma, \mathbf{Q}).$$

0.4. *Pixton's formula*

0.4.1. *Weightings*

Let $\mathrm{A} = (a_1, \ldots, a_n)$ be a vector of double ramification data. Let $\Gamma \in \mathrm{G}_{g,n}$ be a stable graph of genus g with n legs. A *weighting* of Γ is a function on the set of half-edges,

$$w : \mathrm{H}(\Gamma) \to \mathbf{Z},$$

which satisfies the following three properties:

(i) $\forall h_i \in \mathrm{L}(\Gamma)$, corresponding to the marking $i \in \{1, \ldots, n\}$,

$$w(h_i) = a_i,$$

(ii) $\forall e \in \mathrm{E}(\Gamma)$, corresponding to two half-edges $h, h' \in \mathrm{H}(\Gamma)$,

$$w(h) + w\big(h'\big) = 0,$$

(iii) $\forall v \in \mathrm{V}(\Gamma)$,

$$\sum_{v(h)=v} w(h) = 0,$$

where the sum is taken over *all* $n(v)$ half-edges incident to v.

If the graph Γ has cycles, Γ may carry infinitely many weightings.

Let r be a positive integer. A *weighting mod* r of Γ is a function,

$$w : \mathrm{H}(\Gamma) \to \{0, \ldots, r-1\},$$

which satisfies exactly properties (i)–(iii) above, but with the equalities replaced, in each case, by the condition of *congruence mod* r. For example, for (i), we require

$$w(h_i) = a_i \mod r.$$

Let $\mathrm{W}_{\Gamma,r}$ be the set of weightings mod r of Γ. The set $\mathrm{W}_{\Gamma,r}$ is finite, with cardinality $r^{h^1(\Gamma)}$. We view r as a *regularization parameter*.

0.4.2. *Pixton's conjecture*

Let $\mathsf{A} = (a_1, \dots, a_n)$ be a vector of double ramification data. Let r be a positive integer. We denote by $\mathsf{P}_g^{d,r}(\mathsf{A}) \in \mathsf{R}^d(\overline{\mathcal{M}}_{g,n})$ the degree d component of the tautological class

$$\sum_{\Gamma \in \mathsf{G}_{g,n}} \sum_{w \in \mathsf{W}_{\Gamma,r}} \frac{1}{|\mathrm{Aut}(\Gamma)|} \frac{1}{r^{h^1(\Gamma)}} \xi_{\Gamma*} \Bigg[\prod_{i=1}^{n} \exp\big(a_i^2 \psi_{h_i}\big) \times \prod_{e=(h,h') \in \mathsf{E}(\Gamma)} \frac{1 - \exp(-w(h)w(h')(\psi_h + \psi_{h'}))}{\psi_h + \psi_{h'}} \Bigg],$$

in $\mathsf{R}^*(\overline{\mathcal{M}}_{g,n})$.

Inside the push-forward in the above formula, the first product

$$\prod_{i=1}^{n} \exp\big(a_i^2 \psi_{h_i}\big)$$

is over $h \in \mathsf{L}(\Gamma)$ via the correspondence of legs and markings. The second product is over all $e \in \mathsf{E}(\Gamma)$. The factor

$$\frac{1 - \exp(-w(h)w(h')(\psi_h + \psi_{h'}))}{\psi_h + \psi_{h'}}$$

is well-defined since

- the denominator formally divides the numerator,
- the factor is symmetric in h and h'.

No edge orientation is necessary.

The following fundamental polynomiality property of $\mathsf{P}_g^{d,r}(\mathsf{A})$ has been proven by Pixton.

Proposition **3** *(Pixton [35]). — For fixed g, A, and d, the class*

$$\mathsf{P}_g^{d,r}(\mathsf{A}) \in \mathsf{R}^d(\overline{\mathcal{M}}_{g,n})$$

is polynomial in r (for all sufficiently large r).

We denote by $\mathsf{P}_g^{d}(\mathsf{A})$ the value at $r = 0$ of the polynomial associated to $\mathsf{P}_g^{d,r}(\mathsf{A})$ by Proposition 3. In other words, $\mathsf{P}_g^{d}(\mathsf{A})$ is the *constant* term of the associated polynomial in r. For the reader's convenience, Pixton's proof of Proposition 3 is given in the Appendix.

The main result of the paper is a proof of the formula for double ramification cycles conjectured earlier by Pixton [34].

Theorem **1.** — *For $g \geq 0$ and double ramification data* A*, we have*

$$\mathrm{DR}_g(\mathrm{A}) = 2^{-g}\mathrm{P}_g^g(\mathrm{A}) \in \mathrm{R}^g(\overline{\mathcal{M}}_{g,n}).$$

For $d < g$, the classes $\mathrm{P}_g^d(\mathrm{A})$ do not yet have a geometric interpretation. However for $d > g$, the following vanishing conjectured by Pixton is now established.

Theorem **2** *(Clader-Janda [8]).* — *For all $g \geq 0$, double ramification data* A*, and $d > g$, we have*

$$\mathrm{P}_g^d(\mathrm{A}) = 0 \in \mathrm{R}^d(\overline{\mathcal{M}}_{g,n}).$$

Pixton [34] has further proposed a twist of the formula for $\mathrm{P}_g^d(\mathrm{A})$ by $k \in \mathbf{Z}$. The codimension g class in the $k = 1$ case has been (conjecturally) related to the moduli spaces of meromorphic differential in the Appendix of [13]. The vanishing result of Clader-Janda [8] proves Pixton's vanishing conjecture in codimensions $d > g$ for all k. The k-twisted theory will be discussed in Section 1.

In a forthcoming paper [22], we will generalize Pixton's formula to the situation of maps to a $\mathbf{P}^1$-rubber bundle over a target manifold X.

0.5. *Basic examples*

0.5.1. *Genus 0*

Let $g = 0$ and $\mathrm{A} = (a_1, \dots, a_n)$ be a vector of double ramification data. The graphs $\Gamma \in \mathrm{G}_{0,n}$ are trees. Each $\Gamma \in \mathrm{G}_{0,n}$ has a *unique* weighting mod r (for each r). We can directly calculate:

$$\mathrm{P}_0(\mathrm{A}) = \sum_{d\geq 0} \mathrm{P}_0^d(\mathrm{A}) = \exp\left[\sum_{i=1}^n a_i^2\psi_i - \frac{1}{2}\sum_{\substack{\mathrm{I}\sqcup\mathrm{J}=\{1,\dots,n\}\\ |\mathrm{I}|,|\mathrm{J}|\geq 2}} a_\mathrm{I}^2\delta_{\mathrm{I,J}}\right], \tag{7}$$

where $a_\mathrm{I} = \sum_{i\in\mathrm{I}} a_i$.

By Theorem 1, the double ramification cycle is the degree 0 term of (7),

$$\mathrm{DR}_0(\mathrm{A}) = 2^{-0}\mathrm{P}_0^0(\mathrm{A}) = 1.$$

The vanishing of Theorem 2 implies

$$\sum_{i=1}^n a_i^2\psi_i - \frac{1}{2}\sum_{\substack{\mathrm{I}\sqcup\mathrm{J}=\{1,\dots,n\}\\ |\mathrm{I}|,|\mathrm{J}|\geq 2}} a_\mathrm{I}^2\delta_{\mathrm{I,J}} = 0 \in \mathrm{R}^1(\overline{\mathcal{M}}_{0,n}),$$

where $\delta_{\mathrm{I,J}} \in \mathrm{R}^1(\overline{\mathcal{M}}_{0,n})$ is the class of the boundary divisor with marking distribution $\mathrm{I}\sqcup\mathrm{J}$. The above vanishing can be proven here directly by expressing ψ classes in terms of boundary divisors.

0.5.2. *Genus 1*

Let $g = 1$ and $A = (a_1, \dots, a_n)$ be a vector of double ramification data. Denote by $\delta_0 \in R^1(\overline{\mathcal{M}}_{1,n})$ the boundary divisor of singular stable curves with a nonseparating node. Our convention is

$$\delta_0 = \frac{1}{2}\xi_*[\overline{\mathcal{M}}_{0,n+2}], \quad \xi : \overline{\mathcal{M}}_{0,n+2} \to \overline{\mathcal{M}}_{1,n}.$$

The division by the order of the automorphism group is included in the definition of δ_0. Denote by

$$\delta_I \in R^1(\overline{\mathcal{M}}_{1,n})$$

the class of the boundary divisor of curves with a rational component carrying the markings $I \subset \{1, \dots, n\}$ and an elliptic component carrying the markings I^c.

We can calculate directly from the definitions:

$$P_1^1(A) = \sum_{i=1}^{n} a_i^2 \psi_i - \sum_{\substack{I \subset \{1,\dots,n\} \\ |I| \geq 2}} a_I^2 \delta_I - \frac{1}{6}\delta_0. \tag{8}$$

As before, $a_I = \sum_{i \in I} a_i$. By Theorem 1,

$$DR_1(A) = \frac{1}{2} P_1^1(A).$$

In genus 1, the virtual class plays essentially no role (since the moduli spaces of maps to rubber are of expected dimension). The genus 1 formula was already known to Hain [19].

It is instructive to compute the coefficient $-\frac{1}{6}$ of δ_0. The class δ_0 corresponds to the graph Γ with one vertex of genus 0 and one loop,

$$h^1(\Gamma) = 1.$$

According to the definition, the coefficient of δ_0 is the r-free term of the polynomial

$$\frac{1}{r}\left[\sum_{w=1}^{r-1} w(r-w)\right],$$

which is $-B_2 = -\frac{1}{6}$.

0.5.3. *Degree 0*

Let $g \geq 0$ and $\mathrm{A} = (0, \ldots, 0)$, so $\mu = \nu = \emptyset$. The canonical map ϵ from the moduli of stable maps to the moduli of curves is then an isomorphism

$$\epsilon : \overline{\mathcal{M}}_{g,n}\big(\mathbf{P}^1, \emptyset, \emptyset\big)^{\sim} \xrightarrow{\sim} \overline{\mathcal{M}}_{g,n}.$$

By an analysis of the obstruction theory,

$$\mathrm{DR}_g(0, \ldots, 0) = (-1)^g \lambda_g \in \mathrm{R}^g(\overline{\mathcal{M}}_{g,n})$$

where λ_g is the top Chern class of the Hodge bundle

$$\mathbf{E} \to \overline{\mathcal{M}}_{g,n}.$$

Let $\Gamma \in \mathrm{G}_{g,n}$ be a stable graph. By the definition of a *weighting mod r* and the condition $a_i = 0$ for all markings i, the weights

$$w(h), \;\; w\big(h'\big)$$

on the two halves of *every* separating edge e of Γ must both be 0. The factor in Pixton's formula for e,

$$\frac{1 - \exp(-w(h)w(h')(\psi_h + \psi_{h'}))}{\psi_h + \psi_{h'}},$$

then vanishes and kills the contribution of Γ to $\mathrm{P}_g^g(0, \ldots, 0)$.

Let $g \geq 1$. Let ξ be the map to $\overline{\mathcal{M}}_{g,n}$ associated to the divisor of curves with non-separating nodes

$$\xi : \overline{\mathcal{M}}_{g-1,n+2} \to \overline{\mathcal{M}}_{g,n}.$$

Since graphs Γ with separating edges do not contribute to $\mathrm{P}_g^g(0, \ldots, 0)$ and the trivial graph with no edges does not contribute in codimension g,

$$\mathrm{P}_g^g(0, \ldots, 0) = \xi_* \Lambda_{g-1}^{g-1}(0, \ldots, 0)$$

for an explicit tautological class

$$\Lambda_{g-1}^{g-1}(0, \ldots, 0) \in \mathrm{R}^{g-1}(\overline{\mathcal{M}}_{g-1,n+2}).$$

Corollary **3.** — *For $g \geq 1$, we have*

$$\lambda_g = (-2)^{-g} \xi_* \Lambda_{g-1}^{g-1}(0, \ldots, 0) \in \mathrm{R}^g(\overline{\mathcal{M}}_{g,n}).$$

The class λ_g is easily proven to vanish on $\mathcal{M}^{\mathrm{ct}}_{g,n}$ via the Abel-Jacobi map (4) and well-known properties of the moduli space of abelian varieties [38]. Hence, there exists a *Chow* class $\gamma_{g-1,n+2} \in \mathrm{A}^{g-1}(\overline{\mathcal{M}}_{g-1,n+2})$ satisfying

$$\lambda_g = \xi_* \gamma_{g-1,n+2} \in \mathrm{A}^g(\overline{\mathcal{M}}_{g,n}).$$

Corollary 3 is a much stronger statement since $\Lambda^{g-1}_{g-1}(0,\ldots,0)$ is a *tautological* class on $\overline{\mathcal{M}}_{g-1,n+2}$ given by an explicit formula. No such expressions were known before.

