LAGRANGIAN FIBRATIONS ON BLOWUPS OF TORIC VARIETIES AND MIRROR SYMMETRY FOR HYPERSURFACES

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ABSTRACT

We consider mirror symmetry for (essentially arbitrary) hypersurfaces in (possibly noncompact) toric varieties from the perspective of the Strominger-Yau-Zaslow (SYZ) conjecture. Given a hypersurface H in a toric variety V we construct a Landau-Ginzburg model which is SYZ mirror to the blowup of $V \times \mathbf{C}$ along $H \times 0$, under a positivity assumption. This construction also yields SYZ mirrors to affine conic bundles, as well as a Landau-Ginzburg model which can be naturally viewed as a mirror to H. The main applications concern affine hypersurfaces of general type, for which our results provide a geometric basis for various mirror symmetry statements that appear in the recent literature. We also obtain analogous results for complete intersections.

1. Introduction

A number of recent results [3, 17, 25, 32, 47] suggest that the phenomenon of mirror symmetry is not restricted to Calabi-Yau or Fano manifolds. Indeed, while mirror symmetry was initially formulated as a duality between Calabi-Yau manifolds, it was already suggested in the early works of Givental and Batyrev that Fano manifolds also exhibit mirror symmetry. The counterpart to the presence of a nontrivial first Chern class is that the mirror of a compact Fano manifold is not a compact manifold, but rather a *Landau-Ginzburg model*, i.e. a (non-compact) Kähler manifold equipped with a holomorphic function called *superpotential*. A physical explanation of this phenomenon and a number of examples have been given by Hori and Vafa [29]. From a mathematical point of view, Hori and Vafa's construction amounts to a toric duality, and can also be applied to varieties of general type [16, 25, 32, 33].

The Strominger-Yau-Zaslow (SYZ) conjecture [51] provides a geometric interpretation of mirror symmetry for Calabi-Yau manifolds as a duality between (special) Lagrangian torus fibrations. In the language of Kontsevich's homological mirror symmetry [34], the SYZ conjecture reflects the expectation that the mirror can be realized as a moduli space of certain objects in the Fukaya category of the given manifold, namely, a family of Lagrangian tori equipped with rank 1 local systems. Note that this homological perspective eliminates the requirement of finding special Lagrangian fibrations, at the cost of privileging one side of mirror symmetry: in the Calabi-Yau case, the framework we follow produces a degenerating family Y^0 of complex manifolds (B-side) starting with a Lagrangian torus fibration on a symplectic manifold X^0 (A-side).

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Outside of the Calabi-Yau situation, homological mirror symmetry is still expected to hold [35], but the Lagrangian tori bound holomorphic discs, which causes their Floer theory to be obstructed; the mirror superpotential can be interpreted as a weighted count of these holomorphic discs [6, 7, 21, 28]. We call such a mirror a B-*side Landau-Ginzburg model*.

In the Calabi-Yau case, mirror symmetry is expected to be involutive; i.e. when the symplectic form on X^0 is in fact a Kähler form for some degenerating family of complex structures then the mirror Y should be equipped with its own Kähler form which is mirror to these complex structures. Involutivity should hold beyond the Calabi-Yau situation, but requires making sense of a class of potential functions on symplectic manifolds, called A-*side Landau-Ginzburg models*, which have well defined Fukaya categories. The idea for such a definition goes back to Kontsevich [35], and was studied in great depth by Seidel in [46] in the special case of Lefschetz fibrations.

Remark **1.1.** — The general theory of Fukaya categories $\mathcal{F}(X, W^\vee)$ of A-side Landau-Ginzburg models is still under development in different contexts [2, 4, 5]; we shall specifically point out where it is being used in this paper. In fact, we will also need to consider *twisted* versions of A-side Landau-Ginzburg models, where objects of the Fukaya category carry *relatively spin* structures with respect to a background class in $H^2(X, \mathbf{Z}/2)$ (rather than spin structures); see Section 7.

On manifolds of general type (or more generally, whose first Chern class cannot be represented by an effective divisor), the SYZ approach to mirror symmetry seems to fail at first glance due to the lack of a suitable Lagrangian torus fibration. The idea that allows one to overcome this obstacle is to replace the given manifold with another closely related space which does carry an appropriate SYZ fibration. Thus, we make the following definition:

Definition **1.2.** — *We say that a* B-*side Landau-Ginzburg model* (Y, W) *is* SYZ mirror *to a Kähler manifold* X *(resp. an* A-*side Landau-Ginzburg model* $(X, W^\vee)$*) if there exists an open dense subset* X^0 *of* X*, and a Lagrangian torus fibration* $\pi : X^0 \to B$*, such that the following properties hold:*

(1) Y *is a completion of a moduli space of unobstructed torus-like objects of the Fukaya category* $\mathcal{F}(X^0)$ *(resp.* $\mathcal{F}(X^0, W^\vee)$*) containing those objects which are supported on the fibers of* π*;*
(2) *the function* W *restricts to the superpotential induced by the deformation of* $\mathcal{F}(X^0)$ *to* $\mathcal{F}(X)$ *(resp.* $\mathcal{F}(X^0, W^\vee)$ *to* $\mathcal{F}(X, W^\vee)$*) for these objects.*

We say that (Y, W) *is a* generalized SYZ mirror *of* X *if (after shifting* W *by a suitable additive constant) it is an SYZ mirror of a (suitably twisted)* A-*side Landau-Ginzburg model with Morse-Bott superpotential, whose critical locus is isomorphic to* X.

The last part of the definition is motivated by the expectation that the Fukaya category of a Morse-Bott superpotential, twisted by a background class which accounts for the non-triviality of the normal bundle to the critical locus, is equivalent (up to an additive constant shift in the curvature term, which accounts for exceptional curves through the critical locus) to the Fukaya category of the critical locus; see Corollary 7.8 and Proposition 7.10.

Definition 1.2 and the construction of moduli spaces of objects of the Fukaya category are clarified in Section 2 and Appendix A. To understand the first condition in the case of an A-side Landau-Ginzburg model, it is useful to note that every object of the Fukaya category $\mathcal{F}(X^0)$ of compact Lagrangians also defines an object of $\mathcal{F}(X^0, W^\vee)$ since the objects of the latter are Lagrangians satisfying admissibility properties outside a compact set and such properties trivially hold for compact Lagrangians. Hence the fibers of π automatically define objects of $\mathcal{F}(X^0, W^\vee)$; we shall enlarge this space by considering certain non-compact Lagrangians in X^0 which can be seen as limits of compact Lagrangians.

Remark **1.3.** — It is important to note that, even in the absence of superpotentials, the assertion that Y^0 is SYZ mirror to X^0 may not imply that the Fukaya category of X^0 is equivalent to the derived category of Y^0; at a basic level, the example of the Kodaira surface mentioned in [1] shows that there may in general be an analytic gerbe on Y^0 so that the Fukaya category of X^0 is in fact mirror to sheaves twisted by this gerbe. Beyond the Calabi-Yau situation, a complete statement of homological mirror symmetry for SYZ mirrors would have to consider further deformations of the derived category of sheaves by (holomorphic) polyvector fields on Y. The superpotential W should be thought of as the leading order term of this deformation corresponding to discs of Maslov index 2.

More fundamentally, our construction of the analytic completion relies on choices, and it is expected that different choices will given rise to different mirrors. Indeed, this phenomenon would provide a mirror symmetry explanation for the existence of derived equivalent varieties which are birational. Nonetheless, as completely arbitrary choices of completions give rise to varieties which are not derived equivalent (e.g. a blowup), the task of passing from our SYZ mirror statement to homological mirror symmetry would require a more careful understanding of the completions that we have introduced. This paper begins this task by explaining how some of the points that we add should correspond to objects of the Fukaya category supported by immersed or non-compact Lagrangians (see Remark A.12).

In this paper we use this perspective to study mirror symmetry for hypersurfaces (and complete intersections) in toric varieties. If H is a smooth hypersurface in a toric variety V, then one simple way to construct a closely related Kähler manifold with effective first Chern class is to blow up the product $V \times \mathbf{C}$ along the codimension 2 submanifold $H \times 0$. By a result of Bondal and Orlov [9], the derived category of coherent sheaves of the resulting manifold X admits a semi-orthogonal decomposition into subcategories equivalent to $D^b\,\mathrm{Coh}(H)$ and $D^b\,\mathrm{Coh}(V \times \mathbf{C})$; and ideas similar to those of [50] can be used to study the Fukaya category of X, as we explain in Section 7 (cf. Corollary 7.8). Thus, finding a mirror to X is, for many purposes, as good as finding a mirror to H. Accordingly, our main results concern SYZ mirror symmetry for X and, by a slight modification of the construction, for H. Along the way we also obtain descriptions of SYZ mirrors to various related spaces. These results provide a geometric foundation for mirror constructions that have appeared in the recent literature [3, 16, 25, 32, 33, 47, 49].

We focus primarily on the case where V is affine, and other cases which can be handled with the same techniques. The general case requires more subtle arguments in enumerative geometry, which should be the subject of further investigation.

1.1. *Statement of the results.* — Our main result can be formulated as follows (see Section 3 for the details of the notations).

Let $\mathrm{H} = f^{-1}(0)$ be a smooth nearly tropical hypersurface (cf. Section 3.1) in a (possibly noncompact) toric variety V of dimension n, and let X be the blow-up of $\mathrm{V} \times \mathbf{C}$ along $\mathrm{H} \times 0$, equipped with an S^1-invariant Kähler form ω_ϵ for which the fibers of the exceptional divisor have sufficiently small area $\epsilon > 0$ (cf. Section 3.2).

Let Y be the toric variety defined by the polytope $\{(\xi, \eta) \in \mathbf{R}^n \times \mathbf{R} \mid \eta \geq \varphi(\xi)\}$, where φ is the tropicalization of f. Let $w_0 = -\mathrm{T}^\epsilon + \mathrm{T}^\epsilon v_0 \in \mathcal{O}(\mathrm{Y})$, where T is the Novikov parameter and v_0 is the toric monomial with weight $(0, \dots, 0, 1)$, and set $\mathrm{Y}^0 = \mathrm{Y} \setminus w_0^{-1}(0)$. Finally, let $\mathrm{W}_0 = w_0 + w_1 + \cdots + w_r \in \mathcal{O}(\mathrm{Y})$ be the *leading-order superpotential* of Definition 3.10, namely the sum of w_0 and one toric monomial w_i $(1 \leq i \leq r)$ for each irreducible toric divisor of V (see Definition 3.10). We assume:

Assumption **1.4.** — $c_1(\mathrm{V}) \cdot \mathrm{C} > \max(0, \mathrm{H} \cdot \mathrm{C})$ *for every rational curve* $\mathrm{C} \simeq \mathbf{P}^1$ *in* V.

This includes the case where V is an affine toric variety as an important special case. Under this assumption, our main result is the following:

Theorem **1.5.** — *Under Assumption 1.4, the* B*-side Landau-Ginzburg model* $(\mathrm{Y}^0, \mathrm{W}_0)$ *is SYZ mirror to* X.

In the general case, the mirror of X differs from $(\mathrm{Y}^0, \mathrm{W}_0)$ by a correction term which is of higher order with respect to the Novikov parameter (see Remark 6.3).

Equipping X with an appropriate superpotential, given by the affine coordinate of the $\mathbf{C}$ factor, yields an A-side Landau-Ginzburg model whose singularities are of Morse-Bott type. Up to twisting by a class in $\mathrm{H}^2(\mathrm{X}, \mathbf{Z}/2)$, this A-side Landau-Ginzburg model can be viewed as a stabilization of the sigma model with target H.

Theorem **1.6.** — *Assume* V *is affine, and let* $\mathrm{W}_0^{\mathrm{H}} = -v_0 + w_1 + \cdots + w_r \in \mathcal{O}(\mathrm{Y})$ *(see Definition 3.10). Then the* B*-side Landau-Ginzburg model* $(\mathrm{Y}, \mathrm{W}_0^{\mathrm{H}})$ *is a generalized SYZ mirror of* H.

Unlike the other results stated in this introduction, this theorem strictly speaking relies on the assumption that Fukaya categories of Landau-Ginzburg models satisfy certain properties for which we do not provide complete proofs. In Section 7, we give sketches of the proofs of these results, and indicate the steps which are missing from our argument.

A result similar to Theorem 1.6 can also be obtained from the perspective of mirror duality between toric Landau-Ginzburg models [16, 25, 29, 32]. However, the toric approach is much less illuminating, because geometrically it works at the level of the open toric strata in the relevant toric varieties (the total space of $\mathcal{O}(-\mathrm{H}) \to \mathrm{V}$ on one hand, and Y on the other hand), whereas the interesting geometric features of these spaces lie entirely within the toric divisors.

Theorem 1.5 relies on a mirror symmetry statement for open Calabi-Yau manifolds which is of independent interest. Consider the conic bundle

$$\mathrm{X}^0 = \left\{ (\mathbf{x}, y, z) \in \mathrm{V}^0 \times \mathbf{C}^2 \,\middle|\, yz = f(\mathbf{x}) \right\}$$

over the open stratum $\mathrm{V}^0 \simeq (\mathbf{C}^*)^n$ of V, where f is again the defining equation of the hypersurface H. The conic bundle X^0 sits as an open dense subset inside X, see Remark 3.5. Then we have:

Theorem **1.7.** — *The open Calabi-Yau manifold* Y^0 *is SYZ mirror to* X^0.

In the above statements, and in most of this paper, we view X or X^0 as a symplectic manifold, and construct the SYZ mirror Y^0 (with a superpotential) as an algebraic moduli space of objects in the Fukaya category of X or X^0. This is the same direction considered e.g. in [3, 17, 47]. However, one can also work in the opposite direction, starting from the symplectic geometry of Y^0 and showing that it admits X^0 (now viewed as a complex manifold) as an SYZ mirror. For completeness we describe this converse construction in Section 8 (see Theorem 8.4); similar results have also been obtained independently by Chan, Lau and Leung [12].

The methods we use apply in more general settings as well. In particular, the assumption that V be a toric variety is not strictly necessary—it is enough that SYZ mirror symmetry for V be sufficiently well understood. As an illustration, in Section 11 we derive analogues of Theorems 1.5–1.7 for *complete intersections*.

1.2. *A reader's guide.* — The rest of this paper is organized as follows.

First we briefly review (in Section 2) the SYZ approach to mirror symmetry, following [6, 7]. Then in Section 3 we introduce notation and describe the protagonists of our main results, namely the spaces X and Y and the superpotential W_0.

In Section 4 we construct a Lagrangian torus fibration on X^0, similar to those previously considered by Gross [23, 24] and by Castaño-Bernard and Matessi [10, 11]. In Section 5 we study the Lagrangian Floer theory of the torus fibers, which we use to prove Theorem 1.7. In Section 6 we consider the partial compactification of X^0 to X, and prove Theorem 1.5. Theorem 1.6 is then proved in Section 7.

In Section 8 we briefly consider the converse construction, namely we start from a Lagrangian torus fibration on Y^0 and recover X^0 as its SYZ mirror.

Finally, some examples illustrating the main results are given in Section 9, while Sections 10 and 11 discusses various generalizations, including to hypersurfaces in abelian varieties (Theorem 10.4) and complete intersections in toric varieties (Theorem 11.1).

2. Review of SYZ mirror symmetry

In this section, we briefly review SYZ mirror symmetry for Kähler manifolds with effective anticanonical class; the reader is referred to [6, 7] for basic ideas about SYZ, and to Appendix A for technical details.

2.1. *Lagrangian torus fibrations and SYZ mirrors.* — In first approximation, the Strominger-Yau-Zaslow conjecture [51] states that mirror pairs of Calabi-Yau manifolds carry mutually dual Lagrangian torus fibrations (up to "instanton corrections"). A reformulation of this statement in the language of homological mirror symmetry [34] is that a mirror of a Calabi-Yau manifold can be constructed as a moduli space of suitable objects in its Fukaya category (namely, the fibers of an SYZ fibration, equipped with rank 1 local systems); and vice versa. In Appendix A, we explain how ideas of Fukaya [19] yield a precise construction of such a mirror space from local rigid analytic charts glued via the equivalence relation which identifies objects that are quasi-isomorphic in the Fukaya category.

We consider an open Calabi-Yau manifold of the form $X^0 = X \setminus D$, where (X, ω, J) is a Kähler manifold of complex dimension n and $D \subset X$ is an anticanonical divisor (reduced, with normal crossing singularities). X^0 can be equipped with a holomorphic n-form Ω (with simple poles along D), namely the inverse of the defining section of D. The restriction of Ω to an oriented Lagrangian submanifold $L \subset X^0$ is a nowhere vanishing complex-valued n-form on L; the complex argument of this n-form determines the *phase* function $\arg(\Omega_{|L}) : L \to S^1$. Recall that L is said to be *special Lagrangian* if $\arg(\Omega_{|L})$ is constant; a weaker condition is to require the vanishing of the *Maslov class* of L in X^0, i.e. we require the existence of a lift of $\arg(\Omega_{|L})$ to a real-valued function. (The choice of such a real lift then makes L a *graded Lagrangian*, and yields $\mathbf{Z}$-gradings on Floer complexes.)

The main input of the construction of the SYZ mirror of the open Calabi-Yau manifold X^0 is a Lagrangian torus fibration $\pi : X^0 \to B$ (with appropriate singularities) whose fibers have trivial Maslov class. (Physical considerations suggest that one should expect the fibers of π to be special Lagrangian, but such fibrations are hard to produce.)

The base B of the Lagrangian torus fibration π carries a natural real affine structure (with singularities along the locus B^{sing} of singular fibers), i.e. $B \setminus B^{sing}$ can be covered by a set of coordinate charts with transition functions in $GL(n, \mathbf{Z}) \ltimes \mathbf{R}^n$. A smooth fiber $L_0 = \pi^{-1}(b_0)$ and a collection of loops $\gamma_1, \ldots, \gamma_n$ forming a basis of $H_1(L_0, \mathbf{Z})$ determine an affine chart centered at b_0 in the following manner: given $b \in B \setminus B^{sing}$ close enough to b_0, we can isotope L_0 to $L = \pi^{-1}(b)$ among fibers of π. Under such an isotopy, each loop γ_i traces a cylinder Γ_i with boundary in $L_0 \cup L$; the affine coordinates associated to b are then the symplectic areas $(\int_{\Gamma_1} \omega, \dots, \int_{\Gamma_n} \omega)$.

In the examples we will consider, "most" fibers of π do not bound nonconstant holomorphic discs in X^0; we call such Lagrangians *tautologically unobstructed*. Recall that a (graded, spin) Lagrangian submanifold L of X^0 together with a unitary rank one local system ∇ determines an object (L, ∇) of the Fukaya category $\mathcal{F}(X^0)$ [20] whenever certain counts of holomorphic discs cancel; this condition evidently holds if there are no non-constant discs. Thus, given an open subset $U \subset B \setminus B^{sing}$ such that all the fibers in $\pi^{-1}(U)$ are tautologically unobstructed, the moduli space of objects (L, ∇) where $L \subset \pi^{-1}(U)$ is a fiber of π and ∇ is a unitary rank 1 local system on L yields an open subset $U^\vee \subset Y^0$ of the SYZ mirror of X^0.

A word is in order about the choice of coefficient field. A careful definition of Floer homology involves working over the *Novikov field* (here over complex numbers),

$$\Lambda = \left\{ \sum_{i=0}^{\infty} c_i T^{\lambda_i} \,\middle|\, c_i \in \mathbf{C},\ \lambda_i \in \mathbf{R},\ \lambda_i \to +\infty \right\}. \tag{2.1}$$

Recall that the *valuation* of a non-zero element of Λ is the smallest exponent λ_i that appears with a non-zero coefficient; the above-mentioned local systems are required to have holonomy in the multiplicative subgroup

$$U_\Lambda = \left\{ c_0 + \sum c_i T^{\lambda_i} \in \Lambda \,\middle|\, c_0 \neq 0 \text{ and } \lambda_i > 0 \right\}$$

of *unitary elements* (or units) of the Novikov field, i.e. elements whose valuation is zero. The local system $\nabla \in \hom(\pi_1(L), U_\Lambda)$ enters into the definition of Floer-theoretic operations by contributing holonomy terms to the weights of holomorphic discs: a rigid holomorphic disc u with boundary on Lagrangians (L_i, ∇_i) is counted with a weight

$$T^{\omega(u)}\, \mathrm{hol}(\partial u), \tag{2.2}$$

where $\omega(u)$ is the symplectic area of the disc u, and $\mathrm{hol}(\partial u) \in U_\Lambda$ is the total holonomy of the local systems ∇_i along its boundary. (Thus, local systems are conceptually an exponentiated variant of the "bounding cochains" used by Fukaya et al. [20, 21].) Gromov compactness ensures that all structure constants of Floer-theoretic operations are well-defined elements of Λ.

Thus, in general the SYZ mirror of X^0 is naturally an analytic space defined over Λ. However, it is often possible to obtain a complex mirror by treating the Novikov parameter T as a numerical parameter $T = e^{-2\pi t}$ with $t > 0$ sufficiently large; of course it is necessary to assume the convergence of all the power series encountered. The local systems are then taken to be unitary in the usual sense, i.e. $\nabla \in \hom(\pi_1(L), S^1)$, and the weight of a rigid holomorphic disc, still given by (2.2), becomes a complex number. The complex manifolds obtained by varying the parameter t are then understood to be mirrors to the family of Kähler manifolds $(X^0, t\omega)$.

To provide a unified treatment, we denote by $\mathbf{K}$ the coefficient field (Λ or $\mathbf{C}$), by $U_{\mathbf{K}}$ the subgroup of unitary elements (either U_Λ or S^1), and by $\mathrm{val} : \mathbf{K} \to \mathbf{R}$ the valuation (in the case of complex numbers, $\mathrm{val}(z) = -\frac{1}{2\pi t}\log|z|$).

Consider as above a contractible open subset $U \subset B \setminus B^{sing}$ above which all fibers of π are tautologically unobstructed, a reference fiber $L_0 = \pi^{-1}(b_0) \subset \pi^{-1}(U)$, and a basis $\gamma_1, \dots, \gamma_n$ of $H_1(L_0, \mathbf{Z})$. A fiber $L = \pi^{-1}(b) \subset \pi^{-1}(U)$ and a local system $\nabla \in \hom(\pi_1(L), U_{\mathbf{K}})$ determine a point of the mirror, $(L, \nabla) \in U^\vee \subset Y^0$. Identifying implicitly $H_1(L, \mathbf{Z})$ with $H_1(L_0, \mathbf{Z})$, the local system ∇ is determined by its holonomies along the loops $\gamma_1, \dots, \gamma_n$, while the fiber L is determined by the symplectic areas of the cylinders $\Gamma_1, \dots, \Gamma_n$. This yields natural coordinates on $U^\vee \subset Y^0$, identifying it with an open subset of $(\mathbf{K}^*)^n$ via

$$(L, \nabla) \mapsto (z_1, \dots, z_n) = \left(T^{\int_{\Gamma_1}\omega}\nabla(\gamma_1), \dots, T^{\int_{\Gamma_n}\omega}\nabla(\gamma_n)\right). \tag{2.3}$$

One feature of Floer theory that is conveniently captured by this formula is the fact that, in the absence of instanton corrections, the non-Hamiltonian isotopy from L_0 to L is formally equivalent to equipping L_0 with a *non-unitary* local system for which $\mathrm{val}(\nabla(\gamma_i)) = \int_{\Gamma_i}\omega$.

The various regions of B over which the fibers are tautologically unobstructed are separated by *walls* (real hypersurfaces in B, or thickenings of real hypersurfaces) of *potentially obstructed* fibers (i.e. which bound non-constant holomorphic discs), across which the local charts of the mirror (as given by (2.3)) need to be glued together in an appropriate manner to account for "instanton corrections".

The discussion preceding Equation (A.4) makes precise the idea that we can embed the moduli space of Lagrangians equipped with unitary local systems in an analytic space obtained by gluing coordinate charts coming from non-unitary systems. This will be the first step in the construction of the mirror manifold as a completion of the moduli space of Lagrangians.

Consider a potentially obstructed fiber $L = \pi^{-1}(b)$ of π, where $b \in B \setminus B^{sing}$ lies in a wall that separates two tautologically unobstructed chambers. By deforming this fiber to a nearby chamber, we obtain a bounding cochain (with respect to the Floer differential) for the moduli space of holomorphic discs with boundary on L. While local systems on L define objects of $\mathcal{F}(X^0)$, the quasi-isomorphism type of such objects depends on the choice of bounding cochain, which in our case amounts to a choice of this isotopy. In the special situation we are considering, we use this argument to prove in Lemma A.13 that any unitary local system on L can be represented by a non-unitary local system on a fiber lying in a tautologically unobstructed chamber. This implies that the space obtained by gluing the mirrors of the chambers contains the analytic space corresponding to all unitary local systems on smooth fibers of π.

The gluing maps for the mirrors of nearby chambers are given by wall-crossing formulae, with instanton corrections accounting for the disc bubbling phenomena that occur as a Lagrangian submanifold is isotoped across a wall of potentially obstructed Lagrangians (see [6] for an informal discussion, and Appendix A.1 for the relation with the invariance proof of Floer cohomology in this setting [19, 20]). Specifically, consider a Lagrangian isotopy $\{L_t\}_{t\in[0,1]}$ whose end points are tautologically unobstructed and lie in adjacent chambers. Assume that all nonconstant holomorphic discs bounded by the Lagrangians L_t in X^0 represent a single relative homotopy class $\beta \in \pi_2(X^0, L_t)$ (we implicitly identify these groups with each other by means of the isotopy), or its multiples (for non-simple discs). The weight associated to the class β defines a regular function

$$z_\beta = T^{\omega(\beta)}\nabla(\partial\beta) \in \mathcal{O}\big(U_i^\vee\big),$$

in fact a monomial in the coordinates $(z_1, \dots, z_n)$ of (2.3). In this situation, assuming its transversality, the moduli space $\mathcal{M}_1(\{L_t\}, \beta)$ of all holomorphic discs in the class β bounded by L_t as t varies from 0 to 1, with one boundary marked point, is a closed $(n-1)$-dimensional manifold, oriented if we fix a spin structure on L_t. Thus, evaluation at the boundary marked point (combined with identification of the submanifolds L_t via the isotopy) yields a cycle $C_\beta = ev_*[\mathcal{M}_1(\{L_t\}, \beta)] \in H_{n-1}(L_t)$. The instanton corrections to the gluing of the local coordinate charts (2.3) are then of the form

$$z_i \mapsto \big(h(z_\beta)\big)^{C_\beta\cdot\gamma_i} z_i, \tag{2.4}$$

where $h(z_\beta) = 1 + z_\beta + \cdots \in \mathbf{Q}[[z_\beta]]$ is a power series recording the (virtual) contributions of multiple covers of the discs in the class β. In practice, we shall only use the weaker property that these transformations are of the form

$$z_i \mapsto h_i(z_\beta) z_i, \tag{2.5}$$

where $h_i(z_\beta) \in 1 + z_\beta\mathbf{Q}[[z_\beta]]$.

In the examples we consider in this paper, there are only finitely many walls in B, and the above considerations are sufficient to construct the SYZ mirror of X^0 out of instanton-corrected gluings of local charts. In general, intersections between walls lead, via a "scattering" phenomenon, to an infinite number of higher-order instanton corrections; it is conjectured that these Floer-theoretic corrections can be determined using the machinery developed by Kontsevich-Soibelman [36, 37] and Gross-Siebert [26, 27].

Remark **2.1.** — We have discussed how to construct the analytic space Y^0 ("B-model") from the symplectic geometry of X^0 ("A-model"). When Y^0 makes sense as a complex manifold (i.e., assuming convergence), one also expects it to carry a natural Kähler structure for which the A-model of Y^0 is equivalent to the B-model of X^0. We will however not emphasize this feature of mirror symmetry.

2.2. *The superpotential.* — In the previous section we explained the construction of the SYZ mirror Y^0 of an open Calabi-Yau manifold $X^0 = X \setminus D$, where D is an anticanonical divisor in a Kähler manifold (X, ω, J), equipped with a Lagrangian torus fibration $\pi : X^0 \to B$. We now turn to mirror symmetry for X itself.

The Fukaya category of X is a deformation of that of X^0: the Floer cohomology of Lagrangian submanifolds of X^0, when viewed as objects of $\mathcal{F}(X)$, is deformed by the presence of additional holomorphic discs that intersect the divisor D. Let L be a Lagrangian fiber of the SYZ fibration $\pi : X^0 \to B$: since the Maslov class of L in X^0 vanishes, the Maslov index of a holomorphic disc bounded by L in X is equal to twice its algebraic intersection number with D. Following Fukaya, Oh, Ohta, and Ono [20] we associate to L and a rank 1 local system ∇ over it the *obstruction*

$$\mathfrak{m}_0(L, \nabla) = \sum_{\beta \in \pi_2(X,L)\setminus\{0\}} z_\beta(L, \nabla)\, \mathrm{ev}_*\big[\mathcal{M}_1(L, \beta)\big] \in C^*(L; \mathbf{K}), \tag{2.6}$$

where $z_\beta(L, \nabla) = T^{\omega(\beta)} \nabla(\partial\beta)$ is the weight associated to the class β, and $\mathcal{M}_1(L, \beta)$ is the moduli space of holomorphic discs with one boundary marked point in (X, L) representing the class β. In the absence of bubbling, one can achieve regularity, and $[\mathcal{M}_1(L, \beta)]$ can be defined as the fundamental class of the manifold $\mathcal{M}_1(L, \beta)$. To consider a more general situation, we appeal to the work of Fukaya, Oh, Ohta, and Ono who define such a potential for Lagrangian fibers in toric manifolds in [21]. While the examples we consider are not toric, their construction applies more generally whenever the moduli spaces of stable holomorphic discs with non-positive Maslov index contribute trivially to the Floer differential. The situation is therefore simplest when the divisor D is nef, or more generally when the following condition holds:

Assumption **2.2.** — *Every rational curve* $C \simeq \mathbf{P}^1$ *in* X *has non-negative intersection number* $D \cdot C \geq 0$.

Consider first the case of a Lagrangian submanifold L which is tautologically unobstructed in X^0. By positivity of intersections, the minimal Maslov index of a non-constant holomorphic disc with boundary on L is 2 (when $\beta \cdot D = 1$). Gromov compactness implies that the chain $\mathrm{ev}_*[\mathcal{M}_1(L, \beta)]$ is actually a cycle, of dimension $n - 2 + \mu(\beta) = n$, i.e. a scalar multiple $n(L, \beta)[L]$ of the fundamental class of L; whereas the evaluation chains for $\mu(\beta) > 2$ have dimension greater than n and we discard them. Thus (L, ∇) is *weakly unobstructed*, i.e.

$$\mathfrak{m}_0(L, \nabla) = W(L, \nabla) e_L$$

is a multiple of the unit in $H^0(L, \mathbf{K})$, which is Poincaré dual to the fundamental class of L. More generally, Assumption 2.2 excludes discs of negative Maslov index, while the vanishing of the contribution of discs of Maslov index 0 is explained in Appendix A.2.

Given an open subset $U \subset B \setminus B^{sing}$ over which the fibers of π are tautologically unobstructed in X^0, the coordinate chart $U^\vee \subset Y^0$ considered in the previous section now parametrizes weakly unobstructed objects $(L = \pi^{-1}(b), \nabla)$ of $\mathcal{F}(X)$, and the *superpotential*

$$W(L, \nabla) = \sum_{\substack{\beta \in \pi_2(X,L) \\ \beta \cdot D = 1}} n(L, \beta) z_\beta(L, \nabla) \tag{2.7}$$

is a regular function on $U^\vee$. The superpotential represents a curvature term in Floer theory: the differential on the Floer complex of a pair of weakly unobstructed objects (L, ∇) and (L', ∇') squares to $(W(L', \nabla') - W(L, \nabla))\,\mathrm{id}$. In particular, the family Floer cohomology [18] of an unobstructed Lagrangian submanifold of X with the fibers of the SYZ fibration over U is expected to yield no longer an object of the derived category of coherent sheaves over $U^\vee$ but rather a *matrix factorization* of the superpotential W.

In order to construct the mirror of X globally, we again have to account for instanton corrections across the walls of potentially obstructed fibers of π. As before, these corrections are needed in order to account for the bubbling of holomorphic discs of Maslov index 0 as one crosses a wall, and encode weighted counts of such discs. Under Assumption 2.2, positivity of intersection implies that all the holomorphic discs of Maslov index 0 are contained in X^0; therefore the instanton corrections are exactly the same for X as for X^0, i.e. the moduli space of objects of $\mathcal{F}(X)$ that we construct out of the fibers of π is again Y^0 (the SYZ mirror of X^0).

A key feature of the instanton corrections is that the superpotential defined by (2.7) naturally glues to a regular function on Y^0; this is because, by construction, the gluing via wall-crossing transformations identifies quasi-isomorphic objects of $\mathcal{F}(X)$, for which the obstructions $\mathfrak{m}_0$ have to match, as explained in Corollary A.11. Thus, the mirror of X is the B-side Landau-Ginzburg model (Y^0, W), where Y^0 is the SYZ mirror of X^0 and $W \in \mathcal{O}(Y^0)$ is given by (2.7). (However, see Remark 1.3.)

Remark **2.3.** — The regularity of the superpotential W is a useful feature for the construction of the SYZ mirror of X^0. Namely, rather than directly computing the instanton corrections by studying the enumerative geometry of holomorphic discs in X^0, it is often easier to determine them indirectly, by considering either X or some other partial compactification of X^0 (satisfying Assumption 2.2), computing the mirror superpotential in each chamber of $B \setminus B^{sing}$, and matching the expressions on either side of a wall via a coordinate change of the form (2.4).

When Assumption 2.2 fails, the instanton corrections to the SYZ mirror of X might differ from those for X^0 (hence the difference between the mirrors might be more subtle than simply equipping Y^0 with a superpotential). However, this only happens if the (virtual) counts of Maslov index 0 discs bounded by potentially obstructed fibers of π in X differ from the corresponding counts in X^0. Fukaya-Oh-Ohta-Ono have shown that this issue never arises for toric varieties [21, Corollary 11.5]. In that case, the deformation of the Fukaya category which occurs upon (partially) compactifying X^0 to X (due to the presence of additional holomorphic discs) is accurately reflected by the deformation of the mirror B-model given by the superpotential W (i.e., considering matrix factorizations rather than the usual derived category).

Unfortunately, the argument of [21] does not adapt immediately to our setting; thus for the time being we only consider settings in which Assumption 2.2 holds. This will be the subject of further investigation.

The situation is in fact symmetric: just as partially compactifying X^0 to X is mirror to equipping Y^0 with a superpotential, equipping X^0 or X with a superpotential is mirror to partially compactifying Y^0. One way to justify this claim would be to switch to the other direction of mirror symmetry, reconstructing X^0 as an SYZ mirror of Y^0 equipped with a suitable Kähler structure (cf. Remark 2.1). However, in simple cases this statement can also be understood directly. The following example will be nearly sufficient for our purposes (in Section 7 we will revisit and generalize it):

Example **2.4.** — Let $X^0 = \mathbf{C}^*$, whose mirror $Y^0 \simeq \mathbf{K}^*$ parametrizes objects (L, ∇) of $\mathcal{F}(X^0)$, where L is a simple closed curve enclosing the origin (up to Hamiltonian isotopy) and ∇ is a unitary rank 1 local system on L. The natural coordinate on Y^0, as given by (2.3), tends to zero as the area enclosed by L tends to infinity. Equipping X^0 with the superpotential $W(x) = x$, the Fukaya category $\mathcal{F}(X^0, W)$ also contains "admissible" non-compact Lagrangian submanifolds, i.e. properly embedded Lagrangians whose image under W is only allowed to tend to infinity in the direction of the positive real axis. Denote by L_∞ a properly embedded arc which connects $+\infty$ to itself by passing around the origin (and encloses an infinite amount of area). An easy calculation in $\mathcal{F}(X^0, W)$ shows that $HF^*(L_\infty, L_\infty) \simeq H^*(S^1; \mathbf{K})$; so L_∞ behaves Floer cohomologically like a torus. In particular, L_∞ admits a one-parameter family of deformations in $\mathcal{F}(X^0, W)$; these are represented by equipping L_∞ with a bounding cochain in $HF^1(L_\infty, L_\infty) = \mathbf{K}$ of sufficiently large valuation (with our conventions, the valuation of 0 is $+\infty$). Given a point $cT^\lambda \in \mathbf{K}$, the Floer differential on the Floer complex of (L_∞, cT^λ) with another Lagrangian counts, in addition to the usual strips, triangles with one boundary puncture converging to a time 1 chord of an appropriate Hamiltonian (equal to a positive multiple of $\mathrm{Re}(x)$ near $+\infty$) with ends on L_∞ (this is the implementation of the Fukaya category $\mathcal{F}(X^0, W)$ appearing in [48]); these triangles are counted with Novikov weights equal to their topological energy.

Except for the case $c = 0$, these additional objects of the Fukaya category turn out to be isomorphic to simple closed curves (enclosing the origin) with rank 1 local systems. More precisely, let L_λ be the fiber enclosing an additional amount of area $\lambda \in \mathbf{R}$ compared to a suitable reference Lagrangian L_0, and ∇_c the local system with holonomy c. (Fixing a Liouville 1-form θ, we choose L_0 so that $\int_{L_0} \theta$ is equal to the action $\mathcal{A}$ of the Hamiltonian chord from L_∞ to itself; so $\int_{L_\lambda} \theta = \mathcal{A} + \lambda$.) Then an easy computation shows that the pairs $(\mathrm{L}_\infty, c\mathrm{T}^\lambda)$ and $(\mathrm{L}_\lambda, \nabla_c)$ represent quasi-isomorphic objects of $\mathcal{F}(\mathbf{C}^*, \mathrm{W})$. Thus, in $\mathcal{F}(\mathbf{C}^*, \mathrm{W})$ the previously considered moduli space of objects contains an additional point L_∞; this naturally extends the mirror from $\mathrm{Y}^0 \simeq \mathbf{K}^*$ to $\mathrm{Y} \simeq \mathbf{K}$, and the coordinate coming from identifying bounding cochains on L_∞ with local systems on closed curves defines an analytic structure near this point.