- For $\overline{\mathcal{M}}_{1,n}$, Corollary 3 yields

$$\lambda_1 = -\frac{1}{2}\xi_*\Lambda^{1-1}_{1-1}(0,\ldots,0) \in \mathrm{R}^1(\overline{\mathcal{M}}_{1,n}).$$

Moreover, by calculation (8),

$$\Lambda^{1-1}_{1-1}(0,\ldots,0) = -\frac{1}{12}[\overline{\mathcal{M}}_{0,n+2}],$$

since δ_0 already carries a $\frac{1}{2}$ for the graph automorphism. Hence, we recover the well-known formula

$$\lambda_1 = \frac{1}{24}\xi_*[\overline{\mathcal{M}}_{0,n+2}] \in \mathrm{R}^1(\overline{\mathcal{M}}_{1,n}).$$

- For $\overline{\mathcal{M}}_2$, Corollary 3 yields

$$\lambda_2 = \frac{1}{4}\xi_*\Lambda^{2-1}_{2-1}(\emptyset) \in \mathrm{R}^2(\overline{\mathcal{M}}_2).$$

Denote by α the class of the boundary stratum of genus 0 curves with two self-intersections. We include the division by the order of the automorphism group in the definition,

$$\alpha = \frac{1}{8}\xi_*[\overline{\mathcal{M}}_{0,4}].$$

Denote by β the class of the boundary divisor of genus 1 curves with one self-intersection multiplied by the ψ-class at one of the branches. Then we have

$$\xi_*\Lambda^{2-1}_{2-1}(\emptyset) = \frac{1}{36}\alpha + \frac{1}{60}\beta. \tag{9}$$

The classes α and β form a basis of $\mathrm{R}^2(\overline{\mathcal{M}}_2)$. The expression

$$\lambda_2 = \frac{1}{144}\alpha + \frac{1}{240}\beta$$

can be checked by intersection with the two boundary divisors of $\overline{\mathcal{M}}_2$.

The coefficient of α in (9) is obtained from the r-free term of

$$\frac{1}{r^2}\sum_{1\leq w_1,w_2\leq r-1} w_1w_2(r-w_1)(r-w_2).$$

The answer is $\mathrm{B}_2^2=\frac{1}{36}$. The coefficient of β is the r-free term of

$$-\frac{1}{2r}\sum_{w=1}^{r-1} w^2(r-w)^2$$

given by $-\frac{1}{2}\mathrm{B}_4=\frac{1}{60}$.

0.6. *Strategy of proof*

The proof of Proposition 2 given in [11] uses the Gromov-Witten theory of $\mathbf{P}^1$ relative to the point $\infty\in\mathbf{P}^1$. The localization relations of [11] intertwine Hodge classes on the moduli spaces of curves with cycles obtained from maps to rubber. The interplay is very complicated with many auxiliary classes (not just the double ramification cycles).

Our proof of Theorem 1 is obtained by studying the Gromov-Witten theory of $\mathbf{P}^1$ with an orbifold[8] point $\mathrm{B}\mathbf{Z}_r$ at $0\in\mathbf{P}^1$ and a relative point $\infty\in\mathbf{P}^1$. Let $(\mathbf{P}^1[r],\infty)$ denote the resulting orbifold/relative geometry.[9] The proof may be explained conceptually as studying the localization relations for all $(\mathbf{P}^1[r],\infty)$ *simultaneously*. For large r, the localization relations (after appropriate rescaling) may be viewed as having polynomial coefficients in r. Then the $r=0$ specialization of the linear algebra exactly yields Theorem 1. The regularization parameter r in the definition of Pixton's formula in Section 0.4 is related to the orbifold structure $\mathrm{B}\mathbf{Z}_r$.

Our proof is inspired by the structure of Pixton's formula—especially the existence of the regularization parameter r. The argument requires two main steps:

• Let $\overline{\mathcal{M}}^r_{g;a_1,\ldots,a_n}$ be the moduli space of rth roots (with respect to the tensor product) of the bundles $\mathcal{O}_{\mathrm{C}}(\sum_{i=1}^n a_ix_i)$ associated to pointed curves $[\mathrm{C},p_1,\ldots,p_n]$. Let

$$\pi:\mathcal{C}^r_{g;a_1,\ldots,a_n}\to\overline{\mathcal{M}}^r_{g;a_1,\ldots,a_n}$$

be the universal curve, and let $\mathcal{L}$ be the universal rth root over the universal curve. Chiodo's formulas [6] allow us to prove that the push-forward of $r\cdot c_g(-\mathrm{R}^*\pi_*\mathcal{L})$ to $\overline{\mathcal{M}}_{g,n}$ and $2^{-g}\mathrm{P}_g^{g,r}(\mathrm{A})$ are polynomials in r with the *same* constant term. Pixton's formula therefore has a geometric interpretation in terms of the intersection theory of the moduli space of rth roots.

[8] $\mathrm{B}\mathbf{Z}_r$ is the quotient stack obtained by the trivial action of $\mathbf{Z}_r\cong\frac{\mathbf{Z}}{r\mathbf{Z}}$ on a point.

[9] The target $(\mathbf{P}^1[r],\infty)$ was previously studied in [24] in the context of Hurwitz-Hodge integrals.

• We use the localization formula [15] for the virtual class of the moduli space of stable maps to the orbifold/relative geometry $(\mathbf{P}^1[r], \infty)$. The monodromy conditions at 0 are given by μ and the ramification profile over ∞ is given by ν. The push-forward of the localization formula to $\overline{\mathcal{M}}_{g,n}$ is a Laurent series in the equivariant parameter t and in r. The coefficient of $t^{-1}r^0$ must vanish by geometric considerations. We prove the relation obtained from the coefficient of $t^{-1}r^0$ has *only two terms*. The first is the constant term in r of the push-forward of $r \cdot c_g(-\mathrm{R}^*\pi_*\mathcal{L})$ to $\overline{\mathcal{M}}_{g,n}$. The second term is the double ramification cycle $\mathrm{DR}_g(\mathrm{A})$ with a minus sign. The vanishing of the sum of the two terms yields Theorem 1.

1. Moduli of stable maps to $\mathrm{B}\mathbf{Z}_r$

1.1. *Twisting by k*

Let $k \in \mathbf{Z}$. A vector $\mathrm{A} = (a_1, \ldots, a_n)$ of *k-twisted double ramification data* for genus $g \geq 0$ is defined by the condition[10]

$$\sum_{i=1}^{n} a_i = k(2g - 2 + n). \tag{10}$$

For $k = 0$, the condition (10) does *not* depend upon on the genus g and specializes to the balancing condition of the double ramification data of Section 0.2.1.

We recall the definition [34] of Pixton's k-twisted cycle $\mathrm{P}_g^{d,k}(\mathrm{A}) \in \mathrm{R}^d(\overline{\mathcal{M}}_{g,n})$. In case $k = 0$,

$$\mathrm{P}_g^{d,0}(\mathrm{A}) = \mathrm{P}_g^{d}(\mathrm{A}) \in \mathrm{R}^d(\overline{\mathcal{M}}_{g,n}).$$

Let $\Gamma \in \mathrm{G}_{g,n}$ be a stable graph of genus g with n legs. A *k-weighting mod r* of Γ is a function on the set of half-edges,

$$w : \mathrm{H}(\Gamma) \to \{0, \ldots, r-1\},$$

which satisfies the following three properties:

(i) $\forall h_i \in \mathrm{L}(\Gamma)$, corresponding to the marking $i \in \{1, \ldots, n\}$,

$$w(h_i) = a_i \mod r,$$

(ii) $\forall e \in \mathrm{E}(\Gamma)$, corresponding to two half-edges $h, h' \in \mathrm{H}(\Gamma)$,

$$w(h) + w\left(h'\right) = 0 \mod r$$

[10] We *always* assume the stability condition $2g - 2 + n > 0$ holds.

(iii) $\forall v \in \mathrm{V}(\Gamma)$,

$$\sum_{v(h)=v} w(h) = k\big(2\mathrm{g}(v) - 2 + \mathrm{n}(v)\big) \mod r,$$

where the sum is taken over *all* $n(v)$ half-edges incident to v.

We denote by $\mathrm{W}_{\Gamma,r,k}$ the set of all k-weightings mod r of Γ.

Let A be a vector of k-twisted ramification data for genus g. For each positive integer r, let $\mathrm{P}_g^{d,r,k}(\mathrm{A}) \in \mathrm{R}^d(\overline{\mathcal{M}}_{g,n})$ be the *degree d* component of the tautological class

$$\sum_{\Gamma\in \mathrm{G}_{g,n}} \sum_{w\in \mathrm{W}_{\Gamma,r,k}} \frac{1}{|\mathrm{Aut}(\Gamma)|} \frac{1}{r^{h^1(\Gamma)}} \xi_{\Gamma *}\Bigg[\prod_{v\in \mathrm{V}(\Gamma)} e^{-k^2\kappa_1(v)} \prod_{i=1}^n e^{a_i^2\psi_{h_i}} \times \prod_{e=(h,h')\in \mathrm{E}(\Gamma)} \frac{1-e^{-w(h)w(h')(\psi_h+\psi_{h'})}}{\psi_h+\psi_{h'}}\Bigg],$$

in $\mathrm{R}^*(\overline{\mathcal{M}}_{g,n})$.

All the conventions here are the same as in the definitions of Section 0.4.2. A new factor appears: $\kappa_1(v)$ is the first κ class,[11]

$$\kappa_1(v) \in \mathrm{R}^1(\overline{\mathcal{M}}_{\mathrm{g}(v),\mathrm{n}(v)}),$$

on the moduli space corresponding to a vertex v of the graph Γ. As in the untwisted case, polynomiality holds (see the Appendix).

Proposition **3′** *(Pixton [35]). — For fixed g, A, k, and d, the class*

$$\mathrm{P}_g^{d,r,k}(\mathrm{A}) \in \mathrm{R}^d(\overline{\mathcal{M}}_{g,n})$$

is polynomial in r (for all sufficiently large r).

We denote by $\mathrm{P}_g^{d,k}(\mathrm{A})$ the value at $r=0$ of the polynomial associated to $\mathrm{P}_g^{d,r,k}(\mathrm{A})$ by Proposition 3′. In other words, $\mathrm{P}_g^{d,k}(\mathrm{A})$ is the *constant* term of the associated polynomial in r.

In case $k=0$, Proposition 3′ specializes to Proposition 3. Pixton's proof is given in the Appendix.

[11] Our convention is $\kappa_i = \pi_*(\psi_{\mathrm{n}(v)+1}^{i+1}) \in \mathrm{R}^i(\overline{\mathcal{M}}_{\mathrm{g}(v),\mathrm{n}(v)})$ where

$$\pi : \overline{\mathcal{M}}_{\mathrm{g}(v),\mathrm{n}(v)+1} \to \overline{\mathcal{M}}_{\mathrm{g}(v),\mathrm{n}(v)}$$

is the map forgetting the marking $\mathrm{n}(v)+1$. For a review of κ and cotangent ψ classes, see [16].

1.2. *Generalized r-spin structures*

1.2.1. *Overview*

Let $[C, p_1, \ldots, p_n] \in \mathcal{M}_{g,n}$ be a nonsingular curve with distinct markings and canonical bundle ω_C. Let

$$\omega_{\log} = \omega_C(p_1 + \cdots + p_n)$$

be the *log canonical* bundle.

Let k and $a_1, \ldots, a_n$ be integers for which $k(2g - 2 + n) - \sum a_i$ is divisible by a positive integer r. Then rth roots L of the line bundle

$$\omega_{\log}^{\otimes k}\Big(-\sum a_i p_i\Big)$$

on C *exist*. The space of such rth tensor roots possesses a natural compactification $\overline{\mathcal{M}}^{r,k}_{g;a_1,\ldots,a_n}$ constructed in [5, 23]. The moduli space of rth roots carries a universal curve

$$\pi : \mathcal{C}^{r,k}_{g;a_1,\ldots,a_n} \to \overline{\mathcal{M}}^{r,k}_{g;a_1,\ldots,a_n}$$

and a universal line bundle

$$\mathcal{L} \to \mathcal{C}^{r,k}_{g;a_1,\ldots,a_n}$$

equipped with an isomorphism

$$\mathcal{L}^{\otimes r} \xrightarrow{\sim} \omega_{\log}^{\otimes k}\Big(-\sum a_i x_i\Big) \tag{11}$$

on $\mathcal{C}^{r,k}_{g;a_1,\ldots,a_n}$. In case $k = 0$, $\overline{\mathcal{M}}^{r,0}_{g;a_1,\ldots,a_n}$ is the space of stable maps [1] to the orbifold $B\mathbf{Z}_r$.

We review here the basic properties of the moduli spaces $\overline{\mathcal{M}}^{r,k}_{g;a_1,\ldots,a_n}$ which we require and refer the reader to [5, 23] for a foundational development of the theory.

1.2.2. *Covering the moduli of curves*

The forgetful map to the moduli of curves

$$\epsilon : \overline{\mathcal{M}}^{r,k}_{g;a_1,\ldots,a_n} \to \overline{\mathcal{M}}_{g,n}$$

is an r^{2g}-sheeted covering ramified over the boundary divisors. The degree of ϵ equals r^{2g-1} because each rth root L has a $\mathbf{Z}_r$ symmetry group obtained from multiplication by rth roots of unity in the fibers.[12]

[12] The $\mathbf{Z}_r$-action respects the isomorphism (11).

To transform ϵ to an *unramified* covering in the orbifold sense, the orbifold structure of $\overline{\mathcal{M}}_{g,n}$ must be altered. We introduce an extra $\mathbf{Z}_r$ stabilizer for each node of each stable curve, see [5]. The new orbifold thus obtained is called the moduli space of *r-stable curves*.