Alternatively, one can geometrically recover the Lagrangians L_λ (together with a trivial noncompact component which is quasi-isomorphic to zero) as self-surgeries of the immersed Lagrangian obtained by deforming L_∞ to a curve with one self-intersection, enclosing the same amount of area as L_λ. This self-intersection corresponds to a generator in $\mathrm{HF}^1(\mathrm{L}_\infty, \mathrm{L}_\infty)$, giving rise to a bounding cochain. The Floer-theoretic isomorphisms between bounding cochains on admissible Lagrangians and embedded Lagrangians then become an instance of the surgery formula of [22].

3. Notations and constructions

3.1. *Hypersurfaces near the tropical limit.* — Let V be a (possibly non-compact) toric variety of complex dimension n, defined by a fan $\Sigma_\mathrm{V} \subseteq \mathbf{R}^n$. We denote by $\sigma_1, \dots, \sigma_r$ the primitive integer generators of the rays of Σ_V. We consider a family of smooth algebraic hypersurfaces $\mathrm{H}_\tau \subset \mathrm{V}$ (where $\tau \to 0$), transverse to the toric divisors in V, and degenerating to the "tropical" limit. Namely, in affine coordinates $\mathbf{x} = (x_1, \dots, x_n)$ over the open stratum $\mathrm{V}^0 \simeq (\mathbf{C}^*)^n \subset \mathrm{V}$, H_τ is defined by an equation of the form

$$f_\tau = \sum_{\alpha \in \mathrm{A}} c_\alpha \tau^{\rho(\alpha)} \mathbf{x}^\alpha = 0, \tag{3.1}$$

where A is a finite subset of the lattice $\mathbf{Z}^n$ of characters of the torus V^0, $c_\alpha \in \mathbf{C}^*$ are arbitrary constants, and $\rho : \mathrm{A} \to \mathbf{R}$ satisfies a certain convexity property.

More precisely, f_τ is a section of a certain line bundle $\mathcal{L}$ over V, determined by a convex piecewise linear function $\lambda : \Sigma_\mathrm{V} \to \mathbf{R}$ with integer linear slopes. (Note that $\mathcal{L}$ need not be ample; however the convexity assumption forces it to be nef.) The polytope P associated to $\mathcal{L}$ is the set of all $v \in \mathbf{R}^n$ such that $\langle v, \cdot \rangle + \lambda$ takes everywhere non-negative values; more concretely, $\mathrm{P} = \{v \in \mathbf{R}^n \mid \langle \sigma_i, v\rangle + \lambda(\sigma_i) \geq 0 \ \forall 1 \leq i \leq r\}$. It is a classical fact that the integer points of P give a basis of the space of sections of $\mathcal{L}$. The condition that H_τ be transverse to each toric stratum of V is then equivalent to the requirement that $\mathrm{A} \subseteq \mathrm{P} \cap \mathbf{Z}^n$ intersects nontrivially the closure of each face of P (i.e., in the compact case, A should contain every vertex of P).

Consider a polyhedral decomposition $\mathcal{P}$ of the convex hull $\mathrm{Conv}(\mathrm{A}) \subseteq \mathbf{R}^n$, whose set of vertices is exactly $\mathcal{P}^{(0)} = \mathrm{A}$. We will mostly consider the case where the decomposition $\mathcal{P}$ is *regular*, i.e. every cell of $\mathcal{P}$ is congruent under the action of $GL(n, \mathbf{Z})$ to a standard simplex. We say that $\rho : \mathrm{A} \to \mathbf{R}$ is *adapted* to the polyhedral decomposition $\mathcal{P}$ if it is the restriction to A of a convex piecewise linear function $\bar{\rho} : \mathrm{Conv}(\mathrm{A}) \to \mathbf{R}$ whose maximal domains of linearity are exactly the cells of $\mathcal{P}$.

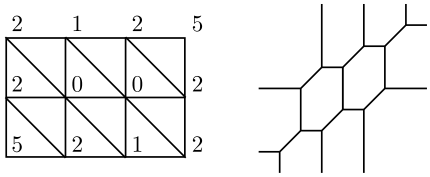

FIG. 1. — A regular decomposition of the polytope for $\mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^1}(3,2)$, and the corresponding tropical hypersurface

Definition **3.1.** — *The family of hypersurfaces* $\mathrm{H}_\tau \subset \mathrm{V}$ *has a* maximal degeneration *for* $\tau \to 0$ *if it is given by equations of the form* (3.1) *where* ρ *is adapted to a regular polyhedral decomposition* $\mathcal{P}$ *of* Conv(A).

The logarithm map $\mathrm{Log}_\tau : \mathbf{x} = (x_1, \ldots, x_n) \mapsto \frac{1}{|\log\tau|}(\log|x_1|, \ldots, \log|x_n|)$ maps H_τ to its *amoeba* $\Pi_\tau = \mathrm{Log}_\tau(\mathrm{H}_\tau \cap \mathrm{V}^0)$; it is known [41, 44] that, for $\tau \to 0$, the amoeba $\Pi_\tau \subset \mathbf{R}^n$ converges to the *tropical hypersurface* $\Pi_0 \subset \mathbf{R}^n$ defined by the tropical polynomial

$$\varphi(\xi) = \max\Big\{\langle\alpha, \xi\rangle - \rho(\alpha) \,\Big|\, \alpha \in \mathrm{A}\Big\} \tag{3.2}$$

(namely, Π_0 is the set of points where the maximum is achieved more than once). Combinatorially, Π_0 is the dual cell complex of $\mathcal{P}$; in particular the connected components of $\mathbf{R}^n \setminus \Pi_0$ can be naturally labelled by the elements of $\mathcal{P}^{(0)} = \mathrm{A}$, according to which term achieves the maximum in (3.2).

Example **3.2.** — The toric variety $\mathrm{V} = \mathbf{P}^1 \times \mathbf{P}^1$ is defined by the fan $\Sigma \subseteq \mathbf{R}^2$ whose rays are generated by $\sigma_1 = (1, 0)$, $\sigma_2 = (0, 1)$, $\sigma_3 = (-1, 0)$, $\sigma_4 = (0, -1)$. The piecewise linear function $\lambda : \Sigma \to \mathbf{R}$ with $\lambda(\sigma_1) = \lambda(\sigma_2) = 0$, $\lambda(\sigma_3) = 3$, and $\lambda(\sigma_4) = 2$ defines the line bundle $\mathcal{L} = \mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^1}(3, 2)$, whose associated polytope is $\mathrm{P} = \{(v_1, v_2) \in \mathbf{R}^2 : \ 0 \le v_1 \le 3,\ \ 0 \le v_2 \le 2\}$. Let $\mathrm{A} = \mathrm{P} \cap \mathbf{Z}^2$. The regular decomposition of P shown in Figure 1 (left) is induced by the function $\rho : \mathrm{A} \to \mathbf{R}$ whose values are given in the figure. The corresponding tropical hypersurface $\Pi_0 \subseteq \mathbf{R}^2$ is shown in Figure 1 (right); Π_0 is the limit of the amoebas of a maximally degenerating family of smooth genus 2 curves $\mathrm{H}_\tau \subset \mathrm{V}$ as $\tau \to 0$.

When the toric variety V is non-compact, P is unbounded, and the convex hull of A is only a proper subset of P. For instance, Figure 1 also represents a maximally degenerating family of smooth genus 2 curves in $\mathrm{V}^0 \simeq (\mathbf{C}^*)^2$ (where now $\mathrm{P} = \mathbf{R}^2$).

We now turn to the symplectic geometry of the situation we just considered. Assume that V is equipped with a complete toric Kähler metric, with Kähler form ω_V. The torus $\mathrm{T}^n = (\mathrm{S}^1)^n$ acts on $(\mathrm{V}, \omega_\mathrm{V})$ by Hamiltonian diffeomorphisms; we denote by $\mu_\mathrm{V} : \mathrm{V} \to \mathbf{R}^n$ the corresponding moment map. It is well-known that the image of μ_V is a convex polytope $\Delta_\mathrm{V} \subset \mathbf{R}^n$, dual to the fan Σ_V. The preimage of the interior of Δ_V is the open stratum $\mathrm{V}^0 \subset \mathrm{V}$; over V^0 the logarithm map Log_τ and the moment map μ_V are related by some diffeomorphism $g_\tau : \mathbf{R}^n \xrightarrow{\sim} \mathrm{int}(\Delta_\mathrm{V})$.

For a fixed Kähler form ω_V, the diffeomorphism g_τ gets rescaled by a factor of $|\log \tau|$ as τ varies; in particular, the moment map images $\mu_V(H_\tau) = \overline{g_\tau(\Pi_\tau)} \subseteq \Delta_V$ of a degenerating family of hypersurfaces collapse towards the boundary of Δ_V as $\tau \to 0$. This can be avoided by considering a varying family of Kähler forms $\omega_{V,\tau}$, obtained from the given ω_V by symplectic inflation along all the toric divisors of V, followed by a rescaling so that $[\omega_{V,\tau}] = [\omega_V]$ is independent of τ. (To be more concrete, one could e.g. consider a family of toric Kähler forms which are multiples of the standard complete Kähler metric of $(\mathbf{C}^*)^n$ over increasingly large open subsets of V^0.)

Throughout this paper, we will consider smooth hypersurfaces that are close enough to the tropical limit, namely hypersurfaces of the form considered above with τ sufficiently close to 0. The key requirement we have for "closeness" to the tropical limit is that the amoeba should lie in a sufficiently small neighborhood of the tropical hypersurface Π_0, so that the complements have the same combinatorics. Since we consider a single hypersurface rather than the whole family, we will omit τ from the notation.

Definition **3.3.** — *A smooth hypersurface* $H = f^{-1}(0)$ *in a toric variety* V *is* nearly tropical *if it is a member of a maximally degenerating family of hypersurfaces as above, with the property that the amoeba* $\Pi = \mathrm{Log}(H) \subset \mathbf{R}^n$ *is entirely contained inside a neighborhood of the tropical hypersurface* Π_0 *which retracts onto* Π_0.

In particular, each element $\alpha \in A$ determines a non-empty open component of $\mathbf{R}^n \setminus \Pi$; we will (abusively) refer to it as the component over which the monomial of f with weight α dominates.

We equip V with a toric Kähler form ω_V of the form discussed above, and denote by μ_V and Δ_V the moment map and its image. Let $\delta > 0$ be a constant such that a standard symplectic tubular neighborhood U_H of H of size δ embeds into V and the complement of the moment map image $\mu_V(U_H)$ has a non-empty component for each element of A (i.e. for each monomial in f).

Remark **3.4.** — The assumption that the degeneration is maximal is made purely for convenience, and to ensure that the toric variety Y constructed in Section 3.3 below is smooth. However, all of our arguments work equally well in the case of non-maximal degenerations.

3.2. *Blowing up.* — Our main goal is to study SYZ mirror symmetry for the blow-up X of $V \times \mathbf{C}$ along $H \times 0$, equipped with a suitable Kähler form.

Recalling that the defining equation f of H is a section of a line bundle $\mathcal{L} \to V$, the normal bundle to $H \times 0$ in $V \times \mathbf{C}$ is the restriction of $\mathcal{L} \oplus \mathcal{O}$, and we can construct explicitly X as a hypersurface in the total space of the $\mathbf{P}^1$-bundle $\mathbf{P}(\mathcal{L} \oplus \mathcal{O}) \to V \times \mathbf{C}$. Namely, the defining section of $H \times 0$ projectivizes to a section $\mathbf{s}(\mathbf{x}, y) = (f(\mathbf{x}) : y)$ of $\mathbf{P}(\mathcal{L} \oplus \mathcal{O})$ over the complement of $H \times 0$; and X is the closure of the graph of $\mathbf{s}$. In other terms,

$$\text{(3.3)} \qquad X = \big\{ \big(\mathbf{x}, y, (u:v)\big) \in \mathbf{P}(\mathcal{L} \oplus \mathcal{O}) \,\big|\, f(\mathbf{x})v = yu \big\}.$$

In this description it is clear that the projection $p : X \to V \times \mathbf{C}$ is a biholomorphism outside of the exceptional divisor $E = p^{-1}(H \times 0)$.

The S^1-action on $V \times \mathbf{C}$ by rotation of the $\mathbf{C}$ factor preserves $H \times 0$ and hence lifts to an S^1-action on X. This action preserves the exceptional divisor E, and acts by rotation in the standard manner on each fiber of the $\mathbf{P}^1$-bundle $p_{|E} : E \to H \times 0$. In coordinates, we can write this action in the form:

$$\text{(3.4)} \qquad e^{i\theta} \cdot \big(\mathbf{x}, y, (u:v)\big) = \big(\mathbf{x}, e^{i\theta} y, \big(u : e^{i\theta} v\big)\big).$$

Thus, the fixed point set of the S^1-action on X consists of two disjoint strata: the proper transform $\tilde{V}$ of $V \times 0$ (corresponding to $y = 0,\ v = 0$ in the above description), and the section $\tilde{H}$ of p over $H \times 0$ given by the line subbundle $\mathcal{O}$ of the normal bundle (i.e., the point $(0:1)$ in each fiber of $p_{|E}$).

The open stratum $V^0 \times \mathbf{C}^*$ of the toric variety $V \times \mathbf{C}$ carries a holomorphic $(n+1)$-form $\Omega_{V\times\mathbf{C}} = i^{n+1} \prod_j d\log x_j \wedge d\log y$, which has simple poles along the toric divisor $D_{V\times\mathbf{C}} = (V \times 0) \cup (D_V \times \mathbf{C})$ (where $D_V = V \setminus V^0$ is the union of the toric divisors in V). The pullback $\Omega = p^*(\Omega_{V\times\mathbf{C}})$ has simple poles along the proper transform of $D_{V\times\mathbf{C}}$, namely the anticanonical divisor $D = \tilde{V} \cup p^{-1}(D_V \times \mathbf{C})$. The complement $X^0 = X \setminus D$, equipped with the S^1-invariant holomorphic $(n+1)$-form Ω, is an open Calabi-Yau manifold.

Remark **3.5.** — $X \setminus \tilde{V}$ corresponds to $v \neq 0$ in (3.3), so it is isomorphic to an affine conic bundle over V, namely the hypersurface in the total space of $\mathcal{O} \oplus \mathcal{L}$ given by

$$\text{(3.5)} \qquad \big\{ (\mathbf{x}, y, z) \in \mathcal{O} \oplus \mathcal{L} \,\big|\, f(\mathbf{x}) = yz \big\}.$$

Further removing the fibers over D_V, we conclude that X^0 is a conic bundle over the open stratum $V^0 \simeq (\mathbf{C}^*)^n$, given again by the equation $\{f(\mathbf{x}) = yz\}$.

We equip X with an S^1-invariant Kähler form ω_ϵ for which the fibers of the exceptional divisor have a sufficiently small area $\epsilon > 0$. Specifically, we require that $\epsilon \in (0, \delta/2)$, where δ is the size of the standard tubular neighborhood of H that embeds in (V, ω_V). The most natural way to construct such a Kähler form would be to equip $\mathcal{L}$ with a Hermitian metric, which determines a Kähler form on $\mathbf{P}(\mathcal{L} \oplus \mathcal{O})$ and, by restriction, on X; on the complement of E the resulting Kähler form is given by

$$\text{(3.6)} \qquad p^*\omega_{V\times\mathbf{C}} + \frac{i\epsilon}{2\pi} \partial\bar{\partial} \log\big(|f(\mathbf{x})|^2 + |y|^2\big),$$

where $\omega_{V\times\mathbf{C}}$ is the product Kähler form on $V \times \mathbf{C}$ induced by the toric Kähler form ω_V on V and the standard area form of $\mathbf{C}$.

However, from a symplectic perspective the blowup operation amounts to deleting from $V \times \mathbf{C}$ a standard symplectic tubular neighborhood of $H \times 0$ and collapsing its boundary (an S^3-bundle over H) onto E by the Hopf map. Thus, X and $V \times \mathbf{C}$ are symplectomorphic away from neighborhoods of E and $H \times 0$; to take full advantage of this, we will choose ω_ϵ in such a way that the projection $p : X \to V \times \mathbf{C}$ is a symplectomorphism away from a neighborhood of the exceptional divisor. Namely, we set

$$\omega_\epsilon = p^*\omega_{V\times\mathbf{C}} + \frac{i\epsilon}{2\pi}\partial\bar\partial\big(\chi(\mathbf{x},y)\log\big(|f(\mathbf{x})|^2 + |y|^2\big)\big), \tag{3.7}$$

where χ is a suitably chosen S^1-invariant smooth cut-off function supported in a tubular neighborhood of $H \times 0$, with $\chi = 1$ near $H \times 0$. It is clear that (3.7) defines a Kähler form provided ϵ is small enough; specifically, ϵ needs to be such that a standard symplectic neighborhood of size ϵ of $H \times 0$ can be embedded (S^1-equivariantly) into the support of χ. For simplicity, we assume that χ is chosen so that the following property holds:

Property **3.6.** — *The support of χ is contained inside $p^{-1}(U_H \times B_\delta)$, where $U_H \subset V$ is a standard symplectic δ-neighborhood of* H *and $B_\delta \subset \mathbf{C}$ is the disc of radius δ.*

Remark **3.7.** — ω_ϵ lies in the same cohomology class $[\omega_\epsilon] = p^*[\omega_{V\times\mathbf{C}}] - \epsilon[E]$ as the Kähler form defined by (3.6), and is equivariantly symplectomorphic to it.

3.3. *The mirror* B*-side Landau-Ginzburg model.* — Using the same notations as in the previous section, we now describe a B-side Landau-Ginzburg model which we claim is SYZ mirror to X (with the Kähler form ω_ϵ, and relatively to the anticanonical divisor D).

Recall that the hypersurface $H \subset X$ has a defining equation of the form (3.1), involving toric monomials whose weights range over a finite subset $A \subset \mathbf{Z}^n$, forming the vertices of a polyhedral complex $\mathcal{P}$ (cf. Definition 3.1).

We denote by Y the (noncompact) $(n+1)$-dimensional toric variety defined by the fan $\Sigma_Y = \mathbf{R}_{\geq 0} \cdot (\mathcal{P} \times \{1\}) \subseteq \mathbf{R}^{n+1} = \mathbf{R}^n \oplus \mathbf{R}$. Namely, the integer generators of the rays of Σ_Y are the vectors of the form $(-\alpha, 1)$, $\alpha \in A$, and the vectors $(-\alpha_1, 1), \dots, (-\alpha_k, 1)$ span a cone of Σ_Y if and only if $\alpha_1, \dots, \alpha_k$ span a cell of $\mathcal{P}$.

Dually, Y can be described by a (noncompact) polytope $\Delta_Y \subseteq \mathbf{R}^{n+1}$, defined in terms of the tropical polynomial $\varphi : \mathbf{R}^n \to \mathbf{R}$ associated to H (cf. (3.2)) by

$$\Delta_Y = \big\{(\xi, \eta) \in \mathbf{R}^n \oplus \mathbf{R} \,\big|\, \eta \geq \varphi(\xi)\big\}. \tag{3.8}$$

Remark **3.8.** — The polytope Δ_Y also determines a Kähler class $[\omega_Y]$ on Y. While in this paper we focus on the A-model of X and the B-model of Y, it can be shown that the family of complex structures on X obtained by blowing up $V \times \mathbf{C}$ along the maximally degenerating family $H_\tau \times 0$ (cf. Section 3.1) corresponds to a family of Kähler forms asymptotic to $|\log \tau|[\omega_Y]$ as $\tau \to 0$.

Remark **3.9.** — Even though deforming the hypersurface H inside V does not modify the symplectic geometry of X, the topology of Y depends on the chosen polyhedral decomposition $\mathcal{P}$ (i.e., on the combinatorial type of the tropical hypersurface defined by φ). However, the various possibilities for Y are related to each other by crepant birational transformations, and hence are expected to yield equivalent B-models. (The A-model of Y, on the other hand, is affected by these birational transformations and does depend on the tropical polynomial φ, as explained in the previous remark.)

The facets of Δ_Y correspond to the maximal domains of linearity of φ. Thus the irreducible toric divisors of Y are in one-to-one correspondence with the connected components of $\mathbf{R}^n \setminus \Pi_0$, and the combinatorics of the toric strata of Y can be immediately read off the tropical hypersurface Π_0 (see Example 3.12 below).

It is advantageous for our purposes to introduce a collection of affine charts on Y indexed by the elements of A (i.e., the facets of Δ_Y, or equivalently, the connected components of $\mathbf{R}^n \setminus \Pi_0$).

For each $\alpha \in A$, let $Y_\alpha = (\mathbf{K}^*)^n \times \mathbf{K}$, with coordinates $\mathbf{v}_\alpha = (v_{\alpha,1}, \dots, v_{\alpha,n}) \in (\mathbf{K}^*)^n$ and $v_{\alpha,0} \in \mathbf{K}$ (as before, $\mathbf{K}$ is either Λ or $\mathbf{C}$). Whenever $\alpha, \beta \in A$ are connected by an edge in the polyhedral decomposition $\mathcal{P}$ (i.e., whenever the corresponding components of $\mathbf{R}^n \setminus \Pi_0$ share a top-dimensional facet, with primitive normal vector $\beta - \alpha$), we glue Y_α to Y_β by the coordinate transformations

$$(\mathbf{3.9}) \qquad \begin{cases} v_{\alpha,i} = v_{\beta,0}^{\beta_i - \alpha_i} v_{\beta,i} \quad (1 \le i \le n), \\ v_{\alpha,0} = v_{\beta,0}. \end{cases}$$

These charts cover the complement in Y of the codimension 2 strata (as Y_α covers the open stratum of Y and the open stratum of the toric divisor corresponding to α). In terms of the standard basis of toric monomials indexed by weights in $\mathbf{Z}^{n+1}$, $v_{\alpha,0}$ is the monomial with weight $(0, \dots, 0, 1)$, and for $i \ge 1$ $v_{\alpha,i}$ is the monomial with weight $(0, \dots, -1, \dots, 0, -\alpha_i)$ (the i-th entry is -1).

Denoting by T the Novikov parameter (treated as an actual complex parameter when $\mathbf{K} = \mathbf{C}$), and by v_0 the common coordinate $v_{\alpha,0}$ for all charts, we set

$$(\mathbf{3.10}) \qquad w_0 = -T^\epsilon + T^\epsilon v_0.$$

With this notation, the above coordinate transformations can be rewritten as

$$v_{\alpha,i} = \left(1 + T^{-\epsilon} w_0\right)^{\beta_i - \alpha_i} v_{\beta,i}, \qquad 1 \le i \le n.$$

More generally, for $m = (m_1, \dots, m_n) \in \mathbf{Z}^n$ we set $\mathbf{v}_\alpha^m = v_{\alpha,1}^{m_1} \cdots v_{\alpha,n}^{m_n}$. Then

$$(\mathbf{3.11}) \qquad \mathbf{v}_\alpha^m = \left(1 + T^{-\epsilon} w_0\right)^{\langle \beta - \alpha, m \rangle} \mathbf{v}_\beta^m.$$

We shall see that w_0 and the transformations (3.11) have a natural interpretation in terms of the enumerative geometry of holomorphic discs in X.

Next, recall from Section 3.1 that the inward normal vectors to the facets of the moment polytope Δ_V associated to (V, ω_V) are the primitive integer generators $\sigma_1, \dots, \sigma_r$ of the rays of Σ_V. Thus, there exist constants $\varpi_1, \dots, \varpi_r \in \mathbf{R}$ such that

$$\Delta_V = \left\{ u \in \mathbf{R}^n \,\middle|\, \langle \sigma_i, u \rangle + \varpi_i \geq 0 \ \forall 1 \leq i \leq r \right\}. \tag{3.12}$$

Then for $i = 1, \dots, r$ we set

$$w_i = T^{\varpi_i} \mathbf{v}_{\alpha_i}^{\sigma_i} \tag{3.13}$$

where $\alpha_i \in A$ is chosen to lie on the facet of P defined by σ_i, i.e. so that $\langle \sigma_i, \alpha_i \rangle$ is minimal. Hence, by the conditions imposed in Section 3.1, $\langle \sigma_i, \alpha_i \rangle + \lambda(\sigma_i) = 0$, where $\lambda : \Sigma_V \to \mathbf{R}$ is the piecewise linear function defining $\mathcal{L} = \mathcal{O}(H)$. By (3.11), the choice of α_i satisfying the required condition is irrelevant: in all cases $\mathbf{v}_{\alpha_i}^{\sigma_i}$ is simply the toric monomial with weight $(-\sigma_i, \lambda(\sigma_i)) \in \mathbf{Z}^n \oplus \mathbf{Z}$. Moreover, this weight pairs non-negatively with all the rays of the fan Σ_Y, therefore w_i defines a regular function on Y.

With all the notation in place, we can at last make the following definition, which clarifies the statements of Theorems 1.5 and 1.6:

Definition **3.10.** — *We denote by* Y^0 *the complement of the hypersurface* $D_Y = w_0^{-1}(0)$ *in the toric* $(n+1)$*-fold* Y, *and define the* leading-order superpotential

$$W_0 = w_0 + w_1 + \cdots + w_r = -T^{\epsilon} + T^{\epsilon} v_0 + \sum_{i=1}^{r} T^{\varpi_i} \mathbf{v}_{\alpha_i}^{\sigma_i} \in \mathcal{O}(Y). \tag{3.14}$$

We also define

$$W_0^H = -v_0 + w_1 + \cdots + w_r = -v_0 + \sum_{i=1}^{r} T^{\varpi_i} \mathbf{v}_{\alpha_i}^{\sigma_i} \in \mathcal{O}(Y). \tag{3.15}$$

Remark **3.11.** — Since there are no convergence issues, we can think of (Y^0, W_0) and (Y, W_0^H) either as B-side Landau-Ginzburg models defined over the Novikov field or as one-parameter families of complex B-side Landau-Ginzburg models defined over $\mathbf{C}$.

Example **3.12.** — When H is the genus 2 curve of Example 3.2, the polytope Δ_Y has 12 facets (2 of them compact and the 10 others non-compact), corresponding to the 12 components of $\mathbf{R}^n \setminus \Pi_0$, and intersecting exactly as pictured on Figure 1 (right). The edges of the figure correspond to the configuration of $\mathbf{P}^1$'s and $\mathbf{A}^1$'s along which the toric divisors of the 3-fold Y intersect.

Label the irreducible toric divisors by $D_{a,b}$ $(0 \leq a \leq 3, 0 \leq b \leq 2)$, corresponding to the elements $(a, b) \in A$. Then the leading-order superpotential W_0 consists of five terms: $w_0 = -T^{\epsilon} + T^{\epsilon} v_0$, where v_0 is the toric monomial of weight $(0, 0, 1)$, which vanishes with multiplicity 1 on each of the 12 toric divisors; and up to constant factors, w_1 is the toric monomial with weight $(-1,0,0)$, which vanishes with multiplicity a on $D_{a,b}$; w_2 is the toric monomial with weight $(0,-1,0)$, vanishing with multiplicity b on $D_{a,b}$; w_3 is the monomial with weight $(1,0,3)$, with multiplicity $(3-a)$ on $D_{a,b}$; and w_4 is the monomial with weight $(0,1,2)$, with multiplicity $(2-b)$ on $D_{a,b}$. In particular, the compact divisors $D_{1,1}$ and $D_{2,1}$ are components of the singular fiber $\{W_0=-T^{\epsilon}\}\subset Y^0$ (which also has a third, non-compact component); and similarly for $\{W_0^H=0\}\subset Y$.

(In general the order of vanishing of w_i on a given divisor is equal to the intersection number with Π_0 of a semi-infinite ray in the direction of $-\sigma_i$ starting from a generic point in the relevant component of $\mathbf{R}^n\setminus\Pi_0$.)

This example does not satisfy Assumption 1.4, and in this case the actual mirror of X differs from (Y^0,W_0) by higher-order correction terms. On the other hand, if we consider the genus 2 curve with 10 punctures $H\cap V^0$ in the open toric variety $V^0\simeq(\mathbf{C}^*)^2$, which does fall within the scope of Theorem 1.5, the construction yields the same toric 3-fold Y, but now we simply have $W_0=w_0$ (resp. $W_0^H=-v_0$).

4. Lagrangian torus fibrations on blowups of toric varieties

As in Section 3.2, we consider a smooth nearly tropical hypersurface $H=f^{-1}(0)$ in a toric variety V of dimension n, and the blow-up X of $V\times\mathbf{C}$ along $H\times 0$, equipped with the S^1-invariant Kähler form ω_ϵ given by (3.7). Our goal in this section is to construct an S^1-invariant Lagrangian torus fibration $\pi: X^0\to B$ (with appropriate singularities) on the open Calabi-Yau manifold $X^0=X\setminus D$, where D is the proper transform of the toric anticanonical divisor of $V\times\mathbf{C}$. (Similar fibrations have been previously considered by Gross [23, 24] and by Castaño-Bernard and Matessi [10, 11].) The key observation is that S^1-invariance forces the fibers of π to be contained in the level sets of the moment map of the S^1-action. Thus, we begin by studying the geometry of the reduced spaces.

4.1. *The reduced spaces.* — The S^1-action (3.4) on X is Hamiltonian with respect to the Kähler form ω_ϵ given by (3.7), and its moment map $\mu_X: X\to\mathbf{R}$ can be determined explicitly. Outside of the exceptional divisor, we identify X with $V\times\mathbf{C}$ via the projection p, and observe that $\mu_X(\mathbf{x},y)=\int_{D(\mathbf{x},y)}\omega_\epsilon$, where $D(\mathbf{x},y)$ is a disc bounded by the orbit of $(\mathbf{x},y)$, namely the total transform of $\{\mathbf{x}\}\times D^2(|y|)\subset V\times\mathbf{C}$. (We normalize μ_X so that it takes the constant value 0 over the proper transform of $V\times 0$; also, our convention differs from the usual one by a factor of 2π.)

Hence, for given $\mathbf{x}$ the quantity $\mu_X(\mathbf{x},y)$ is a strictly increasing function of $|y|$. Moreover, applying Stokes' theorem we find that

$$\mu_X(\mathbf{x},y)=\pi|y|^2+\frac{\epsilon}{2}|y|\frac{\partial}{\partial|y|}\big(\chi(\mathbf{x},y)\log(|f(\mathbf{x})|^2+|y|^2)\big). \tag{4.1}$$

In the regions where χ is constant this simplifies to:

$$\mu_X(\mathbf{x}, y) = \begin{cases} \pi|y|^2 + \epsilon\frac{|y|^2}{|f(\mathbf{x})|^2+|y|^2} & \text{where } \chi \equiv 1 \text{ (near E)}, \\ \pi|y|^2 & \text{where } \chi \equiv 0 \text{ (away from E)}. \end{cases} \tag{4.2}$$

(Note that the first expression extends naturally to a smooth function over E.)

The critical points of μ_X are the fixed points of the S^1-action. Besides $\tilde{V} = \mu_X^{-1}(0)$, the fixed points occur along $\tilde{H}$, which lies in the level set $\mu_X^{-1}(\epsilon)$; in particular, all the other level sets of μ_X are smooth. Since for any given $\mathbf{x}$ the moment map μ_X is a strictly increasing function of $|y|$, each level set of μ_X intersects $p^{-1}(\{\mathbf{x}\} \times \mathbf{C})$ along a single S^1-orbit. Hence, for $\lambda > 0$, the natural projection to V (obtained by composing p with projection to the first factor) yields a natural identification of the reduced space $X_{red,\lambda} = \mu_X^{-1}(\lambda)/S^1$ with V.

For $\lambda \gg \epsilon$, $\mu_X^{-1}(\lambda)$ is disjoint from the support of the cut-off function χ, and the reduced Kähler form $\omega_{red,\lambda}$ on $X_{red,\lambda} \cong V$ coincides with the toric Kähler form ω_V. As λ becomes closer to ϵ, $\omega_{red,\lambda}$ differs from ω_V near H but remains cohomologous to it. At the critical level $\lambda = \epsilon$, the reduced form $\omega_{red,\epsilon}$ is singular along H (but its singularities are fairly mild, see Lemma B.1). Finally, for $\lambda < \epsilon$ the Kähler form $\omega_{red,\lambda}$ differs from ω_V in a tubular neighborhood of H, inside which the normal direction to H has been symplectically *deflated*. In particular, one easily checks that

$$[\omega_{red,\lambda}] = [\omega_V] - \max(0, \epsilon - \lambda)[H]. \tag{4.3}$$

Our goal is to exploit the toric structure of V to construct families of Lagrangian tori in $X_{red,\lambda}$. The Kähler form $\omega_{red,\lambda}$ on $X_{red,\lambda} \cong V$ is not T^n-invariant near H; in fact it isn't even smooth along H for $\lambda = \epsilon$. However, there exist (smooth) toric Kähler forms $\omega'_{V,\lambda}$, depending piecewise smoothly on λ, with $[\omega'_{V,\lambda}] = [\omega_{red,\lambda}]$; see (B.5) for an explicit construction. The following result will be proved in Appendix B.

Lemma **4.1.** — *There exists a family of homeomorphisms* $(\phi_\lambda)_{\lambda\in\mathbf{R}_+}$ *of* V *such that:*

(1) ϕ_λ *preserves the toric divisor* $D_V \subset V$*;*
(2) *the restriction of* ϕ_λ *to* V^0 *is a diffeomorphism for* $\lambda \neq \epsilon$*, and a diffeomorphism outside of* H *for* $\lambda = \epsilon$*;*
(3) ϕ_λ *intertwines the reduced Kähler form* $\omega_{red,\lambda}$ *and the toric Kähler form* $\omega'_{V,\lambda}$*;*
(4) $\phi_\lambda = \mathrm{id}$ *at every point whose* T^n*-orbit is disjoint from the support of* χ*;*
(5) ϕ_λ *depends on* λ *in a continuous manner, and smoothly except at* $\lambda = \epsilon$.

The diffeomorphism (singular along H for $\lambda = \epsilon$) ϕ_λ given by Lemma 4.1 yields a preferred Lagrangian torus fibration on the open stratum $X^0_{red,\lambda} = (\mu_X^{-1}(\lambda) \cap X^0)/S^1$ of $X_{red,\lambda}$ (naturally identified with V^0 under the canonical identification $X_{red,\lambda} \cong V$), namely the preimage by ϕ_λ of the standard fibration of $(V^0, \omega'_{V,\lambda})$ by T^n-orbits:

Definition **4.2.** — *We denote by* $\pi_\lambda : X^0_{red,\lambda} \to \mathbf{R}^n$ *the composition* $\pi_\lambda = \mathrm{Log} \circ \phi_\lambda$, *where* $\mathrm{Log} : V^0 \cong (\mathbf{C}^*)^n \to \mathbf{R}^n$ *is the logarithm map* $(x_1, \dots, x_n) \mapsto \frac{1}{|\log \tau|}(\log|x_1|, \dots, \log|x_n|)$, *and* $\phi_\lambda : (X_{red,\lambda}, \omega_{red,\lambda}) \to (V, \omega'_{V,\lambda})$ *is as in Lemma 4.1.*

Remark **4.3.** — By construction, the natural affine structure (see Section 2.1) on the base of the Lagrangian torus fibration π_λ identifies it with the interior of the moment polytope $\Delta_{V,\lambda}$ associated to the cohomology class $[\omega'_{V,\lambda}] = [\omega_{red,\lambda}] \in H^2(V, \mathbf{R})$.

4.2. *A Lagrangian torus fibration on* X^0. — We now assemble the Lagrangian torus fibrations π_λ on the reduced spaces into a (singular) Lagrangian torus fibration on X^0:

Definition **4.4.** — *We denote by* $\pi : X^0 \to B = \mathbf{R}^n \times \mathbf{R}_+$ *the map which sends the point* $x \in \mu_X^{-1}(\lambda) \cap X^0$ *to* $\pi(x) = (\pi_\lambda(\bar{x}), \lambda)$, *where* $\bar{x} \in X^0_{red,\lambda}$ *is the* S^1*-orbit of* x.

The map π is continuous, and smooth away from $\lambda = \epsilon$. The fiber of π above $(\xi, \lambda) \in B$ is obtained by lifting the Lagrangian torus $\pi_\lambda^{-1}(\xi) \subset X_{red,\lambda}$ to $\mu_X^{-1}(\lambda)$ and "spinning" it by the S^1-action.

Away from the fixed points of the S^1-action, $\mu_X^{-1}(\lambda)$ is a coisotropic manifold with isotropic foliation given by the S^1-orbits. Moreover, the S^1-bundle $\mu_X^{-1}(\lambda) \to X_{red,\lambda}$ is topologically trivial for $\lambda > \epsilon$ (setting $y \in \mathbf{R}_+$ gives a global section), trivial over the complement of H for $\lambda = \epsilon$, and the circle bundle associated to the line bundle $\mathcal{O}(-H)$ for $\lambda < \epsilon$; in any case, its restriction to a fiber of π_λ is topologically trivial. The fibers of π_λ are smooth Lagrangian tori (outside of H when $\lambda = \epsilon$, which corresponds precisely to the S^1-fixed points); therefore, we conclude that the fibers of π are smooth Lagrangian tori unless they contain fixed points of the S^1-action.

The only fixed points occur for $\lambda = \epsilon$, when $\mu_X^{-1}(\lambda)$ contains the stratum of fixed points $\tilde{H}$. The identification of the reduced space with V maps $\tilde{H}$ to the hypersurface H, so the singular fibers map to

$$B^{sing} = \Pi' \times \{\epsilon\} \subset B, \tag{4.4}$$

where $\Pi' = \pi_\epsilon(H \cap V^0) \subset \mathbf{R}^n$ is essentially the amoeba of the hypersurface H (up to the fact that π_ϵ differs from the logarithm map by ϕ_ϵ). The fibers above the points of B^{sing} differ from the regular fibers in that, where a smooth fiber $\pi^{-1}(\xi, \lambda) \simeq T^{n+1}$ is a trivial S^1-bundle over $\pi_\lambda^{-1}(\xi) \simeq T^n \subset V^0$, for $\lambda = \epsilon$ some of the S^1 fibers (namely those which lie over points of H) are collapsed to points.