1.2.3. *Boundary strata*

The rth roots of the trivial line bundle over the cylinder are classified by a pair of integers

$$0 \le a', a'' \le r-1, \qquad a' + a'' = 0 \mod r \tag{12}$$

attached to the borders of the cylinder. Therefore, the nodes of stable curves in the compactification $\overline{\mathcal{M}}^{r,k}_{g;a_1,\ldots,a_n}$ carry the data (12) with a' and a'' assigned to the two branches meeting at the node.

In case the node is *separating*, the data (12) are determined uniquely by the way the genus and the marked points are distributed over the two components of the curve. In case the node is *nonseparating*, the data can be arbitrary and describes different boundary divisors.

More generally, the boundary strata of $\overline{\mathcal{M}}^{r,k}_{g;a_1,\ldots,a_n}$ are described by stable graphs $\Gamma \in \mathsf{G}_{g,n}$ together with a k-weighting mod r. The data (12) is obtained from the weighting w of the half-edges of Γ.

1.2.4. *Normal bundles of boundary divisors*

The normal bundle[13] of a boundary divisor in $\overline{\mathcal{M}}_{g,n}$ has first Chern class

$$-(\psi' + \psi''),$$

where ψ' and ψ'' are the ψ-classes associated to the branches of the node. In the moduli space of r-stable curves *and* in the space $\overline{\mathcal{M}}^{r,k}_{g;a_1,\ldots,a_n}$, the normal bundle to a boundary divisor has first Chern class

$$-\frac{\psi' + \psi''}{r}, \tag{13}$$

where the classes ψ' and ψ'' are pulled back from $\overline{\mathcal{M}}_{g,n}$.

The two standard boundary maps

$$\overline{\mathcal{M}}_{g-1,n+2} \xrightarrow{i} \overline{\mathcal{M}}_{g,n}$$

and

$$\overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \xrightarrow{j} \overline{\mathcal{M}}_{g,n}$$

[13] More precisely, the normal bundle of the associated gluing morphism ξ.

are, in the space of rth roots, replaced by diagrams

$$(\mathbf{14}) \qquad \overline{\mathcal{M}}^{r,k}_{g-1;a_1,\dots,a_n,a',a''} \xleftarrow{\phi} \mathcal{B}^{\mathrm{nonsep}}_{a',a''} \xrightarrow{i} \overline{\mathcal{M}}^{r,k}_{g;a_1,\dots,a_n}$$

and

$$(\mathbf{15}) \qquad \overline{\mathcal{M}}^{r,k}_{g_1;a_1,\dots,a_{n_1},a'} \times \overline{\mathcal{M}}^{r,k}_{g_2;a'',a_{n_1+1},\dots,a_n} \xleftarrow{\varphi} \mathcal{B}^{\mathrm{sep}}_{a',a''} \xrightarrow{j} \overline{\mathcal{M}}^{r,k}_{g;a_1,\dots,a_n}.$$

The maps ϕ and φ are of degree 1, but in general are not isomorphisms, see [7], Section 2.3.

Remark **1.** — There are two natural ways to introduce the orbifold structure on the space of rth roots. In the first, every node of the curve contributes an extra $\mathbf{Z}/r\mathbf{Z}$ in the stabilizer. In the second, a node of type (a', a'') contributes an extra $\mathbf{Z}/\gcd(a', r)\mathbf{Z}$ to the stabilizer. The first is used in [5], the second in [1]. Here, we follow the first convention to avoid gcd factors appearing the computations. In Remark 2, we indicate the changes which must be made in the computations if the second convention is followed. The final result is the *same* (independent of the choice of convention).

1.2.5. *Intersecting boundary strata*

Let (Γ_A, w_A) and (Γ_B, w_B) be two stable graphs in $G_{g,n}$ equipped with k-weightings mod r. The intersection of the two corresponding boundary strata of $\overline{\mathcal{M}}^{r,k}_{g;a_1,\dots,a_n}$ can be determined as follows.[14] Enumerate all pairs (Γ, w) of stable graphs with k-weightings mod r whose edges are marked with letters A, B or both in such a way that contracting all edges outside A yields (Γ_A, w_A) and contracting all edges outside B yields (Γ_B, w_B). Assign a factor

$$-\frac{1}{r}(\psi' + \psi'')$$

to every edge that is marked by both A and B. The sum of boundary strata given by these graphs multiplied by the appropriate combinations of ψ-classes represents the intersection of the boundary components (Γ_A, w_A) and (Γ_B, w_B).

1.3. *Chiodo's formula*

Let g be the genus, and let $k \in \mathbf{Z}$ and $a_1, \dots, a_n \in \mathbf{Z}$ satisfy

$$k(2g-2+n) - \sum a_i = 0 \mod r$$

for an integer $r > 0$.

[14] See the Appendix of [16] for a detailed discussion in the case of $\overline{\mathcal{M}}_{g,n}$.

Following the notation of Section 1.2, let

$$\pi : \mathcal{C}^{r,k}_{g;a_1,\ldots,a_n} \to \overline{\mathcal{M}}^{r,k}_{g;a_1,\ldots,a_n}$$

be the universal curve and $\mathcal{L} \to \mathcal{C}^{r,k}_{g;a_1,\ldots,a_n}$ the universal rth root. Our aim here is to describe the total Chern class

$$c\big(-\mathrm{R}^*\pi_*\mathcal{L}\big) \in \mathrm{A}^*\big(\overline{\mathcal{M}}^{r,k}_{g;a_1,\ldots,a_n}\big)$$

following the work of A. Chiodo [6].

Denote by $\mathrm{B}_m(x)$ the m-th Bernoulli polynomial,

$$\sum_{n=0}^{\infty} \mathrm{B}_n(x)\frac{t^n}{n!} = \frac{te^{xt}}{e^t - 1}.$$

Proposition **4.** — *The total Chern class $c(-\mathrm{R}^*\pi_*\mathcal{L})$ is equal to*

$$\sum_{\Gamma\in \mathrm{G}_{g,n}} \sum_{w\in \mathrm{W}_{\Gamma,r,k}} \frac{1}{|\mathrm{Aut}(\Gamma)|} r^{|\mathrm{E}(\Gamma)|} (\xi_{\Gamma,w})_* \Bigg[\prod_{v\in \mathrm{V}(\Gamma)} e^{-\sum_{m\geq 1}(-1)^{m-1}\frac{\mathrm{B}_{m+1}(k/r)}{m(m+1)}\kappa_m(v)}$$
$$\times \prod_{i=1}^{n} e^{\sum_{m\geq 1}(-1)^{m-1}\frac{\mathrm{B}_{m+1}(a_i/r)}{m(m+1)}\psi_{h_i}^m}$$
$$\times \prod_{\substack{e\in \mathrm{E}(\Gamma)\\ e=(h,h')}} \frac{1 - e^{\sum_{m\geq 1}(-1)^{m-1}\frac{\mathrm{B}_{m+1}(w(h)/r)}{m(m+1)}[(\psi_h)^m - (-\psi_{h'})^m]}}{\psi_h + \psi_{h'}} \Bigg]$$

in $\mathrm{A}^*(\overline{\mathcal{M}}^{r,k}_{g;a_1,\ldots,a_n})$.

Here, $\xi_{\Gamma,w}$ is the map of the boundary stratum corresponding to (Γ, w) to the moduli space $\overline{\mathcal{M}}^{r,k}_{g;a_1,\ldots,a_n}$,

$$\xi_{\Gamma,w} : \overline{\mathcal{M}}_{\Gamma,w} \to \overline{\mathcal{M}}^{r,k}_{g;a_1,\ldots,a_n}.$$

Every factor in the last product is symmetric in h and h' since

$$\mathrm{B}_{m+1}\left(\frac{w}{r}\right) = (-1)^{m+1}\mathrm{B}_{m+1}\left(\frac{r-w}{r}\right)$$

for $m \geq 1$.

Proof. — The proposition follows from Chiodo's formula [6] for the Chern characters of $R^*\pi_*\mathcal{L}$,

$$\mathrm{ch}_m(r,k;a_1,\dots,a_n) = \frac{B_{m+1}(\frac{k}{r})}{(m+1)!}\kappa_m - \sum_{i=1}^{n}\frac{B_{m+1}(\frac{a_i}{r})}{(m+1)!}\psi_i^m + \frac{r}{2}\sum_{w=0}^{r-1}\frac{B_{m+1}(\frac{w}{r})}{(m+1)!}\xi_{w*}\left[\frac{(\psi')^m-(-\psi'')^m}{\psi'+\psi''}\right], \tag{16}$$

and from the universal relation

$$c(-E^\bullet) = \exp\left(\sum_{m\geq 1}(-1)^m(m-1)!\,\mathrm{ch}_m(E^\bullet)\right), \qquad E^\bullet \in D^b.$$

The factor $r^{|E(\Gamma)|}$ in the proposition is due to the factor of r in the last term of Chiodo's formula.[15] When two such terms corresponding to the same edge are multiplied, the factor r^2 partly cancels with the factor $\frac{1}{r}$ in the first Chern class of the normal bundle $-\frac{1}{r}(\psi'+\psi'')$. Thus only a single factor of r per edge remains. $\square$

Corollary **4.** — *The push-forward $\epsilon_* c(-R^*\pi_*\mathcal{L}) \in R^*(\overline{\mathcal{M}}_{g,n})$ is equal to*

$$\sum_{\Gamma\in G_{g,n}}\sum_{w\in W_{\Gamma,r,k}}\frac{r^{2g-1-h^1(\Gamma)}}{|\mathrm{Aut}(\Gamma)|}\xi_{\Gamma*}\Bigg[\prod_{v\in V(\Gamma)} e^{-\sum_{m\geq 1}(-1)^{m-1}\frac{B_{m+1}(k/r)}{m(m+1)}\kappa_m(v)}$$
$$\times\prod_{i=1}^{n} e^{\sum_{m\geq 1}(-1)^{m-1}\frac{B_{m+1}(a_i/r)}{m(m+1)}\psi_{h_i}^m}$$
$$\times\prod_{\substack{e\in E(\Gamma)\\ e=(h,h')}}\frac{1-e^{\sum_{m\geq 1}(-1)^{m-1}\frac{B_{m+1}(w(h)/r)}{m(m+1)}[(\psi_h)^m-(-\psi_{h'})^m]}}{\psi_h+\psi_{h'}}\Bigg].$$

Proof. — Because the maps ϕ and φ of Equations (14) and (15) have degree 1, the degree of the map

$$\epsilon : \overline{\mathcal{M}}^{r,k}_{g;a_1,\dots,a_n} \to \overline{\mathcal{M}}_{g,n} \tag{17}$$

restricted to the stratum associated to (Γ, w) is equal to the product of degrees over vertices, that is,

$$r^{\sum_{v\in V(\Gamma)}(2g_v-1)}.$$

[15] The factor in the last term is $\frac{r}{2}$, but the $\frac{1}{2}$ accounts for the degree 2 of the boundary map ξ obtained from the ordering of the nodes.

The formula of the corollary is obtained by push-forward from the formula of Proposition 4. The power of r is given by

$$\begin{aligned}|\mathrm{E}| + \sum_{v\in \mathrm{V}(\Gamma)} (2g_v - 1) &= |\mathrm{E}| - |\mathrm{V}| + 2\sum_{v\in \mathrm{V}(\Gamma)} g_v \\ &= \big(h^1(\Gamma) - 1\big) + 2\big(g - h^1(\Gamma)\big) \\ &= 2g - 1 - h^1(\Gamma).\end{aligned}$$ □

1.4. *Reduction modulo r*

Let g be the genus, and let $k \in \mathbf{Z}$ and $a_1, \dots, a_n \in \mathbf{Z}$ satisfy

$$k(2g-2+n) - \sum a_i = 0 \mod r$$

for an integer $r > 0$. Let

$$\mathrm{A} = (a_1, \dots, a_n)$$

be the associated vector of k-twisted double ramification data.

Proposition **3**″ *(Pixton [35]). — Let* $\Gamma \in \mathrm{G}_{g,n}$ *be a fixed stable graph. Let* Q *be a polynomial with* **Q***-coefficients in variables corresponding to the set of half-edges* $\mathrm{H}(\Gamma)$*. Then the sum*

$$\mathrm{F}(r) = \sum_{w\in \mathrm{W}_{\Gamma,r,k}} \mathrm{Q}\big(w(h_1), \dots, w(h_{|\mathrm{H}(\Gamma)|})\big)$$

over all possible k-weightings mod r is (for all sufficiently large r) a polynomial *in r divisible by* $r^{h^1(\Gamma)}$.

Proposition 3″ implies Proposition 3′ (which implies Proposition 3 under the specialization $k = 0$). Proposition 3″ is the basic statement. See the Appendix for the proof.

We will require Proposition 3″ to prove the following result. Let

$$\epsilon : \overline{\mathcal{M}}^{r,k}_{g;a_1,\dots,a_n} \to \overline{\mathcal{M}}_{g,n}$$

be the map to the moduli of curves. Let $d \geq 0$ be a degree and let c_d denote the d-th Chern class.

Proposition **5.** — *The two cycle classes*

$$r^{2d-2g+1}\epsilon_* c_d\big(-\mathrm{R}^*\pi_*\mathcal{L}\big) \in \mathrm{R}^d(\overline{\mathcal{M}}_{g,n}) \quad \textit{and} \quad 2^{-d}\mathrm{P}^{d,r,k}_g(\mathrm{A}) \in \mathrm{R}^d(\overline{\mathcal{M}}_{g,n})$$

are polynomials in r (for r sufficiently large). Moreover, the two polynomials have the same constant term.