Because the fibration π has non-trivial monodromy around B^{sing}, the only globally defined affine coordinate on B is the last coordinate λ (the moment map of the S^1-action); other affine coordinates are only defined over subsets of $B \setminus B^{sing}$, i.e. in the complement of certain cuts. Our preferred choice for such a description relates the affine structure on B to the moment polytope $\Delta_V \times \mathbf{R}_+$ of $V \times \mathbf{C}$. Namely, away from a tubular neighborhood of $\Pi' \times (0, \epsilon)$ the Lagrangian torus fibration π coincides with the standard toric fibration on $V \times \mathbf{C}$:

Proposition **4.5.** — *Outside of the support of* χ *(a tubular neighborhood of the exceptional divisor* E*), the Kähler form* ω_ϵ *is equal to* $p^*\omega_{V\times\mathbf{C}}$*, and the moment map of the* S^1*-action is the standard one* $\mu_X(\mathbf{x}, y) = \pi|y|^2$*. Moreover, outside of* $\pi(\operatorname{supp}\chi)$*, the fibers of the Lagrangian fibration* π *are standard product tori, i.e. they are the preimages by* p *of the orbits of the* T^{n+1}*-action in* $V \times \mathbf{C}$.

Proof. — The first statement follows immediately from formulas (3.7) and (4.1). The second one is then a direct consequence of the manner in which π was constructed and condition (3) in Lemma 4.1. □

Recall that the support of χ is constrained by Property 3.6. Thus, the fibration π is standard (coincides with the standard toric fibration on $V \times \mathbf{C}$) over a large subset $B^{std} = (\mathbf{R}^n \times \mathbf{R}_+) \setminus (\mathrm{Log}(U_H) \times (0, \delta))$ of B. Since $\omega_\epsilon = p^*\omega_{V\times\mathbf{C}}$ over $\pi^{-1}(B^{std})$, we conclude that over B^{std} the affine structure of B agrees with that for the standard toric fibration of $V \times \mathbf{C}$, i.e. as an affine manifold B^{std} can be naturally identified with the complement of $\mu_V(U_H) \times (0, \delta)$ inside $\mathrm{int}(\Delta_V) \times \mathbf{R}_+$.

This description of the affine structure on $B \setminus B^{sing}$ can be extended from B^{std} to the complement of a set of codimension 1 cuts. Recall from Section 2.1 that the affine coordinates of $b \in B \setminus B^{sing}$ relative to some reference point b_0 are given by the symplectic areas of certain relative 2-cycles $(\Gamma_1, \dots, \Gamma_{n+1})$ with boundary on $\pi^{-1}(b) \cup \pi^{-1}(b_0)$; the above identification of B^{std} with a subset of $\Delta_V \times \mathbf{R}_+$ arises from taking the boundaries of Γ_i to be (homologous to) orbits of the various S^1 factors of the T^{n+1}-action on $V \times \mathbf{C}$.

When b and b_0 have the same last coordinate $\lambda > \epsilon$, we can choose $\Gamma_1, \dots, \Gamma_n$ to be contained in $\mu_X^{-1}(\lambda)$, and obtained as the lifts of relative 2-cycles $\Gamma_{i,red}$ in $X_{red,\lambda}$ with boundary on fibers of π_λ; we can fix such lifts by requiring that $y \in \mathbf{R}_+$ on Γ_i. Since $\int_{\Gamma_i} \omega_\epsilon = \int_{\Gamma_{i,red}} \omega_{red,\lambda}$, the affine structure on the level set $\mathbf{R}^n \times \{\lambda\} \subset B$ is the same as that on the base of the fibration π_λ on the reduced space $X_{red,\lambda}$, which can be identified via the diffeomorphism ϕ_λ with the standard toric fibration on $(V, \omega'_{V,\lambda})$. For $\lambda > \epsilon$ we have $[\omega_{red,\lambda}] = [\omega'_{V,\lambda}] = [\omega_V]$, so the base is naturally identified with the interior of the moment polytope Δ_V; moreover, this identification is consistent with our previous description of the affine structure over B^{std}, since in that region the various Kähler forms agree pointwise.

In other terms, over $\mathbf{R}^n \times (\epsilon, \infty) \subset B$, the affine structure is globally a product $\mathrm{int}(\Delta_V) \times (\epsilon, \infty)$ of the affine structure on the moment polytope of (V, ω_V) and the interval (ϵ, ∞), in a manner that extends the previous description over B^{std}.

For $\lambda < \epsilon$, the affine structure on $\mathbf{R}^n \times \{\lambda\} \subset B$ can be described similarly, by choosing relative 2-cycles $\Gamma_{i,red}$ in $X_{red,\lambda}$ with boundary on fibers of π_λ and lifting them to relative 2-cycles Γ'_i in $\mu_X^{-1}(\lambda)$. Since the lifts may intersect the exceptional divisor E, we cannot require $y \in \mathbf{R}_+$ as in the case $\lambda > \epsilon$. Instead, we use the monomial $\mathbf{x}^{\alpha_0}$ for some $\alpha_0 \in A$ to fix a trivialization of $\mathcal{L} = \mathcal{O}(H)$ over V^0, and choose the lifts so that $\mathbf{x}^{-\alpha_0} z = \mathbf{x}^{-\alpha_0} f(\mathbf{x})/y \in \mathbf{R}_+$ on Γ'_i. Since $\int_{\Gamma'_i} \omega_\epsilon = \int_{\Gamma_{i,red}} \omega_{red,\lambda}$, the affine structure on the level set $\mathbf{R}^n \times \{\lambda\} \subset B$ is again identical to that on the base of the fibration π_λ on $X_{red,\lambda}$, or

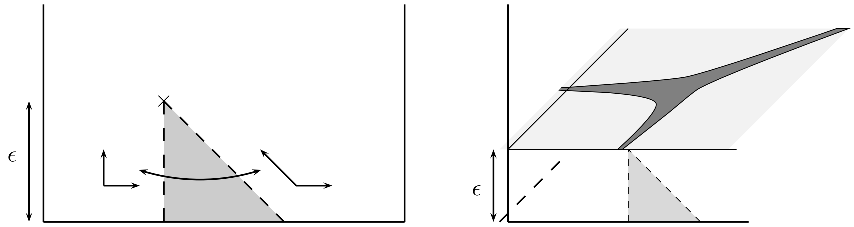

FIG. 2. — The base of the Lagrangian torus fibration $\pi : X^0 \to B$. *Left*: $H = \{\text{point}\} \subset \mathbf{CP}^1$. *Right*: $H = \{x_1 + x_2 = 1\} \subset \mathbf{C}^2$

equivalently via ϕ_λ, the standard toric fibration on $(V, \omega'_{V,\lambda})$. Thus, the affine structure identifies $\mathbf{R}^n \times \{\lambda\} \subset B$ with the interior of the moment polytope $\Delta_{V,\lambda}$ associated to the Kähler class $[\omega'_{V,\lambda}] = [\omega_{red,\lambda}] = [\omega_V] - \max(0, \epsilon - \lambda)[H]$. However, this description is no longer consistent with that previously given for B^{std}, because the boundary of Γ'_i does not represent the expected homology class in $\pi^{-1}(b)$.

Specifically, assume b_0 and $b \in (\mathbf{R}^n \setminus \operatorname{Log}(U_H)) \times \{\lambda\}$ lie in the connected components corresponding to α_0 and $\alpha \in A$ respectively. Then the boundary of Γ'_i in $\pi^{-1}(b_0)$ does represent the homology class of the orbit of the i-th S^1-factor, while the boundary in $\pi^{-1}(b)$ differs from it by $\alpha_i - \alpha_{0,i}$ times the orbit of the last S^1-factor. Moreover,

$$\int_{\Gamma_{i,red}} \omega_V - \int_{\Gamma_{i,red}} \omega_{red,\lambda} = (\epsilon - \lambda)(\Gamma_{i,red} \cdot H) = (\epsilon - \lambda)(\alpha_i - \alpha_{0,i}).$$

This formula also gives the difference between the ω_ϵ-areas of the relative cycles Γ'_i and the relative cycles $\Gamma_i \subset \pi^{-1}(B^{std})$ previously used to determine affine coordinates over B^{std}. Hence, the affine coordinates determined by the relative cycles Γ'_i differ from those constructed previously over B^{std} by a shear

$$(\zeta_1, \dots, \zeta_n, \lambda) \mapsto \big(\zeta_1 + (\epsilon - \lambda)(\alpha_1 - \alpha_{0,1}), \dots, \zeta_n + (\epsilon - \lambda)(\alpha_n - \alpha_{0,n}), \lambda\big) \tag{4.5}$$

or more succinctly, $(\zeta, \lambda) \mapsto (\zeta + (\epsilon - \lambda)(\alpha - \alpha_0), \lambda)$.

More globally, over $\mathbf{R}^n \times (0, \epsilon) \subset B$ the affine structure can be identified (using the relative cycles Γ'_i to define coordinates) with a piece of the moment polytope for the total space of the line bundle $\mathcal{O}(-H)$ over V (equipped with a toric Kähler form in the class $[\omega_V] - \epsilon[H]$), consistent with the fact that the normal bundle to $\tilde{V}$ inside X is $\mathcal{O}(-H)$; but this description is not consistent with the one we have given over B^{std}.

On the other hand, the shears (4.5) map the complement of the amoeba of H in $\Delta_{V,\lambda}$ to the complement of a standard $(\epsilon - \lambda)$-neighborhood of the amoeba of H in Δ_V. Thus, making cuts along the projection of the exceptional divisor, we can extend the affine coordinates previously described over B^{std}, and identify the affine structure on $B \setminus (\Pi' \times (0, \epsilon])$ with an open subset of $\operatorname{int}(\Delta_V) \times \mathbf{R}_+$, obtained by deleting an $(\epsilon - \lambda)$-neighborhood of the amoeba of H from $\operatorname{int}(\Delta_V) \times \{\lambda\}$ for all $\lambda \in (0, \epsilon]$.

This is the picture of B that we choose to emphasize, depicting it as the complement of a set of “triangular” cuts inside $\Delta_V \times \mathbf{R}_+$; see Figure 2.

Remark **4.6.** — While the fibration we construct is merely Lagrangian, it is very reasonable to conjecture that in fact X^0 carries an S^1-invariant *special Lagrangian* fibration over B. The holomorphic $(n+1)$-form $\Omega = p^*\Omega_{V\times\mathbf{C}}$ on X^0 is S^1-invariant, and induces a holomorphic n-form on the reduced space $X^0_{red,\lambda}$, which turns out to coincide with the standard toric form $\Omega_V = i^n \prod_j d\log x_j$. Modifying the construction of the fibration $\pi_\lambda : X^0_{red,\lambda} \to \mathbf{R}^n$ so that its fibers are special Lagrangian with respect to Ω_V would then be sufficient to ensure that the fibers of π are special Lagrangian with respect to Ω. In dimension 1 this is easy to accomplish by elementary methods. In higher dimensions, making π_λ special Lagrangian requires the use of analysis, as the deformation of product tori in V^0 (which are special Lagrangian with respect to $\omega'_{V,\lambda}$ and Ω_V) to tori which are special Lagrangian for $\omega_{red,\lambda}$ and Ω_V is governed by a first-order elliptic PDE [40] (see also [30, §9] or [6, Prop. 2.5]). If one were to argue as in the proof of Lemma 4.1 (cf. Appendix B), the 1-forms used to construct ϕ_λ should be chosen not only to satisfy the usual condition for Moser's trick, but also to be co-closed with respect to a suitable rescaling of the Kähler metric induced by $\omega_{t,\lambda}$. When $V = (\mathbf{C}^*)^n$ this does not seem to pose any major difficulties, but in general it is not obvious that one can ensure the appropriate behavior along the toric divisors.

5. SYZ mirror symmetry for X^0

In this section we apply the procedure described in Section 2 to the Lagrangian torus fibration $\pi : X^0 \to B$ of Section 4 in order to construct the SYZ mirror to the open Calabi-Yau manifold X^0 and prove Theorem 1.7. The key observation is that, by Proposition 4.5, most fibers of π are mapped under the projection p to standard product tori in the toric variety $V \times \mathbf{C}$; therefore, the holomorphic discs bounded by these fibers can be understood by reducing to the toric case, which is well understood (see e.g. [15]).

Proposition **5.1.** — *The fibers of $\pi : X^0 \to B$ which bound holomorphic discs in X^0 are those which intersect the subset $p^{-1}(H \times \mathbf{C})$.*

Moreover, the simple holomorphic discs in X^0 bounded by such a fiber contained in $\mu_X^{-1}(\lambda)$ have Maslov index 0 and symplectic area $|\lambda - \epsilon|$, and their boundary represents the homology class of an S^1-orbit if $\lambda > \epsilon$ and its negative otherwise.

Proof. — Let $L \subset X^0$ be a smooth fiber of π, contained in $\mu_X^{-1}(\lambda)$ for some $\lambda \in \mathbf{R}_+$, and let $u : (D^2, \partial D^2) \to (X^0, L)$ be a holomorphic disc with boundary in L. Denote by L' the projection of L to V (i.e., the image of L by the composition p_V of p and the projection to the first factor). The restriction of p_V to $\mu_X^{-1}(\lambda)$ coincides with the quotient map to the reduced space $X_{red,\lambda} \simeq V$; thus, L' is in fact a fiber of π_λ, i.e. a Lagrangian torus in $(V^0, \omega_{red,\lambda})$, smoothly isotopic to a product torus inside $V^0 \simeq (\mathbf{C}^*)^n$.

Since the relative homotopy group $\pi_2(V^0, L') \simeq \pi_2((\mathbf{C}^*)^n, (S^1)^n)$ vanishes, the holomorphic disc $p_V \circ u : (D^2, \partial D^2) \to (V^0, L')$ is necessarily constant. Hence the image of the disc u is contained inside a fiber $p_V^{-1}(\mathbf{x})$ for some $\mathbf{x} \in V^0$.

If $\mathbf{x} \notin H$, then $p_V^{-1}(\mathbf{x}) \cap X^0 = p^{-1}(\{\mathbf{x}\} \times \mathbf{C}^*) \simeq \mathbf{C}^*$, inside which $p_V^{-1}(\mathbf{x}) \cap L$ is a circle centered at the origin (an orbit of the S^1-action). The maximum principle then implies that the map u is necessarily constant.

On the other hand, when $\mathbf{x} \in H$, $p_V^{-1}(\mathbf{x}) \cap X^0$ is the union of two affine lines intersecting transversely at one point: the proper transform of $\{\mathbf{x}\} \times \mathbf{C}$, and the fiber of E above $\mathbf{x}$ (minus the point where it intersects $\tilde{V}$). Now, $p_V^{-1}(\mathbf{x}) \cap L$ is again an S^1-orbit, i.e. a circle inside one of these two components (depending on whether $\lambda > \epsilon$ or $\lambda < \epsilon$); either way, $p_V^{-1}(\mathbf{x}) \cap L$ bounds exactly one non-constant embedded holomorphic disc in X^0 (and all of its multiple covers). The result follows. □

Denote by $B^{reg} \subset B$ the set of those fibers of π which do not intersect $p^{-1}(U_H \times \mathbf{C})$. From Property 3.6 and Propositions 4.5 and 5.1, we deduce:

Corollary **5.2.** — *The fibers of π above the points of B^{reg} are tautologically unobstructed in X^0, and project under p to standard product tori in $V^0 \times \mathbf{C}$.*

With respect to the affine structure, $B^{reg} = (\mathbf{R}^n \setminus \mathrm{Log}(U_H)) \times \mathbf{R}_+$ is naturally isomorphic to $(\Delta_V \setminus \mu_V(U_H)) \times \mathbf{R}_+$.

Definition **5.3.** — *The* chamber U_α *is the connected component of B^{reg} over which the monomial of weight α dominates all other monomials in the defining equation of H.*

Remark **5.4.** — By construction, the complement of $\mathrm{Log}(U_H)$ is a deformation retract of the complement of the amoeba of H inside $\mathbf{R}^n$; so the set of tautologically unobstructed fibers of π retracts onto $B^{reg} = \bigsqcup U_\alpha$.

As explained in Section 2.1, U_α determines an affine coordinate chart $U_\alpha^\vee$ for the SYZ mirror of X^0, with coordinates of the form (2.3).

Specifically, fix a reference point $b^0 \in U_\alpha$, and observe that, since $L^0 = \pi^{-1}(b^0)$ is the lift of an orbit of the T^{n+1}-action on $V \times \mathbf{C}$, its first homology carries a preferred basis $(\gamma_1, \dots, \gamma_n, \gamma_0)$ consisting of orbits of the various S^1 factors. Consider $b \in U_\alpha$, with coordinates $(\zeta_1, \dots, \zeta_n, \lambda)$ (here we identify $U_\alpha \subset B^{reg}$ with a subset of the moment polytope $\Delta_V \times \mathbf{R}_+ \subset \mathbf{R}^{n+1}$ for the T^{n+1}-action on $V \times \mathbf{C}$), and denote by $(\zeta_1^0, \dots, \zeta_n^0, \lambda^0)$ the coordinates of b^0. Then the valuations of the coordinates given by (2.3), i.e., the areas of the cylinders $\Gamma_1, \dots, \Gamma_n, \Gamma_0$ bounded by L^0 and $L = \pi^{-1}(b)$, are $\zeta_1 - \zeta_1^0, \dots, \zeta_n - \zeta_n^0$, and $\lambda - \lambda^0$ respectively. In order to eliminate the dependence on the choice of L^0, we rescale each coordinate by a suitable power of T, and equip $U_\alpha^\vee$ with the coordinate system

$$(L, \nabla) \mapsto (v_{\alpha,1}, \dots, v_{\alpha,n}, w_{\alpha,0}) = \big(T^{\zeta_1}\nabla(\gamma_1), \dots, T^{\zeta_n}\nabla(\gamma_n), T^{\lambda}\nabla(\gamma_0)\big). \tag{5.1}$$

(Compare with (2.3), noting that $\zeta_i = \zeta_i^0 + \int_{\Gamma_i} \omega_\epsilon$ and $\lambda = \lambda^0 + \int_{\Gamma_0} \omega_\epsilon$.)

As in Section 3.3, we set $\mathbf{v}_\alpha = (v_{\alpha,1}, \dots, v_{\alpha,n})$, and for $m \in \mathbf{Z}^n$ we write $\mathbf{v}_\alpha^m = v_{\alpha,1}^{m_1} \cdots v_{\alpha,n}^{m_n}$. Moreover, we write w_0 for $w_{\alpha,0}$; this is *a priori* ambiguous, but we shall see shortly that the gluings between the charts preserve the last coordinate.

The "naive" gluings between these coordinate charts (i.e., those which describe the geometry of the space of (L, ∇) up to Hamiltonian isotopy without accounting for instanton corrections) are governed by the global affine structure of $\mathrm{B} \setminus \mathrm{B}^{sing}$. Their description is instructive, even though it is not necessary for our argument.

For $\lambda > \epsilon$ the affine structure is globally that of $\Delta_\mathrm{V} \times (\epsilon, \infty)$. Therefore, (5.1) makes sense and is consistent with (2.3) even when b does not lie in U_α; thus, for $\lambda > \epsilon$ the naive gluing is the identity map ($\mathbf{v}_\alpha = \mathbf{v}_\beta$, and $w_{\alpha,0} = w_{\beta,0}$).

On the other hand, for $\lambda \in (0, \epsilon)$ we argue as in Section 4.2 (cf. Equation (4.5) and the preceding discussion). When $b = (\zeta_1, \dots, \zeta_n, \lambda)$ lies in a different chamber U_β from that containing the reference point b^0 (i.e., U_α), the intersection number of a cylinder Γ_i' constructed as previously with the exceptional divisor E is equal to $\beta_i - \alpha_i$, and its symplectic area differs from $\zeta_i - \zeta_i^0$ by $(\beta_i - \alpha_i)(\epsilon - \lambda)$. Moreover, due to the monodromy of the fibration, the bases of first homology used in U_α and U_β differ by $\gamma_i \mapsto \gamma_i + (\beta_i - \alpha_i)\gamma_0$ for $i = 1, \dots, n$. Thus, for $\lambda < \epsilon$ the naive gluing between the charts $\mathrm{U}_\alpha^\vee$ and $\mathrm{U}_\beta^\vee$ corresponds to setting

$$v_{\alpha,i} = \mathrm{T}^{-(\beta_i - \alpha_i)(\epsilon - \lambda)} \nabla(\gamma_0)^{\beta_i - \alpha_i} v_{\beta,i} = \left(\mathrm{T}^{-\epsilon} w_0\right)^{\beta_i - \alpha_i} v_{\beta,i}, \quad 1 \leq i \leq n.$$

The naive gluing formulas for the two cases ($\lambda > \epsilon$ and $\lambda < \epsilon$) are inconsistent. As seen in Section 2.1, this is not unexpected: the actual gluing between the coordinate charts $\{\mathrm{U}_\alpha^\vee\}_{\alpha \in \mathrm{A}}$ differs from these formulas by instanton corrections which account for the bubbling of holomorphic discs as L is isotoped across a wall of potentially obstructed fibers.

Given a potentially obstructed fiber $\mathrm{L} \subset \mu_\mathrm{X}^{-1}(\lambda)$, the simple holomorphic discs bounded by L are classified by Proposition 5.1. For $\lambda > \epsilon$, the symplectic area of these discs is $\lambda - \epsilon$, and their boundary loop represents the class $\gamma_0 \in \mathrm{H}_1(\mathrm{L})$ (the orbit of the S^1-action), so the corresponding weight is $\mathrm{T}^{\lambda - \epsilon} \nabla(\gamma_0)$ $(= \mathrm{T}^{-\epsilon} w_0)$; while for $\lambda < \epsilon$ the symplectic area is $\epsilon - \lambda$ and the boundary loop represents $-\gamma_0$, so the weight is $\mathrm{T}^{\epsilon - \lambda} \nabla(\gamma_0)^{-1}$ $(= \mathrm{T}^{\epsilon} w_0^{-1})$. As explained in Section 2.1, we therefore expect the instanton corrections to the gluings to be given by power series in $(\mathrm{T}^{-\epsilon} w_0)^{\pm 1}$.

While the direct calculation of the multiple cover contributions to the instanton corrections would require sophisticated machinery, Remark 2.3 provides a way to do so by purely elementary techniques. Namely, we study the manner in which counts of Maslov index 2 discs in partial compactifications of X^0 vary between chambers. The reader is referred to Example 3.1.2 of [7] for a simple motivating example (corresponding to the case where $\mathrm{H} = \{\text{point}\}$ in $\mathrm{V} = \mathbf{C}$).

Recall that a point of $\mathrm{U}_\alpha^\vee$ corresponds to a pair (L, ∇), where $\mathrm{L} = \pi^{-1}(b)$ is the fiber of π above some point $b \in \mathrm{U}_\alpha$, and ∇ is a unitary rank 1 local system on L. Given a partial compactification X' of X^0 (satisfying Assumption 2.2), (L, ∇) is a weakly unobstructed object of $\mathcal{F}(X')$, i.e. $\mathfrak{m}_0(L, \nabla) = W_{X'}(L, \nabla)e_L$, where $W_{X'}(L, \nabla)$ is a weighted count of Maslov index 2 holomorphic discs bounded by L in X'. Varying (L, ∇), these weighted counts define regular functions on each chart $U_\alpha^\vee$, and by Corollary A.11, they glue into a global regular function on the SYZ mirror of X^0.

We first use this idea to verify that the coordinate $w_0 = w_{\alpha,0}$ is preserved by the gluing maps, by interpreting it as a weighted count of discs in the partial compactification X_+^0 of X^0 obtained by adding the open stratum $\tilde{V}^0$ of the divisor $\tilde{V}$.

Lemma **5.5.** — *Let* $X_+^0 = p^{-1}(V^0 \times \mathbf{C}) = X^0 \cup \tilde{V}^0 \subset X$. *Then any point* (L, ∇) *of* $U_\alpha^\vee$ *defines a weakly unobstructed object of* $\mathcal{F}(X_+^0)$, *with*

$$W_{X_+^0}(L, \nabla) = w_{\alpha,0}. \tag{5.2}$$

Proof. — Let $u : (D^2, \partial D^2) \to (X_+^0, L)$ be a holomorphic disc in X_+^0 with boundary on L whose Maslov index is 2. The image of u by the projection p is a holomorphic disc in $V^0 \times \mathbf{C} \simeq (\mathbf{C}^*)^n \times \mathbf{C}$ with boundary on the product torus $p(L) = S^1(r_1) \times \cdots \times S^1(r_0)$. Thus, the first n components of $p \circ u$ are constant by the maximum principle, and we can write $p \circ u(z) = (x_1, \dots, x_n, r_0\gamma(z))$, where $|x_1| = r_1, \dots, |x_n| = r_n$, and $\gamma : D^2 \to \mathbf{C}$ maps the unit circle to itself. Moreover, the Maslov index of u is twice its intersection number with $\tilde{V}$. Therefore γ is a degree 1 map of the unit disc to itself, i.e. a biholomorphism; so the choice of $(x_1, \dots, x_n)$ determines u uniquely up to reparametrization.

We conclude that each point of L lies on the boundary of a unique Maslov index 2 holomorphic disc in X_+^0, namely the preimage by p of a disc $\{\mathbf{x}\} \times D^2(r_0)$. These discs are easily seen to be regular, by reduction to the toric case [15]; their symplectic area is λ (by definition of the moment map μ_X, see the beginning of Section 4.1), and their boundary represents the homology class $\gamma_0 \in H_1(L)$ (the orbit of the S^1-action on X). Thus, their weight is $T^{\omega(u)}\nabla(\partial u) = T^\lambda \nabla(\gamma_0) = w_{\alpha,0}$, which completes the proof. ☐

Lemma 5.5 implies that the local coordinates $w_{\alpha,0} \in \mathcal{O}(U_\alpha^\vee)$ glue to a globally defined regular function w_0 on the mirror of X^0 (hence we drop α from the notation).

Next, we consider monomials in the remaining coordinates $\mathbf{v}_\alpha$. First, let $\sigma \in \mathbf{Z}^n$ be a primitive generator of a ray of the fan Σ_V, and denote by D_σ^0 the open stratum of the corresponding toric divisor in V. We will presently see that the monomial $\mathbf{v}_\alpha^\sigma$ is related to a weighted count of discs in the partial compactification X'_σ of X^0 obtained by adding $p^{-1}(D_\sigma^0 \times \mathbf{C})$:

$$X'_\sigma = p^{-1}\big((V^0 \cup D_\sigma^0) \times \mathbf{C}\big) \setminus \tilde{V} \subset X. \tag{5.3}$$

Let $\varpi \in \mathbf{R}$ be the constant such that the corresponding facet of Δ_V has equation $\langle \sigma, u \rangle + \varpi = 0$, and let $\alpha_{\min} \in A$ be such that $\langle \sigma, \alpha_{\min} \rangle$ is minimal.

Lemma **5.6.** — *Any point* (L, ∇) *of* $\mathrm{U}_\alpha^\vee$ $(\alpha \in \mathrm{A})$ *defines a weakly unobstructed object of* $\mathcal{F}(\mathrm{X}'_\sigma)$, *with*

$$\mathrm{W}_{\mathrm{X}'_\sigma}(\mathrm{L}, \nabla) = \left(1 + \mathrm{T}^{-\epsilon} w_0\right)^{\langle \alpha - \alpha_{\min}, \sigma \rangle} \mathrm{T}^{\varpi} \mathbf{v}_\alpha^\sigma. \tag{5.4}$$

Proof. — After performing dual monomial changes of coordinates on V^0 and on $\mathrm{U}_\alpha^\vee$ (i.e., replacing the coordinates $(x_1, \dots, x_n)$ by $(\mathbf{x}^{\tau_1}, \dots, \mathbf{x}^{\tau_n})$ where $\langle \sigma, \tau_i \rangle = \delta_{i,1}$, and $(v_{\alpha,1}, \dots, v_{\alpha,n})$ by $(\mathbf{v}_\alpha^\sigma, \dots))$, we can reduce to the case where $\sigma = (1, 0, \dots, 0)$, and $\mathrm{V}^0 \cup \mathrm{D}_\sigma^0 \simeq \mathbf{C} \times (\mathbf{C}^*)^{n-1}$.

With this understood, let $u : (\mathrm{D}^2, \partial \mathrm{D}^2) \to (\mathrm{X}'_\sigma, \mathrm{L})$ be a Maslov index 2 holomorphic disc with boundary on L. The composition of u with the projection p is a holomorphic disc in $(\mathrm{V}^0 \cup \mathrm{D}_\sigma^0) \times \mathbf{C} \simeq \mathbf{C} \times (\mathbf{C}^*)^{n-1} \times \mathbf{C}$ with boundary on the product torus $p(\mathrm{L}) = \mathrm{S}^1(r_1) \times \cdots \times \mathrm{S}^1(r_0)$. Thus, all the components of $p \circ u$ except for the first and last ones are constant by the maximum principle. Moreover, since the Maslov index of u is twice its intersection number with D_σ^0, the first component of $p \circ u$ has a single zero, i.e. it is a biholomorphism from D^2 to the disc of radius r_1. Therefore, up to reparametrization we have $p \circ u(z) = (r_1 z, x_2, \dots, x_n, r_0 \gamma(z))$, where $|x_2| = r_2, \dots, |x_n| = r_n$, and $\gamma : \mathrm{D}^2 \to \mathbf{C}$ maps the unit circle to itself.

A further constraint is given by the requirement that the image of u be disjoint from $\tilde{\mathrm{V}}$ (the proper transform of $\mathrm{V} \times 0$). Thus, the last component $\gamma(z)$ is allowed to vanish only when $(r_1 z, x_2, \dots, x_n) \in \mathrm{H}$, and its vanishing order at such points is constrained as well. We claim that the intersection number k of the disc $\mathbf{D} = \mathrm{D}^2(r_1) \times \{(x_2, \dots, x_n)\}$ with H is equal to $\langle \alpha - \alpha_{\min}, \sigma \rangle$. Indeed, with respect to the chosen trivialization of $\mathcal{O}(\mathrm{H})$ over V^0, near $p_{\mathrm{V}}(\mathrm{L})$ the dominating term in the defining section of H is the monomial $\mathbf{x}^\alpha$, whose values over the circle $\mathrm{S}^1(r_1) \times \{(x_2, \dots, x_n)\}$ wind $\alpha_1 = \langle \alpha, \sigma \rangle$ times around the origin; whereas near D_σ^0 (i.e., in the chambers which are unbounded in the direction of $-\sigma$) the dominating terms have winding number $\langle \alpha_{\min}, \sigma \rangle$. Comparing these winding numbers we obtain that $k = \langle \alpha - \alpha_{\min}, \sigma \rangle$.

Assume first that $(x_2, \dots, x_n)$ are generic, in the sense that $\mathbf{D}$ intersects H transversely at k distinct points $(r_1 a_i, x_2, \dots, x_n)$, $i = 1, \dots, k$ (with $a_i \in \mathrm{D}^2$). The condition that u avoids $\tilde{\mathrm{V}}$ implies that γ is allowed to have at most simple zeroes at $a_1, \dots, a_k$. Denote by $\mathrm{I} \subseteq \{1, \dots, k\}$ the set of those a_i at which γ does have a zero, and let

$$\gamma_{\mathrm{I}}(z) = \prod_{i \in \mathrm{I}} \frac{z - a_i}{1 - \bar{a}_i z}.$$

Then γ_{I} maps the unit circle to itself, and its zeroes in the disc are the same as those of γ, so that $\gamma_{\mathrm{I}}^{-1} \gamma$ is a holomorphic function on the unit disc, without zeroes, and mapping the unit circle to itself, i.e. a constant map. Thus $\gamma(z) = e^{i\theta} \gamma_{\mathrm{I}}(z)$, and

$$p \circ u(z) = \left(r_1 z, x_2, \dots, x_n, r_0 e^{i\theta} \gamma_{\mathrm{I}}(z)\right) \tag{5.5}$$

for some $I \subseteq \{1, \dots, k\}$ and $e^{i\theta} \in S^1$. We conclude that there are 2^k holomorphic discs of Maslov index 2 in (X'_σ, L) whose boundary passes through a given generic point of L. It is not hard to check that these discs are all regular, using e.g. the same argument as in the proof of Lemma 7 in [8]. Succinctly: observing that u does not intersect $\tilde{H}$, projection to V decomposes (via a short exact sequence) the Cauchy-Riemann operator for u into a $\bar\partial$ operator on the trivial holomorphic line bundle with trivial real boundary condition (along the fibers of the projection), and the $\bar\partial$ operator for the "standard" disc $\mathbf{D}$ in $\mathbf{C} \times (\mathbf{C}^*)^{n-1}$ (which itself splits into a direct sum of line bundles and is easily checked to be surjective); this implies surjectivity.

When the disc $\mathbf{D}$ is not transverse to H, we can argue in exactly the same manner, except that $a_1, \dots, a_k \in D^2$ are no longer distinct; and γ may have a multiple zero at a_i as long as its order of vanishing does not exceed the multiplicity of $(r_1 a_i, x_2, \dots, x_n)$ as an intersection of $\mathbf{D}$ with H. We still conclude that $p \circ u$ is of the form (5.5). These discs are not all distinct (or regular), but we can argue by continuity as follows. There are diffeomorphisms arbitrarily C^∞-close to identity which fix a neighborhood of H and map $S^1(r_1) \times \{(x_2, \dots, x_n)\}$ to a nearby circle $S^1(r'_1) \times \{(x'_2, \dots, x'_n)\}$ contained in a generic fiber. The moduli space of holomorphic discs with respect to the pullback of the standard complex structure by such a diffeomorphism is canonically identified with the moduli space of holomorphic discs for the standard complex structure with boundary on the nearby generic fiber. This provides an explicit regularization of the moduli space, and we conclude that the enumeration of holomorphic discs is as in the transverse case (i.e., discs which can be written in the form (5.5) in more than one way should be counted with a multiplicity equal to the number of such expressions).

All that remains is to calculate the weights (2.2) associated to the holomorphic discs we have identified. Denote by $(\zeta_1, \dots, \zeta_n, \lambda)$ the affine coordinates of $\pi(L) \in U_\alpha$ introduced above, and consider a disc given by (5.5) with $|I| = \ell \in \{0, \dots, k\}$. Then the relative homology class represented by $p \circ u(D^2)$ in $\mathbf{C} \times (\mathbf{C}^*)^{n-1} \times \mathbf{C} \subset V \times \mathbf{C}$ is equal to $[D^2(r_1) \times \{pt\}] + \ell[\{pt\} \times D^2(r_0)]$. By elementary toric geometry, the symplectic area of the disc $D^2(r_1) \times \{pt\}$ with respect to the toric Kähler form $\omega_{V\times\mathbf{C}}$ is equal to $\langle \sigma, \mu_V \rangle + \varpi = \zeta_1 + \varpi$, while that of $\{pt\} \times D^2(r_0)$ is equal to λ. Thus, the symplectic area of the disc $p \circ u(D^2)$ with respect to $\omega_{V\times\mathbf{C}}$ is $\zeta_1 + \varpi + \ell\lambda$. The disc we are interested in, $u(D^2) \subset X'_\sigma$, is the proper transform of $p \circ u(D^2)$ under the blowup map; since its intersection number with the exceptional divisor E is equal to $|I| = \ell$, we conclude that

$$\int_{D^2} u^*\omega_\epsilon = \left(\int_{D^2} (p \circ u)^* \omega_{V\times\mathbf{C}} \right) - \ell\epsilon = \zeta_1 + \varpi + \ell(\lambda - \epsilon). \tag{5.6}$$

On the other hand, the degree of $\gamma_{I|S^1} : S^1 \to S^1$ is equal to $|I| = \ell$, so in $H_1(L, \mathbf{Z})$ we have $[u(S^1)] = \gamma_1 + \ell\gamma_0$. Thus the weight of u is

$$T^{\omega_\epsilon(u)} \nabla(\partial u) = T^{\zeta_1 + \varpi + \ell(\lambda-\epsilon)} \nabla(\gamma_1) \nabla(\gamma_0)^\ell = \left(T^{-\epsilon} w_0\right)^\ell T^{\varpi} v_{\alpha,1}.$$

Summing over the $\binom{k}{\ell}$ families of discs with $|\mathrm{I}| = \ell$ for each $\ell = 0, \dots, k$, we find that

$$\mathrm{W}_{\mathrm{X}'_\sigma}(\mathrm{L}, \nabla) = \sum_{\ell=0}^{k} \binom{k}{\ell} \big(\mathrm{T}^{-\epsilon} w_0\big)^{\ell} \mathrm{T}^{\varpi} v_{\alpha,1} = \big(1 + \mathrm{T}^{-\epsilon} w_0\big)^{k} \mathrm{T}^{\varpi} v_{\alpha,1}. \qquad \square$$

Next we extend Lemma 5.6 to the case of general monomials in the coordinates $\mathbf{v}_\alpha$. Let σ be *any* primitive element of $\mathbf{Z}^n$, and denote again by $\alpha_{\min}$ an element of A such that $\langle \alpha_{\min}, \sigma \rangle$ is minimal. Denote by $\mathrm{V}'_\sigma = \mathrm{V}^0 \cup \mathrm{D}^0_\sigma$ the toric partial compactification of V^0 obtained by adding a single toric divisor D^0_σ in the direction of the ray $-\sigma$. The hypersurface H^0 admits a natural partial compactification H'_σ inside V'_σ.

We claim that H'_σ is smooth for τ sufficiently small in (3.1). Indeed, rescaling f_τ by a factor of $\mathbf{x}^{-\alpha_{\min}}$ if necessary, we can assume without loss of generality that $\langle \alpha_{\min}, \sigma \rangle = 0$. Then f_τ extends to a regular function on V'_σ, whose restriction to D^0_σ is again a maximally degenerating family of Laurent polynomials, associated to the regular polyhedral decomposition $\mathcal{P} \cap \sigma^\perp$ of the convex hull of $\mathrm{A} \cap \sigma^\perp$. This implies that for sufficiently small τ the restriction of f_τ to D^0_σ vanishes transversely; the smoothness of H'_σ follows.

By blowing up $\mathrm{V}'_\sigma \times \mathbf{C}$ along $\mathrm{H}'_\sigma \times 0$ and removing the proper transform of $\mathrm{V}'_\sigma \times 0$, we obtain a partial compactification X'_σ of X^0. While X'_σ does not necessarily embed into X, we can equip V'_σ (resp. X'_σ) with a toric (resp. S^1-invariant) Kähler form which agrees with ω_{V} (resp. ω_ϵ) everywhere outside of an arbitrarily small neighborhood of the compactification divisor.

Denote by $\mathrm{L} \subset \mathrm{X}^0$ a smooth fiber of π which lies in the region where the Kähler forms agree (so that L is Lagrangian in X'_σ as well).

Lemma **5.7.** — *The Maslov index 0 holomorphic discs bounded by* L *inside* X'_σ *are all contained in* X^0 *and described by Proposition 5.1.*

Moreover, if L *is tautologically unobstructed in* X^0 *and lies over the chamber* U_α*, then the points* $(\mathrm{L}, \nabla) \in \mathrm{U}^\vee_\alpha$ *define weakly unobstructed objects of* $\mathcal{F}(\mathrm{X}'_\sigma)$*, with*

$$\mathrm{W}_{\mathrm{X}'_\sigma}(\mathrm{L}, \nabla) = \big(1 + \mathrm{T}^{-\epsilon} w_0\big)^{\langle \alpha - \alpha_{\min}, \sigma \rangle} \mathrm{T}^{\varpi} \mathbf{v}^\sigma_\alpha \tag{5.7}$$

for some $\varpi \in \mathbf{R}$.

Proof. — The Maslov index of a disc in X'_σ with boundary on L is twice its intersection number with the compactification divisor, and Assumption 2.2 is satisfied (in fact X'_σ is affine). Thus all Maslov index 0 holomorphic discs are contained in the open stratum X^0, and Proposition 5.1 holds. (Since L lies away from the compactification divisor, the symplectic area of these discs remains the same as for ω_ϵ.)

Thus, whenever L lies over a chamber U_α it does not bound any holomorphic discs of Maslov index zero or less in X'_σ, and the Maslov index 2 discs can be classified exactly as in the proof of Lemma 5.6. The only difference is that, since we evaluate the symplectic areas of these discs with respect to the Kähler form on X'_σ rather than X, the constant term ϖ in the area formula (5.6) now depends on the choice of the toric Kähler form on V'_σ near the compactification divisor. □

By Remark 2.3 (see also Corollary A.11), the expressions (5.7) determine globally defined regular functions on the mirror of X^0. Thus, we can use Lemma 5.7 to determine the wall-crossing transformations between the affine charts of the mirror.

Consider two adjacent chambers U_α and U_β separated by a wall of potentially obstructed fibers of π, i.e. assume that $\alpha, \beta \in A$ are connected by an edge in the polyhedral decomposition $\mathcal{P}$. Then we have:

Proposition **5.8.** — *The instanton-corrected gluing between the coordinate charts* $U_\alpha^\vee$ *and* $U_\beta^\vee$ *preserves the coordinate* w_0*, and matches the remaining coordinates via*

$$\mathbf{v}_\alpha^\sigma = \left(1 + T^{-\epsilon} w_0\right)^{\langle \beta - \alpha, \sigma \rangle} \mathbf{v}_\beta^\sigma \quad \textit{for all } \sigma \in \mathbf{Z}^n. \tag{5.8}$$

Proof. — Let $\{L_t\}_{t\in[0,1]}$ be a path among smooth fibers of π, with L_0 and L_1 tautologically unobstructed and lying over the chambers U_α and U_β respectively. We consider the partial compactifications X^0_+ and X'_σ of X^0 introduced in Lemmas 5.5–5.7; in the case of X'_σ we choose the Kähler form to agree with ω_ϵ over a large open subset which contains the path L_t, so as to be able to apply Lemma 5.7.

Since these partial compactifications satisfy Assumption 2.2, the moduli spaces of Maslov index 0 holomorphic discs bounded by the Lagrangians L_t in X^0_+, X'_σ, and X^0 are the same, and the corresponding wall-crossing transformations are identical (see Appendix A). Noting that the expressions (5.2) and (5.7) are manifestly convergent over the whole completions $(\mathbf{K}^*)^{n+1}$ of $U_\alpha^\vee$ and $U_\beta^\vee$, we appeal to Lemma A.10, and conclude that these expressions for the superpotentials $W_{X^0_+}$ and $W_{X'_\sigma}$ over the chambers $U_\alpha^\vee$ and $U_\beta^\vee$ match under the wall-crossing transformation. Thus w_0 is preserved, and for primitive $\sigma \in \mathbf{Z}^n$ the monomials $\mathbf{v}_\alpha^\sigma$ and $\mathbf{v}_\beta^\sigma$ are related by (5.8). (The case of non-primitive σ follows obviously from the primitive case.) □

This completes the proof of Theorem 1.7. Indeed, the instanton-corrected gluing maps (5.8) coincide with the coordinate change formulas (3.11) between the affine charts for the toric variety Y introduced in Section 3.3. Therefore, the SYZ mirror of X^0 embeds inside Y, by identifying the completion of the local chart $U_\alpha^\vee$ with the subset of Y_α where w_0 is non-zero. It follows that the SYZ mirror of X^0 is the subset of Y where w_0 is nonzero, namely Y^0.

6. Proof of Theorem 1.5

We now turn to the proof of Theorem 1.5. We begin with an elementary observation:

Lemma **6.1.** — *If Assumption 1.4 holds, then every rational curve* $C \simeq \mathbf{P}^1$ *in* X *satisfies* $D \cdot C = c_1(X) \cdot C > 0$*; so in particular Assumption 2.2 holds.*

Proof. — $c_1(X) = p_V^* c_1(V) - [E]$, where p_V is the projection to V and $E = p^{-1}(H \times 0)$ is the exceptional divisor. Consider a rational curve C in X (i.e., the image of a nonconstant holomorphic map from $\mathbf{P}^1$ to X), and denote by $C' = p_V(C)$ the rational curve in V obtained by projecting C to V. Applying the maximum principle to the projection to the last coordinate $y \in \mathbf{C}$, we conclude that C is contained either in $p^{-1}(V \times 0) = \tilde{V} \cup E$, or in $p^{-1}(V \times \{y\})$ for some nonzero value of y.

When $C \subset p^{-1}(V \times \{y\})$ for $y \neq 0$, the curve C is disjoint from E and its projection C' is nonconstant, so $c_1(X) \cdot [C] = c_1(V) \cdot [C'] > 0$ by Assumption 1.4.

When C is contained in $\tilde{V}$, the curve C' is again nonconstant, and since the normal bundle of $\tilde{V}$ in X is $\mathcal{O}(-H)$, we have $c_1(X) \cdot [C] = c_1(V) \cdot [C'] - [H] \cdot [C']$, which is positive by Assumption 1.4.

Finally, we consider the case where C is contained in E but not in $\tilde{V}$. Then

$$\begin{aligned} c_1(X) \cdot [C] = [D] \cdot [C] &= [\tilde{V}] \cdot [C] + \left[p^{-1}(D_V)\right] \cdot [C] \\ &= [\tilde{V}] \cdot [C] + c_1(V) \cdot \left[C'\right]. \end{aligned}$$

The first term is non-negative by positivity of intersection; and by Assumption 1.4 the second one is positive unless C' is a constant curve, and non-negative in any case. However C' is constant only when C is (a cover of) a fiber of the $\mathbf{P}^1$-bundle $p_{|E} : E \to H \times 0$; in that case $[\tilde{V}] \cdot [C] > 0$, so $c_1(X) \cdot [C] > 0$ in all cases. □

As explained in Section 2.2, this implies that the tautologically unobstructed fibers of $\pi : X^0 \to B$ remain weakly unobstructed in X, and that the SYZ mirror of X is just Y^0 (the SYZ mirror of X^0) equipped with a superpotential W_0 which counts Maslov index 2 holomorphic discs bounded by the fibers of π. Indeed, the conclusion of Lemma 6.1 implies that any component which is a sphere contributes at least 2 to the Maslov index of a stable genus 0 holomorphic curve bounded by a fiber of π. Thus, Maslov index 0 configurations are just discs contained in X^0, and Maslov index 2 configurations are discs intersecting D transversely in a single point.

Observe that each Maslov index 2 holomorphic disc intersects exactly one of the components of the divisor D. Thus, the superpotential W_0 can be expressed as a sum over the components of $D = \tilde{V} \cup p^{-1}(D_V \times \mathbf{C})$, in which each term counts those discs which intersect a particular component. It turns out that the necessary calculations have been carried out in the preceding section: Lemma 5.5 describes the contribution from discs which only hit $\tilde{V}$, and Lemma 5.6 describes the contributions from discs which hit the various components of $p^{-1}(D_V \times \mathbf{C})$. Summing these, and using the notations of

Section 3.3, we obtain that, for any point (L, ∇) of $\mathrm{U}_\alpha^\vee$ $(\alpha \in \mathrm{A})$,

$$\mathrm{W}_0(\mathrm{L}, \nabla) = w_{\alpha,0} + \sum_{i=1}^{r} \big(1 + \mathrm{T}^{-\epsilon} w_0\big)^{\langle \alpha - \alpha_i, \sigma_i \rangle} \mathrm{T}^{\varpi_i} \mathbf{v}_\alpha^{\sigma_i} = w_0 + \sum_{i=1}^{r} w_i.$$

Hence W_0 is precisely the leading-order superpotential (3.14). This completes the proof of Theorem 1.5.

Remark **6.2.** — In the proofs of Lemmas 5.5 and 5.6 we have not discussed in any detail the orientations of moduli spaces of discs, which determine the signs of the various terms appearing in the superpotential. The fact that those are all positive follows from two ingredients.

The first is that, for a standard product torus in a toric variety, equipped with the standard spin structure, the contributions of the various families of Maslov index 2 holomorphic discs to the superpotential are all positive. See [13] for a detailed calculation in the case of the Clifford torus. The fact that all the signs are the same is not surprising, since a monomial change of variables can be used to reduce to a single example, namely the family of discs $\mathrm{D}^2 \times \{pt\}$ bounded by a product torus in $\mathbf{C} \times (\mathbf{C}^*)^n$ equipped with the standard spin structure. The same argument also applies to the discs in Lemma 5.5 since those can also be reduced to the toric case.

The second ingredient is a comparison of the orientations of moduli spaces of discs in V and their lifts to X (as in Lemma 5.6). A short calculation shows that, for the standard spin structure, the orientation of the moduli space of lifted discs in X agrees with that induced by the orientation of the moduli space of discs in V and the natural orientation of the orbits of the S^1-action. See the proof of Corollary 8 in [8] for a similar argument. The positivity of the signs in Lemma 5.6 follows.

Remark **6.3.** — When Assumption 1.4 does not hold, the SYZ mirror of X differs from $(\mathrm{Y}^0, \mathrm{W}_0)$, since the enumerative geometry of discs is modified by the presence of stable genus 0 configurations of total Maslov index 0 or 2. A borderline case that remains fairly easy is when the strict inequality in Assumption 1.4 is relaxed to

$$c_1(\mathrm{V}) \cdot \mathrm{C} \geq \max(0, \mathrm{H} \cdot \mathrm{C}).$$

(This includes the situation where H is a Calabi-Yau hypersurface in a toric Fano variety as an important special case.)

In this case, Assumption 2.2 still holds, so the mirror of X remains Y^0; the only modification is that the superpotential should also count the contributions of configurations consisting of a Maslov index 2 disc together with one or more rational curves satisfying $c_1(\mathrm{X}) \cdot \mathrm{C} = 0$. Thus, we now have

$$\mathrm{W} = (1 + c_0) w_0 + (1 + c_1) w_1 + \cdots + (1 + c_r) w_r,$$

where $c_0, \ldots, c_r \in \Lambda$ are constants (determined by the genus 0 Gromov-Witten theory of X), with $\mathrm{val}(c_i) > 0$.

7. From the blowup X to the hypersurface H

The goal of this section is to prove Theorem 1.6. As a first step, we establish:

Theorem **7.1.** — *Under Assumption 1.4, the* B*-side Landau-Ginzburg model* (Y, W_0) *is SYZ mirror to the* A*-side Landau-Ginzburg model* $(X, W^\vee = y)$ *(with the Kähler form* ω_ϵ*).*

(Recall that y is the coordinate on the second factor of $V \times \mathbf{C}$.)

Sketch of proof. — This result follows from Theorem 1.5 by the same considerations as in Example 2.4. Specifically, equipping X with the superpotential $W^\vee = y$ enlarges its Fukaya category by adding admissible non-compact Lagrangian submanifolds, i.e., properly embedded Lagrangian submanifolds of X whose image under $W^\vee$ is only allowed to tend to infinity in the direction of the positive real axis; in other terms, the y coordinate is allowed to be unbounded, but only in the positive real direction.

Let $a_0 \subset \mathbf{C}$ be a properly embedded arc which connects $+\infty$ to itself by passing around the origin, encloses an infinite amount of area, and stays away from the projection to $\mathbf{C}$ of the support of the cut-off function χ used to construct ω_ϵ. Then we can supplement the family of Lagrangian tori in X^0 constructed in Section 4 by considering product Lagrangians of the form $L = p^{-1}(L' \times a_0)$, where L' is an orbit of the T^n-action on V. Indeed, by Proposition 4.5, away from the exceptional divisor the fibers of $\pi : X^0 \to B$ are lifts to X of product tori $L' \times S^1(r) \subset V \times \mathbf{C}$. For large enough r, the circles $S^1(r)$ can be deformed by Hamiltonian isotopies in $\mathbf{C}$ to simple closed curves that approximate a_0 as $r \to \infty$; moreover, the induced isotopies preserve the tautological unobstructedness in X^0 of the fibers of π which do not intersect $p^{-1}(H \times \mathbf{C})$. In this sense, $p^{-1}(L' \times a_0)$ is naturally a limit of the tori $p^{-1}(L' \times S^1(r))$ as $r \to \infty$. The analytic structure near this point is obtained by equation (2.3), which naturally extends as in Example 2.4.

To be more specific, let $L' = \mu_V^{-1}(\zeta_1, \ldots, \zeta_n)$ for $(\zeta_1, \ldots, \zeta_n)$ a point in the component of $\Delta_V \setminus \mu_V(U_H)$ corresponding to the weight $\alpha \in A$, and equip $L = p^{-1}(L' \times a_0)$ with a local system $\nabla \in \hom(\pi_1(L), U_{\mathbf{K}})$. The maximum principle implies that any holomorphic disc bounded by L in X^0 must be contained inside a fiber of the projection to V (see the proof of Proposition 5.1). Thus L is tautologically unobstructed in X^0, and (L, ∇) defines an object of the Fukaya category $\mathcal{F}(X^0, W^\vee)$, and a point in some partial compactification of the coordinate chart $U_\alpha^\vee$ considered in Section 5. Denoting by $\gamma_1, \ldots, \gamma_n$ the standard basis of $H_1(L) \simeq H_1(L')$ given by the various S^1 factors, in the coordinate chart (5.1) the object (L, ∇) corresponds to

$$(v_{\alpha,1}, \ldots, v_{\alpha,n}, w_{\alpha,0}) = \big(T^{\zeta_1}\nabla(\gamma_1), \ldots, T^{\zeta_n}\nabla(\gamma_n), 0\big).$$

Thus, equipping X^0 with the superpotential $W^\vee$ extends the moduli space of objects under consideration from $Y^0 = Y \setminus w_0^{-1}(0)$ to Y.

Under Assumption 1.4, (L, ∇) remains a weakly unobstructed object of the Fukaya category $\mathcal{F}(X, W^\vee)$. We now study the families of Maslov index 2 holomorphic discs bounded by L in X, in order to determine the corresponding value of the superpotential and show that it agrees with (3.14). Under projection to the y coordinate, any holomorphic disc $u : (D^2, \partial D^2) \to (X, L)$ maps to a holomorphic disc in $\mathbf{C}$ with boundary on the arc a_0, which is necessarily constant; hence the image of u is contained inside $p^{-1}(V \times \{y\})$ for some $y \in a_0$. Moreover, inside the toric variety $p^{-1}(V \times \{y\}) \simeq V$ the holomorphic disc u has boundary on the product torus L'.

Thus, the holomorphic discs bounded by L in X can be determined by reduction to the toric case of (V, L'). For each toric divisor of V there is a family of Maslov index 2 discs which intersect it transversely at a single point and are disjoint from all the other toric divisors; these discs are all regular, and exactly one of them passes through each point of L [15]. The discs which intersect the toric divisor corresponding to a facet of Δ_V with equation $\langle \sigma, \cdot \rangle + \varpi = 0$ have area $\langle \sigma, \zeta \rangle + \varpi$ and weight $T^{\varpi} \mathbf{v}_\alpha^\sigma$. Summing over all facets of Δ_V, we conclude that

$$W_0(L, \nabla) = \sum_{i=1}^{r} T^{\varpi_i} \mathbf{v}_\alpha^{\sigma_i}. \tag{7.1}$$

Moreover, because $w_0 = 0$ at the point (L, ∇), the coordinate transformations (3.11) simplify to $\mathbf{v}_{\alpha_i}^{\sigma_i} = \mathbf{v}_\alpha^{\sigma_i}$. Thus the expression (7.1) agrees with (3.14). □

Remark **7.2.** — In order to fill the details of this sketch, we would need a sufficient development of Fukaya categories of A-side Landau-Ginzburg models in order to verify the existence of the analytic charts at infinity. The most straightforward way to do this is to introduce non-compact Lagrangians which are mirror to the powers of an ample line bundle on Y, and check that (i) these Lagrangians generate the Fukaya category and (ii) when r is sufficiently large, the product Lagrangian $L' \times S^1(r) \subset V \times \mathbf{C}$ defines a module over the Floer cochains of this generating family which is equivalent to the one associated to the product of L' with an admissible arc in $\mathbf{C}$ equipped with a bounding cochain which is a multiple of a degree 1 generator coming from a self-intersection at infinity.

Our next observation is that $W^\vee : X \to \mathbf{C}$ has a particularly simple structure. The following statement is a direct consequence of the construction:

Proposition **7.3.** — $W^\vee = y : X \to \mathbf{C}$ *is a Morse-Bott fibration, with* 0 *as its only critical value; in fact the singular fiber* ${W^\vee}^{-1}(0) = \tilde{V} \cup E \subset X$ *has normal crossing singularities along* $\mathrm{crit}(W^\vee) = \tilde{V} \cap E \simeq H$.

Remark **7.4.** — However, the Kähler form on $\mathrm{crit}(W^\vee) \simeq H$ is not that induced by ω_V, but rather that induced by the restriction of ω_ϵ, which represents the cohomology class $[\omega_V] - \epsilon[H]$. To compensate for this, in the proof of Theorem 1.6 we will actually replace $[\omega_V]$ by $[\omega_V] + \epsilon[H]$.

Proposition 7.3 allows us to relate the Fukaya category of $(X, W^\vee)$ to that of H, using the ideas developed by Seidel in [46], adapted to the Morse-Bott case (see [53]).

Remark **7.5.** — Strictly speaking, the literature does not include any definition of the Fukaya category of a superpotential without assuming that it is a Lefschetz fibration. The difficulty resides not in defining the morphisms and the compositions, but in defining the higher order products in a coherent way. These technical problems were resolved by Seidel in [48], by introducing a method of defining Fukaya categories of Lefschetz fibration that generalizes in a straightforward way to the Morse-Bott case we are considering. This construction will be revisited in [5]. As the reader will see, in the only example where we shall study such a Fukaya category, the precise nature of the construction of higher products will not enter.

Outside of its critical locus, the Morse-Bott fibration $W^\vee$ carries a natural horizontal distribution given by the ω_ϵ-orthogonal to the fiber. Parallel transport with respect to this distribution induces symplectomorphisms between the smooth fibers; in fact, parallel transport along the real direction is given by (a rescaling of) the Hamiltonian flow generated by $\mathrm{Im}\, W^\vee$, or equivalently, the gradient flow of $\mathrm{Re}\, W^\vee$ (for the Kähler metric).

Given a Lagrangian submanifold $\ell \subset \mathrm{crit}(W^\vee) \simeq H$, parallel transport by the positive gradient flow of $\mathrm{Re}\, W^\vee$ yields an admissible Lagrangian *thimble* $L_\ell \subset X$ (topologically a disc bundle over ℓ). Moreover, any local system ∇ on ℓ induces by pullback a local system $\tilde{\nabla}$ on L_ℓ. However, there is a subtlety related to the nontriviality of the normal bundle to H inside X:

Lemma **7.6.** — *The thimble* L_ℓ *is naturally diffeomorphic to the restriction of the complex line bundle* $\mathcal{L} = \mathcal{O}(H)$ *to* $\ell \subset H$.

Proof. — First note that, for the Lefschetz fibration $f(x, y) = xy$ on $\mathbf{C}^2$ equipped with its standard Kähler form, the thimble associated to the critical point at the origin is $\{(x, \bar{x}),\ x \in \mathbf{C}\} \subset \mathbf{C}^2$. Indeed, parallel transport preserves the quantity $|x|^2 - |y|^2$, so that the thimble consists of the points (x, y) where $|x| = |y|$ and $xy \in \mathbf{R}_{\geq 0}$, i.e. $y = \bar{x}$. In particular, the thimble projects diffeomorphically onto either of the two $\mathbf{C}$ factors (the two projections induce opposite orientations).

Now we consider the Morse-Bott fibration $W^\vee : X \to \mathbf{C}$. The normal bundle to the critical locus $\mathrm{crit}\, W^\vee = \tilde{V} \cap E \simeq H$ is isomorphic to $\mathcal{L} \oplus \mathcal{L}^{-1}$ (where $\mathcal{L}$ is the normal bundle to H inside $\tilde{V}$, while $\mathcal{L}^{-1}$ is its normal bundle inside E). Moreover, $W^\vee$ is locally given by the product of the fiber coordinates on the two line subbundles. The local calculation then shows that, by projecting to either subbundle, a neighborhood of ℓ in L_ℓ can be identified diffeomorphically with a neighborhood of the zero section in either $\mathcal{L}_{|\ell}$ or $\mathcal{L}^{-1}_{|\ell}$. $\square$

Lemma 7.6 implies that, even when $\ell \subset H$ is spin, $L_\ell \subset X$ need not be spin; indeed, $w_2(TL_\ell) = w_2(T\ell) + w_2(\mathcal{L}_{|\ell})$. Rather, L_ℓ is *relatively spin*, i.e. its second Stiefel-Whitney class is the restriction of the *background class* $s \in H^2(X, \mathbf{Z}/2)$ Poincaré dual to $[\tilde{V}]$ (or equivalently to $[E]$). Hence, applying the thimble construction to an object of the Fukaya category $\mathcal{F}(H)$ does not determine an object of $\mathcal{F}(X, W^\vee)$, but rather an object of the s-twisted Fukaya category $\mathcal{F}_s(X, W^\vee)$ (we shall verify in Proposition 7.10 that thimbles are indeed weakly unobstructed objects of this category).

Remark **7.7.** — While it has not appeared in the literature, the notion of weak unobstructedness of an admissible Lagrangian L is a straightforward generalization of the case of closed Lagrangians. There is a Floer-theoretic A_∞-structure on the ordinary cohomology of L, and a natural A_∞-homomorphism from the ordinary cohomology of L equipped with this A_∞-structure to the endomorphisms of L as an object of the Fukaya category of the potential. This homomorphism is not necessarily an isomorphism, but it is always unital and preserves the curvature $\mathfrak{m}_0$. We say that L is *weakly unobstructed* if the curvature is a multiple of the unit in $H^0(L)$. In the case of thimbles, radial parallel transport allows one to lift Maurer-Cartan elements and bounding cochains from an arbitrarily small neighborhood of the critical fiber to the total space. This implies that an admissible thimble which bounds no holomorphic disc of Maslov index less than 2 in a neighborhood of the critical fiber is weakly unobstructed; and the curvature is then the product of the unit with the count of Maslov index 2 discs passing through a generic point near the critical fiber.

Corollary **7.8.** — *Under Assumption 1.4, there is a fully faithful A_∞-functor from the Fukaya category $\mathcal{F}(H)$ to $\mathcal{F}_s(X, W^\vee)$, which at the level of objects maps (ℓ, ∇) to the thimble $(L_\ell, \tilde{\nabla})$.*

Sketch of proof. — Let ℓ_1, ℓ_2 be two Lagrangian submanifolds of $\mathrm{crit}(W^\vee) \simeq H$, assumed to intersect transversely (otherwise transversality is achieved by Hamiltonian perturbations, which may be needed to achieve regularity of holomorphic discs in any case), and denote by $L_1, L_2 \subset X$ the corresponding thimbles. (For simplicity we drop the local systems from the notations; we also postpone the discussion of relatively spin structures until further below.)

Recall that $\hom_{\mathcal{F}_s(X,W^\vee)}(L_1, L_2)$ is defined by perturbing L_1, L_2 to Lagrangians $\tilde{L}_1, \tilde{L}_2$ whose images under $W^\vee$ are half-lines which intersect transversely and such that the first one lies above the second one near infinity; so for example, fixing a small angle $\theta > 0$, we can take $\tilde{L}_1$ (resp. $\tilde{L}_2$) to be the Lagrangian obtained from ℓ_1 (resp. ℓ_2) by the gradient flow of $\mathrm{Re}(e^{-i\theta}W^\vee)$ (resp. $\mathrm{Re}(e^{i\theta}W^\vee)$). (A more general approach would be to perturb the holomorphic curve equation by a Hamiltonian vector field generated by a suitable rescaling of the real part of $W^\vee$, instead of perturbing the Lagrangian boundary conditions; in our case the two approaches are equivalent.)

We now observe that $\tilde{L}_1$ and $\tilde{L}_2$ intersect transversely, with all intersections lying in the singular fiber ${W^\vee}^{-1}(0)$, and in fact $\tilde{L}_1 \cap \tilde{L}_2 = \ell_1 \cap \ell_2$. Thus, $\hom_{\mathcal{F}(H)}(\ell_1, \ell_2)$ and $\hom_{\mathcal{F}_s(X,W^\vee)}(L_1, L_2)$ are naturally isomorphic. Moreover, the maximum principle applied to the projection $W^\vee$ implies that all holomorphic discs bounded by the (perturbed) thimbles in X are contained in $(W^\vee)^{-1}(0) = \tilde{V} \cup E$ (and hence their boundary lies on $\ell_1 \cup \ell_2 \subset H \subset \tilde{V} \cup E$).

After quotienting by a suitable reference section, we can view the defining section of H as a meromorphic function on $\tilde{V}$, with $f^{-1}(0) = H$. Since $f = 0$ at the boundary, and since a meromorphic function on the disc which vanishes at the boundary is everywhere zero, any holomorphic disc in $\tilde{V}$ with boundary in $\ell_1 \cup \ell_2$ must lie entirely inside $f^{-1}(0) = H$. By the same argument, any holomorphic disc in E with boundary in $\ell_1 \cup \ell_2$ must stay inside H as well. Finally, Lemma 6.1 implies that stable curves with both disc and sphere components cannot contribute to the Floer differential (since each sphere component contributes at least 2 to the total Maslov index).

This implies that the Floer differentials on $\hom_{\mathcal{F}(H)}(\ell_1, \ell_2)$ and $\hom_{\mathcal{F}_s(X,W^\vee)}(L_1, L_2)$ count the same holomorphic discs. The same argument applies to Floer products and higher structure maps.

To complete the proof it only remains to check that the orientations of the relevant moduli spaces of discs agree. Recall that a relatively spin structure on a Lagrangian submanifold L with background class s is the same thing as a stable trivialization of the tangent bundle of L over its 2-skeleton, i.e. a trivialization of $TL_{|L^{(2)}} \oplus E_{|L^{(2)}}$, where E is a vector bundle over the ambient manifold with $w_2(E) = s$; such a stable trivialization in turn determines orientations of the moduli spaces of holomorphic discs with boundary on L (see [20, Chapter 8], noting that the definition of spin structures in terms of stable trivializations goes back to Milnor [42]).

In our case, we are considering discs in H with boundary on Lagrangian submanifolds $\ell_i \subset H$, and the given spin structures on ℓ_i determine orientations of the moduli spaces for the structure maps in $\mathcal{F}(H)$. If we consider the same holomorphic discs in the context of the thimbles $L_i \subset X$, the spin structure of ℓ_i does not induce a spin structure on $TL_i \simeq T\ell_i \oplus \mathcal{L}_{|\ell_i}$ (what would be needed instead is a relatively spin structure on ℓ_i with background class $w_2(\mathcal{L}_{|H})$). On the other hand, the normal bundle to H inside X, namely $\mathcal{L} \oplus \mathcal{L}^{-1}$, is an $SU(2)$-bundle and hence has a canonical isotopy class of trivialization over the 2-skeleton. Thus, the spin structure on ℓ_i induces a trivialization of $TL_i \oplus \mathcal{L}^{-1}$ over the 2-skeleton of L_i, i.e. a relative spin structure on L_i with background class $w_2(\mathcal{L}^{-1}_{|L_i}) = s_{|L_i}$. Furthermore, because $w_2(\mathcal{L} \oplus \mathcal{L}^{-1}) = 0$, stabilizing by this rank 2 bundle does not affect the orientation of the moduli space of discs [20, Proposition 8.1.16]. Hence the structure maps of $\mathcal{F}(H)$ and $\mathcal{F}_s(X, W^\vee)$ involve the same moduli spaces of holomorphic discs, oriented in the same manner, which completes the proof. $\square$

Remark **7.9.** — The reason the above is only a sketch of proof is that the construction of the two Fukaya categories requires choices of perturbations, and we have not discussed how to arrange for these choices to yield the same answer. A model for such arguments in a related situation is provided by Seidel in [46, Section (14c)].

Implicit in the statement of Corollary 7.8 is the fact that, if (ℓ, ∇) is weakly unobstructed in $\mathcal{F}(H)$, then $(L_\ell, \tilde{\nabla})$ is weakly unobstructed in $\mathcal{F}_s(X, W^\vee)$. In our setting, the values of the superpotentials for objects of $\mathcal{F}(H)$ and their images in $\mathcal{F}_s(X, W^\vee)$ differ by an additive constant δ. This constant is easiest to determine if we assume that V is affine:

Proposition **7.10.** — *Under the assumption that* V *is affine, the functor of Corollary 7.8 increases the value of the superpotential by* $\delta = T^\epsilon$.

Sketch of proof. — Consider a weakly unobstructed object (ℓ, ∇) of $\mathcal{F}(H)$ and the corresponding thimble $L_\ell \subset X$. Holomorphic discs bounded by L_ℓ in X are contained in the level sets of $W^\vee = y$ (by the maximum principle). By Remark 7.7, we only need to study the moduli spaces of such discs for small values of y.

For $y > 0$, the intersection L_ℓ^y of L_ℓ with $(W^\vee)^{-1}(y) \simeq V$ is a circle bundle over ℓ, lying in the boundary of a standard symplectic tubular neighborhood of size ϵ of H in $(W^\vee)^{-1}(y)$ equipped with the restriction of ω_ϵ. Indeed, as $y \to 0$, the fibers of $W^\vee$ degenerate to the normal crossing divisor $\tilde{V} \cup E$. Symplectic parallel transport identifies the standard disc bundle $E \setminus (\tilde{V} \cap E) \simeq H \times D^2(\epsilon)$ inside $(W^\vee)^{-1}(0)$ with a standard symplectic neighborhood U^y of H inside $(W^\vee)^{-1}(y)$ for $y > 0$. The boundary of U^y (a trivial S^1-bundle over H) consists of all points in $(W^\vee)^{-1}(y)$ whose parallel transport converges to $\tilde{V} \cap E \simeq H$ as $y \to 0$, and in particular it contains L_ℓ^y.

However, while the restriction of ω_ϵ to $(W^\vee)^{-1}(y) \simeq V$ is cohomologous to ω_V for all $y > 0$ and agrees with it pointwise for y sufficiently large, the actual forms differ near H for small y. Under the identification $(W^\vee)^{-1}(y) \simeq V$, the neighborhoods U^y are small tubular neighborhoods of H, increasing in size along a suitably normalized gradient flow of $|f|$ as y increases, and agreeing with a standard ω_V-neighborhood of H of size ϵ for $y \gg \epsilon^{1/2}$.

Using that V is affine, H is the vanishing locus of the globally defined holomorphic function f, and the maximum principle applied to f implies that, for small enough y (or for all y if ϵ is small enough), all holomorphic discs bounded by L_ℓ^y in V lie in a neighborhood U'^y of H (possibly larger than U^y).

The complex structure on the neighborhood U'^y of H in V is not biholomorphic to the standard product complex structure on a domain in $H \times \mathbf{C}$, but agrees with it along H. Thus, for small enough y, an arbitrarily C^∞-small perturbation of the almost-complex structure on V (preserving the holomorphicity of f) ensures the existence of a holomorphic projection map $\pi_H : U'^y \to H$, without affecting counts of holomorphic discs; without loss of generality, we can further assume that π_H maps L_ℓ^y to ℓ as an S^1-bundle, with $|f|$ constant in the S^1 fiber over each point of ℓ.

Holomorphic discs with boundary on L_ℓ^y can then be classified by using the projection to H. The Maslov index of a disc $u : D^2 \to (V, L_\ell^y)$ (with image contained in U'^y) is the sum of the Maslov index of $\pi_H \circ u$ and twice the intersection number of u with H. Thus, the weak unobstructedness of ℓ in H implies that of L_ℓ^y, and there are two types of Maslov index 2 discs to consider:

- $\pi_H \circ u$ is a Maslov index 2 disc in H, and u avoids H;
- $\pi_H \circ u$ is constant, and u intersects H transversely once.

In the first case, we observe that, given a point $\hat{p} \in L_\ell^y$, each holomorphic disc $v : D^2 \to (H, \ell)$ through $p = \pi_H(\hat{p})$ has a unique lift u through $\hat{p}$ that avoids H. Indeed, v determines the value of $\log |f|$ along the boundary of the disc u; the (unique) harmonic extension of this function to the entire disc can be expressed as the real part of some holomorphic function g, unique up to a pure imaginary additive constant. We then find that necessarily $f \circ u = \exp(g)$ up to some constant factor which is determined by requiring that the marked point map to $\hat{p}$. This, together with $\pi_H \circ u = v$, determines u. Recalling that L_ℓ^y lives on the boundary of a standard symplectic neighborhood of H, and using that u is disjoint from H, we further observe that the symplectic area of u in $(W^\vee)^{-1}(y)$ is equal to that of v in H, and the holonomy of $\tilde{\nabla}$ along the boundary of u equals that of ∇ along the boundary of v. Moreover, the same argument as in the proof of Corollary 7.8 shows that the orientations of the moduli spaces match. Thus, the total contribution of all these discs corresponds exactly to the superpotential in $\mathcal{F}(H)$.

In the second case, denoting $\pi_H \circ u = p \in \ell$, by construction L_ℓ^y intersects $\pi_H^{-1}(p)$ in a circle which bounds a disc of symplectic area ϵ, and u necessarily maps D^2 biholomorphically onto this disc. These small discs of size ϵ in the normal slices to H are regular, and contribute positively to the superpotential: indeed, their deformation theory splits into that of constant discs in H and that of the standard disc in the complex plane with boundary on a circle with the trivial spin structure (the triviality of the spin structure is due to the twist by the background class s). Thus, these discs are responsible for the additional term T^ϵ in the superpotential for L_ℓ.

For the sake of completeness, we also consider the case $y = 0$, where the intersection of L_ℓ with $(W^\vee)^{-1}(0) = \tilde{V} \cup E$ is simply ℓ. The argument in the proof of Corollary 7.8 then shows that holomorphic discs bounded by ℓ in $\tilde{V} \cup E$ lie entirely within H; however, there is a nontrivial contribution of Maslov index 2 configurations consisting of a constant disc together with a rational curve contained in E, namely the $\mathbf{P}^1$ fiber of the exceptional divisor over a point of $\ell \subset H$. (These exceptional spheres are actually the limits of the area ϵ discs discussed above as $y \to 0$.) □

Remark **7.11.** — The assumption that V is affine can be weakened somewhat: for Proposition 7.10 to hold it is sufficient to assume that the minimal Chern number of a rational curve contained in $\tilde{V}$ is at least 2. When this assumption does not hold, the discrepancy δ between the two superpotentials includes additional contributions from the enumerative geometry of rational curves of Chern number 1 in $\tilde{V}$.

Remark **7.12.** — The A_∞-functor from $\mathcal{F}(H)$ to $\mathcal{F}_s(X, W^\vee)$ is induced by a Lagrangian correspondence in the product $H \times X$, namely the set of all $(p, q) \in H \times X$ such that parallel transport of q by the gradient flow of $-\operatorname{Re} W^\vee$ converges to $p \in \operatorname{crit} W^\vee$. This Lagrangian correspondence is admissible with respect to $\mathrm{pr}_2^* W^\vee$, and weakly unobstructed with $\mathfrak{m}_0 = \delta$. While the Ma'u-Wehrheim-Woodward construction of A_∞-functors from Lagrangian correspondences [39] has not yet been developed in the setting considered here, it is certainly the right conceptual framework in which Corollary 7.8 should be understood.

By analogy with the case of Lefschetz fibrations [46], it is expected that the Fukaya category of a Morse-Bott fibration is generated by thimbles, at least under the assumption that the Fukaya category of the critical locus admits a resolution of the diagonal. The argument is expected to be similar to that in [46], except in the Morse-Bott case the key ingredient becomes the long exact sequence for fibered Dehn twists [53]. Thus, it is reasonable to expect that the A_∞-functor of Corollary 7.8 is in fact a quasi-equivalence.

Similar statements are also expected to hold for the *wrapped Fukaya category* of H and the *partially wrapped Fukaya category* of $(X, W^\vee)$ (twisted by s); however, this remains speculative, as the latter category has not been suitably constructed yet.

In any case, Corollary 7.8 and Proposition 7.10 motivate the terminology introduced in Definition 1.2.

Proof of Theorem 1.6. — While Theorem 7.1 provides an SYZ mirror to the Landau-Ginzburg model $(X, W^\vee)$, in light of the above discussion several adjustments are necessary in order to arrive at a generalized SYZ mirror to H.

(1) As noted in Remark 7.4, the restriction of ω_ϵ to $\operatorname{crit}(W^\vee)$ does not agree with the restriction of ω_V to H. To remedy this, in our main construction V should be equipped with a Kähler form in the class $[\omega_V] + \epsilon[H]$ rather than $[\omega_V]$. This ensures that the critical locus of $W^\vee$ is indeed isomorphic to H equipped with the restriction of the Kähler form ω_V.

(2) In light of Corollary 7.8, the A-side Landau-Ginzburg model $(X, W^\vee)$ should be twisted by the background class $s = \mathrm{PD}([\tilde{V}]) \in H^2(X, \mathbf{Z}/2)$. Namely, the tori we consider in our main argument should be viewed as objects of $\mathcal{F}_s(X, W^\vee)$ rather than $\mathcal{F}(X, W^\vee)$. This modifies the sign conventions for counting discs and hence the mirror superpotential.