Proof. — Choose a stable graph Γ decorated with κ classes on the vertices and ψ classes on half-edges,

$$[\Gamma, \gamma], \quad \gamma = \prod_{v \in \mathrm{V}(\Gamma)} \mathrm{M}_v\big(\kappa(v)\big) \cdot \prod_{h \in \mathrm{H}(\Gamma)} \psi_h^{m_h}.$$

Here, M_v is a monomial in the κ classes at v and m_h is a non-negative integer for each h. The decorated graph data $[\Gamma, \gamma]$ represents a tautological class of degree d. We will study the coefficient of the decorated graph in the formula for $\epsilon_* c_d(-\mathrm{R}^*\pi_*\mathcal{L})$ given by Corollary 4 for r sufficiently large.

According to Proposition 3″, the coefficient of $[\Gamma, \gamma]$ in the formula for the push-forward $\epsilon_* c_d(-\mathrm{R}^*\pi_*\mathcal{L})$ is a Laurent polynomial in r. Indeed, the formula of Corollary 4 is a combination of a finite number of sums of the type of Proposition 3″ multiplied by positive and negative powers of r. Each contributing sum is obtained by expanding the exponentials in the formula of Corollary 4 and selecting a degree d monomial.

To begin with, let us determine the terms of lowest power in r in the Laurent polynomial obtained by summation in the formula of Corollary 4. The $(m+1)$st Bernoulli polynomial B_{m+1} has degree $m+1$. Therefore, in the formula of Corollary 4, *every* class of degree m, be it κ_m, $\psi_{h_i}^m$ or an edge[16] with a class ψ^{m-1}, appears with a power of r equal to $\frac{1}{r^{m+1}}$ or larger.[17] If we take a product of q coefficients of expanded exponentials of Bernoulli polynomials, we will obtain a power of r equal to $\frac{1}{r^{d+q}}$ or larger. Thus the lowest possible power of r is obtained if we take a product of d classes of degree 1 each. Then, the power of r is equal to $\frac{1}{r^{2d}}$ which is the lowest possible value.

After we perform the summation over all k-weightings mod r of the graph Γ, we obtain a polynomial in r divisible by $r^{h^1(\Gamma)}$ by Proposition 3″. The lowest possible power of r after summation becomes $r^{h^1(\Gamma)-2d}$. Finally, formula of Corollary 4 carries a global factor of $r^{2g-1-h^1(\Gamma)}$. We conclude $\epsilon_* c_d(-\mathrm{R}^*\pi_*\mathcal{L})$ is a Laurent polynomial in r with lowest power of r equal to $r^{2g-2d-1}$. In other words, the product

$$r^{2d-2g+1}\epsilon_* c_d\big(-\mathrm{R}^*\pi_*\mathcal{L}\big)$$

is a polynomial in r.

We now identify more precisely the terms of lowest power in r. As we have seen above, the lowest possible power of r in the formula of Corollary 4 is obtained by choosing only the terms that contain a degree 1 class multiplied by the second Bernoulli polynomial,

$$\mathrm{B}_2\left(\frac{w}{r}\right) = \frac{w^2}{r^2} - \frac{w}{r} + \frac{1}{6}.$$

[16] The edge contributes 1 to the degree of the class.

[17] The power of $\frac{1}{r^{m+1}}$ is $-m-1$. By *larger* here, we mean larger power (so smaller pole or polynomial terms).

Moreover, in the Bernoulli polynomial only the first term $\frac{w^2}{r^2}$ will contribute to the lowest power of r. After dropping all the terms which do not contribute to the lowest power of r, the formula of Corollary 4 simplifies to

$$\sum_{\Gamma \in \mathrm{G}_{g,n}} \sum_{w \in \mathrm{W}_{\Gamma,r,k}} \frac{r^{2g-1-h^1(\Gamma)}}{|\mathrm{Aut}(\Gamma)|} \xi_{\Gamma *} \Bigg[\prod_{v \in \mathrm{V}(\Gamma)} \exp\Big[-\frac{k^2}{2r^2}\kappa_1(v)\Big] \prod_{i=1}^{n} \exp\Big[\frac{a_i^2}{2r^2}\psi_{h_i}\Big]$$
$$\times \prod_{e=(h,h') \in \mathrm{E}(\Gamma)} \frac{1-\exp[\frac{w(h)^2}{2r^2}(\psi_h+\psi_{h'})]}{\psi_h+\psi_{h'}} \Bigg].$$

We transform $w(h)^2$ in the last factor to $-w(h)w(h')$ for symmetry. Indeed,

$$w(h)^2 = w(h)\big(r - w\big(h'\big)\big) = -w(h)w\big(h'\big) \mod r,$$

so the lowest degree in r is not affected. After factoring out the $2r^2$ in the denominators, we obtain

$$r^{2g-1-2d} \cdot 2^{-d} \cdot \sum_{\Gamma \in \mathrm{G}_{g,n}} \sum_{w \in \mathrm{W}_{\Gamma,r,k}} \frac{r^{-h^1(\Gamma)}}{|\mathrm{Aut}(\Gamma)|} \xi_{\Gamma *} \Bigg[\prod_{v \in \mathrm{V}(\Gamma)} e^{-k^2\kappa_1(v)} \prod_{i=1}^{n} e^{a_i^2\psi_{h_i}}$$
$$\times \prod_{e=(h,h') \in \mathrm{E}(\Gamma)} \frac{1-e^{-w(h)w(h')(\psi_h+\psi_{h'})}}{\psi_h+\psi_{h'}} \Bigg].$$

Multiplication by $r^{2d-2g+1}$ then yields exactly $2^{-d}\mathrm{P}_g^{d,k,r}(\mathrm{A})$ as claimed. □

2. Localization analysis

2.1. *Overview*

Let $\mathrm{A} = (a_1, \ldots, a_n)$ be a vector of double ramification data as defined in Section 0.2.1,

$$\sum_{i=1}^{n} a_i = 0.$$

From the positive and negative parts of A, we obtain two partitions μ and ν of the same size $|\mu| = |\nu|$. The double ramification cycle

$$\mathrm{DR}_g(\mathrm{A}) \in \mathrm{R}^g(\overline{\mathcal{M}}_{g,n})$$

is defined via the moduli space of stable maps to rubber $\overline{\mathcal{M}}_{g,\mathrm{I}}(\mathbf{P}^1, \mu, \nu)^\sim$ where I corresponds to the 0 parts of A. We prove here the claim of Theorem 1,

$$\mathrm{DR}_g(\mathrm{A}) = 2^{-g}\mathrm{P}_g^g(\mathrm{A}) \in \mathrm{R}^g(\overline{\mathcal{M}}_{g,n}).$$

2.2. *Target geometry*

Following the notation of [24], let $\mathbf{P}^1[r]$ be the projective line with an orbifold point $B\mathbf{Z}_r$ at $0 \in \mathbf{P}^1$. Let

$$\overline{\mathcal{M}}_{g,I,\mu}(\mathbf{P}^1[r], \nu)$$

be the moduli space of stable maps to the orbifold/relative pair $(\mathbf{P}^1[r], \infty)$. The moduli space parameterizes connected, semistable, twisted curves C of genus g with I markings together with a map

$$f : C \to P$$

where P is a destabilization of $\mathbf{P}^1[r]$ over $\infty \in \mathbf{P}^1[r]$.

We refer the reader to [1, 24] for the definitions of the moduli space of stable maps to $(\mathbf{P}^1[r], \infty)$. The following conditions are required to hold over $0, \infty \in \mathbf{P}^1[r]$:

- The stack structure of the domain curve C occurs only at the nodes over $0 \in \mathbf{P}^1[r]$ and the markings corresponding to μ (which must to be mapped to $0 \in \mathbf{P}^1[r]$). The monodromies associated to the latter markings are specified by the parts μ_i of μ. For each part μ_i, let

$$\mu_i = \overline{\mu}_i \mod r \quad \text{where } 0 \leq \overline{\mu}_i \leq r-1.$$

- The map f is finite over $\infty \in \mathbf{P}^1[r]$ on the last component of P with ramification data given by ν. The $\ell(\nu)$ ramification points are marked. The map f satisfies the ramification matching condition over the internal nodes of the destabilization P.

The moduli space $\overline{\mathcal{M}}_{g,I,\mu}(\mathbf{P}^1[r], \nu)$ has a perfect obstruction theory and a virtual class of dimension

$$\dim_{\mathbf{C}}\left[\overline{\mathcal{M}}_{g,I,\mu}(\mathbf{P}^1[r], \nu)\right]^{\mathrm{vir}} = 2g-2+n+\frac{|\nu|}{r} - \sum_{i=1}^{\ell(\mu)} \frac{\overline{\mu}_i}{r}, \tag{18}$$

see [24, Section 1.1]. Recall n is the length of A,

$$n = \ell(\mu) + \ell(\nu) + \ell(I).$$

Since $|\mu| = |\nu|$, the virtual dimension is an integer.

We will be most interested in the case where $r > |\mu| = |\nu|$. Then,

$$\mu_i = \overline{\mu}_i$$

for all the parts of μ, and formula (18) for the virtual dimension simplifies to

$$\dim_{\mathbf{C}}\left[\overline{\mathcal{M}}_{g,I,\mu}(\mathbf{P}^1[r], \nu)\right]^{\mathrm{vir}} = 2g-2+n.$$

2.3. $\mathbf{C}^*$*-fixed loci*

The standard $\mathbf{C}^*$-action on $\mathbf{P}^1$ defined by

$$\xi \cdot [z_0, z_1] = [z_0, \xi z_1]$$

lifts canonically to $\mathbf{C}^*$-actions on $\mathbf{P}^1[r]$ and $\overline{\mathcal{M}}_{g,\mathrm{I},\mu}(\mathbf{P}^1[r], \nu)$. We describe here the $\mathbf{C}^*$-fixed loci of $\overline{\mathcal{M}}_{g,\mathrm{I},\mu}(\mathbf{P}^1[r], \nu)$.

The $\mathbf{C}^*$-fixed loci of $\overline{\mathcal{M}}_{g,\mathrm{I},\mu}(\mathbf{P}^1[r], \nu)$ are labeled by decorated graphs Φ. The vertices $v \in \mathrm{V}(\Phi)$ are decorated with a genus $\mathrm{g}(v)$ and the legs are labeled with markings in $\mu \cup \mathrm{I} \cup \nu$. Each vertex v is labeled by

$$0 \in \mathbf{P}^1[r] \quad \text{or} \quad \infty \in \mathbf{P}^1[r].$$

Each edge $e \in \mathrm{E}(\Phi)$ is decorated with a degree d_e that corresponds to the d_e-th power map

$$\mathbf{P}^1[r] \to \mathbf{P}^1[r].$$

The vertex labeling endows the graph Φ with a *bipartite* structure.

A vertex of Φ over $0 \in \mathbf{P}^1[r]$ corresponds to a stable map contracted to 0 given by an element of $\overline{\mathcal{M}}_{\mathrm{g}(v),\mathrm{I}(v),\mu(v)}(\mathrm{B}\mathbf{Z}_r)$ with monodromies $\mu(v)$ specified by the corresponding entries of μ *and* the negatives of the degrees d_e of the incident edges (modulo r). We will use the notation

$$\overline{\mathcal{M}}_v^r = \overline{\mathcal{M}}^r_{\mathrm{g}(v);\mathrm{I}(v),\mu(v)} = \overline{\mathcal{M}}_{\mathrm{g}(v),\mathrm{I}(v),\mu(v)}(\mathrm{B}\mathbf{Z}_r).$$

The fundamental class $[\overline{\mathcal{M}}_v^r]$ of $\overline{\mathcal{M}}_v^r$ is of dimension

$$\dim_{\mathbf{C}}\left[\overline{\mathcal{M}}_v^r\right] = 3\mathrm{g}(v) - 3 + \ell\big(\mathrm{I}(v)\big) + \ell\big(\mu(v)\big).$$

As explained in Section 1.2.2,

$$\epsilon : \overline{\mathcal{M}}_v^r \to \overline{\mathcal{M}}_{\mathrm{g}(v),\mathrm{n}(v)}$$

is a finite covering of the moduli space $\overline{\mathcal{M}}_{\mathrm{g}(v),\mathrm{n}(v)}$ where

$$\mathrm{n}(v) = \ell\big(\mu(v)\big) + \ell\big(\mathrm{I}(v)\big)$$

is the total number of markings and edges incident to v.

The stable map over $\infty \in \mathbf{P}^1[r]$ can take two different forms. If the target does not expand, then the stable map has $\ell(\nu)$ preimages of $\infty \in \mathbf{P}^1[r]$. Each preimage is described by an unstable vertex of Φ labeled by ∞. If the target expands, then the stable map is a possibly disconnected rubber map. The ramification data is given by incident edges over 0 of the rubber and by elements of ν over ∞ of the rubber. In this case every vertex v over $\infty \in \mathbf{P}^1[r]$ describes a connected component of the rubber map.

Let $\mathrm{g}(\infty)$ be the genus of the possibly disconnected domain of the rubber map. We will denote[18] the moduli space of stable maps to rubber by $\overline{\mathcal{M}}_{\infty}^{\sim}$ and the virtual fundamental class by $[\overline{\mathcal{M}}_{\infty}^{\sim}]^{\mathrm{vir}}$,

$$\dim_{\mathbf{C}}\left[\overline{\mathcal{M}}_{\infty}^{\sim}\right]^{\mathrm{vir}} = 2\mathrm{g}(\infty) - 3 + \mathrm{n}(\infty),$$

where $\mathrm{n}(\infty)$ is the total number of markings and edges incident to vertices over ∞,

$$\mathrm{n}(\infty) = \ell\big(\mathrm{I}(\infty)\big) + \ell(\nu) + |\mathrm{E}(\Phi)|.$$

The image of the virtual fundamental class $[\overline{\mathcal{M}}_{\infty}^{\sim}]^{\mathrm{vir}}$ in the moduli space of (not necessarily connected[19]) stable curves is denoted by DR_{∞}.

A vertex $v \in \mathrm{V}(\Phi)$ is *unstable* when $2\mathrm{g}(v) - 2 + \mathrm{n}(v) \leq 0$. There are four types of unstable vertices:

(i) $v \mapsto 0$, $\mathrm{g}(v) = 0$, v carries no markings and one incident edge,
(ii) $v \mapsto 0$, $\mathrm{g}(v) = 0$, v carries no markings and two incident edges,
(iii) $v \mapsto 0$, $\mathrm{g}(v) = 0$, v carries one marking and one incident edge,
(iv) $v \mapsto \infty$, $\mathrm{g}(v) = 0$, v carries one marking and one incident edge.

The target of the stable map expands iff there is at least one stable vertex over ∞.

A stable map in the $\mathbf{C}^*$-fixed locus corresponding to Φ is obtained by gluing together maps associated to the vertices $v \in \mathrm{V}(\Phi)$ with Galois covers associated to the edges. Denote by $\mathrm{V}^0_{\mathrm{st}}(\Phi)$ the set of stable vertices of Φ over 0. Then the $\mathbf{C}^*$-fixed locus corresponding to Φ is isomorphic to the product

$$\overline{\mathcal{M}}_{\Phi} = \begin{cases} \prod_{v\in \mathrm{V}^0_{\mathrm{st}}(\Phi)} \overline{\mathcal{M}}^r_v \times \overline{\mathcal{M}}^{\sim}_{\infty}, & \text{if the target expands,} \\ \prod_{v\in \mathrm{V}^0_{\mathrm{st}}(\Phi)} \overline{\mathcal{M}}^r_v, & \text{if the target does not expand,} \end{cases}$$

quotiented by the automorphism group of Φ *and* the product of cyclic groups $\mathbf{Z}_{d_e}$ associated to the Galois covers of the edges.

Thus the natural morphism corresponding to Φ,

$$\iota : \overline{\mathcal{M}}_{\Phi} \to \overline{\mathcal{M}}_{g,\mathrm{I},\mu}\big(\mathbf{P}^1[r], \nu\big),$$

is of degree

$$\big|\mathrm{Aut}(\Phi)\big| \prod_{e\in \mathrm{E}(\Phi)} d_e,$$

onto the image $\iota(\overline{\mathcal{M}}_{\Phi})$.

Recall that $\mathrm{D} = |\mu| = |\nu|$ is the degree of the stable map.

---

[18] We omit the data $\mathrm{g}(\infty)$, $\mathrm{I}(\infty)$, ν, and $\delta(\infty)$ in the notation.

[19] By [3, Lemma 2.3], a double ramification cycle vanishes as soon as there are at least two nontrivial components of the domain. However, we will *not* require the vanishing result here. Our analysis will naturally avoid disconnected domains mapping to the rubber over $\infty \in \mathbf{P}^1[r]$.

Lemma **6.** — *For* $r > \mathrm{D}$, *the unstable vertices of type* (i) *and* (ii) *can* not *occur.*

Proof. — At each stable vertex $v \in \mathrm{V}(\Phi)$ over $0 \in \mathbf{P}^1[r]$, the condition

$$|\mu_v| = 0 \mod r$$

has to hold in order for $\overline{\mathcal{M}}_{g_v, \mathrm{I}_v, \mu_v}(\mathrm{B}\mathbf{Z}_r)$ to be non-empty. Because of the balancing condition at the nodes, the parts of μ at v have to add up to the sum of the degrees of the incident edges modulo r. Since $r > \mathrm{D}$, the parts of μ at v must add up to *exactly* the sum of the degrees of the incident edges. So the incident edges of the stable vertices and the type (iii) unstable vertices must account for the *entire* degree D. Thus, there is no remaining edge degree for unstable vertices of type (i) and (ii). □

2.4. *Localization formula*

We write the $\mathbf{C}^*$-equivariant Chow ring of a point as

$$\mathrm{A}^*_{\mathbf{C}^*}(\bullet) = \mathbf{Q}[t],$$

where t is the first Chern class of the standard representation.

For the localization formula, we will require the inverse of the $\mathbf{C}^*$-equivariant Euler class of the virtual normal bundle in $\overline{\mathcal{M}}_{g,\mathrm{I},\mu}(\mathbf{P}^1[r], \nu)$ to the $\mathbf{C}^*$-fixed locus corresponding to Φ. Let

$$f : \mathrm{C} \to \mathbf{P}^1[r], \qquad [f] \in \overline{\mathcal{M}}_\Phi.$$

The inverse Euler class is represented by

$$\frac{1}{e(\mathrm{Norm}^{\mathrm{vir}})} = \frac{e(\mathrm{H}^1(\mathrm{C}, f^*\mathrm{T}_{\mathbf{P}^1[r]}(-\infty)))}{e(\mathrm{H}^0(\mathrm{C}, f^*\mathrm{T}_{\mathbf{P}^1[r]}(-\infty)))} \frac{1}{\prod_i e(\mathrm{N}_i)} \frac{1}{e(\mathrm{N}_\infty)}. \tag{19}$$

Formula (19) has several terms which require explanation. We assume

$$r > |\mu| = |\nu|,$$

so unstable vertices over $0 \in \mathbf{P}^1[r]$ of type (i) and (ii) do not occur in Φ. First consider the leading factor of (19),

$$\frac{e(\mathrm{H}^1(\mathrm{C}, f^*\mathrm{T}_{\mathbf{P}^1[r]}(-\infty)))}{e(\mathrm{H}^0(\mathrm{C}, f^*\mathrm{T}_{\mathbf{P}^1[r]}(-\infty)))}. \tag{20}$$

To compute this factor one uses the normalization exact sequence for the domain C tensored with the line bundle $f^*\mathrm{T}_{\mathbf{P}^1[r]}(-\infty)$. The associated long exact sequence in cohomology decomposes the leading factor into a product of vertex, edge, and node contributions.

- Let $v \in \mathrm{V}(\Phi)$ be a stable vertex over $0 \in \mathbf{P}^1[r]$ corresponding to a contracted component C_v. From the resulting stable map to the orbifold point

$$\mathrm{C}_v \to \mathrm{B}\mathbf{Z}_r \subset \mathbf{P}^1[r],$$

we obtain an orbifold line bundle

$$\mathrm{L} = f^*\mathrm{T}_{\mathbf{P}^1[r]}|_0$$

on C_v which is an rth root of $\mathcal{O}_{\mathrm{C}_v}$. Therefore, the contribution

$$\frac{e(\mathrm{H}^1(\mathrm{C}_v, f^*\mathrm{T}_{\mathbf{P}^1[r]}|_0))}{e(\mathrm{H}^0(\mathrm{C}_v, f^*\mathrm{T}_{\mathbf{P}^1[r]}|_0))}$$

yields the class

$$c_{\mathrm{rk}}\left(\left(-\mathrm{R}^*\pi_*\mathcal{L}\right) \otimes \mathcal{O}^{(1/r)}\right) \in \mathrm{A}^*\left(\overline{\mathcal{M}}^r_{\mathrm{g}(v),\mathrm{I}(v),\mu(v)}\right) \otimes \mathbf{Q}\left[t, \frac{1}{t}\right],$$

where $\mathcal{L} \to \overline{\mathcal{M}}^r_{\mathrm{g}(v),\mathrm{I}(v),\mu(v)}$ is the universal rth root, $\mathcal{O}^{(1/r)}$ is a trivial line bundle with a $\mathbf{C}^*$-action of weight $\frac{1}{r}$, and

$$\mathrm{rk} = g_v - 1 + \left|\mathrm{E}(v)\right|$$

is the virtual rank of $-\mathrm{R}^*\pi_*\mathcal{L}$.

Unstable vertices of type (iii) over $0 \in \mathbf{P}^1[r]$ and the vertices over ∞ contribute factors of 1.

- The edge contribution is trivial since the degree $\frac{d_i}{r}$ of $f^*\mathrm{T}_{\mathbf{P}^1[r]}(-\infty)$ is less than 1, see [24, Section 2.2].
- The contribution of a node N over $0 \in \mathbf{P}^1[r]$ is trivial. The space of sections $\mathrm{H}^0(\mathrm{N}, f^*\mathrm{T}_{\mathbf{P}^1[r]}(-\infty))$ vanishes because N must be stacky, and $\mathrm{H}^1(\mathrm{N}, f^*\mathrm{T}_{\mathbf{P}^1[r]}(-\infty))$ is trivial for dimension reasons.

Nodes over $\infty \in \mathbf{P}^1[r]$ contribute 1.

Consider next the last two factors of (19),

$$\frac{1}{\prod_i e(\mathrm{N}_i)} \frac{1}{e(\mathrm{N}_\infty)}.$$

- The product $\prod_i e(\mathrm{N}_i)^{-1}$ is over all nodes over $0 \in \mathbf{P}^1[r]$ of the domain C which are *forced* by the graph Φ. These are nodes where the edges of Φ are attached to the vertices. If N is a node over 0 forced by the edge $e \in \mathrm{E}(\Phi)$ and the associated vertex v is stable, then

$$e(\mathrm{N}) = \frac{t}{rd_e} - \frac{\psi_e}{r}. \tag{21}$$

This expression corresponds to the smoothing of the node N of the domain curve: $e(\mathrm{N})$ is the first Chern classe of the normal bundle of the divisor of nodal domain curves. The first Chern classes of the cotangent lines to the branches at the node are divided by r because of the orbifold twist, see Section 1.2.

In the case of an unstable vertex of type (iii), the associated edge does *not* produce a node of the domain. The type (iii) edge incidences do *not* appear in $\prod_i e(\mathrm{N}_i)^{-1}$.

- N_∞ corresponds to the target degeneration over $\infty \in \mathbf{P}^1[r]$. The factor $e(\mathrm{N}_\infty)$ is 1 if the *target* $(\mathbf{P}^1[r], \infty)$ does not degenerate and
$$e(\mathrm{N}_\infty) = -\frac{t+\psi_\infty}{\prod_{e\in \mathrm{E}(\Phi)} d_e}$$
if the target does degenerate [17]. Here, ψ_∞ is the first Chern class of the cotangent line bundle to the bubble in the target curve at the attachment point to ∞ of $\mathbf{P}^1[r]$.

Summing up our analysis, we write out the outcome of the virtual localization formula [15] in $\mathrm{A}_*(\overline{\mathcal{M}}_{g,\mathrm{I},\mu}(\mathbf{P}^1[r], \nu)) \otimes \mathbf{Q}[t, \frac{1}{t}]$. We have

$$[\overline{\mathcal{M}}_{g,\mathrm{I},\mu}(\mathbf{P}^1[r], \nu)]^{\mathrm{vir}} = \sum_\Phi \frac{1}{|\mathrm{Aut}(\Phi)|}\frac{1}{\prod_{e\in\mathrm{E}(\Phi)} d_e} \cdot \iota_*\left(\frac{[\overline{\mathcal{M}}_\Phi]^{\mathrm{vir}}}{e(\mathrm{Norm}^{\mathrm{vir}})}\right), \tag{22}$$

where $\frac{[\overline{\mathcal{M}}_\Phi]^{\mathrm{vir}}}{e(\mathrm{Norm}^{\mathrm{vir}})}$ is the product of the following factors:

- $\prod_{e\in\mathrm{E}(v)} \frac{r}{\frac{t}{d_e}-\psi_e} \cdot \sum_{d\geq 0} c_d(-\mathrm{R}^*\pi_*\mathcal{L})(\frac{t}{r})^{\mathrm{g}(v)-1+|\mathrm{E}(v)|-d}$ for each stable vertex $v \in \mathrm{V}(\Phi)$ over 0,
- $-\frac{\prod_{e\in\mathrm{E}(\Phi)} d_e}{t+\psi_\infty}$ if the target degenerates.

Remark **2.** — As we mentioned in Remark 1, there are two conventions for the orbifold structure on the space of rth roots. In the paper, we follow the convention which assigns to each node of the curve the stabilizer $\mathbf{Z}/r\mathbf{Z}$. It is also possible to assign to a node of type (a', a'') the stabilizer $\mathbf{Z}/q\mathbf{Z}$, where q is the order of a' and a'' in $\mathbf{Z}/r\mathbf{Z}$. We list here the required modifications of our formulas when the latter convention is followed.