(3) By Proposition 7.10, the additive constant $\delta = T^{\epsilon}$ should be subtracted from the superpotential, since the natural A_∞-functor from $\mathcal{F}(H)$ to $\mathcal{F}_s(X, W^\vee)$ increases $\mathfrak{m}_0$ by that amount.

Thus, the mirror space remains the toric variety Y, but the superpotential is no longer

$$W_0 = w_0 + \sum_{i=1}^{r} T^{\varpi_i} \mathbf{v}_{\alpha_i}^{\sigma_i}; \tag{7.2}$$

we now make explicit how each of the above changes affects the potential.

Replacing $[\omega_V]$ by $[\omega_V] + \epsilon[H]$ amounts to changing the equations of the facets of the moment polytope Δ_V from $\langle \sigma_i, \cdot \rangle + \varpi_i = 0$ to $\langle \sigma_i, \cdot \rangle + \varpi_i + \epsilon\lambda(\sigma_i) = 0$ (where $\lambda : \Sigma_V \to \mathbf{R}$ is the piecewise linear function defining $\mathcal{L} = \mathcal{O}(H)$). Accordingly, each exponent ϖ_i in (7.2) should be changed to $\varpi_i + \epsilon\lambda(\sigma_i)$.

Next, we twist by the background class $s = \mathrm{PD}([\tilde{V}])$, and view the tori studied in Section 5 as objects of $\mathcal{F}_s(X, W^\vee)$ rather than $\mathcal{F}(X, W^\vee)$. Specifically, s lifts to a class in $H^2(X, L; \mathbf{Z}/2)$ (dual to $[\tilde{V}] \in H_{2n}(X \setminus L)$), and twisting the standard spin structure by this lift of s yields a relatively spin structure on L. By [20, Proposition 8.1.16], this twist affects the signed count of holomorphic discs in a given class $\beta \in \pi_2(X, L)$ by a factor of $(-1)^k$ where $k = \beta \cdot [\tilde{V}]$. Recall from Section 6 that, of the various families of holomorphic discs that contribute to the superpotential, the only ones that intersect $\tilde{V}$ are those described by Lemma 5.5; thus the only effect of the twisting by the background class s is to change the first term of W_0 from w_0 to $-w_0$.

Finally, we subtract $\delta = T^\epsilon$ from the superpotential, and find that the appropriate superpotential to consider on Y is given by

$$W_0' = -T^\epsilon - w_0 + \sum_{i=1}^{r} T^{\varpi_i + \epsilon\lambda(\sigma_i)} \mathbf{v}_{\alpha_i}^{\sigma_i} = -T^\epsilon v_0 + \sum_{i=1}^{r} T^{\varpi_i} T^{\epsilon\lambda(\sigma_i)} \mathbf{v}_{\alpha_i}^{\sigma_i}.$$

Finally, recall from Section 3.3 that the weights of the toric monomials v_0 and $\mathbf{v}_{\alpha_i}^{\sigma_i}$ are respectively $(0, 1)$ and $(-\sigma_i, \lambda(\sigma_i)) \in \mathbf{Z}^n \oplus \mathbf{Z}$. Therefore, a rescaling of the last coordinate by a factor of T^ϵ changes v_0 to $T^\epsilon v_0$ and $\mathbf{v}_{\alpha_i}^{\sigma_i}$ to $T^{\epsilon\lambda(\sigma_i)} \mathbf{v}_{\alpha_i}^{\sigma_i}$. This change of variables eliminates the dependence on ϵ (as one would expect for the mirror to H) and replaces W_0' by the simpler expression

$$-v_0 + \sum_{i=1}^{r} T^{\varpi_i} \mathbf{v}_{\alpha_i}^{\sigma_i},$$

which is exactly W_0^H (see Definition 3.10). □

Remark **7.13.** — Another way to produce an A_∞-functor from the Fukaya category of H to that of X (more specifically, the idempotent closure of $\mathcal{F}_s(X)$) is the following construction considered by Ivan Smith in [50, Section 4.5].

Given a Lagrangian submanifold $\ell \subset \mathrm{H}$, first lift it to the boundary of the ϵ-tubular neighborhood of H inside V, to obtain a Lagrangian submanifold $\mathrm{C}_\ell \subset \mathrm{V}$ which is a circle bundle over ℓ; then, identifying V with the reduced space $\mathrm{X}_{red,\epsilon} = \mu_{\mathrm{X}}^{-1}(\epsilon)/\mathrm{S}^1$, lift C_ℓ to $\mu_{\mathrm{X}}^{-1}(\epsilon)$ and "spin" it by the S^1-action, to obtain a Lagrangian submanifold $\mathrm{T}_\ell \subset \mathrm{X}$ which is a T^2-bundle over ℓ. Then T_ℓ formally splits into a direct sum $\mathrm{T}_\ell^+ \oplus \mathrm{T}_\ell^-$; the A_∞-functor is constructed by mapping ℓ to either summand.

The two constructions are equivalent: in $\mathcal{F}_s(\mathrm{X}, \mathrm{W}^\vee)$ the summands $\mathrm{T}_\ell^\pm$ are isomorphic to the thimble L_ℓ (up to a shift). One benefit of Smith's construction is that, unlike L_ℓ, the Lagrangian submanifold T_ℓ is entirely contained inside X^0, which makes its further study amenable to T-duality arguments involving X^0 and Y^0.

8. The converse construction

As a consequence of Theorem 1.7, the mirror Y^0 of X^0 can be defined as a variety not only over the Novikov field, but also over the complex numbers. In this section, we impose the maximal degeneration condition (cf. Definition 3.1) which implies that Y^0 is smooth. We then reverse our viewpoint from the preceding discussion: treating T as a numerical parameter and equipping Y^0 with a Kähler form, we shall reconstruct X^0 (as an analytic space that also happens to be defined over complex numbers) as an SYZ mirror. Along the way, we also obtain another perspective on how compactifying Y^0 to the toric variety Y amounts to equipping X^0 with a superpotential. We omit any discussion of Y or Y^0 equipped with A-side Landau-Ginzburg models, which would require a deeper understanding of the corresponding Fukaya categories.

(Note: many of the results in this section were also independently obtained by Chan, Lau and Leung [12].)

To begin our construction, observe that $\mathrm{Y}^0 = \mathrm{Y} \setminus w_0^{-1}(0)$ carries a natural T^n-action, given in the coordinates introduced in Section 3.3 by

$$\left(e^{i\theta_1}, \ldots, e^{i\theta_n}\right) \cdot (v_{\alpha,1}, \ldots, v_{\alpha,n}, v_{\alpha,0}) = \left(e^{i\theta_1} v_{\alpha,1}, \ldots, e^{i\theta_n} v_{\alpha,n}, v_{\alpha,0}\right).$$

This torus is a subgroup of the $(n+1)$-dimensional torus which acts on the toric variety Y, namely the stabilizer of the regular function $w_0 = -\mathrm{T}^\epsilon + \mathrm{T}^\epsilon v_0$.

We equip Y^0 with a T^n-invariant Kähler form ω_{Y}. To make things concrete, take ω_{Y} to be the restriction of a complete toric Kähler form on Y, with moment polytope

$$\Delta_{\mathrm{Y}} = \left\{ (\xi, \eta) \in \mathbf{R}^n \oplus \mathbf{R} \,\middle|\, \eta \geq \varphi(\xi) = \max_{\alpha \in \mathrm{A}} \big(\langle \alpha, \xi \rangle - \rho(\alpha) \big) \right\}$$

(cf. (3.8)). We denote by $\tilde{\mu}_{\mathrm{Y}} : \mathrm{Y} \to \mathbf{R}^{n+1}$ the moment map for the T^{n+1}-action on Y, and by $\mu_{\mathrm{Y}} : \mathrm{Y}^0 \to \mathbf{R}^n$ the moment map for the T^n-action on Y^0. Observing that μ_{Y} is obtained from $\tilde{\mu}_{\mathrm{Y}}$ by restricting to Y^0 and projecting to the first n components, the critical locus of μ_{Y} is the union of all codimension 2 toric strata, and the set of critical values of μ_{Y} is precisely the tropical hypersurface $\Pi_0 \subset \mathbf{R}^n$ defined by φ. Finally, we also equip Y^0 with the T^n-invariant holomorphic $(n+1)$-form given in each chart by

$$\Omega_Y = d\log v_{\alpha,1} \wedge \cdots \wedge d\log v_{\alpha,n} \wedge d\log w_0.$$

Note that this holomorphic volume form scales with ϵ.

Lemma **8.1.** — *The map* $\pi_Y = (\mu_Y, |w_0|) : Y^0 \to B_Y = \mathbf{R}^n \times \mathbf{R}_+$ *defines a* T^n*-invariant special Lagrangian torus fibration on* Y^0. *Moreover,* $\pi_Y^{-1}(\xi, r)$ *is singular if and only if* $(\xi, r) \in \Pi_0 \times \{T^\epsilon\}$, *and obstructed if and only if* $r = T^\epsilon$.

This fibration is analogous to some of the examples considered in [10, 11, 23, 24]; see also Example 3.3.1 in [7].

The statement that $\pi_Y^{-1}(\xi, r)$ is special Lagrangian follows immediately from the observation that Ω_Y descends to the holomorphic 1-form $d\log w_0$ on the reduced space $\mu_Y^{-1}(\xi)/T^n \simeq \mathbf{C}^*$; thus the circle $|w_0| = r$ is special Lagrangian in the reduced space, and its lift to $\mu_Y^{-1}(\xi)$ is special Lagrangian in Y^0.

A useful way to think of these tori is to consider the projection of Y^0 to the coordinate w_0, whose fibers are all isomorphic to $(\mathbf{C}^*)^n$ except for $w_0^{-1}(-T^\epsilon) = v_0^{-1}(0)$ which is the union of all toric strata in Y. In this projection, $\pi_Y^{-1}(\xi, r)$ fibers over the circle of radius r centered at the origin, and intersects each of the fibers $w_0^{-1}(re^{i\theta})$ in a standard product torus (corresponding to the level ξ of the moment map). In particular, $\pi_Y^{-1}(\xi, r)$ is singular precisely when $r = T^\epsilon$ and $\xi \in \Pi_0$.

By the maximum principle, any holomorphic disc in Y^0 bounded by $\pi_Y^{-1}(\xi, r)$ must lie entirely within a fiber of the projection to w_0. Since the regular fibers of w_0 are isomorphic to $(\mathbf{C}^*)^n$, inside which product tori do not bound any nonconstant holomorphic discs, $\pi_Y^{-1}(\xi, r)$ is tautologically unobstructed for $r \neq T^\epsilon$. When $r = T^\epsilon$, $\pi_Y^{-1}(\xi, r)$ intersects one of the components of $w_0^{-1}(-T^\epsilon)$ (i.e. one of the toric divisors of Y) in a product torus, which bounds various families of holomorphic discs as well as configurations consisting of holomorphic discs and rational curves in the toric strata. This completes the proof of Lemma 8.1.

The maximum principle applied to w_0 also implies that every rational curve in Y is contained in $w_0^{-1}(-T^\epsilon)$ (i.e. the union of all toric strata), hence disjoint from the anticanonical divisor $w_0^{-1}(0)$, and thus satisfies $c_1(Y) \cdot C = 0$; in fact Y is a toric Calabi-Yau variety. So Assumption 2.2 holds, and partially compactifying Y^0 to Y does not modify the enumerative geometry of Maslov index 0 discs bounded by the fibers of π_Y. Hence the SYZ mirror of Y is just the mirror of Y^0 equipped with an appropriate superpotential, and we determine both at the same time.

The wall $r = T^\epsilon$ divides the fibration $\pi_Y : Y^0 \to B_Y$ into two chambers; accordingly, the SYZ mirror of Y^0 (and Y) is constructed by gluing together two coordinate charts U' and U'' via a transformation which accounts for the enumerative geometry of discs bounded by the potentially obstructed fibers of π_{Y}. We now define coordinate systems for both charts and determine the superpotential (for the mirror of Y) in terms of those coordinates. For notational consistency and to avoid confusion, we now denote by τ (rather than T) the Novikov parameter recording areas with respect to ω_{Y}.

We start with the chamber $r > \mathrm{T}^{\epsilon}$, over which the fibers of π_{Y} can be deformed into product tori in Y (i.e., orbits of the T^{n+1}-action) by a Hamiltonian isotopy that does not intersect $w_0^{-1}(-\mathrm{T}^{\epsilon})$ (from the perspective of the projection to w_0, the isotopy amounts simply to deforming the circle of radius r centered at 0 to a circle of the appropriate radius centered at $-\mathrm{T}^{\epsilon}$).

Fix a reference fiber $\mathrm{L}^0 = \pi_{\mathrm{Y}}^{-1}(\xi^0, r^0)$, where $\xi^0 \in \mathbf{R}^n$ and $r^0 > \mathrm{T}^{\epsilon}$, and choose a basis $(\gamma_1, \dots, \gamma_n, \gamma_0')$ of $\mathrm{H}_1(\mathrm{L}^0, \mathbf{Z})$, where $-\gamma_1, \dots, -\gamma_n$ correspond to the factors of the T^n-action on L^0, and $-\gamma_0'$ corresponds to an orbit of the last S^1 factor of T^{n+1} acting on a product torus $\tilde{\mu}_{\mathrm{Y}}^{-1}(\xi^0, \eta^0)$ which is Hamiltonian isotopic to L^0 in Y. (The signs are motivated by consistency with the notations used for X^0.)

A point of the chart U′ mirror to the chamber $\{r > \mathrm{T}^{\epsilon}\}$ corresponds to a pair (L, ∇), where $\mathrm{L} = \pi_{\mathrm{Y}}^{-1}(\xi, r)$ is a fiber of π_{Y} (with $r > \mathrm{T}^{\epsilon}$), Hamiltonian isotopic to a product torus $\tilde{\mu}_{\mathrm{Y}}^{-1}(\xi, \eta)$ in Y, and $\nabla \in \hom(\pi_1(\mathrm{L}), \mathrm{U}_{\mathbf{K}})$. We rescale the coordinates given by (2.3) to eliminate the dependence on the base point (ξ^0, r^0), i.e. we identify U′ with a domain in $(\mathbf{K}^*)^{n+1}$ via

$$(\mathbf{8.1}) \qquad (\mathrm{L}, \nabla) \mapsto \left(x_1', \dots, x_n', z'\right) = \left(\tau^{-\xi_1} \nabla(\gamma_1), \dots, \tau^{-\xi_n} \nabla(\gamma_n), \tau^{-\eta} \nabla\left(\gamma_0'\right)\right).$$

(Compare with (2.3), noting that $-\xi_i = -\xi_i^0 + \int_{\Gamma_i} \omega_{\mathrm{Y}}$ and $-\eta = -\eta^0 + \int_{\Gamma_0'} \omega_{\mathrm{Y}}$.)

Lemma **8.2.** — *In the chart* U′*, the superpotential for the mirror to* Y *is given by*

$$(\mathbf{8.2}) \qquad \mathrm{W}^{\vee}\left(x_1', \dots, x_n', z'\right) = \sum_{\alpha \in \mathrm{A}} (1 + \kappa_\alpha) \tau^{\rho(\alpha)} x_1'^{\alpha_1} \cdots x_n'^{\alpha_n} z'^{-1},$$

where $\kappa_\alpha \in \mathbf{K}$ *are constants with* $\operatorname{val}(\kappa_\alpha) > 0$.

Proof. — Consider a point $(\mathrm{L}, \nabla) \in \mathrm{U}'$, where $\mathrm{L} = \pi_{\mathrm{Y}}^{-1}(\xi, r)$ is Hamiltonian isotopic to the product torus $\mathrm{L}' = \tilde{\mu}_{\mathrm{Y}}^{-1}(\xi, \eta)$ in Y. As explained above, the isotopy can be performed without intersecting the toric divisors of Y, i.e. without wall-crossing; therefore, the isotopy provides a cobordism between the moduli spaces of Maslov index 2 holomorphic discs bounded by L and L′ in Y.

It is well-known that the families of Maslov index 2 holomorphic discs bounded by the standard product torus L′ in the toric manifold Y are in one-to-one correspondence with the codimension 1 toric strata of Y. Namely, for each codimension 1 stratum, there is a unique family of holomorphic discs which intersect this stratum transversely at a single point and do not intersect any of the other strata. Moreover, every point of L′ lies on the boundary of exactly one disc of each family, and these discs are all regular [15] (see also [6, §4]).

The toric divisors of Y, or equivalently the facets of Δ_Y, are in one-to-one correspondence with the elements of A. The symplectic area of a Maslov index 2 holomorphic disc in (Y, L') which intersects the divisor corresponding to $\alpha \in A$ (and whose class we denote by β_α) is equal to the distance from the point (ξ, η) to that facet of Δ_Y, namely $\eta - \langle \alpha, \xi \rangle + \rho(\alpha)$, whilst the boundary of the disc represents the class $\partial\beta_\alpha = \sum \alpha_i \gamma_i - \gamma_0' \in H_1(L', \mathbf{Z})$. The weight associated to such a disc is therefore

$$\begin{aligned} z_{\beta_\alpha}\big(L', \nabla\big) &= \tau^{\eta - \langle \alpha, \xi \rangle + \rho(\alpha)} \nabla(\gamma_1)^{\alpha_1} \cdots \nabla(\gamma_n)^{\alpha_n} \nabla\big(\gamma_0'\big)^{-1} \\ &= \tau^{\rho(\alpha)} {x_1'}^{\alpha_1} \cdots {x_n'}^{\alpha_n} z'^{-1}. \end{aligned}$$

Using the isotopy between L and L', we conclude that the contributions of Maslov index 2 holomorphic discs in (Y, L) to the superpotential $W^\vee$ add up to

$$\sum_{\alpha \in A} z_{\beta_\alpha}(L, \nabla) = \sum_{\alpha \in A} \tau^{\rho(\alpha)} {x_1'}^{\alpha_1} \cdots {x_n'}^{\alpha_n} z'^{-1}.$$

However, the superpotential $W^\vee$ also includes contributions from (virtual) counts of stable genus 0 configurations of discs and rational curves of total Maslov index 2. These configurations consist of a single Maslov index 2 disc (in one of the above families) together with one or more rational curves contained in the toric divisors of Y (representing a total class $C \in H_2(Y, \mathbf{Z})$). The enumerative invariant $n(L, \beta_\alpha + C)$ giving the (virtual) count of such configurations whose boundary passes through a generic point of L can be understood in terms of genus 0 Gromov-Witten invariants of suitable partial compactifications of Y (see e.g. [12]). However, all that matters to us is the general form of the corresponding terms of the superpotential. Since the rational components contribute a multiplicative factor $\tau^{[\omega_Y]\cdot C}$ to the weight, we obtain that

$$W^\vee = \sum_{\alpha \in A} \Bigg(1 + \sum_{\substack{C \in H_2(Y, \mathbf{Z}) \\ [\omega_Y]\cdot C > 0}} n(L, \beta_\alpha + C) \tau^{[\omega_Y]\cdot C} \Bigg) \tau^{\rho(\alpha)} {x_1'}^{\alpha_1} \cdots {x_n'}^{\alpha_n} z'^{-1},$$

which is of the expected form (8.2). □

Next we look at the other chart U'', which corresponds to the chamber $r < T^\epsilon$ of the fibration π_Y. Fix again a reference fiber $L^0 = \pi_Y^{-1}(\xi^0, r^0)$, where $\xi^0 \in \mathbf{R}^n$ and $r^0 < T^\epsilon$, and choose a basis $(\gamma_1, \ldots, \gamma_n, \gamma_0'')$ of $H_1(L^0, \mathbf{Z})$, where $-\gamma_1, \ldots, -\gamma_n$ correspond to the factors of the T^n-action on L^0, and γ_0'' can be represented by a loop in L^0 over which w_0 runs counterclockwise around the circle of radius r^0 while $v_{\alpha,1}, \ldots, v_{\alpha,n} \in \mathbf{R}_+$ (for some arbitrary choice of α). Note that the fibration $w_0 : Y \to \mathbf{C}$ is trivial over the disc of radius r^0; in fact the coordinates $(w_0, v_{\alpha,1}, \ldots, v_{\alpha,n})$ (for any α) give a biholomorphism from the subset $\{|w_0| \leq r^0\}$ of Y to $D^2(r^0) \times (\mathbf{C}^*)^n$. Then γ_0'' can be characterized as the unique element of $H_1(L^0, \mathbf{Z})$ which arises as the boundary of a section of $w_0 : Y \to \mathbf{C}$ over the disc of radius r^0; we denote by β_0 the relative homotopy class of this section.

A point of U'' corresponds to a pair (L, ∇) where $L = \pi_Y^{-1}(\xi, r)$ is a fiber of π_Y (with $r < T^{\epsilon}$), and $\nabla \in \hom(\pi_1(L), U_{\mathbf{K}})$. As before, we rescale the coordinates given by (2.3) to eliminate the dependence on the base point (ξ^0, r^0), i.e. we identify U'' with a domain in $(\mathbf{K}^*)^{n+1}$ via

$$(L, \nabla) \mapsto \big(x_1'', \dots, x_n'', y''\big) = \big(\tau^{-\xi_1}\nabla(\gamma_1), \dots, \tau^{-\xi_n}\nabla(\gamma_n), \tau^{[\omega_Y]\cdot\beta_0}\nabla\big(\gamma_0''\big)\big). \tag{8.3}$$

Lemma **8.3.** — *In the chart* U''*, the superpotential for the mirror to* Y *is given by*

$$W^{\vee}\big(x_1'', \dots, x_n'', y''\big) = y''. \tag{8.4}$$

Proof. — By the maximum principle applied to the projection to w_0, any holomorphic disc bounded by $L = \pi_Y^{-1}(\xi, r)$ in Y must be contained in the subset $\{|w_0| \leq r\} \subset Y$, which is diffeomorphic to $D^2 \times (\mathbf{C}^*)^n$. Thus, for topological reasons, any holomorphic disc bounded by L must represent a multiple of the class β_0. Since the Maslov index is equal to twice the intersection number with $w_0^{-1}(0)$, Maslov index 2 discs are holomorphic sections of $w_0 : Y \to \mathbf{C}$ over the disc of radius r, representing β_0.

The formula (8.4) now follows from the claim that the number of such sections passing through a given point of L is $n(L, \beta_0) = 1$. This can be viewed as an enumerative problem for holomorphic sections of a trivial Lefschetz fibration with a Lagrangian boundary condition, easily answered by applying the powerful methods of [45, §2]. An alternative, more elementary approach is to deform ω_Y among toric Kähler forms in its cohomology class to ensure that, for some $\xi^0 \in \mathbf{R}^n$, $\mu_Y^{-1}(\xi^0)$ is given in one of the coordinate charts Y_α of Section 3.3 by equations of the form $|v_{\alpha,1}| = \rho_1, \dots, |v_{\alpha,n}| = \rho_n$. (In fact, many natural choices for ω_Y cause this property to hold immediately.) When this property holds, the maximum principle applied to $v_{\alpha,1}, \dots, v_{\alpha,n}$ implies that the holomorphic Maslov index 2 discs bounded by $L^0 = \pi_Y^{-1}(\xi^0, r^0)$ are given by letting w_0 vary in the disc of radius r^0 while the other coordinates $v_{\alpha,1}, \dots, v_{\alpha,n}$ are held constant. All these discs are regular, and there is precisely one disc passing through each point of L^0. It follows that $n(L^0, \beta_0) = 1$. This completes the proof, since the invariant $n(L^0, \beta_0)$ is not affected by the deformation of ω_Y to the special case we have considered, and the value of $n(L, \beta_0)$ is the same for all the fibers of π_Y over the chamber $r < T^{\epsilon}$. $\square$

We can now formulate and prove the main result of this section:

Theorem **8.4.** — *The rigid analytic manifold*

$$\mathcal{X}^0 = \big\{(x_1, \dots, x_n, y, z) \in \big(\mathbf{K}^*\big)^n \times \mathbf{K}^2 \,\big|\, yz = \tilde{f}(x_1, \dots, x_n)\big\}, \tag{8.5}$$

where $\tilde{f}(x_1, \dots, x_n) = \sum_{\alpha \in A}(1 + \kappa_\alpha)\tau^{\rho(\alpha)}x_1^{\alpha_1}\cdots x_n^{\alpha_n}$*, is SYZ mirror to* (Y^0, ω_Y).

Moreover, the B*-side Landau-Ginzburg model* $(\mathcal{X}^0, W^{\vee} = y)$ *is SYZ mirror to* (Y, ω_Y).

Proof. — The two charts U' and U'' are glued to each other by a coordinate transformation which accounts for the Maslov index 0 holomorphic discs bounded by the potentially obstructed fibers of π_Y. There are many families of such discs, all contained in $w_0^{-1}(-T^\epsilon) = v_0^{-1}(0)$. However we claim that the first n coordinates of the charts (8.1) and (8.3) are not affected by these instanton corrections, so that the gluing satisfies $x_1'' = x_1', \dots, x_n'' = x_n'$.

One way to argue is based on the observation that all Maslov index 0 configurations are contained in $w_0^{-1}(-T^\epsilon)$. Consider as in Section 2.1 a Lagrangian isotopy $\{L_t\}_{t\in[0,1]}$ between fibers of π_Y in the two chambers (with L_{t_0} the only potentially obstructed fiber), and the cycles $C_\alpha = ev_*[\mathcal{M}_1(\{L_{t_0}\}, \alpha)] \in H_{n-1}(L_{t_0})$ corresponding to the various classes $\alpha \in \pi_2(Y, L_t)$ that may contain Maslov index 0 configurations. The fact that each C_α is supported on $L_{t_0} \cap w_0^{-1}(-T^\epsilon)$ implies readily that $C_\alpha \cdot \gamma_1 = \cdots = C_\alpha \cdot \gamma_n = 0$. Since the overall gluing transformation is given by a composition of elementary transformations of the type (2.4), the first n coordinates are not affected.

By Corollary A.11, a more down-to-earth way to see that the gluing preserves $x_i'' = x_i'$ $(i = 1, \dots, n)$ is to consider the partial compactification Y_i' of Y^0 given by the moment polytope $\Delta_Y \cap \{\xi_i \le K\}$ for some constant $K \gg 0$ (still removing $w_0^{-1}(0)$ from the resulting toric variety). From the perspective of the projection $w_0 : Y^0 \to \mathbf{C}^*$, this simply amounts to a toric partial compactification of each fiber, where the generic fiber $(\mathbf{C}^*)^n$ is partially compactified along the i-th factor to $(\mathbf{C}^*)^{n-1} \times \mathbf{C}$. The Maslov index 2 holomorphic discs bounded by $L = \pi_Y^{-1}(\xi, r)$ inside Y_i' are contained in the fibers of w_0 by the maximum principle; requiring that the boundary of the disc pass through a given point $p \in L$ (where we assume $w_0 \neq -T^\epsilon$), we are reduced to the fiber of w_0 containing p, which L intersects in a standard product torus $(S^1)^n \subset (\mathbf{C}^*)^{n-1} \times \mathbf{C}$ (where the radii of the various S^1 factors depend on ξ). Thus, there is exactly one Maslov index 2 holomorphic disc in (Y_i', L) through a generic point $p \in L$ (namely a disc over which all coordinates except the i-th one are constant). The superpotential is equal to the weight of this disc, i.e. $\tau^{K-\xi_i}\nabla(\gamma_i)$, which can be rewritten as $\tau^K x_i'$ if $r > T^\epsilon$, and $\tau^K x_i''$ if $r < T^\epsilon$. Comparing these two expressions, we see that the gluing between U' and U'' identifies $x_i' = x_i''$.

The gluing transformation between the coordinates y'' and z' is more complicated, but is now determined entirely by a comparison between (8.2) and (8.4): since the two formulas for $W^\vee$ must glue to a regular function on the mirror, y'' must equal the right-hand side of (8.2), hence

$$y''z' = \sum_{\alpha\in A}(1+\kappa_\alpha)\tau^{\rho(\alpha)}x_1'^{\alpha_1}\cdots x_n'^{\alpha_n} = \tilde{f}\big(x_1', \dots, x_n'\big).$$

This completes the proof of the theorem. ∎

The first part of Theorem 8.4 is a statement of SYZ mirror symmetry in the opposite direction from Theorem 1.7; the two results taken together relate the symplectic topology and algebraic geometry of the spaces X^0 and Y^0 to each other. More precisely, we would like to treat τ as a fixed complex number and view the mirror of (Y^0, ω_Y) as a complex manifold. The convergence of the function $\tilde{f}$ depends only on that of the constants κ_α, which is unknown in general but holds in practice for a number of examples (see [12] and other work by the same authors). Even when convergence is not an issue, the result reveals the need for care in constructing the mirror map: while our main construction is essentially independent of the coefficients c_α appearing in (3.1) (which do not affect the symplectic geometry of X^0), the direction considered here requires the complex structure of X^0 to be chosen carefully to match with the Kähler class $[\omega_Y]$, specifically we have to take $c_\alpha = 1 + \kappa_\alpha$.

The second part of Theorem 8.4 gives a mirror symmetric interpretation of the partial compactification of Y^0 to Y, in terms of equipping X^0 with the superpotential $W^\vee = y$. From the perspective of our main construction (viewing X^0 as a symplectic manifold and Y^0 as its SYZ mirror), we saw the same phenomenon in Section 7.

9. Examples

9.1. *Hyperplanes and pairs of pants.* — We consider as our first example the (higher dimensional) pair of pants H defined by the equation

$$x_1 + \cdots + x_n + 1 = 0 \tag{9.1}$$

in $V = (\mathbf{C}^*)^n$. (The case $n = 2$ corresponds to the ordinary pair of pants; in general H is the complement of $n+1$ hyperplanes in general position in $\mathbf{CP}^{n-1}$.)

The tropical polynomial corresponding to (9.1) is $\varphi(\xi) = \max(\xi_1, \dots, \xi_n, 0)$; the polytope Δ_Y defined by (3.8) is equivalent via $(\xi_1, \dots, \xi_n, \eta) \mapsto (\eta - \xi_1, \dots, \eta - \xi_n, \eta)$ to the orthant $(\mathbf{R}_{\geq 0})^{n+1} \subset \mathbf{R}^{n+1}$. Thus $Y \simeq \mathbf{C}^{n+1}$. In terms of the coordinates $(z_1, \dots, z_{n+1})$ of $\mathbf{C}^{n+1}$, the monomial v_0 is given by $v_0 = z_1 \cdots z_{n+1}$. Thus, in this example our main results are:

(1) the open Calabi-Yau manifold $Y^0 = \mathbf{C}^{n+1} \setminus \{z_1 \cdots z_{n+1} = 1\}$ is SYZ mirror to the conic bundle $X^0 = \{(x_1, \dots, x_n, y, z) \in (\mathbf{C}^*)^n \times \mathbf{C}^2 \mid yz = x_1 + \cdots + x_n + 1\}$;

(2) the B-side Landau-Ginzburg model $(Y^0, W_0 = -T^\epsilon + T^\epsilon z_1 \cdots z_{n+1})$ is SYZ mirror to the blowup X of $(\mathbf{C}^*)^n \times \mathbf{C}$ along $H \times 0$, where

$$H = \left\{(x_1, \dots, x_n) \in \left(\mathbf{C}^*\right)^n \,\middle|\, x_1 + \cdots + x_n + 1 = 0\right\};$$

(3) the B-side Landau-Ginzburg model $(\mathbf{C}^{n+1}, W_0^H = -z_1 \cdots z_{n+1})$ is a generalized SYZ mirror of H.

The last statement in particular has been verified in the sense of homological mirror symmetry by Sheridan [49]; see also [3] for a more detailed result in the case $n = 2$ (the usual pair of pants).

If instead we consider the same Equation (9.1) to define (in an affine chart) a hyperplane $\mathrm{H} \simeq \mathbf{CP}^{n-1}$ inside $\mathrm{V} = \mathbf{CP}^n$, with a Kähler form such that $\int_{\mathbf{CP}^1} \omega_{\mathrm{V}} = \mathrm{A}$, then our main result becomes that the B-side Landau-Ginzburg model consisting of $\mathrm{Y}^0 = \mathbf{C}^{n+1} \setminus \{z_1 \cdots z_{n+1} = 1\}$ equipped with the superpotential

$$\mathrm{W}_0 = -\mathrm{T}^\epsilon + \mathrm{T}^\epsilon z_1 \cdots z_{n+1} + z_1 + \cdots + z_n + \mathrm{T}^{\mathrm{A}} z_{n+1}$$

is SYZ mirror to the blowup X of $\mathbf{CP}^n \times \mathbf{C}$ along $\mathrm{H} \times 0 \simeq \mathbf{CP}^{n-1} \times 0$.

Even though $\mathbf{CP}^{n-1}$ is not affine, Theorem 1.6 still holds for this example if we assume that $n \geq 2$, by Remark 7.11. In this case, the mirror we obtain for $\mathbf{CP}^{n-1}$ (viewed as a hyperplane in $\mathbf{CP}^n$) is the B-side Landau-Ginzburg model

$$\big(\mathbf{C}^{n+1}, \mathrm{W}_0^{\mathrm{H}} = -z_1 \cdots z_{n+1} + z_1 + \cdots + z_n + \mathrm{T}^{\mathrm{A}} z_{n+1}\big).$$

Rewriting the superpotential as

$$\begin{aligned} \mathrm{W}_0^{\mathrm{H}} &= z_1 + \cdots + z_n + z_{n+1}\big(\mathrm{T}^{\mathrm{A}} - z_1 \cdots z_n\big) \\ &= \tilde{\mathrm{W}}(z_1, \ldots, z_n) + z_{n+1} g(z_1, \ldots, z_n) \end{aligned}$$

makes it apparent that this B-side Landau-Ginzburg model is equivalent (e.g. in the sense of Orlov's generalized Knörrer periodicity [43]) to the B-side Landau-Ginzburg model consisting of $g^{-1}(0) = \{(z_1, \ldots, z_n) \in \mathbf{C}^n \mid z_1 \cdots z_n = \mathrm{T}^{\mathrm{A}}\}$ equipped with the superpotential $\tilde{\mathrm{W}} = z_1 + \cdots + z_n$, which is the classical toric mirror of $\mathbf{CP}^{n-1}$.

9.2. *ALE spaces.* — Let $\mathrm{V} = \mathbf{C}$, and let $\mathrm{H} = \{x_1, \ldots, x_{k+1}\} \subset \mathbf{C}^*$ consist of $k+1$ points, $k \geq 0$, with $|x_1| \ll \cdots \ll |x_{k+1}|$ (so that the defining polynomial of H, $f_{k+1}(x) = (x - x_1) \cdots (x - x_{k+1}) \in \mathbf{C}[x]$, is near the tropical limit).

The conic bundle $\mathrm{X}^0 = \{(x, y, z) \in \mathbf{C}^* \times \mathbf{C}^2 \mid yz = f_{k+1}(x)\}$ is the complement of the regular conic $x = 0$ in the A_k-Milnor fiber

$$\mathrm{X}' = \big\{(x, y, z) \in \mathbf{C}^3 \,\big|\, yz = f_{k+1}(x)\big\}.$$

In fact, X' is the main space of interest here, rather than its open subset X^0 or its partial compactification X (note that $\mathrm{X}' = \mathrm{X} \setminus \tilde{\mathrm{V}}$). However the mirror of X' differs from that of X simply by excluding the term w_0 (which accounts for those holomorphic discs that intersect $\tilde{\mathrm{V}}$) from the mirror superpotential.

The tropical polynomial $\varphi : \mathbf{R} \to \mathbf{R}$ corresponding to f_{k+1} is a piecewise linear function whose slope takes the successive integer values $0, 1, \ldots, k+1$. Thus the toric variety Y determined by the polytope $\Delta_{\mathrm{Y}} = \{(\xi, \eta) \in \mathbf{R}^2 \mid \eta \geq \varphi(\xi)\}$ is the resolution of the A_k singularity $\{st = u^{k+1}\} \subset \mathbf{C}^3$. The $k+2$ edges of Δ_{Y} correspond to the toric strata of Y, namely the proper transforms of the coordinate axes $s = 0$ and $t = 0$ and the k rational (-2)-curves created by the resolution. Specifically, Y is covered by $k+1$ affine coordinate charts U_α with coordinates $(s_\alpha = v_{\alpha,1}, t_\alpha = v_{\alpha+1,1}^{-1})$, $0 \leq \alpha \leq k$; denoting the toric coordinate $v_{\alpha,0}$ by u, Equation (3.9) becomes $s_\alpha t_\alpha = u$, and the regular functions $s = s_0$, $t = t_k$, $u \in \mathcal{O}(Y)$ satisfy the relation $st = u^{k+1}$.

Since $w_0 = -T^\epsilon + T^\epsilon v_0 = -T^\epsilon + T^\epsilon u$, the space Y^0 is the complement of the curve $u = 1$ inside Y. With this understood, our main results become:

(1) the complement Y^0 of the curve $u = 1$ in the resolution Y of the A_k singularity $\{st = u^{k+1}\} \subset \mathbf{C}^3$ is SYZ mirror to the complement X^0 of the curve $x = 0$ in the Milnor fiber $X' = \{(x, y, z) \in \mathbf{C}^3 \mid yz = f_{k+1}(x)\}$ of the A_k singularity;
(2) the B-side Landau-Ginzburg model $(Y^0, W_0 = s)$ is SYZ mirror to X';
(3) the Landau-Ginzburg models $(Y, W_0 = s)$ and $(X', W^\vee = y)$ are SYZ mirror to each other.

These results show that the oft-stated mirror symmetry relation between the smoothing and the resolution of the A_k singularity (or, specializing to the case $k = 1$, between the affine quadric T^*S^2 and the total space of the line bundle $\mathcal{O}(-2) \to \mathbf{P}^1$) needs to be corrected either by removing smooth curves from each side, or by equipping both sides with superpotentials.

One final comment that may be of interest to symplectic geometers is that $W_0 = s$ vanishes to order $k + 1$ along the t coordinate axis, and to orders $1, 2, \dots, k$ along the exceptional curves of the resolution. The higher derivatives of the superpotential encode information about the A_∞-products on the Floer cohomology of the Lagrangian torus fiber of the SYZ fibration, and the high-order vanishing of W_0 along the toric divisors of Y^0 indicates that the A_k Milnor fiber contains Lagrangian tori whose Floer cohomology is isomorphic to the usual cohomology of T^2 as an algebra, but carries non-trivial A_∞-operations. (See also [38] for related considerations.)