Denote $p = \gcd(r, a') = \gcd(r, a'')$. We then have $pq = r$. In Section 1.2.4, the first Chern class of the normal line bundle to a boundary divisor in the space of rth roots becomes $-(\psi' + \psi'')/q$ rather than $-(\psi' + \psi'')/r$. The degrees of the maps ϕ and φ in Equations (14) and (15) are equal to p instead of 1. The factor $\frac{r}{2}$ in the last term of Chiodo's formula (16) becomes $\frac{q}{2}$. Because of the above, the factor $r^{|\mathrm{E}(\Gamma)|}$ in Proposition 4 is transformed into $q^{|\mathrm{E}(\Gamma)|}$. The statement of Corollary 4 is unchanged, but in the proof the factor $r^{|\mathrm{E}(\Gamma)|}$ is now obtained as a product of $q^{|\mathrm{E}(\Gamma)|}$ from Proposition 4 and $p^{|\mathrm{E}(\Gamma)|}$ from the degree of ϕ and φ.

In the localization formula, the Euler classes of the normal bundles at the nodes over 0 become $\frac{t}{qd_e} - \frac{\psi_e}{q}$ instead of $\frac{t}{rd_e} - \frac{\psi_e}{r}$. On the other hand, at each node there are p ways to reconstruct an rth root bundle from its restrictions to the two branches. Thus the contribution of each node $\frac{p}{e(\mathrm{N})}$ still has an r in the numerator.

2.5. *Extracting the double ramification cycle*

2.5.1. *Three operations*

We will now perform three operations on the localization formula (22) for the virtual class $[\overline{\mathcal{M}}_{g,\mathrm{I},\mu}(\mathbf{P}^1[r],\nu)]^{\mathrm{vir}}$:

(i) the $\mathbf{C}^*$-equivariant push-forward via

$$\epsilon : \overline{\mathcal{M}}_{g,\mathrm{I},\mu}\big(\mathbf{P}^1[r],\nu\big) \to \overline{\mathcal{M}}_{g,n} \tag{23}$$

to the moduli space $\overline{\mathcal{M}}_{g,n}$ with trivial $\mathbf{C}^*$-action,

(ii) extraction of the coefficient of t^{-1} after push-forward by ϵ_*,

(iii) extraction of the coefficient of r^0.

After push-forward by ϵ_*, the coefficient of t^{-1} is equal to 0 because

$$\epsilon_*\big[\overline{\mathcal{M}}_{g,\mathrm{I},\mu}\big(\mathbf{P}^1[r],\nu\big)\big]^{\mathrm{vir}} \in \mathrm{A}^*(\overline{\mathcal{M}}_{g,n}) \otimes \mathbf{Q}[t].$$

Using the result of Section 1.4, all terms of the t^{-1} coefficient will be seen to be polynomials in r, so operation (iii) will be well-defined. After operations (i)–(iii), only two nonzero terms will remain. The cancellation of the two remaining terms will prove Theorem 1.

To perform (i)–(iii), we multiply the ϵ-push-forward of the localization formula (22) by t and extract the coefficient of $t^0 r^0$. To simplify the computations, we introduce the new variable

$$s = tr.$$

Then, instead of extracting the coefficient of $t^0 r^0$, we extract the coefficient of $s^0 r^0$.

2.5.2. *Push-forward to* $\overline{\mathcal{M}}_{g,n}$

For each vertex $v \in \mathrm{V}(\Phi)$, we have

$$\overline{\mathcal{M}}^r_v = \overline{\mathcal{M}}^r_{\mathrm{g}(v);\mathrm{I}(v),\mu(v)}.$$

Following Section 1.2.2, we denote the map to moduli by

$$\epsilon : \overline{\mathcal{M}}^r_{\mathrm{g}(v);\mathrm{I}(v),\mu(v)} \to \overline{\mathcal{M}}_{\mathrm{g}(v),\mathrm{n}(v)}. \tag{24}$$

The use of the notation ϵ in (23) and (24) is compatible. Denote by

$$\widehat{c}_d = r^{2d-2g(v)+1}\epsilon_* c_d\big(-\mathrm{R}^*\pi_*\mathcal{L}\big) \in \mathrm{R}^d(\overline{\mathcal{M}}_{g(v),n(v)}). \tag{25}$$

By Proposition 5, $\widehat{c}_d$ is a polynomial in r for r sufficiently large.

From (22) and the contribution calculus for Φ presented in Section 2.5.2, we have a complete formula for the $\mathbf{C}^*$-equivariant push-forward of t times the virtual class:

$$\begin{aligned}\epsilon_*\Big(t\big[\overline{\mathcal{M}}_{g,\mathrm{I},\mu}\big(\mathbf{P}^1[r],\nu\big)\big]^{\mathrm{vir}}\Big)& \\ = \frac{s}{r}\cdot\sum_{\Phi}\frac{1}{|\mathrm{Aut}(\Phi)|}\frac{1}{\prod_{e\in\mathrm{E}(\Phi)}d_e}&\cdot\epsilon_*\iota_*\left(\frac{[\overline{\mathcal{M}}_\Phi]^{\mathrm{vir}}}{e(\mathrm{Norm}^{\mathrm{vir}})}\right),\end{aligned} \tag{26}$$

where $\epsilon_*\iota_*\frac{[\overline{\mathcal{M}}_\Phi]^{\mathrm{vir}}}{e(\mathrm{Norm}^{\mathrm{vir}})}$ is the product of the following factors:

- a factor

$$\frac{r}{s}\cdot\prod_{e\in\mathrm{E}(v)}\frac{d_e}{1-\frac{rd_e}{s}\psi_e}\cdot\sum_{d\geq 0}\widehat{c}_d s^{g(v)-d} \in \mathrm{R}^d(\overline{\mathcal{M}}_{g(v),n(v)})\otimes\mathbf{Q}\Big[s,\frac{1}{s}\Big]$$

  for each stable vertex $v\in\mathrm{V}(\Phi)$ over 0,
- a factor

$$-\frac{r}{s}\cdot\frac{\prod_{e\in\mathrm{E}(\Phi)}d_e}{1+\frac{r}{s}\psi_\infty}\cdot\mathrm{DR}_\infty$$

  if the target degenerates.

2.5.3. *Extracting coefficients*

Extracting the coefficient of r^0. — By Proposition 5, the classes $\widehat{c}_d$ are polynomial in r for r sufficiently large. We have an r in the denominator in the prefactor on the right side of (26) which comes from the multiplication by t on the left side. However, in all other factors, we only have positive powers of r, with at least one r per stable vertex of the graph over 0 and one more r if the target degenerates. The *only* graphs Φ which contribute to the coefficient of r^0 are those with *exactly one r in the numerator*. There are only two graphs which have exactly one r factor in the numerator:

- the graph Φ' with a stable vertex of full genus g over 0 and $\ell(\nu)$ type (iv) unstable vertices over ∞,
- the graph Φ'' with a stable vertex of full genus g over ∞ and $\ell(\mu)$ type (iii) unstable vertices over 0.

No terms involving ψ classes contribute to the r^0 coefficient of either Φ' or Φ'' since every ψ class in the localization formula comes with an extra factor of r. We can now write the r^0 coefficient of the right side of (26):

$$\text{Coeff}_{r^0}\Big[\epsilon_*\Big(t\big[\overline{\mathcal{M}}_{g,\mathrm{I},\mu}\big(\mathbf{P}^1[r],\nu\big)\big]^{\mathrm{vir}}\Big)\Big] = \text{Coeff}_{r^0}\Big[\sum_{d\geq 0}\widehat{c}_d s^{g-d}\Big] - \mathrm{DR}_g(\mathrm{A}) \tag{27}$$

in $\mathrm{R}^d(\overline{\mathcal{M}}_{g,n})\otimes\mathbf{Q}[s,\frac{1}{s}]$.

Extracting the coefficient of s^0. — The remaining powers of s in (27) appear only with the classes $\widehat{c}_d$ in the contribution of the graph Φ'. In order to obtain s^0, we take to take $d=g$,

$$\text{Coeff}_{s^0r^0}\Big[\epsilon_*\Big(t\big[\overline{\mathcal{M}}_{g,\mathrm{I},\mu}\big(\mathbf{P}^1[r],\nu\big)\big]^{\mathrm{vir}}\Big)\Big] = \text{Coeff}_{r^0}\big[\widehat{c}_g\big] - \mathrm{DR}_g(\mathrm{A}) \tag{28}$$

in $\mathrm{R}^d(\overline{\mathcal{M}}_{g,n})$.

2.5.4. *Final relation*

Since $\text{Coeff}_{s^0r^0}[\epsilon_*(t[\overline{\mathcal{M}}_{g,\mathrm{I},\mu}(\mathbf{P}^1[r],\nu)]^{\mathrm{vir}})]$ vanishes, we can rewrite (28) as

$$\mathrm{DR}_g(\mathrm{A}) = \text{Coeff}_{r^0}\Big[r^{2g-2g+1}\epsilon_* c_g\big(-\mathrm{R}^*\pi_*\mathcal{L}\big)\Big]\in \mathrm{R}^g(\overline{\mathcal{M}}_{g,n}), \tag{29}$$

using definition (25) of $\widehat{c}_d$. By Proposition 5 applied to the right side of relation (29),

$$\mathrm{DR}_g(\mathrm{A}) = 2^{-g}\mathrm{P}_g^g(\mathrm{A})\in \mathrm{R}^g(\overline{\mathcal{M}}_{g,n}).$$

The proof of Theorem 1 is complete. □

3. A few applications

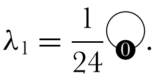
3.1. *A new expression for* λ_g

We discuss here a few more examples of the formula for Corollary 3: a formula for the class λ_g supported on the boundary divisor with a nonseparating node. We give the formulas for the class λ_g in $\overline{\mathcal{M}}_g$ up to genus 4.

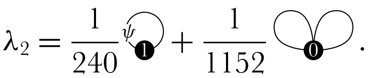
The answer is expressed as a sum of labeled stable graphs with coefficients where each stable graph has at least one nontrivial cycle. The genus of each vertex is written inside the vertex. The powers of ψ-classes are written on the corresponding half-edges (zero powers are omitted). The expressions do not involve κ-classes.

Each labeled graph Γ describes a moduli space $\overline{\mathcal{M}}_\Gamma$ (a product of moduli spaces associated with its vertices), a tautological class $\alpha\in\mathrm{R}^*(\overline{\mathcal{M}}_\Gamma)$ and a natural map $\pi:\overline{\mathcal{M}}_\Gamma\to\overline{\mathcal{M}}_g$. Our convention is that Γ then represents the cycle class $\pi_*\alpha$. For instance, if the graph carries no ψ-classes, the class α equals 1 and the map π is of degree $|\mathrm{Aut}(\Gamma)|$. The cycle class represented by Γ is then $|\mathrm{Aut}(\Gamma)|$ times the class of the image of π.

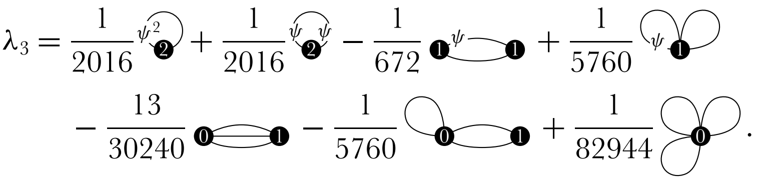
Genus 1.



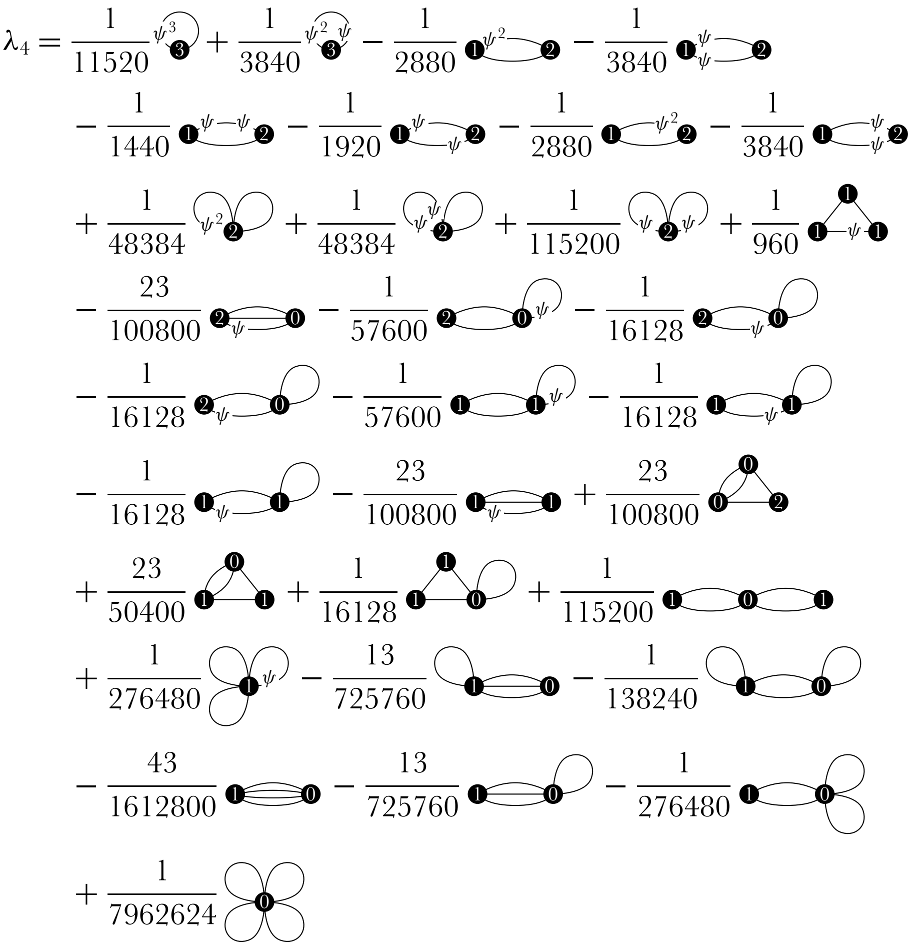
Genus 2.



Genus 3.



Genus 4.



All these expressions are obtained by taking $n=0$ in the formula for the double ramification cycle.

3.2. *Hodge integrals*

Corollary 3 may be applied to any Hodge integral which contains the top Chern class of the Hodge bundle. As an example, we consider the following Hodge integral (first calculated in [10]) related to the constant map contribution in the Gromov-Witten theory of Calabi-Yau threefolds.

Proposition **7.** — *For* $g \geq 1$*, we have*

$$\int_{\overline{\mathcal{M}}_{g+1}} \lambda_{g+1}\lambda_g\lambda_{g-1} = -\frac{1}{2}\frac{1}{(2g)!}\frac{B_{2g}}{2g}\frac{B_{2g+2}}{2g+2}.$$

Proof. — We compute the integral by replacing λ_{g+1} with the expression obtained from Corollary 3.

In Pixton's formula for $\mathrm{DR}_{g+1}(\emptyset) = (-1)^{g+1}\lambda_{g+1}$, we only have to consider graphs with no separating edges. On the other hand, $\lambda_g\lambda_{g-1}$ vanishes on strata as soon as there are at least *two* nonseparating edges. Thus we are left with just one graph: the graph with a vertex of genus g and one loop.

The edge term in Pixton's formula for $(-1)^{g+1}\mathrm{DR}_{g+1}(\emptyset) = \lambda_{g+1}$ for the unique remaining graph is

$$(-1)^{g+1}\frac{1-e^{\frac{a^2}{2}(\psi'+\psi'')}}{\psi'+\psi''} = (-1)^g\sum_{k\geq 0}\frac{a^{2k+2}}{2^{k+1}(k+1)!}\left(\psi'+\psi''\right)^k.$$

The contribution of the graph is therefore the r-free term of

$$\frac{(-1)^g}{2r}\sum_{a=0}^{r-1}\frac{a^{2g+2}}{2^{g+1}(g+1)!}\cdot\int_{\overline{\mathcal{M}}_{g,2}}(\psi_1+\psi_2)^g\lambda_g\lambda_{g-1}.$$

The factor of $\frac{1}{2}$ comes from the automorphism group of the graph, and the factor of $\frac{1}{r}$ comes from the first Betti number of graph. Since

$$\sum_{a=0}^{r-1} a^{2g+2} = B_{2g+2}r + O\left(r^2\right),$$

we see that the r-free term equals

$$(-1)^g\frac{B_{2g+2}}{2^{g+2}(g+1)!}\int_{\overline{\mathcal{M}}_{g,2}}(\psi_1+\psi_2)^g\lambda_g\lambda_{g-1}.$$

Using Lemma 8 below we can rewrite this expression as

$$(-1)^g \frac{\mathrm{B}_{2g+2}}{2^{g+2}(g+1)!} \cdot (-1)^{g+1} \frac{\mathrm{B}_{2g}}{2g} \frac{1}{(2g-1)!!} = -\frac{1}{2} \frac{1}{(2g)!} \frac{\mathrm{B}_{2g}}{2g} \frac{\mathrm{B}_{2g+2}}{2g+2},$$

which completes the derivation. □

Lemma **8.** — *For $g \geq 1$, we have*

$$\int_{\overline{\mathcal{M}}_{g,2}} (\psi_1 + \psi_2)^g \lambda_g \lambda_{g-1} = (-1)^{g+1} \frac{\mathrm{B}_{2g}}{2g} \frac{1}{(2g-1)!!}.$$

Proof. — According to Faber's socle formula [9, 14], we have

$$\int_{\overline{\mathcal{M}}_{g,2}} \psi_1^p \psi_2^q \lambda_g \lambda_{g-1} = \frac{(-1)^{g+1} \mathrm{B}_{2g}}{2^{2g} g (2p-1)!!(2q-1)!!} \tag{30}$$

whenever $p + q = g$. The formula holds even if p or q is equal to 0.[20] After expanding $(\psi_1 + \psi_2)^g$ and applying (30), we obtain

$$\int_{\overline{\mathcal{M}}_{g,2}} (\psi_1 + \psi_2)^g \lambda_g \lambda_{g-1} = (-1)^{g+1} \frac{\mathrm{B}_{2g}}{g} \frac{g!}{2^{2g}} \sum_{p+q=g} \frac{1}{p!(2p-1)!!q!(2q-1)!!}.$$

The right side is, up to a factor, the sum of even binomial coefficients in the $2g$-th line of the Pascal triangle:

$$\begin{aligned} \sum_{p+q=g} \frac{1}{p!(2p-1)!!q!(2q-1)!!} = \sum_{p+q=g} \frac{2^g}{(2p)!(2q)!} = \sum_{p=0}^{g} \frac{2^g}{(2g)!} \binom{2g}{2p} \\ = \frac{2^{3g-1}}{(2g)!}. \end{aligned}$$

Finally, we have

$$(-1)^{g+1} \frac{\mathrm{B}_{2g}}{g} \frac{g!}{2^{2g}} \cdot \frac{2^{3g-1}}{(2g)!} = (-1)^{g+1} \frac{\mathrm{B}_{2g}}{2g} \frac{1}{(2g-1)!!}$$

as claimed. □

[20] The socle formula is true if at most one power of a ψ class vanishes while the powers of all other ψ classes are positive.

3.3. *Local theory of curves*

The local Gromov-Witten theory of curves as developed in [2] starts with a fundamental pairing with a double ramification cycle (the resulting matrix then also appears in the quantum cohomology of the Hilbert scheme of points of $\mathbf{C}^2$ [30]). We rederive the double ramification cycle pairing as an almost immediate consequence of Theorem 1.

Proposition **9.** — *For* $g \geq 1$, *we have*

$$\int_{\mathrm{DR}_g(a,-a)} \lambda_g \lambda_{g-1} = (-1)^{g+1} \frac{a^{2g}}{(2g)!} \frac{\mathrm{B}_{2g}}{2g}.$$

Proposition 9 is essentially equivalent to the basic calculation of [2, Theorem 6.5] in the Gromov-Witten theory of local curves. There are two minor differences. First, our answer differs from [2, Theorem 6.5] by a factor of $2g$ coming from the choice of branch point in [2, Theorem 6.5]. Second, if we write out the generating series obtained from Proposition 9 then we only obtain the $\cot(u)$ summand in the formula of [2, Theorem 6.5] after a standard correction accounting for the geometric difference between the space of rubber maps and the space of maps relative to three points.

Proof. — We need only compute the contributions to $\mathrm{DR}_g(a, -a)$ of all rational tails type graphs since $\lambda_g \lambda_{g-1}$ vanishes on all other strata. Since we just have two marked points, there are exactly two rational tails type graphs:

- the graph with a single vertex,
- the graph with a vertex of genus g, a vertex of genus 0 containing both marked points, and exactly one edge joining the two vertices.

The contribution of the first graph is (using Lemma 8)

$$\begin{aligned}\frac{a^{2g}}{2^g g!} \int_{\overline{\mathcal{M}}_{g,2}} (\psi_1 + \psi_2)^g \lambda_g \lambda_{g-1} &= \frac{a^{2g}}{2^g g!} \cdot (-1)^{g+1} \frac{\mathrm{B}_{2g}}{2g} \frac{1}{(2g-1)!!} \\ &= (-1)^{g+1} \frac{a^{2g}}{(2g)!} \frac{\mathrm{B}_{2g}}{2g}.\end{aligned}$$

The contribution of the second graph vanishes because the weight of the edge is equal to 0. $\square$

For the above computation, we only require Hain's formula for the restriction of the double ramification cycle to $\overline{\mathcal{M}}^{\mathrm{ct}}_{g,n}$ (the extension to $\overline{\mathcal{M}}_{g,n}$ is not needed). In the forthcoming paper [22], we will study the Gromov-Witten theory of the rubber over the resolution of the A_n-singularity where the full structure is needed.

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Appendix A: Polynomiality *by A. Pixton*

A.1 Overview

Our goal here is to present a self contained proof of Proposition 3″ of Section 1.4. Let

$$A = (a_1, \dots, a_n)$$

be a vector of k-twisted double ramification data for genus $g \geq 0$. A longer treatment, in the context of the deeper polynomiality of $P_g^{d,k}(A)$ in the *parts a_i of* A, will be given in [35].

Proposition **3**″ *(Pixton [35]). — Let $\Gamma \in G_{g,n}$ be a fixed stable graph. Let Q be a polynomial with* **Q**-*coefficients in variables corresponding to the set of half-edges* $H(\Gamma)$. *Then the sum*

$$F(r) = \sum_{w \in W_{\Gamma,r,k}} Q\big(w(h_1), \dots, w(h_{|H(\Gamma)|})\big)$$

over all possible k-weightings mod r is (for all sufficiently large r) a polynomial *in r divisible by* $r^{h^1(\Gamma)}$.

A.2 Polynomiality of $F(r)$

A.2.1 Ehrhart polynomials

The polynomiality of $F(r)$ for r sufficiently large will be derived as a consequence of a small variation of the standard theory [37] of Ehrhart polynomials of integral convex polytopes.

Let M be a $\mathrm{v}\times\mathrm{h}$ matrix with $\mathbf{Q}$-coefficients satisfying the following two properties:

(i) M is *totally unimodular*: every square minor of M has determinant 0, 1, or -1.
(ii) If $\mathrm{x}=(x_j)_{1\le j\le \mathrm{h}}$ is a column vector satisfying

$$\mathrm{Mx}=0 \quad \text{and} \quad x_j\ge 0 \quad \text{for all } j,$$

then $\mathrm{x}=0$.

We will use the notation $\mathrm{x}\ge 0$ to signify $x_j\ge 0$ for all j. Let

$$\mathbf{Z}^{\mathrm{h}}_{\ge 0}=\left\{\mathrm{x}\mid \mathrm{x}\in\mathbf{Z}^{\mathrm{h}} \text{ and } \mathrm{x}\ge 0\right\}.$$

Proposition **A1.** — *Let* $\mathrm{a},\mathrm{b}\in\mathbf{Z}^{\mathrm{v}}$ *be column vectors with integral coefficients. Let* $\mathrm{Q}\in\mathbf{Q}[x_1,\dots,x_{\mathrm{h}}]$ *be a polynomial. Then, the sum over lattice points*

$$\mathrm{S}(r)=\sum_{\mathrm{x}\in\mathbf{Z}^{\mathrm{h}}_{\ge 0},\quad \mathrm{Mx}=\mathrm{a}+r\mathrm{b}}\mathrm{Q}(\mathrm{x})$$

is a polynomial *in the integer parameter r (for all sufficiently large r).*

Proof. — We follow the treatment of Ehrhart theory[21] in [37, Sections 4.5–4.6].

To start, we may assume all coefficients a_i of a are nonnegative: if not, multiply the negative coefficients of a and the corresponding parts of M and b by -1.

Next, we define a generating function of $\mathrm{h}+\mathrm{v}+1$ variables:

$$\mathrm{R}(\mathrm{X}_1,\dots,\mathrm{X}_{\mathrm{h}},\mathrm{Y}_1,\dots,\mathrm{Y}_{\mathrm{v}},\mathrm{Z}) = \sum_{\mathrm{x}\in\mathbf{Z}^{\mathrm{h}}_{\ge 0},\quad \mathrm{y}\in\mathbf{Z}^{\mathrm{v}}_{\ge 0},\quad z\in\mathbf{Z}_{\ge 0},\quad \mathrm{Mx}=\mathrm{y}+z\mathrm{b}}\mathrm{X}^{\mathrm{x}}\mathrm{Y}^{\mathrm{y}}\mathrm{Z}^{z}.$$

By [37, Theorem 4.5.11], R represents a rational function of the variables X, Y, and Z with denominator given by a product of factors of the form

$$1-\mathrm{Monomial}(\mathrm{X},\mathrm{Y},\mathrm{Z}).$$

The monomials in the denominator factors correspond to the extremal rays of the cone in $\mathbf{Z}^{\mathrm{h}+\mathrm{v}+1}_{\ge 0}$ specified by the condition

$$\mathrm{Mx}=\mathrm{y}+z\mathrm{b}.$$

Because M is totally unimodular, these extremal rays all have integral generators with z-coordinate equal to 0 or 1. Therefore, the denominator is a product of factors of the form

$$1-\mathrm{Monomial}(\mathrm{X},\mathrm{Y}) \quad \text{and} \quad 1-\mathrm{Monomial}(\mathrm{X},\mathrm{Y})\cdot\mathrm{Z}.$$

[21] The standard situation for the Ehrhart polynomial occurs when $\mathrm{Q}=1$ is the constant polynomial and $\mathrm{a}=0$. The variation required here is not very significant, but we were not able to find an adequate reference.