Corollary **9.1.** — *For $\alpha \in \{2, \dots, k + 1\}$, let $r \in \mathbf{R}_+$ be such that exactly α of the points $x_1, \dots, x_{k+1}$ satisfy $|x_i| < r$. Then the Floer cohomology of the Lagrangian torus $T_r = \{(x, y, z) \in X' \mid |x| = r,\ |y| = |z|\}$ in the A_k Milnor fiber X', equipped with a suitable spin structure, is $HF^*(T_r, T_r) \simeq H^*(T^2; \Lambda)$, equipped with an A_∞-structure for which the generators a, b of $HF^1(T_r, T_r)$ satisfy the relations $\mathfrak{m}_2(a, b) + \mathfrak{m}_2(b, a) = 0$; $\mathfrak{m}_i(a, \dots, a) = 0$ for all i; $\mathfrak{m}_i(b, \dots, b) = 0$ for $i \le \alpha - 1$; and $\mathfrak{m}_\alpha(b, \dots, b) \neq 0$.*

Proof. — The condition $|x| = r$ implies that the torus T_r yields a point in the chamber U_α, while the condition that $|y| = |z|$ implies that it lies on the critical locus of W_0: this shows that T_r is a critical point of W_0 of order $\alpha + 1$.

By a construction which is standard in the toric case (see [14]), the restriction of W_0 to a chart of Y modeled after a domain in $H^1(T_r, \Lambda^*)$ (identified with $(\Lambda^*)^2$ by choosing the basis (a, b)) agrees with the map

$$\big(\exp(\lambda_a), \exp(\lambda_b)\big) \mapsto \sum_k \mathfrak{m}_k(\lambda_a a + \lambda_b b, \dots, \lambda_a a + \lambda_b b). \tag{9.2}$$

Choosing a to correspond to the generator which vanishes on loops whose projection to $\mathbf{C}$ is constant, the result follows immediately. $\square$

9.3. *Plane curves.* — For $p, q \geq 2$, consider a smooth Riemann surface H of genus $g = (p-1)(q-1)$ embedded in $\mathrm{V} = \mathbf{P}^1 \times \mathbf{P}^1$, defined as the zero set of a suitably chosen polynomial of bidegree (p, q). (The case of a genus 2 curve of bidegree $(3, 2)$ was used in Section 3 to illustrate the general construction, see Examples 3.2 and 3.12.)

Namely, in affine coordinates f is given by

$$f(x_1, x_2) = \sum_{a=0}^{p} \sum_{b=0}^{q} c_{a,b} \tau^{\rho(a,b)} x_1^a x_2^b,$$

where $c_{a,b} \in \mathbf{C}^*$ are arbitrary, $\rho(a, b) \in \mathbf{R}$ satisfy a suitable convexity condition, and $\tau \ll 1$. The corresponding tropical polynomial

$$\varphi(\xi_1, \xi_2) = \max\left\{a\xi_1 + b\xi_2 - \rho(a, b) \,\middle|\, 0 \leq a \leq p,\ 0 \leq b \leq q\right\} \tag{9.3}$$

defines a tropical curve $\Pi_0 \subset \mathbf{R}^2$; see Figure 1. We also denote by H', resp. H^0, the genus g curves with $p + q$ (resp. $2(p + q)$) punctures obtained by intersecting H with the affine subset $\mathrm{V}' = \mathbf{C}^2 \subset \mathrm{V}$, resp. $\mathrm{V}^0 = (\mathbf{C}^*)^2$.

The polytope $\Delta_{\mathrm{Y}} = \{(\xi_1, \xi_2, \eta) \mid \eta \geq \varphi(\xi_1, \xi_2)\}$ has $(p + 1)(q + 1)$ facets, corresponding to the regions where a particular term in (9.3) realizes the maximum. Thus the 3-fold Y has $(p + 1)(q + 1)$ irreducible toric divisors $\mathrm{D}_{a,b}$ $(0 \leq a \leq p,\ 0 \leq b \leq q)$ (we label each divisor by the weight of the dominant monomial). The moment polytopes for these divisors are exactly the components of $\mathbf{R}^2 \setminus \Pi_0$, and the tropical curve Π_0 depicts the moment map images of the codimension 2 strata where they intersect (a configuration of $\mathbf{P}^1$'s and $\mathbf{A}^1$'s); see Figure 3 (left) (and compare with Figure 1 (right)).

The leading-order superpotential W_0 of Definition 3.10 consists of five terms: $w_0 = -\mathrm{T}^\epsilon + \mathrm{T}^\epsilon v_0$, where v_0 is the toric monomial of weight $(0, 0, 1)$, which vanishes with multiplicity 1 on each of the toric divisors $\mathrm{D}_{a,b}$; and four terms $w_1, \ldots, w_4$ corresponding to the facets of Δ_{V}. Up to constant factors, w_1 is the toric monomial with weight $(-1, 0, 0)$, which vanishes with multiplicity a on $\mathrm{D}_{a,b}$; w_2 is the toric monomial with weight $(0, -1, 0)$, vanishing with multiplicity b on $\mathrm{D}_{a,b}$; w_3 is the monomial with weight $(1, 0, p)$, with multiplicity $(p - a)$ on $\mathrm{D}_{a,b}$; and w_4 is the monomial with weight $(0, 1, q)$, with multiplicity $(q - b)$ on $\mathrm{D}_{a,b}$ (compare Example 3.12).

Our main results for the open curve $\mathrm{H}^0 \subset \mathrm{V}^0 = (\mathbf{C}^*)^2$ are the following:

(1) the complement Y^0 of $w_0^{-1}(0) \simeq (\mathbf{C}^*)^2$ in the toric 3-fold Y is SYZ mirror to the conic bundle $\mathrm{X}^0 = \{(x_1, x_2, y, z) \in (\mathbf{C}^*)^2 \times \mathbf{C}^2 \mid yz = f(x_1, x_2)\}$;
(2) the B-side Landau-Ginzburg model (Y^0, w_0) is SYZ mirror to the blowup of $(\mathbf{C}^*)^2 \times \mathbf{C}$ along $\mathrm{H}^0 \times 0$;
(3) the B-side Landau-Ginzburg model $(\mathrm{Y}, -v_0)$ is a generalized SYZ mirror to the open genus g curve H^0.

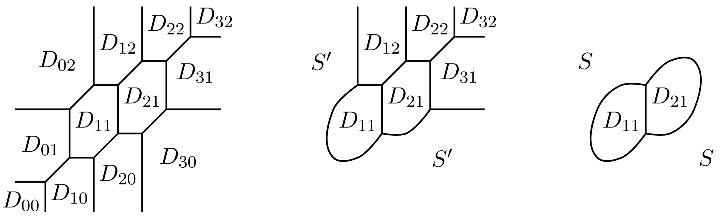

FIG. 3. — The singular fibers of the mirrors to $H^0 = H \cap (\mathbf{C}^*)^2$ (*left*) and $H' = H \cap \mathbf{C}^2$ (*middle*), and of the leading-order terms of the mirror to H (*right*). Here H is a genus 2 curve of bidegree $(3, 2)$ in $\mathbf{P}^1 \times \mathbf{P}^1$

The B-side Landau-Ginzburg models (Y^0, w_0) and $(Y, -v_0)$ have regular fibers isomorphic to $(\mathbf{C}^*)^2$, while the singular fiber $w_0^{-1}(-T^\epsilon) = v_0^{-1}(0)$ is the union of all the toric divisors $D_{a,b}$. In particular, the singular fiber consists of $(p+1)(q+1)$ toric surfaces intersecting pairwise along a configuration of $\mathbf{P}^1$'s and $\mathbf{A}^1$'s (the 1-dimensional strata of Y), themselves intersecting at triple points (the 0-dimensional strata of Y); the combinatorial structure of the trivalent configuration of $\mathbf{P}^1$'s and $\mathbf{A}^1$'s is exactly given by the tropical curve Π_0. (See Figure 3 (left).)

If we partially compactify to $V' = \mathbf{C}^2$, then we get:

- (2′) the B-side Landau-Ginzburg model $(Y^0, w_0 + w_1 + w_2)$ is SYZ mirror to the blowup of $\mathbf{C}^3$ along $H' \times 0$;
- (3′) the B-side Landau-Ginzburg model $(Y, -v_0 + w_1 + w_2)$ is mirror to H'.

Adding $w_1 + w_2$ to the superpotential results in a partial smoothing of the singular fiber; namely, the singular fiber is now the union of the toric surfaces $D_{a,b}$ where $a > 0$ and $b > 0$ (over which $w_1 + w_2$ vanishes identically) and a single noncompact surface $S' \subset Y$, which can be thought of as a smoothing (or partial smoothing) of $S'_0 = (\bigcup_a D_{a,0}) \cup (\bigcup_b D_{0,b})$.

By an easy calculation in the toric affine charts of Y, the critical locus of $W_{H'} = -v_0 + w_1 + w_2$ (i.e. the pairwise intersections of components of $W_{H'}^{-1}(0)$ and the possible self-intersections of S') is again a union of $\mathbf{P}^1$'s and $\mathbf{A}^1$'s meeting at triple points; the combinatorics of this configuration is obtained from the planar graph Π_0 (which describes the critical locus of $W_{H^0} = -v_0$) by deleting all the unbounded edges in the directions of $(-1, 0)$ and $(0, -1)$, then inductively collapsing the bounded edges that connect to univalent vertices and merging the edges that meet at bivalent vertices (see Figure 3 middle); this construction can be understood as a sequence of "tropical modifications" applied to the tropical curve Π_0.

The closed genus g curve H does not satisfy Assumption 1.4, so our main results do not apply to it. However, it is instructive to consider the leading-order mirrors (Y^0, W_0) to the blowup X of $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{C}$ along $H \times 0$ and (Y, W_0^H) to the curve H itself. Indeed, in this case the additional instanton corrections (i.e., virtual counts of configurations that include exceptional rational curves in $\tilde{V}$) are expected to only have a mild effect on the mirror: specifically, they should not affect the topology of the critical locus, but merely deform it in a way that can be accounted for by corrections to the mirror map. We will return to this question in a forthcoming paper.

The zero set of the leading-order superpotential $W_0^H = -v_0 + w_1 + w_2 + w_3 + w_4$ is the union of the compact toric surfaces $D_{a,b}$, $0 < a < p$, $0 < b < q$, with a single noncompact surface $S \subset Y$, which can be thought of as a smoothing (or partial smoothing) of the union S_0 of the noncompact toric divisors of Y. (There may also be new critical points which do not lie over 0; we shall not discuss them.)

Here again, an easy calculation in the toric affine charts shows that the singular locus of $(W_0^H)^{-1}(0)$ (i.e., the pairwise intersections of components and the possible self-intersections of S) forms a configuration of $3g - 3$ $\mathbf{P}^1$'s meeting at triple points. Combinatorially, this configuration is obtained from the planar graph Π_0 by deleting all the unbounded edges, then inductively collapsing the bounded edges that connect to univalent vertices and merging the edges that meet at bivalent vertices (see Figure 3 (right)); this can be understood as a sequence of tropical modifications turning Π_0 into a closed genus g tropical curve (i.e., a trivalent graph without unbounded edges).

(The situation is slightly different when $p = q = 2$ and $g = 1$: in this case $(W_0^H)^{-1}(0) = D_{1,1} \cup S$, and the critical locus $D_{1,1} \cap S$ is a smooth elliptic curve. In this case, the higher instanton corrections are easy to analyze, and simply amount to rescaling the first term $-v_0$ of the superpotential by a multiplicative factor which encodes certain genus 0 Gromov-Witten invariants of $\mathbf{P}^1 \times \mathbf{P}^1$.)

10. Generalizations

In this section we mention (without details) a couple of straightforward generalizations of our construction.

10.1. *Non-maximal degenerations.* — In our main construction we have assumed that the hypersurface $H \subset V$ is part of a maximally degenerating family $(H_\tau)_{\tau\to 0}$ (see Definition 3.1). This was used for two purposes: (1) to ensure that, for each weight $\alpha \in A$, there exists a connected component of $\mathbf{R}^n \setminus \mathrm{Log}(H)$ over which the corresponding monomial in the defining equation (3.1) dominates all other terms, and (2) to ensure that the toric variety Y associated to the polytope (3.8) is smooth.

(Note that the regularity of $\mathcal{P}$ also ensures the smoothness of H throughout, and of H'_σ in the discussion before Lemma 5.7; without the regularity assumption, smoothness can still be achieved by making generic choices of the coefficients c_α in (3.1).)

In general, removing the assumption of maximal degeneration, some of the terms in the tropical polynomial

$$\varphi(\xi) = \max\left\{\langle\alpha, \xi\rangle - \rho(\alpha) \,\middle|\, \alpha \in A\right\}$$

may not achieve the maximum under any circumstances; denote by A_{red} the set of those weights which do achieve the maximum for some value of ξ. Equivalently, those are exactly the vertices of the polyhedral decomposition $\mathcal{P}$ of Conv(A) induced by the function

$\rho : \mathrm{A} \to \mathbf{R}$. Then the elements of $\mathrm{A} \setminus \mathrm{A}_{red}$ do not give rise to connected components of the complement of the tropical curve, nor to facets of Δ_{Y}, and should be discarded altogether. Thus, the main difference with the maximal degeneration case is that the rays of the fan Σ_{Y} are the vectors $(-\alpha, 1)$ for $\alpha \in \mathrm{A}_{red}$, and the toric variety Y is usually singular.

Indeed, the construction of the Lagrangian torus fibration $\pi : \mathrm{X}^0 \to \mathrm{B}$ proceeds as in Section 4, and the arguments in Sections 4 to 6 remain valid, the only difference being that only the weights $\alpha \in \mathrm{A}_{red}$ give rise to chambers U_α of tautologically unobstructed fibers of π, and hence to affine coordinate charts $\mathrm{U}_\alpha^\vee$ for the SYZ mirror Y^0 of X^0. Replacing A by A_{red} throughout the arguments addresses this issue.

The smooth mirrors obtained from maximal degenerations are crepant resolutions of the singular mirrors obtained from non-maximal ones. Starting from a non-maximal polyhedral decomposition $\mathcal{P}$, the various ways in which it can be refined to a regular decomposition correspond to different choices of resolution. We give a few examples.

Example **10.1.** — Revisiting the example of the A_k-Milnor fiber considered in Section 9.2, we now consider the case where the roots of the polynomial f_{k+1} satisfy $|x_1| = \cdots = |x_{k+1}|$, for example $f_{k+1}(x) = x^{k+1} - 1$, which gives

$$\mathrm{X}' = \left\{(x, y, z) \in \mathbf{C}^3 \,\middle|\, yz = x^{k+1} - 1\right\}.$$

Then the tropical polynomial $\varphi : \mathbf{R} \to \mathbf{R}$ is $\varphi(\xi) = \max(0, (k+1)\xi)$, and the polytope $\Delta_{\mathrm{Y}} = \{(\xi, \eta) \in \mathbf{R}^2 \mid \eta \geq \varphi(\xi)\}$ determines the singular toric variety $\{st = u^{k+1}\} \subset \mathbf{K}^3$, i.e. the A_k singularity, rather than its resolution as previously.

Geometrically, the Lagrangian torus fibration π normally consists of $k+2$ chambers, depending on how many of the roots of f_{k+1} lie inside the projection of the fiber to the x coordinate plane. In the case considered here, all the walls lie at $|x| = 1$, and the fibration π only consists of two chambers ($|x| < 1$ and $|x| > 1$).

In fact, $\mathbf{Z}/(k+1)$ acts freely on $\mathrm{X}_k^0 = \{(x, y, z) \in \mathbf{C}^* \times \mathbf{C}^2 \mid yz = x^{k+1} - 1\}$, making it an unramified cover of $\mathrm{X}_0^0 = \{(\hat{x}, y, z) \in \mathbf{C}^* \times \mathbf{C}^2 \mid yz = \hat{x} - 1\} \simeq \mathbf{C}^2 \setminus \{yz = -1\}$ via the map $(x, y, z) \mapsto (x^{k+1}, y, z)$. The Lagrangian tori we consider on X_k^0 are simply the preimages of the SYZ fibration on X_0^0, which results in the mirror being the quotient of the mirror of X_0^0 (namely, $\{(\hat{s}, \hat{t}, u) \in \mathbf{K}^3 \mid \hat{s}\hat{t} = u,\ u \neq 1\}$) by a $\mathbf{Z}/(k+1)$-action (namely $\zeta \cdot (\hat{s}, \hat{t}, u) = (\zeta\hat{s}, \zeta^{-1}\hat{t}, u)$). As expected, the quotient is nothing other than $\mathrm{Y}_k^0 = \{(s, t, u) \in \mathbf{K}^3 \mid st = u^{k+1},\ u \neq 1\}$ (via the map $(\hat{s}, \hat{t}, u) \mapsto (\hat{s}^{k+1}, \hat{t}^{k+1}, u)$).

Example **10.2.** — The higher-dimensional analogue of the previous example is that of Fermat hypersurfaces in $(\mathbf{C}^*)^n$ or in $\mathbf{CP}^n$. Let H be the Fermat hypersurface in $\mathbf{CP}^n$ given by the equation $\sum \mathrm{X}_i^d = 0$ in homogeneous coordinates, i.e. $x_1^d + \cdots + x_n^d + 1 = 0$ in affine coordinates, and let X be the blowup of $\mathbf{CP}^n \times \mathbf{C}$ at $\mathrm{H} \times 0$. In this case, the open Calabi-Yau manifold X^0 is

$$\mathrm{X}^0 = \left\{(x_1, \ldots, x_n, y, z) \in \left(\mathbf{C}^*\right)^n \times \mathbf{C}^2 \,\middle|\, yz = x_1^d + \cdots + x_n^d + 1\right\}.$$

The tropical polynomial corresponding to H is $\varphi(\xi_1,\dots,\xi_n)=\max(d\xi_1,\dots,d\xi_n,0)$, which is highly degenerate. Thus the toric variety Y associated to the polytope Δ_Y given by (3.8) is singular, in fact it can be described as

$$\mathrm{Y}=\left\{(z_1,\dots,z_{n+1},v)\in\mathbf{K}^{n+2}\,\middle|\,z_1\cdots z_{n+1}=v^d\right\},$$

which can be viewed as the quotient of $\mathbf{K}^{n+1}$ by the diagonal action of $(\mathbf{Z}/d)^n$ (multiplying all coordinates by roots of unity but preserving their product), via the map $(\tilde z_1,\dots,\tilde z_{n+1})\mapsto(\tilde z_1^d,\dots,\tilde z_{n+1}^d,\tilde z_1\cdots\tilde z_{n+1})$. As in the previous example, this is consistent with the observation that X^0 is a $(\mathbf{Z}/d)^n$-fold cover of the conic bundle considered in Section 9.1, where $(\mathbf{Z}/d)^n$ acts diagonally by multiplication on the coordinates $x_1,\dots,x_n$.

(As usual, considering a maximally degenerating family of hypersurfaces of degree d instead of a Fermat hypersurface would yield a crepant resolution of Y.)

By Theorem 1.6, the affine Fermat hypersurface $\mathrm{H}^0=\mathrm{H}\cap(\mathbf{C}^*)^n$ is mirror to the singular B-side Landau-Ginzburg model $(\mathrm{Y},\mathrm{W}_0^{\mathrm{H}}=-v)$ or, in other terms, the quotient of $(\mathbf{K}^{n+1},\tilde{\mathrm{W}}_0^{\mathrm{H}}=-\tilde z_1\cdots\tilde z_{n+1})$ by the action of $(\mathbf{Z}/d)^n$, which is consistent with [49].

Furthermore, by Remark 7.11 the theorem also applies to projective Fermat hypersurfaces of degree $d<n$ in $\mathbf{CP}^n$. Setting $a=\frac{1}{n+1}\int_{\mathbf{CP}^1}\omega_{\mathbf{CP}^n}$, and placing the barycenter of the moment polytope of $\mathbf{CP}^n$ at the origin, we find that

$$\left(\mathrm{Y},\mathrm{W}_0^{\mathrm{H}}=-v+\mathrm{T}^a(z_1+\cdots+z_{n+1})\right)$$

is mirror to H (for $d<n$; otherwise this is only the leading-order approximation to the mirror). Equivalently, this can be viewed as the quotient of

$$\left(\mathbf{K}^{n+1},\tilde{\mathrm{W}}_0^{\mathrm{H}}=-\tilde z_1\cdots\tilde z_{n+1}+\mathrm{T}^a\left(\tilde z_1^d+\cdots+\tilde z_{n+1}^d\right)\right)$$

by the action of $(\mathbf{Z}/d)^n$, which is again consistent with Sheridan's work.

Example **10.3.** — We now revisit the example considered in Section 9.3, where we found the mirrors of nearly tropical plane curves of bidegree (p,q) to be smooth toric 3-folds (equipped with suitable superpotentials) whose topology is determined by the combinatorics of the corresponding tropical plane curve Π_0 (or dually, of the regular triangulation $\mathcal{P}$ of the rectangle $[0,p]\times[0,q]$).

A particularly simple way to modify the combinatorics is to "flip" a pair of adjacent triangles of $\mathcal{P}$ whose union is a unit parallelogram; this affects the toric 3-fold Y by a flip. This operation can be implemented by a continuous deformation of the tropical curve Π_0 in which the length of a bounded edge shrinks to zero, creating a four-valent vertex, which is then resolved by creating a bounded edge in the other direction and increasing its length. The intermediate situation where Π_0 has a 4-valent vertex corresponds to a non-maximal degeneration where $\mathcal{P}$ is no longer a maximal triangulation of $[0,p]\times[0,q]$, instead containing a single parallelogram of unit area; the mirror toric variety Y then acquires an ordinary double point singularity. The two manners in which the four-valent vertex of the tropical curve can be deformed to a pair of trivalent vertices connected by a bounded edge then amount to the two small resolutions of the ordinary double point, and differ by a flip.

10.2. *Hypersurfaces in abelian varieties.* — As suggested to us by Paul Seidel, the methods we use to study hypersurfaces in toric varieties can also be applied to the case of hypersurfaces in abelian varieties. For simplicity, we only discuss the case of abelian varieties V which can be viewed as quotients of $(\mathbf{C}^*)^n$ (with its standard Kähler form) by the action of a real lattice $\Gamma_{\mathrm{B}} \subset \mathbf{R}^n$, where $\gamma \in \Gamma_{\mathrm{B}}$ acts by $(x_1, \dots, x_n) \mapsto (e^{\gamma_1} x_1, \dots, e^{\gamma_n} x_n)$. In other terms, the logarithm map identifies V with the product $\mathrm{T_B} \times \mathrm{T_F}$ of two real Lagrangian tori, the "base" $\mathrm{T_B} = \mathbf{R}^n/\Gamma_{\mathrm{B}}$ and the "fiber" $\mathrm{T_F} = i\mathbf{R}^n/(2\pi\mathbf{Z})^n$ (which corresponds to the orbit of a T^n-action).

Since the T^n-action on V is not Hamiltonian, there is no globally defined $\mathbf{R}^n$-valued moment map. However, there is an analogous map which takes values in a real torus, namely the quotient of $\mathbf{R}^n$ by the lattice spanned by the periods of ω_{V} on $\mathrm{H}_1(\mathrm{T_B}) \times \mathrm{H}_1(\mathrm{T_F})$; due to our choice of the standard Kähler form on $(\mathbf{C}^*)^n$, this period lattice is simply Γ_{B}, and the "moment map" is the logarithm map projecting from V to the real torus $\mathrm{T_B} = \mathbf{R}^n/\Gamma_{\mathrm{B}}$.

A tropical hypersurface $\Pi_0 \subset \mathrm{T_B}$ can be thought of as the image of a Γ_{B}-periodic tropical hypersurface $\tilde{\Pi}_0 \subset \mathbf{R}^n$ under the natural projection $\mathbf{R}^n \to \mathbf{R}^n/\Gamma_{\mathrm{B}} = \mathrm{T_B}$. Such a tropical hypersurface occurs naturally as the limit of the amoebas (moment map images) of a degenerating family of hypersurfaces H_τ inside the degenerating family of abelian varieties V_τ ($\tau \to 0$) corresponding to rescaling the lattice Γ_{B} by a factor of $|\log \tau|$. (We keep the Kähler class $[\omega_{\mathrm{V}}]$ and its period lattice Γ_{B} constant by rescaling the Kähler form of $(\mathbf{C}^*)^n$ by an appropriate factor, so that the moment map is given by the base τ logarithm map, $\mu_{\mathrm{V}} = \mathrm{Log}_\tau : \mathrm{V}_\tau \to \mathrm{T_B}$.) As in Section 3 we call $\mathrm{H}_\tau \subset \mathrm{V}_\tau$ "nearly tropical" if its amoeba $\Pi_\tau = \mathrm{Log}_\tau(\mathrm{H}_\tau) \subset \mathrm{T_B}$ is contained in a tubular neighborhood of the tropical hypersurface Π_0; we place ourselves in the nearly tropical setting, and elide τ from the notation.

Concretely, the hypersurface H is defined by a section of a line bundle $\mathcal{L} \to \mathrm{V}$ whose pullback to $(\mathbf{C}^*)^n$ is trivial; $\mathcal{L}$ can be viewed as the quotient of $(\mathbf{C}^*)^n \times \mathbf{C}$ by Γ_{B}, where $\gamma \in \Gamma_{\mathrm{B}}$ acts by

$$\gamma_\# : (x_1, \dots, x_n, v) \mapsto \big(\tau^{-\gamma_1} x_1, \dots, \tau^{-\gamma_n} x_n, \tau^{\kappa(\gamma)} \mathbf{x}^{\lambda(\gamma)} v\big), \tag{10.1}$$

where $\lambda \in \hom(\Gamma_{\mathrm{B}}, \mathbf{Z}^n)$ is a homomorphism determined by the Chern class $c_1(\mathcal{L})$ (observe that $\hom(\Gamma_{\mathrm{B}}, \mathbf{Z}^n) \simeq \mathrm{H}^1(\mathrm{T_B}, \mathbf{Z}) \otimes \mathrm{H}^1(\mathrm{T_F}, \mathbf{Z}) \subset \mathrm{H}^2(\mathrm{V}, \mathbf{Z})$), and $\kappa : \Gamma_{\mathrm{B}} \to \mathbf{R}$ satisfies a cocycle-type condition in order to make (10.1) a group action. A basis of sections of $\mathcal{L}$ is given by the theta functions

$$\vartheta_\alpha(x_1, \dots, x_n) = \sum_{\gamma \in \Gamma_{\mathrm{B}}} \gamma_\#^*\big(\mathbf{x}^\alpha\big), \quad \alpha \in \mathbf{Z}^n/\lambda(\Gamma_{\mathrm{B}}). \tag{10.2}$$

(Note: for $\gamma \in \Gamma_B$, ϑ_α and $\vartheta_{\alpha+\lambda(\gamma)}$ actually differ by a constant factor.) The defining section f of H is a finite linear combination of these theta functions; equivalently, its lift to $(\mathbf{C}^*)^n$ can be viewed as an infinite Laurent series of the form (3.1), invariant under the action (10.1) (which forces the set of weights A to be $\lambda(\Gamma_B)$-periodic). We note that the corresponding tropical function $\varphi : \mathbf{R}^n \to \mathbf{R}$ is also Γ_B-equivariant, in the sense that $\varphi(\xi+\gamma) = \varphi(\xi) + \langle \lambda(\gamma), \xi\rangle - \kappa(\gamma)$ for all $\gamma \in \Gamma_B$.

Let X be the blowup of $V \times \mathbf{C}$ along $H \times 0$, equipped with an S^1-invariant Kähler form ω_ϵ such that the fibers of the exceptional divisor have area $\epsilon > 0$ (chosen sufficiently small). Denote by $\tilde{V}$ the proper transform of $V \times 0$, and let $X^0 = X \setminus \tilde{V}$. Then X^0 carries an S^1-invariant Lagrangian torus fibration $\pi : X^0 \to B = T_B \times \mathbf{R}_+$, constructed as in Section 4 by assembling fibrations on the reduced spaces of the S^1-action. This allows us to determine SYZ mirrors to X^0 and X as in Sections 5 and 6.

The construction can be understood either directly at the level of X and X^0, or by viewing the whole process as a Γ_B-equivariant construction on the cover $\tilde{X}$, namely the blowup of $(\mathbf{C}^*)^n \times \mathbf{C}$ along $\tilde{H} \times 0$, where $\tilde{H}$ is the preimage of H under the covering map $q : (\mathbf{C}^*)^n \to (\mathbf{C}^*)^n/\Gamma_B = V$. The latter viewpoint makes it easier to see that the enumerative geometry arguments from the toric case extend to this setting.

As in the toric case, each weight $\bar{\alpha} \in \bar{A} := A/\lambda(\Gamma_B)$ determines a connected component of the complement $T_B \setminus \Pi_0$ of the tropical hypersurface Π_0, and hence a chamber $U_{\bar{\alpha}} \subset B^{reg} \subset B$ over which the fibers of π are tautologically unobstructed. Each of these determines an affine coordinate chart $U^\vee_{\bar{\alpha}}$ for the SYZ mirror of X^0, and these charts are glued to each other via coordinate transformations of the form (3.11).

Alternatively, we can think of the mirror as a quotient by Γ_B of a space built from an infinite collection of charts $U^\vee_\alpha$, $\alpha \in A$, where each chart $U^\vee_\alpha$ has coordinates $(v_{\alpha,1}, \dots, v_{\alpha,n}, w_0)$, glued together by (3.11). Specifically, for each element $\gamma = (\gamma_1, \dots, \gamma_n) \in \Gamma_B$, we identify $U^\vee_\alpha$ with $U^\vee_{\alpha+\lambda(\gamma)}$ via the map

$$\gamma^\vee_\# : (v_{\alpha,1}, \dots, v_{\alpha,n}, w_0) \in U^\vee_\alpha \mapsto \left(T^{\gamma_1} v_{\alpha,1}, \dots, T^{\gamma_n} v_{\alpha,n}, w_0\right) \in U^\vee_{\alpha+\lambda(\gamma)}, \tag{10.3}$$

where the multiplicative factors T^{γ_i} account for the amount of symplectic area separating the different lifts to $\tilde{X}$ of a given fiber of π.

Setting $v_0 = 1 + T^{-\epsilon} w_0$, we can again view the SYZ mirror Y^0 of X^0 as the complement of the hypersurface $w_0^{-1}(0) = v_0^{-1}(1)$ in a "locally toric" variety Y covered (outside of codimension 2 strata) by local coordinate charts $Y_\alpha = (\mathbf{K}^*)^n \times \mathbf{K}$ $(\alpha \in A)$ glued together by (3.9) and identified under the action of Γ_B. Namely, for all $\alpha, \beta \in A$ and $\gamma \in \Gamma_B$ we make the identifications

$$(v_1, \dots, v_n, v_0) \in Y_\alpha \sim \left(v_0^{\alpha_1-\beta_1} v_1, \dots, v_0^{\alpha_n-\beta_n} v_n, v_0\right) \in Y_\beta, \tag{10.4}$$

$$(v_1, \dots, v_n, v_0) \in Y_\alpha \sim \left(T^{\gamma_1} v_1, \dots, T^{\gamma_n} v_n, v_0\right) \in Y_{\alpha+\lambda(\gamma)}. \tag{10.5}$$

Finally, the abelian variety V is aspherical, and any holomorphic disc bounded by $\pi^{-1}(b)$, $b \in B^{reg}$ must be entirely contained in a fiber of the projection to V, so that the only contribution to the superpotential is w_0 (as in the case of hypersurfaces in $(\mathbf{C}^*)^n$). With this understood, our main results become:

Theorem **10.4.** — *Let* H *be a nearly tropical hypersurface in an abelian variety* V*, let* X *be the blowup of* $\mathrm{V} \times \mathbf{C}$ *along* $\mathrm{H} \times 0$*, and let* Y *be as above. Then:*

(1) $\mathrm{Y}^0 = \mathrm{Y} \setminus w_0^{-1}(0)$ *is SYZ mirror to* $\mathrm{X}^0 = \mathrm{X} \setminus \tilde{\mathrm{V}}$*;*
(2) *the* B*-side Landau-Ginzburg model* (Y^0, w_0) *is SYZ mirror to* X*;*
(3) *the* B*-side Landau-Ginzburg model* $(\mathrm{Y}, -v_0)$ *is generalized SYZ mirror to* H.

Note that, unlike Theorems 1.5 and 1.6, this result holds without any restrictions: when V is an abelian variety, Assumption 1.4 always holds and there are never any higher-order instanton corrections. On the other hand, the statement of part (3) implicitly uses the properties of Fukaya categories of Landau-Ginzburg models whose proofs are sketched in Section 7 (whereas parts (1) and (2) rely only on familiar versions of the Fukaya category).

The smooth fibers of $-v_0 : \mathrm{Y} \to \mathbf{K}$ (or equivalently up to a reparametrization, $w_0 : \mathrm{Y}^0 \to \mathbf{K}^*$) are all abelian varieties, in fact quotients of $(\mathbf{K}^*)^n$ (with coordinates $\mathbf{v} = (v_1, \dots, v_n)$) by the identification

$$\mathbf{v}^m \sim v_0^{\langle \lambda(\gamma), m\rangle} \mathrm{T}^{\langle \gamma, m\rangle} \mathbf{v}^m \quad \text{for all } m \in \mathbf{Z}^n \text{ and } \gamma \in \Gamma_{\mathrm{B}},$$

while the singular fiber is a union of toric varieties

$$v_0^{-1}(0) = \bigcup_{\bar{\alpha} \in \bar{\mathrm{A}}} \mathrm{D}_{\bar{\alpha}}$$

glued (to each other or to themselves) along toric strata. The moment polytopes for the toric varieties $\mathrm{D}_{\bar{\alpha}}$ are exactly the components of $\mathrm{T}_{\mathrm{B}} \setminus \Pi_0$, and the tropical hypersurface Π_0 depicts the moment map images of the codimension 2 strata of Y along which they intersect.

Example **10.5.** — When H is a set of n points on an elliptic curve V, we find that the fibers of $-v_0 : \mathrm{Y} \to \mathbf{K}$ are a family of elliptic curves, all smooth except $v_0^{-1}(0)$ which is a union of n $\mathbf{P}^1$'s forming a cycle (in the terminology of elliptic fibrations, this is known as an I_n fiber). In this case the superpotential $-v_0$ has n isolated critical points, all lying in the fiber over zero, as expected.

Example **10.6.** — Now consider the case where H is a genus 2 curve embedded in an abelian surface V (for example its Jacobian torus). The tropical genus 2 curve Π_0 is a trivalent graph on the 2-torus T_{B} with two vertices and three edges, see Figure 4 (left). Since $\mathrm{T}_{\mathrm{B}} \setminus \Pi_0$ is connected, the singular fiber $v_0^{-1}(0)$ of the mirror B-side

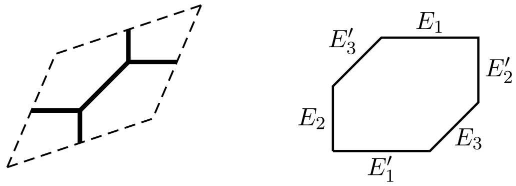

Fig. 4. — A tropical genus 2 curve on the 2-torus (*left*); the singular fiber of the mirror Landau-Ginzburg model is the quotient of the toric Del Pezzo surface shown (*right*) by identifying $E_i \sim E_i'$

Landau-Ginzburg model is irreducible. Specifically, it is obtained from the toric Del Pezzo surface shown in Figure 4 (right), i.e. $\mathbf{P}^2$ blown up in 3 points, by identifying each exceptional curve E_i with the "opposite" exceptional curve E_i' (the proper transform of the line through the two other points). Thus the critical locus of the superpotential is a configuration of three rational curves $E_1 = E_1'$, $E_2 = E_2'$, $E_3 = E_3'$ intersecting at two triple points. (Compare with Section 9.3: the mirrors are very different, but the critical loci are the same.)

11. Complete intersections

In this section we explain (without details) how to extend our main results to the case of complete intersections in toric varieties (under a suitable positivity assumption for rational curves, which always holds in the affine case).

11.1. *Notations and statement of the results.* — Let $H_1, \dots, H_d$ be smooth nearly tropical hypersurfaces in a toric variety V of dimension n, in general position. We denote by f_i the defining equation of H_i, a section of a line bundle $\mathcal{L}_i$ which can be written as a Laurent polynomial (3.1) in affine coordinates $\mathbf{x} = (x_1, \dots, x_n)$; by $\varphi_i : \mathbf{R}^n \to \mathbf{R}$ the corresponding tropical polynomial; and by $\Pi_i \subset \mathbf{R}^n$ the tropical hypersurface defined by φ_i. (To ensure smoothness of the mirror, it is useful to assume that the tropical hypersurfaces $\Pi_1, \dots, \Pi_d$ intersect transversely, though this assumption is actually not necessary.)

We denote by X the blowup of $V \times \mathbf{C}^d$ along the d codimension 2 subvarieties $H_i \times \mathbf{C}_i^{d-1}$, where $\mathbf{C}_i^{d-1} = \{y_i = 0\}$ is the i-th coordinate hyperplane in $\mathbf{C}^d$. (The blowup is smooth since the subvarieties $H_i \times \mathbf{C}_i^{d-1}$ intersect transversely.) Explicitly, X can be a described as a smooth submanifold of the total space of the $(\mathbf{P}^1)^d$-bundle $\prod_{i=1}^d \mathbf{P}(\mathcal{L}_i \oplus \mathcal{O})$ over $V \times \mathbf{C}^d$,

$$X = \left\{ \left(\mathbf{x}, y_1, \dots, y_d, (u_1 : v_1), \dots, (u_d : v_d)\right) \,\middle|\, f_i(\mathbf{x}) v_i = y_i u_i \ \forall i = 1, \dots, d \right\}. \tag{11.1}$$

Outside of the union of the hypersurfaces H_i, the fibers of the projection $p_V : X \to V$ obtained by composing the blowup map $p : X \to V \times \mathbf{C}^d$ with projection to the first factor are isomorphic to $\mathbf{C}^d$; above a point which belongs to k of the H_i, the fiber consists of 2^k components, each of which is a product of $\mathbf{C}$'s and $\mathbf{P}^1$'s.

The action of $T^d = (S^1)^d$ on $V \times \mathbf{C}^d$ by rotation on the last d coordinates lifts to X; we equip X with a T^d-invariant Kähler form for which the exceptional $\mathbf{P}^1$ fibers of the i-th exceptional divisor have area ϵ_i (where $\epsilon_i > 0$ is chosen small enough). As in Section 3.2, we arrange for the Kähler form on X to coincide with that on $V \times \mathbf{C}^d$ away from the exceptional divisors. We denote by $\mu_X : X \to \mathbf{R}^d$ the moment map.

The dense open subset $X^0 \subset X$ over which we can construct an SYZ fibration is the complement of the proper transforms of the toric strata of $V \times \mathbf{C}^d$; it can be viewed as an iterated conic bundle over the open stratum $V^0 \simeq (\mathbf{C}^*)^n \subset V$, namely

$$X^0 \simeq \left\{(\mathbf{x}, y_1, \ldots, y_d, z_1, \ldots, z_d) \in V^0 \times \mathbf{C}^{2d} \,\middle|\, y_i z_i = f_i(\mathbf{x}) \ \forall i = 1, \ldots, d\right\}. \tag{11.2}$$

Consider the polytope $\Delta_Y \subseteq \mathbf{R}^{n+d}$ defined by

$$\Delta_Y = \left\{(\xi, \eta_1, \ldots, \eta_d) \in \mathbf{R}^n \oplus \mathbf{R}^d \,\middle|\, \eta_i \geq \varphi(\xi_i) \ \forall i = 1, \ldots, d\right\}, \tag{11.3}$$

and let Y be the corresponding toric variety. For $i = 1, \ldots, d$, denote by $v_{0,i}$ the monomial with weight $(0, \ldots, 0, 1, \ldots, 0)$ (the $(n+i)$-th entry is 1), and set

$$w_{0,i} = -T^{\epsilon_i} + T^{\epsilon_i} v_{0,i}. \tag{11.4}$$

Denote by A the set of connected components of $\mathbf{R}^n \setminus (\Pi_1 \cup \cdots \cup \Pi_d)$, and index each component by the tuple of weights $\vec{\alpha} = (\alpha^1, \ldots, \alpha^d) \in \mathbf{Z}^{n\times d}$ corresponding to the dominant monomials of $\varphi_1, \ldots, \varphi_d$ in that component. Then for each $\vec{\alpha} \in A$ we have a coordinate chart $Y_{\vec{\alpha}} \simeq (\mathbf{K}^*)^n \times \mathbf{K}^d$ with coordinates $\mathbf{v}_{\vec{\alpha}} = (v_{\vec{\alpha},1}, \ldots, v_{\vec{\alpha},n}) \in (\mathbf{K}^*)^n$ and $(v_{0,1}, \ldots, v_{0,d}) \in \mathbf{K}^d$, where the monomial $\mathbf{v}_{\vec{\alpha}}^m = v_{\vec{\alpha},1}^{m_1} \cdots v_{\vec{\alpha},n}^{m_n}$ is the toric monomial with weight $(-m_1, \ldots, -m_n, \langle \alpha^1, m\rangle, \ldots, \langle \alpha^d, m\rangle) \in \mathbf{Z}^{n+d}$. These charts glue via

$$\mathbf{v}_{\vec{\alpha}}^m = \left(\prod_{i=1}^d \left(1 + T^{-\epsilon_i} w_{0,i}\right)^{\langle \beta^i - \alpha^i, m\rangle}\right) \mathbf{v}_{\vec{\beta}}^m. \tag{11.5}$$

Denoting by $\sigma_1, \ldots, \sigma_r \in \mathbf{Z}^n$ the primitive generators of the rays of the fan Σ_V, and writing the moment polytope of V in the form (3.12), for $j = 1, \ldots, r$ we define

$$w_j = T^{\varpi_j} \mathbf{v}_{\vec{\alpha}_{\min}(\sigma_j)}^{\sigma_j}, \tag{11.6}$$

where $\vec{\alpha}_{\min}(\sigma_j) \in A$ is chosen so that all $\langle \sigma_j, \alpha^i\rangle$ are minimal. In other terms, $\mathbf{v}_{\vec{\alpha}_{\min}(\sigma_j)}^{\sigma_j}$ is the toric monomial with weight $(-\sigma_j, \lambda_1(\sigma_j), \ldots, \lambda_d(\sigma_j)) \in \mathbf{Z}^{n+d}$, where $\lambda_1, \ldots, \lambda_d : \Sigma_V \to \mathbf{R}$ are the piecewise linear functions defining $\mathcal{L}_i = \mathcal{O}(H_i)$.

Finally, define Y^0 to be the subset of Y where $w_{0,1}, \ldots, w_{0,d}$ are all non-zero, and define the leading-order superpotentials

$$W_0 = w_{0,1} + \cdots + w_{0,d} + w_1 + \cdots + w_r = \sum_{i=1}^{d} \big(-T^{\epsilon_i} + T^{\epsilon_i} v_{0,i}\big) + \sum_{i=1}^{r} T^{\varpi_j} \mathbf{v}^{\sigma_j}_{\vec{\alpha}_{\min}(\sigma_j)}, \tag{11.7}$$

$$W_0^{H} = -v_{0,1} - \cdots - v_{0,d} + w_1 + \cdots + w_r = -\sum_{i=1}^{d} v_{0,i} + \sum_{i=1}^{r} T^{\varpi_j} \mathbf{v}^{\sigma_j}_{\vec{\alpha}_{\min}(\sigma_j)}. \tag{11.8}$$

With this understood, the analogue of Theorems 1.5–1.7 is the following

Theorem **11.1.** — *With the above notations:*

(1) Y^0 *is SYZ mirror to the iterated conic bundle* X^0*;*
(2) *assuming that all rational curves in* X *have positive Chern number (e.g. when* V *is affine), the* B*-side Landau-Ginzburg model* (Y^0, W_0) *is SYZ mirror to* X*;*
(3) *assuming that* V *is affine, the* B*-side Landau-Ginzburg model* (Y, W_0^H) *is a generalized SYZ mirror to the complete intersection* $H_1 \cap \cdots \cap H_d \subset V$.

As in Theorem 10.4, part (3) of this theorem relies on the expected properties of Fukaya categories of Landau-Ginzburg models.

Remark **11.2.** — Denoting by X_i the blowup of $V \times \mathbf{C}$ at $H_i \times 0$ and by X_i^0 the corresponding conic bundle over V^0, the space X (resp. X^0) is the fiber product of $X_1, \ldots, X_d$ (resp. $X_1^0, \ldots, X_d^0$) with respect to the natural projections to V. This perspective explains many of the geometric features of the construction.

11.2. *Sketch of proof.* — The argument proceeds along the same lines as for the case of hypersurfaces, of which it is really a straightforward adaptation. We outline the key steps for the reader's convenience.

As in Section 4, a key observation to be made about the T^d-action on X is that the reduced spaces $X_{red,\lambda} = \mu_X^{-1}(\lambda)/T^d$ ($\lambda \in \mathbf{R}^d_{\geq 0}$) are all isomorphic to V via the projection p_V (though the Kähler forms may differ near $H_1 \cup \cdots \cup H_d$). This allows us to build a (singular) Lagrangian torus fibration

$$\pi : X^0 \to B = \mathbf{R}^n \times (\mathbf{R}_+)^d$$

(where the second component is the moment map) by assembling standard Lagrangian torus fibrations on the reduced spaces. The singular fibers of π correspond to the points of X^0 where the T^d-action is not free; therefore

$$\mathrm{B}^{sing} = \bigcup_{i=1}^{d} \Pi'_i \times \big\{(\lambda_1, \dots, \lambda_d) \,\big|\, \lambda_i = \epsilon_i\big\},$$

where $\Pi'_i \subset \mathbf{R}^n$ is essentially the amoeba of H_i. The potentially obstructed fibers of π : $\mathrm{X}^0 \to \mathrm{B}$ are precisely those that intersect $p_{\mathrm{V}}^{-1}(\mathrm{H}_1 \cup \cdots \cup \mathrm{H}_d)$, and for each $\vec{\alpha} \in \mathrm{A}$ we have an open subset $\mathrm{U}_{\vec{\alpha}} \subset \mathrm{B}$ of tautologically unobstructed fibers which project under p to standard product tori in $\mathrm{V}^0 \times \mathbf{C}^d$.

Each of the components $\mathrm{U}_{\vec{\alpha}} \subset \mathrm{B}$ determines an affine coordinate chart $\mathrm{U}^\vee_{\vec{\alpha}}$ in the SYZ mirror to X^0. Namely, for $b \in \mathrm{U}_{\vec{\alpha}} \subset \mathrm{B}$, the Lagrangian torus $\mathrm{L} = \pi^{-1}(b) \subset \mathrm{X}^0$ is the preimage by p of a standard product torus in $\mathrm{V} \times \mathbf{C}^d$. Denoting by $(\zeta_1, \dots, \zeta_n, \lambda_1, \dots, \lambda_d) \in \Delta_{\mathrm{V}} \times \mathbf{R}^d_+$ the corresponding value of the moment map of $\mathrm{V} \times \mathbf{C}^d$, and by $(\gamma_1, \dots, \gamma_n, \gamma_{0,1}, \dots, \gamma_{0,d})$ the natural basis of $\mathrm{H}_1(\mathrm{L}, \mathbf{Z})$, we equip $\mathrm{U}^\vee_{\vec{\alpha}}$ with the coordinate system

$$\begin{aligned}(\mathbf{11.9}) \qquad (\mathrm{L}, \nabla) \mapsto\ & (v_{\vec{\alpha},1}, \dots, v_{\vec{\alpha},n}, w_{0,1}, \dots, w_{0,d}) \\ & := \big(\mathrm{T}^{\zeta_1}\nabla(\gamma_1), \dots, \mathrm{T}^{\zeta_n}\nabla(\gamma_n), \mathrm{T}^{\lambda_1}\nabla(\gamma_{0,1}), \dots, \mathrm{T}^{\lambda_d}\nabla(\gamma_{0,d})\big).\end{aligned}$$

For $b \in \mathrm{U}_{\vec{\alpha}}$, the Maslov index 2 holomorphic discs bounded by $\mathrm{L} = \pi^{-1}(b)$ in X can be determined explicitly as in Section 5, by projecting to $\mathrm{V} \times \mathbf{C}^d$. Specifically, these discs intersect the proper transform of exactly one of the toric divisors transversely in a single point, and there are two cases:

Lemma **11.3.** — *For any* $i = 1, \dots, d$, L *bounds a unique family of Maslov index* 2 *holomorphic discs in* X *which intersect the proper transform of* $\mathrm{V} \times \mathbf{C}^{d-1}_i = \{y_i = 0\}$ *transversely in a single point; the images of these discs under* p *are contained in lines parallel to the* y_i *coordinate axis, and their contribution to the superpotential is* $w_{0,i}$.

Lemma **11.4.** — *For any* $j = 1, \dots, r$, *denote by* D_{σ_j} *the toric divisor in* V *associated to the ray* σ_j *of the fan* Σ_{V}, *and let* $k_i = \langle \alpha^i - \alpha^i_{\min}(\sigma_j), \sigma_j\rangle$ $(i = 1, \dots, d)$. *Then* L *bounds* $2^{k_1+\cdots+k_d}$ *families of Maslov index* 2 *holomorphic discs in* X *which intersect the proper transform of* $\mathrm{D}_{\sigma_j} \times \mathbf{C}^d$ *transversely in a single point (all of which have the same projections to* V*), and their total contribution to the superpotential is*

$$\left(\prod_{i=1}^{d} \big(1 + \mathrm{T}^{-\epsilon_i} w_{0,i}\big)^{k_i}\right) \mathrm{T}^{\varpi_i} \mathbf{v}^{\sigma_j}_{\vec{\alpha}}.$$

The proofs are essentially identical to those of Lemmas 5.5 and 5.6, and left to the reader. As in Section 5, the first lemma implies that the coordinates $w_{0,i}$ agree on all charts $\mathrm{U}^\vee_{\vec{\alpha}}$, and the second one implies that the coordinates $v_{\vec{\alpha},i}$ transform according to (11.5). The first two statements in Theorem 11.1 follow.

The last statement in the theorem follows from equipping X with the superpotential $\mathrm{W}^\vee = y_1 + \cdots + y_d : \mathrm{X} \to \mathbf{C}$, which has Morse-Bott singularities along the intersection of the proper transform of $\mathrm{V} \times 0$ with the d exceptional divisors, i.e. $\mathrm{crit}(\mathrm{W}^\vee) \simeq$

$H_1 \cap \cdots \cap H_d$. As in Section 7, the nontriviality of the normal bundle forces us to twist the Fukaya category of $(X, W^\vee)$ by a background class $s \in H^2(X, \mathbf{Z}/2)$, in this case Poincaré dual to the sum of the exceptional divisors (or equivalently to the sum of the proper transforms of the toric divisors $V \times \mathbf{C}_i^{d-1}$). The thimble construction then provides a fully faithful A_∞-functor from $\mathcal{F}(H_1 \cap \cdots \cap H_d)$ to $\mathcal{F}_s(X, W^\vee)$. The twisting affects the superpotential by changing the signs of the terms $w_{0,1}, \ldots, w_{0,d}$. Moreover, the thimble functor modifies the value of the superpotential by an additive constant, which equals $T^{\epsilon_1} + \cdots + T^{\epsilon_d}$ when V is affine (the i-th term corresponds to a family of small discs of area ϵ_i in the normal direction to H_i). Putting everything together, the result follows by a straightforward adaptation of the arguments in Section 7.

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Appendix A: Moduli of objects in the Fukaya category

A.1 *General theory.* — Let L be an embedded spin Lagrangian of vanishing Maslov class in the Kähler manifold $X^0 = X \setminus D$, where D is an anticanonical divisor which satisfies Assumption 2.2. We begin with a brief overview of the results of [19], which in part implement the constructions of [20] in the setting of de Rham cohomology.

For each positive real number E, Fukaya defines a *curved* A_∞ structure on the de Rham cochains with coefficients in Λ_0/T^E, which we denote by

$$\Omega^*\big(L; \Lambda_0/T^E\Lambda_0\big) \equiv \Omega^*(L; \mathbf{R}) \otimes_{\mathbf{R}} \Lambda_0/T^E\Lambda_0.$$

The operations are obtained from the moduli space of holomorphic discs in $X^0 = X \setminus D$ with boundary on L, whose energy is bounded by E. By induction, one obtains an unbounded sequence of real numbers E_i, together with formal diffeomorphisms on $\Omega^*(L; \Lambda_0/T^{E_i}\Lambda_0)$ which pull back the A_∞ structure constructed from discs of energy bounded by E_i to the projection of the A_∞ structure on $\Omega^*(L; \Lambda_0/T^{E_{i+1}}\Lambda_0)$ modulo T^{E_i}. After applying such a formal diffeomorphism, we may therefore assume that the A_∞ map

$$\Omega^*\big(L; \Lambda_0/T^{E_{i+1}}\Lambda_0\big) \to \Omega^*\big(L; \Lambda_0/T^{E_i}\Lambda_0\big)$$

is defined by projection of coefficient rings. Taking the inverse limit over E_i, we obtain an A_∞ structure on $\Omega^*(L; \Lambda_0)$. By passing to the canonical model (i.e. applying a filtered version of the homological perturbation lemma [31]), we can reduce this A_∞ structure to $H^*(L; \Lambda_0)$.

Fukaya checks that any class $b \in H^1(L; U_\Lambda)$ defines a deformed A_∞ structure on the cohomology. In particular, there is a subset

$$\hat{\mathcal{Y}}_L \subset H^1(L; U_\Lambda)$$

consisting of elements for which this A_∞ structure has vanishing curvature (i.e. solutions to the Maurer-Cartan equation). Gauge transformations [20, Section 4.3] define an equivalence relation on this set; we call the quotient the *moduli space of simple objects supported on* L, which we denote $\mathcal{Y}_L$.

Remark **A.1.** — The original formalism of Fukaya, Oh, Ohta, and Ono [20] considered deformation classes corresponding to $b \in H^1(L; \Lambda_+)$, called bounding cochains, which via exponentiation $\Lambda_+ \to 1 + \Lambda_+$ can also be reinterpreted as local systems. As noted in the discussion following Theorem 1.2 of [19], there are inclusions $1 + \Lambda_+ \subset U_\Lambda \subset \Lambda^*$, and the original construction of Floer cohomology can be generalized to all unitary local systems using an idea of Cho.

The invariance statement of Floer cohomology [20, Theorem 14.1-14.3] asserts that $\mathcal{Y}_L$ does not depend on the choice of auxiliary data (e.g. almost-complex structure) in the following sense: let $\mathcal{Y}_L^1$ and $\mathcal{Y}_L^2$ denote the moduli spaces for different choices of auxiliary structures. A homotopy between the auxiliary data induces an isomorphism

$$\mathcal{Y}_L^1 \cong \mathcal{Y}_L^2 \tag{A.1}$$

which is invariant under homotopies of homotopies.

Assumption **A.2.** — *The* A_∞ *structure on* $H^*(L; \Lambda_0)$ *is isomorphic to the undeformed structure.*

Remark **A.3.** — For most Lagrangians that we consider, this condition holds automatically because there is a choice of almost complex structure for which the Lagrangian bounds no holomorphic discs which are not constant.

In this setting, the Maurer-Cartan equation vanishes identically, and the gauge equivalence relation is trivial. A choice of isomorphism of the Floer-theoretic A_∞-structure with the undeformed structure (e.g. a choice of almost complex structure for which there are no non-constant holomorphic discs) therefore yields an identification of the moduli space $\mathcal{Y}_L$ of simple objects of the Fukaya category supported on L with its first cohomology with coefficients in U_Λ:

$$\mathcal{Y}_L \equiv H^1(L; U_\Lambda).$$

Let L_t be a Hamiltonian path of Lagrangians in X^0 with vanishing Maslov class, and J_t a family of almost complex structures on X which we assume are fixed at infinity.

We describe the isomorphism (A.1) in the special situation which we consider in this paper. We first identify $H_1(L_0; \mathbf{Z}) \cong H_1(L_t; \mathbf{Z})$ via the given path. A basis for this group yields an identification

$$(z_1, \ldots, z_n) : H^1(L_0; U_\Lambda) \to U_\Lambda^n.$$

Assumption **A.4.** — *For the family* (L_t, J_t), *all stable holomorphic discs represent multiples of a given relative homology class* $\beta \in H_2(X, L_0; \mathbf{Z})$.

The wall-crossing map is then of the form

$$z_i \mapsto h_i(z_\beta) z_i, \tag{A.2}$$

where h_i is a power series with $\mathbf{Q}$ coefficients and leading order term equal to 1, and z_β denotes the monomial $T^{\omega(\beta)} z^{[\partial\beta]}$. Equation (A.2) can be extracted from the construction in Section 11 of [19]. For an explicit derivation, see [52, Lemma 4.4]: for bounding cochains, the transformation corresponds to adding a power series in z_β with vanishing constant term, and Equation (A.2) follows by exponentiation.

By Proposition 5.8, the following assumption holds in the geometric setting of the main theorem:

Assumption **A.5.** — *The power series* h_i *is the expansion of a rational function in* z_β.

In this case, the transformation in Equation (A.2) converges away from the zeroes and poles of h_i. This is stronger than the general result proved by Fukaya namely that the transformation converges in an analytic neighborhood of the unitary elements in $H^1(L; \Lambda^*)$.

In order to extend this construction to the non-Hamiltonian setting, we use the main construction of [19] which identifies the moduli space of simple objects supported on Lagrangians near L (but not necessarily Hamiltonian isotopic to it) with an affinoid domain in $H^1(L; \Lambda^*)$ in the sense of Tate.

Given a path $\{L_t\}_{t\in[0,1]}$ between Lagrangians L_0 and L_1 in which there is no wall crossing (e.g. so that no Lagrangian in the family bounds a holomorphic disc), the natural gluing map between these domains is obtained from the flux homomorphism

$$\Phi\big(\{L_t\}\big) \in H^1(L_0; \mathbf{R})$$

and the product on cohomology groups

$$H^1(L_0; \mathbf{R}) \times H^1\big(L_0; \Lambda^*\big) \to H^1\big(L_0; \Lambda^*\big)$$

induced by the map on coefficients $(\lambda, f) \mapsto T^\lambda f$. In the absence of wall crossing we identify $H^1(L_1; \Lambda^*)$ with $H^1(L_0; \Lambda^*)$ via this rescaling map.

Given a general path between Lagrangians L_0 and L_1 (subject to Assumptions A.4 and A.5), this identification is modified by the wall crossing formula given in Equation (A.2), yielding a birational map

$$H^1(L_0;\Lambda^*) \dashrightarrow H^1(L_1;\Lambda^*),$$

defined away from a hypersurface. We glue the moduli spaces of objects supported near L_0 and L_1 using this identification.

Remark **A.6.** — The construction of a map for a Lagrangian path can be reduced to the case of Hamiltonian paths as follows: any path (L_t, J) can be deformed, with fixed endpoints, to a path (L'_t, J_t) which is a concatenation of paths for which the Lagrangian is constant and paths in which there is no wall-crossing. The desired map is then obtained as a composition of the wall-crossing maps for Hamiltonian paths and the rescalings given by the flux homomorphism.

The idea for constructing the deformed path follows the main strategy for proving convergence in [19]. Whenever ϵ is sufficiently small, there is a (compactly supported) diffeomorphism ψ_ϵ taking L_t to $L_{t+\epsilon}$ which preserves the tameness of J. For tautological reasons, there is a path without wall-crossing from (L_t, J) to $(L_{t+\epsilon}, J_{t+\epsilon})$ if $J_{t+\epsilon}$ is the pullback of J by ψ_ϵ. Interpolating between this pullback and $(L_{t+\epsilon}, J)$, via pullbacks of (L_{t+s}, J), we then reach $(L_{t+\epsilon}, J)$ via a path for which the Lagrangian is constant and Assumption A.4 remains satisfied.

Remark **A.7.** — (1) More generally, given a path from L_0 to L_1 that can be decomposed into finitely many sub-paths $\{L_t\}_{t\in[t_j,t_{j+1}]}$, each satisfying Assumption A.4 for some relative class β_j, and for which the wall-crossing transformations are rational functions as in Assumption A.5, we again obtain a wall-crossing map

$$H^1(L_0;\Lambda^*) \dashrightarrow H^1(L_1;\Lambda^*) \tag{A.3}$$

by composing the maps associated to the various sub-paths.

(2) When all the classes β_j have the same boundary in $H_1(L_t, \mathbf{Z})$ and the same symplectic areas, the monomials z_{β_j} are all equal and the birational transformation (A.3) again takes the form of Equation (A.2) up to rescaling of the coefficients.

If we restrict attention to the smooth fibers of a Lagrangian torus fibration, we obtain an embedding of the moduli space $\mathcal{Y}^0_\pi$ of all simple objects supported on such Lagrangians into the rigid analytic space

$$\coprod H^1(L;\Lambda^*)/\sim \tag{A.4}$$

where the equivalence relation identifies points which correspond to each other under the birational wall-crossing transformations of Equation (A.3) induced by all paths among smooth fibers. It does not automatically follow from the above considerations that this quotient is a well-behaved (e.g. separated) analytic space, but in our case this will not be an issue. By the invariance of Floer cohomology [20, Theorem 14.1-14.3], the transformations induced by homotopic paths are equal. The fact that these transformations should in general depend only on the homotopy class of the path in the space of all fibers (i.e. allowing fibers which are not necessarily embedded), is expected to follow as a consequence of forthcoming developments in the study of family Floer cohomology in the presence of singular fibers. In our main example, this independence will be manifest from Proposition 5.8, and the quotient (A.4) can easily be seen to be a smooth analytic space.

Remark **A.8.** — We can think of (A.4) as the natural (analytic) completion of $\mathcal{Y}^0_\pi$. While the points of this completion do not necessarily correspond to unitary local systems on Lagrangians in X^0 with the given Kähler form, in good situations, they can be interpreted as Lagrangians in X^0 equipped with a completed Kähler form. Slightly strengthening Assumption 2.2 by requiring that X^0 be the complement of a nef divisor, we can obtain such a completion by inflation along the divisor at infinity.

It shall be convenient for our purposes to consider a completion which is obtained by gluing only finitely many charts. To this end, assume that $\{L_t\}_{t\in[0,1]}$ is a path of Lagrangians so that the wall-crossing map defines an embedding

$$H^1(L_0; U_\Lambda) \hookrightarrow H^1\big(L_1; \Lambda^*\big). \tag{A.5}$$

In this case, the above construction yields that all elements of $\mathcal{Y}_{L_0}$ can be represented in Equation (A.4) by elements of $H^1(L_1; \Lambda^*)$.

More generally, assume that $\{L_\alpha\}_{\alpha\in A}$ is a collection of fibers with the property that for some fixed almost complex structure J, any smooth fiber L can be connected to some fiber L_α in our collection by a path such that the wall-crossing map defines an embedding $H^1(L; U_\Lambda) \to H^1(L_\alpha; \Lambda^*)$. We define

$$\hat{\mathcal{Y}}^0_\pi \equiv \coprod_{\alpha\in A} H^1\big(L_\alpha; \Lambda^*\big)/\sim . \tag{A.6}$$

Lemma **A.9.** — *There is a natural analytic embedding of* $\mathcal{Y}^0_\pi$ *into* $\hat{\mathcal{Y}}^0_\pi$.

Next, we study the moduli spaces of holomorphic discs in X with boundary on a Lagrangian $L \subset X^0$ of vanishing Maslov class. Since D is an anticanonical divisor, stable holomorphic discs whose intersection number with D is 1 have Maslov index equal to 2. Assumption 2.2 implies that there are no discs of negative Maslov index, and that those of vanishing Maslov index are disjoint from D. For each unitary local system ∇ on L, choice of almost complex structure J, and action cutoff E we obtain a $\Lambda_0/T^E\Lambda_0$-valued de Rham cochain

$$\sum_{\substack{\beta\in\pi_2(X,L)\\ \beta\cdot D=1}} z_\beta(L,\nabla)\mathrm{ev}_*\big[\mathcal{M}_1(L,\beta,J)\big]\in\Omega^0\big(L;\Lambda_0/T^E\Lambda_0\big) \tag{A.7}$$

which is closed with respect to the Floer differential. Passing to the canonical model and to the inverse limit over E we obtain a multiple of the unit in the self-Floer cohomology of (L,∇):

$$\mathfrak{m}_0(L,\nabla,J)=W(L,\nabla,J)e_L\in H^0(L;\Lambda). \tag{A.8}$$

Since the moduli spaces of discs of vanishing Maslov index in X and in $X\setminus D$ agree, the invariance of Floer theory and in particular of the potential function [20, Theorem B], as extended to non-unitary local systems in [19], implies that $W(L,\nabla,J)$ gives rise to a well-defined convergent function on $\mathcal{Y}^0_\pi$. Because of this, we shall henceforth drop J from the notation. For non-unitary local systems, $W(L,\nabla)$ may not in general converge, so we have to impose this as an additional assumption. With this in mind, the proof of the following result follows from the unitary case by Remark A.6.

Lemma **A.10.** — *If for each* $\alpha\in A$, *the map* $\nabla\mapsto W(L_\alpha,\nabla)$ *converges on* $H^1(L_\alpha;\Lambda^*)$, *then* W *defines a regular function on* $\hat{\mathcal{Y}}^0_\pi$.

We record the following consequence:

Corollary **A.11.** — *If* (L_i,∇_i) *and* (L_j,∇_j) *are identified by a wall-crossing gluing map, then* $W(L_i,\nabla_i)=W(L_j,\nabla_j)$.

Remark **A.12.** — Fukaya has announced that rank 1 unitary local systems on immersed Lagrangians which are fibers of π define a rigid analytic space which includes $\hat{\mathcal{Y}}^0_\pi$ as an analytic subset. The general idea is to describe the nearby smooth fibers as the result of Lagrangian surgery, and understand the behavior of holomorphic discs under such surgeries sufficiently explicitly to produce an analytic structure on this neighborhood which can be seen to be compatible with the analytic structure on $\hat{\mathcal{Y}}^0_\pi$.

We expect that, in the presence of a potential function, similar ideas can be applied to associate analytic charts to certain admissible non-compact Lagrangians arising as limits of smooth fibers. While we do not develop the general theory in this paper, Example 2.4 explains how one can use equivalences in the Fukaya category (rather than surgery formulae) to produce the desired charts in the class of examples we encounter.

A.2 *Convergence of the wall-crossing.* — In this section, we verify that the assumptions of Lemma A.9 hold for the smooth fibers of the map $\pi : X^0\to B$ introduced in Definition 4.4. Recall that the moment map μ_X of the S^1-action descends to a natural map from B to $\mathbf{R}_+$; we write $X^0_\lambda = \mu_X^{-1}(\lambda) \cap X^0$. If ϵ is the blowup parameter in the definition of X, then all fibers of π contained in X^0_λ are smooth whenever $\lambda \neq \epsilon$; and the smooth fibers in X^0_ϵ are exactly those whose image under the blowdown map $p : X^0 \to V^0 \times \mathbf{C}$ is disjoint from $H \times \mathbf{C}$.

Assumption A.2 follows immediately from Proposition 5.1 for all fibers of π whose images under p are disjoint from $H \times \mathbf{C}$, since these bound no holomorphic discs. In general, invariance of Floer cohomology shows that Assumption A.2 is independent of the choice of almost complex structure. Moreover, the identification of the A_∞ structure obtained by deforming by an element in $H^1(L; \Lambda_+)$ with the deformed Floer theory for the associated local system in $H^1(L; 1 + \Lambda_+)$ implies that Assumption A.2 holds for the Floer theory of L equipped with unitary local systems as well, since an analytic function vanishing on $1 + \Lambda_+$ must vanish on all of U_Λ. The same argument shows that the A_∞ structure on L equipped with a non-unitary local system is also undeformed, as long as the valuation is sufficiently small. By Fukaya's work on Family Floer cohomology [19], we conclude that the A_∞ structure on a Lagrangian fibre L' sufficiently close to L is undeformed. Here, sufficiently close means that there is a diffeomorphism preserving the tameness of J and moving L to L'; in compact subsets of the space of smooth fibers, there are uniform bounds on the size of such neighborhoods, so we conclude that the condition of having undeformed A_∞ structure is open and closed among smooth fibers of π. Therefore, all smooth fibers of π satisfy Assumption A.2.

We next choose Lagrangians $\{L_\alpha\}_{\alpha \in A}$, labelled by the monomials in the equation defining the hypersurface H. We require that L_α be contained in X^0_ϵ, and that its projection to B lie in the chamber $U_\alpha \subset B$ (see Definition 5.3).

Lemma **A.13.** — *Any smooth fiber* L *of* π *can be connected to some fiber* L_α *so that the wall-crossing map defines an embedding*

$$H^1(L; U_\Lambda) \to H^1\big(L_\alpha; \Lambda^*\big). \tag{A.9}$$

Proof. — There are two cases to consider:

Case 1: Assume that the smooth fiber L lies in X^0_ϵ. Then $\pi_\epsilon(L)$ lies outside of the amoeba of H (cf. Equation (4.4)) and L is tautologically unobstructed (cf. Proposition 5.1). By Remark 5.4, the component of the complement of the amoeba over which L lies determines a chamber U_α, and L can be connected to L_α by a path of tautologically unobstructed fibers. The absence of holomorphic discs in this region implies that there are no non-trivial walls, and hence that the map

$$H^1\big(L; \Lambda^*\big) \to H^1\big(L_\alpha; \Lambda^*\big) \tag{A.10}$$

is given simply by a rescaling of the coefficients (see the discussion following Equation (A.2)). This completes the argument in this case.

Case 2: Assume that L lies in X^0_λ, with $\lambda \neq \epsilon$. Choose a smooth fiber L^λ_α which is also contained in X^0_λ and whose projection lies in some chamber U_α, and consider the concatenation of a path from L to L_α^λ via Lagrangians contained in X_λ^0 with a path from L_α^λ to L_α over the chamber U_α. Since the map associated to the latter path is a simple rescaling as in the previous case, it suffices to show convergence of the wall-crossing map for the path from L to L_α^λ.

To this end, recall from Proposition 5.1 that the simple holomorphic discs bounded by the Lagrangian torus fibers along the path all have the same area $|\lambda-\epsilon|$ and their boundaries all represent the same homology class in $H_1(L_t, \mathbf{Z})$. Thus, the monomials $z_\beta = T^{\omega(\beta)} z^{[\partial\beta]}$ associated to their homology classes are all equal, and by Remark A.7 (2) the wall crossing map is of the form

$$z_i \mapsto h_i(z_\beta) z_i, \tag{A.11}$$

where h_i is a power series with coefficients in $\mathbf{Q}$ and leading order term equal to 1. Whenever we evaluate at a point of $H^1(L; U_\Lambda)$, the valuation of z_β is $|\lambda-\epsilon|>0$, and so $h_i(z_\beta)$ and its inverse both converge and take values in U_Λ. Thus the leading order term of (A.11) is identity, and the wall-crossing map defines an embedding

$$H^1(L; U_\Lambda) \hookrightarrow H^1\big(L_\alpha^\lambda; \Lambda^*\big).$$

Composing this map with the rescaling isomorphism induced by the flux homomorphism of a path over U_α, we arrive at the desired result. □

Appendix B: The geometry of the reduced spaces

In this section we study in more detail the symplectic geometry of the reduced spaces $X_{red,\lambda} = \mu_X^{-1}(\lambda)/S^1$ and prove Lemma 4.1.

Recall from Section 4.1 that the moment map for the S^1-action on X is given by (4.1), and that the only fixed points apart from $\tilde{V} = \mu_X^{-1}(0)$ occur along $\tilde{H}$, which lies in the level set $\mu_X^{-1}(\epsilon)$. Also recall that, for all $\lambda>0$, the natural projection to V (obtained by composing $p: X \to V \times \mathbf{C}$ with projection to the first factor) yields a natural identification of $X_{red,\lambda}$ with V.

We will exploit the toric structure of V to construct families of Lagrangian tori in $X_{red,\lambda}$ equipped with the reduced Kähler form $\omega_{red,\lambda}$. The two obstacles are (1) the lack of smoothness along H at $\lambda=\epsilon$, and (2) the lack of T^n-invariance near H.

We start with the first issue, giving a formula for $\omega_{red,\lambda}$ near $\tilde{H}$ and introducing an explicit family of smoothings. Consider a small neighborhood of $\tilde{H}$ where, without loss of generality, we may assume that $\chi \equiv 1$.

Lemma **B.1.** — *Identifying $X_{red,\lambda}$ with V as above, where $\chi \equiv 1$ we have*

$$\omega_{red,\lambda} = \omega_V - \max(0, \epsilon-\lambda) c_1(\mathcal{L}) + d\alpha_{0,\lambda}, \tag{B.1}$$

where $c_1(\mathcal{L}) = i\mathrm{F}_{\mathcal{L}}/2\pi$ *is the Chern form of the chosen Hermitian metric on* $\mathcal{L}$, *and*

$$\alpha_{0,\lambda} = \frac{\min(\lambda,\epsilon)d^c(|f(\mathbf{x})|^2)}{2(\sqrt{4\pi\epsilon|f(\mathbf{x})|^2 + (\lambda-\epsilon+\pi|f(\mathbf{x})|^2)^2} + \pi|f(\mathbf{x})|^2 + |\lambda-\epsilon|)}. \tag{B.2}$$

Proof. — Recall that away from $\tilde{\mathrm{V}}$ we can write X as a conic bundle $f(\mathbf{x}) = yz$. Where $f \neq 0$ and $\chi \equiv 1$, the restriction of ω_ϵ to $\mu_{\mathrm{X}}^{-1}(\lambda)$ is equal to

$$p_{\mathrm{V}}^*\omega_{\mathrm{V}} + d\left(\frac{1}{4}|y|^2 d^c\big(\log|y|^2\big) + \frac{\epsilon}{4\pi}\frac{|z|^2}{1+|z|^2}d^c\big(\log|z|^2\big)\right).$$

Since $d^c\log|y|^2 + d^c\log|z|^2 = d^c\log|f|^2$, using (4.2) we can rewrite the 1-form in this expression as either

$$\frac{1}{4}|y|^2 d^c\big(\log|f|^2\big) + \frac{\epsilon-\lambda}{4\pi}d^c\big(\log|z|^2\big) \quad \text{or}$$
$$\frac{\epsilon}{4\pi}\frac{|z|^2}{1+|z|^2}d^c\big(\log|f|^2\big) + \frac{\lambda-\epsilon}{4\pi}d^c\big(\log|y|^2\big).$$

Now $dd^c\log|y|^2 = 0$, whereas $dd^c\log|z|^2 = -4\pi p_{\mathrm{V}}^*c_1(\mathcal{L})$, so we find that (still where $f\neq 0$ and $\chi \equiv 1$)

$$\begin{aligned}(\omega_\epsilon)_{|\mu_{\mathrm{X}}^{-1}(\lambda)} &= p_{\mathrm{V}}^*\big(\omega_{\mathrm{V}} + (\lambda-\epsilon)c_1(\mathcal{L})\big) + d\left(\frac{d^c(|f(\mathbf{x})|^2)}{4|z|^2}\right)\\ &= p_{\mathrm{V}}^*\omega_{\mathrm{V}} + d\left(\frac{\epsilon}{4\pi}\frac{d^c(|f(\mathbf{x})|^2)}{|y|^2+|f(\mathbf{x})|^2}\right).\end{aligned} \tag{B.3}$$

The first expression makes sense wherever $z\neq 0$, in particular for $\lambda < \epsilon$; the second one makes sense wherever $y \neq 0$, in particular for $\lambda > \epsilon$. Solving (4.2) for $|y|$, we obtain

$$2\pi|y|^2 = \sqrt{4\pi\epsilon|f(\mathbf{x})|^2 + \big(\lambda-\epsilon+\pi|f(\mathbf{x})|^2\big)^2} - \pi|f(\mathbf{x})|^2 + (\lambda-\epsilon), \quad \text{and}$$
$$2\lambda|z|^2 = \sqrt{4\pi\epsilon|f(\mathbf{x})|^2 + \big(\lambda-\epsilon+\pi|f(\mathbf{x})|^2\big)^2} + \pi|f(\mathbf{x})|^2 - (\lambda-\epsilon).$$

Substituting into (B.3) gives the desired expression. □

We can smooth the singularity of $\omega_{red,\lambda}$ by considering the modified Kähler forms given near H by

$$\omega_{sm,\lambda} = \omega_{\mathrm{V}} - \max(0,\epsilon-\lambda)c_1(\mathcal{L}) + d\alpha_{\kappa,\lambda}$$

where $\kappa > 0$ is an arbitrarily small constant, and

$$\alpha_{t,\lambda} = \frac{\min(\lambda,\epsilon)d^c(|f(\mathbf{x})|^2)}{2(\sqrt{4\pi\epsilon|f(\mathbf{x})|^2 + (\lambda-\epsilon+\pi|f(\mathbf{x})|^2)^2 + t^2\tilde{\chi}} + \pi|f(\mathbf{x})|^2 + |\lambda-\epsilon|)}, \tag{B.4}$$

where $\tilde{\chi} = \tilde{\chi}(|f(\mathbf{x})|, \lambda)$ is a suitable cut-off function which equals 1 near $\tilde{\mathrm{H}}$ and vanishes outside of the region where $\chi \equiv 1$. (We can also assume that $\tilde{\chi}$ vanishes whenever λ is not close to ϵ.) We set $\omega_{sm,\lambda} = \omega_{red,\lambda}$ wherever $\chi \neq 1$. Choosing κ small enough ensures that $\omega_{\mathrm{V}} - \max(0, \epsilon - \lambda)c_1(\mathcal{L}) + d\alpha_{t,\lambda}$ is non-degenerate for all $t \in [0, \kappa]$; it is then a Kähler form, because $\alpha_{t,\lambda}$ can be written as d^c of some function of $|f(\mathbf{x})|$.