The sum $S(r)$ in the statement of Proposition A1 can be obtained from R by executing the following steps:

(i) apply a differential operator in the X variables to R which has the effect of multiplying the coefficient of $X^x Y^y Z^z$ by $Q(x)$,

(ii) apply the differential operator

$$\frac{1}{\prod_{i=1}^{v} a_i!} \frac{\partial^{a_1}}{\partial Y_1^{a_1}} \frac{\partial^{a_2}}{\partial Y_2^{a_2}} \cdots \frac{\partial^{a_v}}{\partial Y_v^{a_v}},$$

(iii) set[22] $X_j = 1$ for all $1 \le j \le h$ and $Y_i = 0$ for all $1 \le i \le v$,

(iv) extract the coefficient of Z^r.

After step (iii) and before step (iv), we have a rational function in Z with denominator equal to a power of $1 - Z$. The Z^r coefficient of such a rational function is (eventually) a polynomial in r. □

A.2.2 Vertex-edge matrix

We now prove the polynomiality of $F(r)$ for all r sufficiently large. Let

$$A = (a_1, \ldots, a_n)$$

be a vector of k-twisted double ramification data for genus $g \ge 0$. Let $\Gamma \in G_{g,n}$ be a fixed stable graph. For all sufficiently large r, the sum over half-edge weightings

$$w \in W_{\Gamma,r,k}, \quad 0 \le w(h) < r \tag{31}$$

satisfying conditions mod r,

$$F(r) = \sum_{w \in W_{\Gamma,r,k}} Q\big(w(h_1), \ldots, w(h_{|H(\Gamma)|})\big), \tag{32}$$

can be split into *finitely* many sums over half-edge weightings satisfying equalities[23] (involving multiples of r).

We study each sum in the splitting of (32) separately. The conditions on the weights $w(h)$ in these sums will be of the following form. The inequalities

$$0 \le w(h) < r$$

continue to hold. Moreover, each leg, edge, and vertex yields an equality:

[22] By property (ii) of the definition of M, the specialization is well defined. There are no denominator factors in R of the form $1 - \text{Monomial}(X)$.

[23] Equalities *not* equalities mod r.

(i) $w(h_i) = a_i + rb_i$ at the ith leg h_i, where

$$b_i = 0 \quad \text{if } a_i \geq 0 \quad \text{and} \quad b_i = 1 \quad \text{if } a_i < 0,$$

(ii) $w(h) + w(h') = 0$ or $w(h) + w(h') = r$ along the edge (h, h'),

(iii) $\sum_{v(h)=v} w(h) = k(2\mathrm{g}(v) - 2 + \mathrm{n}(v)) + rb_v$ at the vertex v with $b_v \in \mathbf{Z}$.

The conditions $w(h_i) < r$ are now redundant with the equalities of type (i) and can be removed. Along an edge (h, h'), the condition $w(h) < r$ only serves to rule out the possibility $w(h) = r$, $w(h') = 0$ (if we have the equality $w(h) + w(h') = r$). Thus the condition $w(h) < r$ can be removed if we subtract off the contributions with $w(h) = r$, $w(h') = 0$. These contributions are given by more sums of the same form (corresponding to the graph formed by deleting the edge (h, h') from Γ), multiplied by powers of r and 0 coming from factors of $w(h)$ and $w(h')$ in the polynomial Q.

Thus the original sum $\mathrm{F}(r)$ can be split into a linear combination (with coefficients being polynomials in r) of sums with weight conditions of the following form: inequalities

$$w(h) \geq 0$$

and equalities of the form (i), (ii), and (iii) associated to legs, edges, and vertices as described above.

We must check that the resulting sums satisfy the hypotheses of Proposition A1.

Consider the graph Γ' obtained from Γ by adding a new vertex in the middle of each edge $e \in \mathrm{E}(\Gamma)$ and at the end of each leg $l \in \mathrm{L}(\Gamma)$. The half-edge set of Γ then corresponds bijectively to the edge set of Γ',

$$\mathrm{H}(\Gamma) \stackrel{\sim}{\longleftrightarrow} \mathrm{E}(\Gamma').$$

We associate *each* linear condition on the half-edge weights $w(h)$ described in (i)–(iii) above to the corresponding *vertex* of Γ'. For types (i) and (ii), the corresponding vertex of Γ' is a new vertex. For (iii), the corresponding vertex already exists in Γ.

The matrix M associated to the linear terms in these linear conditions is precisely the *vertex-edge incidence matrix* of Γ': M is a $\mathrm{v} \times \mathrm{h}$ matrix,

$$\mathrm{v} = |\mathrm{V}(\Gamma')|, \qquad \mathrm{h} = |\mathrm{E}(\Gamma')| = |\mathrm{H}(\Gamma)|,$$

with matrix entry 1 if the corresponding vertex of Γ' meets the corresponding edge of Γ' (and 0 otherwise). Since all the entries of M are nonnegative and every column contains a positive entry, we immediately see

$$(\mathrm{Mx} = 0 \text{ and } \mathrm{x} \geq 0) \quad \Rightarrow \quad \mathrm{x} = 0.$$

Because Γ' is bipartite, the incidence matrix M is totally unimodular by Proposition A2 below.

We may therefore apply Proposition A1 to each sum of the splitting of (32). Each such sum is a polynomial in r for sufficiently large r. Hence, $\mathrm{F}(r)$ is a polynomial in r for sufficiently large r.

Proposition **A2.** — *The vertex-edge matrix of a bipartite graph is totally unimodular.*

A proof of general results concerning the total unimodularity of vertex-edge matrices can be found in [20]. Proposition A2 arises as a special case.

A.3 Divisibility by $r^{h^1(\Gamma)}$

In this section we write

$$\mathrm{b} = h^1(\Gamma)$$

for convenience. We wish to show that the polynomial $\mathrm{F}(r)$ is divisible by r^{b}. This is implied by checking that the valuation condition,

$$\mathrm{val}_p\big(\mathrm{F}(p)\big) \geq \mathrm{b},$$

holds for *all* sufficiently large primes p. Here,

$$\mathrm{val}_p : \mathbf{Q} \to \mathbf{Z}$$

is the p-adic valuation.

Let $\mathrm{h} = |\mathrm{H}(\Gamma)|$ be the number of half-edges of Γ. We write the evaluation $\mathrm{F}(p)$ as

$$\mathrm{F}(p) = \sum_{0 \leq w_1, \ldots, w_{\mathrm{h}} \leq p-1} \mathrm{Q}(w_1, \ldots, w_{\mathrm{h}})$$

where the sum is over h-tuples $(w_1, \ldots, w_{\mathrm{h}})$ subject to $\mathrm{h} - \mathrm{b}$ linear conditions mod p. The polynomial Q has (p-integral) $\mathbf{Q}$-coefficients.

We count the linear conditions mod p as follows. By the definition of a k-weighting mod p, we start with

$$\mathrm{v} = \big|\mathrm{L}(\Gamma)\big| + \big|\mathrm{E}(\Gamma)\big| + \big|\mathrm{V}(\Gamma)\big|$$

linear conditions, but the conditions are not independent since the vector A of k-twisted double ramification data satisfies

$$\sum_{i=1}^{n} a_i = k(2g - 2 + n).$$

After removing a redundant linear relation, we have $\mathrm{v} - 1$ independent linear conditions. Since

$$\mathrm{h} = \big|\mathrm{H}(\Gamma)\big| = \big|\mathrm{L}(\Gamma)\big| + 2\big|\mathrm{E}(\Gamma)\big|,$$

and Γ is connected, we see

$$\mathrm{h} - \mathrm{b} = \mathrm{h} - \big(\mathrm{E}(\Gamma) - \mathrm{V}(\Gamma) + 1\big) = \mathrm{v} - 1.$$

Let $\widehat{\mathrm{Q}}$ be a polynomial with $\mathbf{Q}$-coefficients in h variables, with all coefficients p-integral. We will prove *every* summation

$$\widehat{\mathrm{F}}(p) = \sum_{0 \le w_1, \ldots, w_{\mathrm{h}} \le p-1} \widehat{\mathrm{Q}}(w_1, \ldots, w_{\mathrm{h}}), \tag{33}$$

where the sum is over h-tuples $(w_1, \ldots, w_{\mathrm{h}})$ subject to $\mathrm{h}-\mathrm{b}$ linear conditions mod p, is divisible by p^{b} for $p \ge \deg(\widehat{\mathrm{Q}}) + 2$.

We may certainly take $\widehat{\mathrm{Q}}$ to be a monomial. Without loss of generality,[24] we need only consider

$$\widehat{\mathrm{Q}}(w_1, \ldots, w_{\mathrm{h}}) = \prod_{j=1}^{\mathrm{h}} w_j.$$

We write the linear conditions imposed in the sum (33) as

$$\sum_{j=1}^{\mathrm{h}} \mathrm{m}_{ij} w_j + \ell_i = 0 \mod p$$

for $1 \le i \le \mathrm{h}-\mathrm{b}$. Alternatively, we may impose the conditions by standard p-th root of unity filters:

$$p^{\mathrm{h}-\mathrm{b}} \widehat{\mathrm{F}}(p) = \sum_{z_1, \ldots, z_{\mathrm{h}-\mathrm{b}}} \sum_{0 \le w_1, \ldots, w_{\mathrm{h}} \le p-1} \prod_{j=1}^{\mathrm{h}} w_j \cdot \prod_{\substack{1 \le i \le \mathrm{h}-\mathrm{b} \\ 1 \le j \le \mathrm{h}}} z_i^{\mathrm{m}_{ij} w_j} \cdot \prod_{1 \le i \le \mathrm{h}-\mathrm{b}} z_i^{\ell_i}.$$

The first sum on the right is over all $(\mathrm{h}-\mathrm{b})$-tuples $(z_1, \ldots, z_{\mathrm{h}-\mathrm{b}})$ of p-th roots of unity. The second sum on the right is over *all* h-tuples $(w_1, \ldots, w_{\mathrm{h}})$.

Consider the following polynomial with integer coefficients:

$$\mathrm{G}(\mathrm{Z}) = \sum_{0 \le a \le p-1} a \mathrm{Z}^a = \frac{\mathrm{Z} - p\mathrm{Z}^p + (p-1)\mathrm{Z}^{p+1}}{(1-\mathrm{Z})^2}.$$

We can use G to rewrite the terms inside the first sum in the above expression:

$$p^{\mathrm{h}-\mathrm{b}} \widehat{\mathrm{F}}(p) = \sum_{z_1, \ldots, z_{\mathrm{h}-\mathrm{b}}} \prod_{1 \le j \le \mathrm{h}} \mathrm{G}\Bigg(\prod_{1 \le i \le \mathrm{h}-\mathrm{b}} z_i^{\mathrm{m}_{ij}} \Bigg) \cdot \prod_{1 \le i \le \mathrm{h}-\mathrm{b}} z_i^{\ell_i}. \tag{34}$$

[24] To increase w_1 to w_1^2, add a new variable $w_{\mathrm{h}+1}$ and a new linear condition $w_1 = w_{\mathrm{h}+1}$. If a variable w_j does not appear in $\widehat{\mathrm{Q}}$, then a reduction to a fewer variable case can be made.

For every p-th root of unity z, $\mathrm{G}(z)$ has p-adic valuation[25] at least $\frac{p-2}{p-1}$: either $z = 1$ and $\mathrm{G}(z) = \frac{p(p-1)}{2}$ or

$$z \neq 1 \quad \Rightarrow \quad \mathrm{G}(z) = \frac{p}{z-1} \quad \text{and} \quad \mathrm{val}_p(z-1) = \frac{1}{p-1}.$$

Therefore, every term in the sum on the right side of (34) has p-adic valuation at least $\mathrm{h} \cdot \frac{p-2}{p-1}$. Hence,

$$\mathrm{val}_p\big(\widehat{\mathrm{F}}(p)\big) \geq \mathrm{h} \cdot \frac{p-2}{p-1} - (\mathrm{h} - \mathrm{b}).$$

Since $\mathrm{val}_p(\widehat{\mathrm{F}}(p))$ is an *integer*,

$$\mathrm{val}_p\big(\widehat{\mathrm{F}}(p)\big) \geq \mathrm{b}$$

if $p \geq \mathrm{h} + 2$. □

A.4 Polynomiality of $\mathrm{P}_g^{d,k}(A)$ in the a_i

For double ramification data of fixed length n, consider

$$\mathrm{P}_g^{d,k}(a_1, \ldots, a_n) \in \mathrm{R}^d(\overline{\mathcal{M}}_{g,n})$$

as a function of

$$(a_1, \ldots, a_n) \in \mathbf{Z}^n, \quad \sum_{i=1}^n a_i = k(2g - 2 + n)$$

holding g, d, and k fixed. The following property is proven in [35]:

$\mathrm{P}_g^{d,k}(a_1, \ldots, a_n)$ *is polynomial in the variables* a_i.

By Theorem 1, the double ramification cycle $\mathrm{DR}_g(A)$ is then also polynomial in the variables a_i. Shadows of the polynomiality of the double ramification cycle were proven earlier in [3, 29].

[25] We view the p-th root of unity z as lying in $\overline{\mathbf{Q}}_p$. The p-adic valuation of $x = z - 1$ can be calculated from the minimal polynomial

$$\sum_{i=0}^{p-1} (x+1)^i = 0.$$

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