The Kähler forms $\omega_{sm,\lambda}$ are all smooth, coincide with $\omega_{red,\lambda}$ away from H for all λ, and everywhere when λ is not very close to ϵ. Moreover, $[\omega_{sm,\lambda}] = [\omega_{red,\lambda}]$ by construction, and the dependence of $\omega_{sm,\lambda}$ on λ is piecewise smooth.

Like $\omega_{red,\lambda}$, the Kähler form $\omega_{sm,\lambda}$ is not invariant under the given torus action, but there exist toric Kähler forms in the same cohomology class. Such a Kähler form $\omega'_{\mathrm{V},\lambda}$ can be constructed by averaging $\omega_{sm,\lambda}$ with respect to the standard T^n-action on V:

$$\omega'_{\mathrm{V},\lambda} = \frac{1}{(2\pi)^n} \int_{g \in \mathrm{T}^n} g^* \omega_{sm,\lambda} dg. \tag{B.5}$$

To see that the T^n-orbits are Lagrangian with respect to $\omega'_{\mathrm{V},\lambda}$, we note that the pullback of $\omega_{sm,\lambda}$ to an orbit represents the trivial cohomology class, since the classes $[\omega_{\mathrm{V}}]$ and [H] are both trivial on a torus fibre. The pullback of $\omega'_{\mathrm{V},\lambda}$ is therefore also trivial in cohomology, but since it is invariant, it must vanish pointwise. This in turn implies that the T^n-action not only preserves $\omega'_{\mathrm{V},\lambda}$ but in fact it is Hamiltonian.

We now state again Lemma 4.1 and give its proof:

Lemma **B.2.** — *There exists a family of homeomorphisms* $(\phi_\lambda)_{\lambda \in \mathbf{R}_+}$ *of* V *such that:*

(1) ϕ_λ *preserves the toric divisor* $\mathrm{D}_{\mathrm{V}} \subset \mathrm{V}$*;*
(2) *the restriction of* ϕ_λ *to* V^0 *is a diffeomorphism for* $\lambda \neq \epsilon$*, and a diffeomorphism outside of* H *for* $\lambda = \epsilon$*;*
(3) ϕ_λ *intertwines the reduced Kähler form* $\omega_{red,\lambda}$ *and the toric Kähler form* $\omega'_{\mathrm{V},\lambda}$*;*
(4) $\phi_\lambda = \mathrm{id}$ *at every point whose* T^n*-orbit is disjoint from the support of* χ*;*
(5) ϕ_λ *depends on* λ *in a continuous manner, and smoothly except at* $\lambda = \epsilon$*.*

Proof. — We proceed in two stages, obtaining ϕ_λ as the composition of two maps $\phi_{sm,\lambda}$, taking $\omega_{red,\lambda}$ to $\omega_{sm,\lambda}$, and $\phi_{avg,\lambda}$ taking $\omega_{sm,\lambda}$ to $\omega'_{\mathrm{V},\lambda}$, each satisfying all the other conditions in the statement. The arguments are quite similar in both cases; we start with the construction of $\phi_{avg,\lambda}$ (Steps 1–2), then proceed with $\phi_{sm,\lambda}$ (Steps 3–4).

Step 1. Let $\beta_\lambda = \omega_{sm,\lambda} - \omega'_{\mathrm{V},\lambda}$. Since $\omega'_{\mathrm{V},\lambda}$ is T^n-invariant, for $\theta \in \mathfrak{t}^n \simeq \mathbf{R}^n$ we have

$$\begin{aligned} \exp(\theta)^*\omega_{sm,\lambda} - \omega_{sm,\lambda} = \exp(\theta)^*\beta_\lambda - \beta_\lambda &= \int_0^1 \frac{d}{dt}\big(\exp(t\theta)^*\beta_\lambda\big)dt \\ &= d\left[\int_0^1 \exp(t\theta)^*(\iota_{\theta_\#}\beta_\lambda)dt\right]. \end{aligned}$$

Hence, averaging over all elements of T^n, we see that the 1-form

$$a'_\lambda = \frac{1}{(2\pi)^n}\int_{[-\pi,\pi]^n}\int_0^1 \exp(t\theta)^*(\iota_{\theta_\#}\beta_\lambda)\,dt\,d\theta$$

satisfies $\omega'_{V,\lambda} - \omega_{sm,\lambda} = da'_\lambda$ (i.e., $da'_\lambda = -\beta_\lambda$).

Let $U \subset V$ be the orbit of the support of χ under the standard T^n-action on $X_{red,\lambda} \cong V$. Outside of U, the Kähler forms $\omega_{sm,\lambda} = \omega_{red,\lambda}$ are T^n-invariant, and $\omega_{sm,\lambda}$ and $\omega'_{V,\lambda}$ coincide (in fact they both coincide with ω_V). Therefore, β_λ is supported in U, and consequently so is a'_λ.

Let $\omega'_{t,\lambda} = t\omega'_{V,\lambda} + (1-t)\omega_{sm,\lambda}$ (for $t \in [0,1]$ these are Kähler forms since $\omega'_{V,\lambda}$ and $\omega_{sm,\lambda}$ are Kähler). Denote by v_t the vector field such that $\iota_{v_t}\omega'_{t,\lambda} = -a'_\lambda$ and by ψ_t the flow generated by v_t. Then by Moser's trick,

$$\frac{d}{dt}\left(\psi_t^*\omega'_{t,\lambda}\right) = \psi_t^*\left(L_{v_t}\omega'_{t,\lambda} + \frac{d\omega'_{t,\lambda}}{dt}\right) = \psi_t^*\left(d\iota_{v_t}\omega'_{t,\lambda} + da'_\lambda\right) = 0,$$

so $\psi_t^*\omega'_{t,\lambda} = \omega_{sm,\lambda}$, and the time 1 flow ψ_1 intertwines $\omega_{sm,\lambda}$ and $\omega'_{V,\lambda}$ as desired. Moreover, because a'_λ is supported in U, outside of U we have $\psi_t = \mathrm{id}$. However, it is not clear that the flow preserves the toric divisors of V.

Step 2. To remedy the issue with the flow not preserving the toric divisors, we modify a'_λ in a neighborhood of D_V. Let $f'_{\lambda,t}$ be a family of C^1 real-valued functions (with locally Lipschitz first derivatives), smooth on V^0, with the following properties:

- the support of $f'_{\lambda,t}$ is contained in the intersection of U with a small tubular neighborhood of D_V;
- at every point $x \in D_V$, belonging to a toric stratum $S \subset V$,

(**B.6**) the 1-form $a'_\lambda + df'_{\lambda,t}$ vanishes on $(T_xS)^\perp$,

  where the orthogonal is with respect to $\omega'_{t,\lambda}$;
- $f'_{\lambda,t}$ depends smoothly on t, and piecewise smoothly on λ.

We construct $f'_{\lambda,t}$ by induction over toric strata of increasing dimension, successively constructing functions $f'_{\lambda,t,\le k} : V \to \mathbf{R}$ which satisfy (B.6) for all strata of dimension at most k and are smooth outside of strata of dimension $< k$. We start by setting $f'_{\lambda,t,\le 0} = 0$, which satisfies (B.6) at the fixed points of the torus action since they lie away from the support of a'_λ.

Assume $f'_{\lambda,t,\le k}$ constructed, and consider a stratum S of dimension $k+1$. At each point $x \in S$, the restriction of $a'_\lambda + df'_{\lambda,t,\le k}$ to $(T_xS)^\perp$ is a real-valued linear form, vanishing whenever x belongs to a lower-dimensional stratum, and smooth outside of strata of dimension $< k$. Let $f'^0_{\lambda,t,S}$ be a C^1 function on a neighborhood of S, smooth outside of the strata of dimension $\le k$, which vanishes on S and whose derivative in the normal directions at each point of S satisfies $(df'^0_{\lambda,t,\mathrm{S}})_{|(\mathrm{T}_x\mathrm{S})^\perp} = -(a'_\lambda + df'_{\lambda,t,\le k})_{|(\mathrm{T}_x\mathrm{S})^\perp}$. (For instance, identify a neighborhood of S with a subset of its normal bundle in a manner compatible with the toric structure, and take $f'^0_{\lambda,t,\mathrm{S}}$ to be linear in the fibers.)

Let χ_S be a cut-off function with values in $[0,1]$, defined and smooth outside of the strata of dimension $\le k$, equal to 1 at all points of a neighborhood of S which are much closer to S than to any other $(k+1)$-dimensional stratum, and with support disjoint from those of the corresponding cut-off functions for all other $(k+1)$-dimensional strata. Specifically, picking an auxiliary metric, we take χ_S to be the product of a standard smooth cut-off function supported in a tubular neighborhood of S with functions $\chi_{\mathrm{S}/\Sigma}$ for all strata Σ with $\dim \Sigma \ge k+1$ and $\dim(\Sigma \cap \mathrm{S}) \le k$, chosen so that $\chi_{\mathrm{S}/\Sigma}$ equals 1 except near Σ, where it depends on the ratio between distance to S and distance to Σ, equals 1 at all points that lie much closer to S than to Σ, and vanishes at all points that lie closer to Σ than to S.

We note that near a lower-dimensional stratum S′, the norm of $d\chi_\mathrm{S}$ is bounded by a constant over distance to S′. We then set $f'_{\lambda,t,\mathrm{S}} = \chi_\mathrm{S} f'^0_{\lambda,t,\mathrm{S}}$. By construction, this function is smooth away from strata of dimension $\le k$. Moreover, near a lower-dimensional stratum S′, $f'^0_{\lambda,t,\mathrm{S}}$ is bounded by a constant multiple of distance to S times distance to S′, so the regularity of $f'_{\lambda,t,\mathrm{S}}$ is indeed as desired.

By construction, $f'_{\lambda,t,\le k+1} = f'_{\lambda,t,\le k} + \sum_{\dim \mathrm{S}=k+1} f'_{\lambda,t,\mathrm{S}}$ has the desired properties on all strata of dimension $\le k+1$. (Note that, since a'_λ vanishes outside of U, so do the various functions we construct.) Finally, we let $f'_{\lambda,t} = f'_{\lambda,t,\le n-1}$.

We now use Moser's trick again, replacing a'_λ by $\tilde{a}'_{t,\lambda} = a'_\lambda + df'_{\lambda,t}$. Namely, denote by $\tilde{v}_{t,\lambda}$ the vector field such that $\iota_{\tilde{v}_{t,\lambda}} \omega'_{t,\lambda} = -\tilde{a}_{t,\lambda}$. This vector field is locally Lipschitz continuous along $\mathrm{D_V}$, and smooth on V^0; moreover, by construction it is supported in U and, by (B.6), tangent to each stratum of $\mathrm{D_V}$. We thus obtain $\phi_{avg,\lambda}$ with all the desired properties by considering the time 1 flow generated by $\tilde{v}_{t,\lambda}$. (Note: because we have assumed that ω_V defines a complete Kähler metric on V, it is easy to check that even when V is noncompact the time 1 flow is well-defined.)

Step 3. We now turn to the construction of $\phi_{sm,\lambda}$. We interpolate between $\omega_{red,\lambda}$ and $\omega_{sm,\lambda}$ via the family of Kähler forms $\omega_{t,\lambda}$, $t \in [0,\kappa]$, defined by

$$\omega_{t,\lambda} = \omega_\mathrm{V} - \max(0, \epsilon - \lambda) c_1(\mathcal{L}) + d\alpha_{t,\lambda}$$

where $\chi \equiv 1$ (where $\alpha_{t,\lambda}$ is given by (B.4)) and $\omega_{t,\lambda} = \omega_{red,\lambda}$ wherever $\chi \neq 1$.

These Kähler forms are smooth whenever $t > 0$ or $\lambda \neq \epsilon$. Let $a_{t,\lambda}$ be the 1-form with support contained in the region where $\chi \equiv 1$, and defined by $a_{t,\lambda} = d\alpha_{t,\lambda}/dt$ inside that region. By construction, $d\omega_{t,\lambda}/dt = da_{t,\lambda}$. We use Moser's trick again, and denote by $v_{t,\lambda}$ the vector field such that $\iota_{v_{t,\lambda}} \omega_{t,\lambda} = -a_{t,\lambda}$. This vector field vanishes outside of U, and is smooth except for $t = 0$ and $\lambda = \epsilon$, in which case it is singular along H. We will momentarily check that the flow of $v_{t,\lambda}$ is well-defined even for $\lambda = \epsilon$; the time κ flow then intertwines $\omega_{red,\lambda}$ and $\omega_{sm,\lambda}$ as desired, except it need not preserve the toric divisors of V, an issue which we will address in Step 4 below.

Differentiating (B.4) with respect to t, we have

$$a_{t,\lambda} = \frac{t\tilde{\chi}\min(\lambda,\epsilon)d^c(|f(\mathbf{x})|^2)}{2\sqrt{\Phi}(\sqrt{\Phi}+\pi|f(\mathbf{x})|^2+|\lambda-\epsilon|)^2}, \tag{B.7}$$

where

$$\Phi = 4\pi\epsilon|f(\mathbf{x})|^2 + \left(\lambda-\epsilon+\pi|f(\mathbf{x})|^2\right)^2 + t^2\tilde{\chi}. \tag{B.8}$$

Taking the dual vector field, we find that

$$v_{t,\lambda} = \frac{t\tilde{\chi}\min(\lambda,\epsilon)\nabla^{t,\lambda}(|f(\mathbf{x})|^2)}{2\sqrt{\Phi}(\sqrt{\Phi}+\pi|f(\mathbf{x})|^2+|\lambda-\epsilon|)^2}, \tag{B.9}$$

where $\nabla^{t,\lambda}$ is the gradient with respect to the Kähler metric determined by $\omega_{t,\lambda}$.

We restrict our attention to the neighborhood of $\tilde{\mathrm{H}}$ where $\tilde{\chi}\equiv 1$, since it is clear that $v_{t,\lambda}$ is well-defined and smooth everywhere else. To estimate the norm of $\nabla^{t,\lambda}(|f(\mathbf{x})|^2)$, we differentiate (B.4) to find that, in this region,

$$\begin{aligned} d\alpha_{t,\lambda} = {} & \frac{2\min(\lambda,\epsilon)(\pi(\epsilon+\lambda)|f|^2+(\lambda-\epsilon)^2+t^2+|\lambda-\epsilon|\sqrt{\Phi})d|f|\wedge d^c|f|}{\sqrt{\Phi}(\sqrt{\Phi}+\pi|f|^2+|\lambda-\epsilon|)^2} \\ & - \frac{2\pi\min(\lambda,\epsilon)|f|^2 c_1(\mathcal{L})}{(\sqrt{\Phi}+\pi|f|^2+|\lambda-\epsilon|)}. \end{aligned} \tag{B.10}$$

(Here we have used the fact that $dd^c|f|^2 = -4\pi|f|^2c_1(\mathcal{L}) + 4d|f|\wedge d^c|f|$.)

When $\lambda-\epsilon$ and $|f(\mathbf{x})|^2$ are much smaller than ϵ, we have $\Phi\sim 4\pi\epsilon|f|^2+(\lambda-\epsilon)^2+t^2$. Estimating the various terms in (B.10), we find that the second term tends to zero near H, while the leading-order part of the coefficient of $d|f|\wedge d^c|f|$ is bounded from below by $\epsilon/\sqrt{\Phi}$ (and from above by $4\epsilon/\sqrt{\Phi}$). Hence:

$$d\alpha_{t,\lambda} \gtrsim \frac{\epsilon}{\sqrt{\Phi}}d|f|\wedge d^c|f| \tag{B.11}$$

(where $\gtrsim$ means that the inequality holds up to lower-order terms). In more geometric terms, the Kähler metrics induced by $\omega_{t,\lambda}$ blow up in the normal direction to H, by an amount of the order of $\epsilon/\sqrt{\Phi}$, while remaining well-behaved in the other directions.

This implies in turn that the norms of $d(|f(\mathbf{x})|^2)$ and $\nabla^{t,\lambda}(|f(\mathbf{x})|^2)$ with respect to the Kähler metric $\omega_{t,\lambda}$ are bounded by $2(\sqrt{\Phi}/\epsilon)^{1/2}|f(\mathbf{x})|$; and, more importantly, the norm of $\nabla^{t,\lambda}(|f(\mathbf{x})|^2)$ with respect to a suitable fixed auxiliary metric is locally bounded by a constant multiple of $(\sqrt{\Phi}/\epsilon)|f(\mathbf{x})|$. Plugging into (B.9), we conclude that the norm of $v_{t,\lambda}$ (again with respect to a smooth auxiliary metric) is bounded by a constant multiple of $t|f(\mathbf{x})|/\Phi \leq t|f(\mathbf{x})|/(t^2+4\pi\epsilon|f(\mathbf{x})|^2)$, and hence uniformly bounded. Thus, even though $v_{t,\lambda}$ itself is not continuous at $(t,\lambda,|f(\mathbf{x})|)=(0,\epsilon,0)$, its flow is well-defined and continuous even for $\lambda=\epsilon$, and depends continuously on λ.

Geometrically, for $\lambda - \epsilon$ sufficiently small, near H the leading-order term in $v_{t,\lambda}$ points radially away from H, in the same direction as the gradient of $|f(\mathbf{x})|$ with respect to ω_V, and the time t flow rescales the radial coordinate $r = |f(\mathbf{x})|$ in a suitable manner. A complicated explicit formula for the leading-order term of the rescaling can be obtained by comparing the Kähler areas of small discs in the direction normal to H; for example, for $\lambda = \epsilon$ one finds that the time t flow maps points where $|f(\mathbf{x})| = r_0$ to points where $|f(\mathbf{x})|^2 \approx \frac{1}{2} r_0 (r_0 + (r_0^2 + \frac{1}{\pi\epsilon} t^2)^{1/2})$.

Step 4. We now modify the flow constructed in Step 3 in order to arrange for the toric divisors of V to be preserved. We proceed as in Step 2, i.e. we replace the 1-forms $a_{t,\lambda}$ used in Step 3 with $a_{t,\lambda} + df_{t,\lambda}$ for carefully constructed real-valued functions $f_{t,\lambda}$, smooth on V^0 except for $(t, \lambda) = (0, \epsilon)$, such that:

- the support of $f_{t,\lambda}$ is contained in the intersection of U with a small tubular neighborhood of D_V;
- at every point $x \in D_V$, belonging to a toric stratum $S \subset V$,

$$\text{(B.12)} \qquad \text{the 1-form } a_{t,\lambda} + df_{t,\lambda} \text{ vanishes on } (T_x S)^{\perp},$$

  where the orthogonal is with respect to $\omega_{t,\lambda}$;
- where it is smooth, $f_{t,\lambda}$ depends smoothly on t, and piecewise smoothly on λ.

We construct $f_{t,\lambda}$ inductively to satisfy (B.12) on toric strata of increasing dimension, by exactly the same method as in Step 2. The main new difficulty is that we need to control the behavior of $f_{t,\lambda}$ near H for (t, λ) close to $(0, \epsilon)$.

We begin with a geometric digression. Fix a collection of smooth foliations $\mathcal{F}_S$ of neighborhoods of $H \cap S$ in V for all toric strata $S \subset V$, with the following properties:

- each leaf of $\mathcal{F}_S$ intersects S transversely at a single point;
- $|f|$ is constant on the leaves; in particular the leaves through $H \cap S$ are contained in H;
- given two strata $S' \subset S$, the leaves of $\mathcal{F}_{S'}$ are unions of leaves of $\mathcal{F}_S$.
- given two strata S and Σ which intersect transversely along a stratum $S' = S \cap \Sigma$, the leaves of $\mathcal{F}_S$ through S' foliate Σ.

The existence of $\mathcal{F}_S$ with these properties follows from the transversality of H to all toric strata. Indeed, near a k-dimensional stratum S' and away from all lower-dimensional strata, consider a toric chart of the form $(\mathbf{C}^*)^k \times \mathbf{C}^{n-k}$, and modify the first k coordinates (in a C^∞ manner) so that, near H, $|f|$ only depends on these coordinates, without changing the remaining $n - k$ coordinates. Each stratum $S \supset S'$ is then defined by the vanishing of a certain subset of the last $n - k$ coordinates; we choose the leaves of $\mathcal{F}_S$ to be given by letting these coordinates vary and fixing all others. (More globally, start from a collection of toric charts identifying neighborhoods of strata with toric vector bundles over them, and modify the bundle structures compatibly along H so that $|f|$ is constant in the fibers and the strata containing a given one remain given by distinguished subbundles.)

Henceforth, unless stated otherwise, all estimates (on distances, derivatives, etc.) are with respect to a fixed reference metric (independent of t and λ), rather than the metric $g_{t,\lambda}$ determined by $\omega_{t,\lambda}$; and the notation $\mathrm{O}(\cdots)$ means that an inequality holds up to a constant factor which is uniformly bounded independently of t and λ over any compact subset of V.

Recall that $\omega_{t,\lambda}$ blows up (by a factor of the order of $\epsilon/\sqrt{\Phi}$, cf. (B.11)) in the directions transverse to the complex hyperplane field

$$\Theta = \mathrm{Ker}(d|f|) \cap \mathrm{Ker}\big(d^c|f|\big).$$

In what follows, we will often have better estimates on derivatives along Θ than on arbitrary derivatives. We will call derivatives of order (ℓ, m), denoted by $\mathrm{D}^{(\ell,m)}(\cdots)$, the derivatives of order $\ell+m$ along ℓ vector fields tangent to Θ and m arbitrary vector fields. Since the hyperplane distribution Θ is not integrable, estimates on higher derivatives in the direction of Θ only make sense up to lower-order derivatives along the level sets of $|f|$; however, the curvature of Θ is $\mathrm{O}(|f|^2)$, and the estimates we will obtain below on derivatives of order $(\ell+2, m)$ will generally be no better than $\mathrm{O}(|f|^2)$ times the bounds on derivatives of order $(\ell, m+1)$.

Along a stratum S, denote by $\pi^{\mathrm{S}}_{t,\lambda} : \mathrm{TV}_{|\mathrm{S}} \to \mathrm{TS}^\perp$ the orthogonal projection (with respect to $\omega_{t,\lambda}$). Because S is transverse to H, and hence to Θ near H, the behavior of $\omega_{t,\lambda}$ in the directions transverse to Θ implies that, near $\mathrm{H}\cap\mathrm{S}$, the $\omega_{t,\lambda}$-orthogonal to S becomes nearly tangent to Θ for (t,λ) close to $(0,\epsilon)$. Specifically, near $\mathrm{H}\cap\mathrm{S}$, the maximum angle (with respect to a fixed reference metric) between a unit vector in $\mathrm{TS}^\perp$ and Θ is $\mathrm{O}(\epsilon^{-1}\sqrt{\Phi})$. Thus, denoting by $(\pi^{\mathrm{S}}_{t,\lambda})^{\parallel}$ and $(\pi^{\mathrm{S}}_{t,\lambda})^{\perp}$ the components of $\pi^{\mathrm{S}}_{t,\lambda}$ along Θ and its orthogonal for the reference metric, pointwise we have $(\pi^{\mathrm{S}}_{t,\lambda})^{\perp} = \mathrm{O}(\epsilon^{-1}\sqrt{\Phi})$. This in turns implies that

$$\big|d^c\big(|f|^2\big)\circ\pi^{\mathrm{S}}_{t,\lambda}\big| = \mathrm{O}\big(\epsilon^{-1}|f|\sqrt{\Phi}\big).$$

Along the level sets of $|f|$, the coefficient of $d|f|\wedge d^c|f|$ in (B.10) remains constant, and so the geometric behavior of the $\omega_{t,\lambda}$-orthogonals $\mathrm{TS}^\perp$ can be controlled uniformly. In particular, the derivatives along Θ of $(\pi^{\mathrm{S}}_{t,\lambda})^{\perp}$ are bounded by $\mathrm{O}(\sqrt{\Phi})$ to all orders. On the other hand, the variation of (B.10) in the directions transverse to the level sets of $|f|$ implies that each derivative in those directions worsens the bounds by a factor of $1/\sqrt{\Phi}$. We conclude that $\mathrm{D}^{(\ell,m)}((\pi^{\mathrm{S}}_{t,\lambda})^{\perp}) = \mathrm{O}(\Phi^{(1-m)/2})$. Meanwhile, by a similar reasoning, $\mathrm{D}^{(\ell,m)}((\pi^{\mathrm{S}}_{t,\lambda})^{\parallel}) = \mathrm{O}(\Phi^{-m/2})$.

These estimates on $\pi^{\mathrm{S}}_{t,\lambda}$ (and the inequality $|f| \le (\Phi/4\pi\epsilon)^{1/2}$) in turn imply that

$$\mathrm{D}^{(\ell,m)}\big(d^c\big(|f|^2\big)\circ\pi^{\mathrm{S}}_{t,\lambda}\big) = \mathrm{O}\big(\Phi^{(2-m)/2}\big).$$

Thus, the 1-form $a_{t,\lambda}$ from Step 3 satisfies

$$\begin{aligned}\left|a_{t,\lambda}\circ\pi^{\mathrm{S}}_{t,\lambda}\right| &= \frac{t\tilde\chi\min(\lambda,\epsilon)|d^c(|f|^2)\circ\pi^{\mathrm{S}}_{t,\lambda}|}{2\sqrt{\Phi}(\sqrt{\Phi}+\pi|f|^2+|\lambda-\epsilon|)^2}\\ &= \mathrm{O}\left(\frac{t|f|}{\Phi}\right)=\mathrm{O}\left(\frac{t}{\sqrt{\Phi}}\right)\quad\text{and}\\ \mathrm{D}^{(\ell,m)}\left(a_{t,\lambda}\circ\pi^{\mathrm{S}}_{t,\lambda}\right) &= \mathrm{O}\left(\frac{t}{\Phi^{(m+1)/2}}\right).\end{aligned}$$

We now return to our main construction. Starting with $f_{\lambda,t,\le 0}=0$ as before, assume that we have already constructed $f_{\lambda,t,\le k}$, supported in a neighborhood of the intersection of H with the toric strata of dimension $\le k$, in such a way that (B.12) holds for all strata of dimension $\le k$. We further require that, away from all strata of dimension $\le k-1$, resp. near a stratum S′ of dimension $\le k-1$ (and assuming S′ is the closest such stratum),

$$\mathrm{D}^{(\ell,m)}(f_{t,\lambda,\le k})=\mathrm{O}\left(\frac{t}{\Phi^{(m+1)/2}}\right),\quad \text{resp. }\mathrm{O}\left(\frac{t}{\Phi^{(m+1)/2}}\operatorname{dist}_{\mathrm{S}'}^{\min(0,2-\ell-m)}\right),\tag{B.13}$$

where $\operatorname{dist}_{\mathrm{S}'}$ is the distance to S′ with respect to the fixed reference metric.

Let S be a stratum of dimension $k+1$. The above estimates on the derivatives of $\pi^{\mathrm{S}}_{t,\lambda}$, together with (B.13), imply that at any point of S which lies away from the strata of dimension $\le k-1$, resp. near (and closest to) such a stratum S′,

$$\begin{aligned}\mathrm{D}^{(\ell,m)}\left((a_{t,\lambda}+df_{t,\lambda,\le k})\circ\pi^{\mathrm{S}}_{t,\lambda}\right)&=\mathrm{O}\left(\frac{t}{\Phi^{(m+1)/2}}\right),\\ &\text{resp. }\mathrm{O}\left(\frac{t}{\Phi^{(m+1)/2}}\operatorname{dist}_{\mathrm{S}'}^{\min(0,1-\ell-m)}\right).\end{aligned}\tag{B.14}$$

(Note that, while the quantity in (B.14) involves an additional derivative of $f_{t,\lambda,\le k}$, the extra factor of $\Phi^{-1/2}$ when this derivative is taken in a direction transverse to Θ is offset by the factor of $\Phi^{1/2}$ in the estimates for the transverse component of $\pi^{\mathrm{S}}_{t,\lambda}$.)

Near a stratum $\mathrm{S}'\subset\mathrm{S}$ with $\dim\mathrm{S}'\le k$, condition (B.12) for $f_{t,\lambda,\le k}$ along S′ implies that $(a_{t,\lambda}+df_{t,\lambda,\le k})\circ\pi^{\mathrm{S}}_{t,\lambda}$ vanishes along S′. Since Θ is transverse to S′, (B.14) for $(\ell,m)=(1,0)$ in turn implies that, at all points of S which lie near S′,

$$|(a_{t,\lambda}+df_{t,\lambda,\le k})\circ\pi^{\mathrm{S}}_{t,\lambda}|=\mathrm{O}\left(\frac{t\operatorname{dist}_{\mathrm{S}'}}{\sqrt{\Phi}}\right).\tag{B.15}$$

Meanwhile, since the distance to the nearest k-dimensional stratum is no greater than the distance to the nearest lower-dimensional stratum, the bounds in the second part of

(B.14) also hold when $\dim S' = k$. Hence, at any point of S which lies near (and closest) to a stratum $S' \subset S$ of dimension $\leq k$,

$$\mathrm{D}^{(\ell,m)}\big((a_{t,\lambda} + df_{t,\lambda,\leq k}) \circ \pi^{S}_{t,\lambda}\big) = \mathrm{O}\bigg(\frac{t}{\Phi^{(m+1)/2}} \operatorname{dist}_{S'}^{1-\ell-m}\bigg). \tag{B.16}$$

Define a function $f^0_{\lambda,t,S}$ on a neighborhood of the given $(k+1)$-dimensional stratum S, smooth outside of the leaves of $\mathcal{F}_S$ through strata of dimension $\leq k-1$ (and H if $(\lambda, t) = (\epsilon, 0)$), which vanishes on S and whose derivative at each point of S satisfies

$$df^0_{\lambda,t,S} = -(a_{t,\lambda} + df_{\lambda,t,\leq k}) \circ \pi^{S}_{t,\lambda}. \tag{B.17}$$

Specifically, we identify the leaves of $\mathcal{F}_S$ with open subsets in the fibers of the normal bundle to S, and take $f^0_{\lambda,t,S}$ to be linear in the fibers. We then define $f_{\lambda,t,S} = \chi_S f^0_{\lambda,t,S}$, where χ_S is the same cut-off function as in Step 2.

By construction, $f^0_{t,\lambda,S} = \mathrm{O}(t \operatorname{dist}_S / \sqrt{\Phi})$. Moreover, using (B.15), along the leaf of $\mathcal{F}_S$ through a point $x \in S$ which lies near a lower-dimensional stratum S' we have $f^0_{t,\lambda,S} = \mathrm{O}(t \operatorname{dist}_{S'}(x) \operatorname{dist}_S / \sqrt{\Phi})$.

The derivative of $f^0_{\lambda,t,S}$ along the leaves of $\mathcal{F}_S$ is the constant extension of (B.17) along $\mathcal{F}_S$; whereas its derivative in the directions transverse to $\mathcal{F}_S$ is a cross-term which grows linearly with distance to S and involves the dependence of (B.17) on the point of S. Moreover, the leaves of $\mathcal{F}_S$ are tangent to the level sets of $|f|$ near H, and hence nearly tangent to Θ: the maximum angle between vectors in $T\mathcal{F}_S$ and Θ is $\mathrm{O}(|f|)$. It then follows from (B.14) that, away from $(k-1)$-dimensional strata,

$$\mathrm{D}^{(\ell,m)}\big(f^0_{\lambda,t,S}\big) = \mathrm{O}\bigg(\frac{t}{\Phi^{(m+1)/2}}\bigg). \tag{B.18}$$

Meanwhile, along the leaf of $\mathcal{F}_S$ through a point $x \in S$ which lies near (and closest to) a stratum $S' \subset S$ with $\dim S' \leq k$, (B.16) implies that

$$\mathrm{D}^{(\ell,m)}\big(f^0_{\lambda,t,S}\big) = \mathrm{O}\bigg(\frac{t}{\Phi^{(m+1)/2}}\big(\operatorname{dist}_{S'}(x)^{2-\ell-m} + \operatorname{dist}_{S'}(x)^{1-\ell-m} \operatorname{dist}_S(\cdot)\big)\bigg).$$

The leaf of $\mathcal{F}_S$ through x locally stays close to a leaf through S', which by construction is contained in some other stratum of D_V. In particular, as soon as the distance to S is sufficiently large compared to $\operatorname{dist}_{S'}(x)$, points on the leaf through x lie closer to some other stratum Σ of dimension $\geq k+1$ (and not containing S) than to S, and so the cut-off function χ_S vanishes identically. Thus, over the support of χ_S, $\operatorname{dist}_{S'}(\cdot)$ and $\operatorname{dist}_{S'}(x)$ are within bounded factors of each other. Since $\operatorname{dist}_S \leq \operatorname{dist}_{S'}$, we conclude that, at all points of the support of χ_S which lie near (and closest to) S',

$$\mathrm{D}^{(\ell,m)}\big(f^0_{\lambda,t,S}\big) = \mathrm{O}\bigg(\frac{t}{\Phi^{(m+1)/2}} \operatorname{dist}_{S'}^{2-\ell-m}\bigg). \tag{B.19}$$

Now we observe that the derivatives of the cut-off function χ_S are $O(1)$ away from strata of dimension $\leq k$, and near a stratum $S' \subset S$ of dimension $\leq k$ the derivatives of order r are $O(1/\operatorname{dist}^r_{S'})$. Thus, (B.18) and (B.19) imply that away from k-dimensional strata, resp. near (and closest to) $S' \subset S$ with $\dim S' \leq k$,

$$\textbf{(B.20)} \qquad D^{(\ell,m)}(f_{\lambda,t,S}) = O\left(\frac{t}{\Phi^{(m+1)/2}}\right), \quad \text{resp. } O\left(\frac{t}{\Phi^{(m+1)/2}} \operatorname{dist}_{S'}^{2-\ell-m}\right).$$

We now set

$$f_{t,\lambda,\leq k+1} = f_{t,\lambda,\leq k} + \sum_{\dim S = k+1} f_{t,\lambda,S}.$$

By construction, $f_{t,\lambda,\leq k+1}$ is supported in a neighborhood of the intersection of H with the strata of dimension at most $k+1$, and satisfies (B.12) for all strata of dimension $\leq k+1$. Indeed, by (B.20), $df_{t,\lambda,S}$ vanishes along strata of dimension $\leq k$, so (B.12) continues to hold for those; whereas, over the interior of the stratum S, $df_{t,\lambda,S} = df^0_{t,\lambda,S}$, and the contributions from other $(k+1)$-dimensional strata vanish.

Moreover, $f_{t,\lambda,\leq k+1}$ satisfies the estimate (B.13) (with $k+1$ instead of k), as needed for the induction to proceed. Indeed, this follows immediately from the estimates (B.13) for $f_{t,\lambda,\leq k}$ (note that the second estimate also holds near k-dimensional strata, since the distance to the nearest k-dimensional stratum is no greater than that to the nearest lower-dimensional stratum), and (B.20) for $f_{t,\lambda,S}$.

Thus, we can indeed carry out the construction of $f_{t,\lambda,\leq k}$ with the desired properties by induction on k. Finally, we let $f_{t,\lambda} = f_{t,\lambda,\leq n-1}$.

As a consequence of the estimates (B.20) on individual terms, $f_{t,\lambda}$ is C^1 with locally Lipschitz first derivatives, and smooth on V^0, except along H for $(t,\lambda) = (0,\epsilon)$. By construction, it is supported in the intersection of U with a neighborhood of D_V, and satisfies (B.12) for all toric strata.

By (B.13), $|df_{t,\lambda}| = O(t/\Phi)$, while $|df_{t,\lambda|\Theta}| = O(t/\sqrt{\Phi})$.

Because the Kähler form $\omega_{t,\lambda}$ blows up like $\epsilon/\sqrt{\Phi}$ in the directions transverse to Θ, we conclude that the Hamiltonian vector field of $f_{t,\lambda}$ with respect to $\omega_{t,\lambda}$ is bounded by $O(t/\sqrt{\Phi})$ (again with respect to the fixed reference metric), hence locally uniformly bounded. (Recall that $\sqrt{\Phi} \geq t$ wherever $\tilde{\chi} \equiv 1$, while the other terms are bounded below wherever $\tilde{\chi} < 1$.) Moreover, the regularity of $f_{t,\lambda}$ implies that this vector field is locally Lipschitz continuous, and smooth on V^0, except along H for $(t,\lambda) = (0,\epsilon)$.

Combining this with the outcome of Step 3, we find that the vector field $\tilde{v}_{t,\lambda}$ defined by $\iota_{\tilde{v}_{t,\lambda}}\omega'_{t,\lambda} = -a_{t,\lambda} - df_{t,\lambda}$ is smooth on V^0 (and locally Lipschitz continuous along D_V), except along H for $(t,\lambda) = (0,\epsilon)$, and its norm (again with respect to a smooth reference metric) is bounded by $O(t/\sqrt{\Phi})$, hence locally uniformly bounded. Thus, even though $\tilde{v}_{t,\lambda}$ is not continuous along H for $(t,\lambda) = (0,\epsilon)$, its flow is well-defined and continuous even for $\lambda = \epsilon$. We then obtain $\phi_{sm,\lambda}$ with all the desired properties by considering the time κ flow generated by $\tilde{v}_{t,\lambda}$. $\square$

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